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ORIGINAL PAPER

Yi Zhang

Adiabatic invariants and Lie symmetries on time scales for nonholonomic systems of non-Chetaev type

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Abstract Time-scale dynamics integrates the differential equations of continuous systems and the difference equations of discrete systems. It can not only reveal the similarities and differences between continuous and discrete systems, but also describe the physical nature of continuous and discrete systems and other complex dynamical systems more clearly and accurately. Therefore, it has been widely used in many fields of science and engineering in recent years. In this paper, we investigate Lie symmetries and invariants of nonholonomic systems of non-Chetaev type on time scales. First, we present and prove the Lie symmetry theorem for undisturbed nonholonomic systems of non-Chetaev type on time scales. The study shows that if the Lie symmetry satisfies the structural equation, it will lead to the conserved quantity, which is the exact invariant of the system. Secondly, considering that the system is subjected to small disturbance, we present and prove the adiabatic invariant theorem of Lie symmetry for nonholonomic systems of non-Chetaev type on time scales. Due to the arbitrariness of the time scale, the method and results of this paper are of universal significance. An example is given to illustrate the validity of the results.

1 Introduction

Symmetry is a very important and universal property of dynamical systems. The invariants of dynamical systems are intrinsically related to the symmetries. For a complex dynamical system, one of the effective ways to find invariants is to study its symmetries. Lie symmetry is the invariance of differential equations under an infinitesimal transformation [1]. Since Lutzky [2] introduced the Lie method into dynamical systems and established the relationship between the invariance of the differential equations of motion under infinitesimal transformations and the invariants of the systems, important progress has been made in the study of Lie symmetry of constrained mechanical systems [3–9].

Under the action of small disturbance, the change in symmetry and its invariant are closely related to the integrability of dynamical systems. Therefore, it is important to study the perturbation of symmetries and adiabatic invariants of the systems. The classical adiabatic invariant refers to a physical quantity that changes slower compared to the slow change in a parameter of the system [10]. In fact, slow change in parameters is equivalent to small disturbance. Some results have been presented in the studies on the perturbation of symmetries and adiabatic invariants of constrained mechanical systems [11–16].

The time scale is any nonempty closed subset of the real number set. Time-scale calculus integrates continuous analysis, discrete analysis and quantum analysis into a whole, and provides a powerful mathematical tool for the study of complex dynamical systems [17–20]. Bartosiewicz and Torres [21] first studied and proved the Noether theorem on time scales. In recent years, the study of Noether theorems and integral methods of

Y. Zhang (✉)
College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, Jiangsu, People's Republic of China
E-mail: zhy@mail.usts.edu.cn

constrained mechanical systems on time scales has attracted great attention, including Lagrange systems, Hamilton systems, nonholonomic systems, Birkhoff systems, etc., and some results have been obtained [22–29]. Recently, we proposed and studied Lie symmetries of Lagrange systems and Hamilton systems on time scales [30]. However, up to now, there are few studies on Lie symmetries of nonholonomic systems on time scales, and perturbation of Lie symmetries under small disturbance and adiabatic invariants. This is the motivation of the research to be carried out in this paper.

2 Differential equations of motion for nonholonomic systems of non-Chetaev type on time scales

On time scales, the d'Alembert–Lagrange principle can be expressed as [31]

$$\left(Q_s - \frac{\Delta}{\Delta t}\frac{\partial T}{\partial q_s^{\Delta}} + \frac{\partial T}{\partial q_s^{\sigma}}\right)\delta q_s^{\sigma} = 0 \tag{1}$$

where $T = T\left(t, q_s^{\sigma}(t), q_s^{\Delta}(t)\right)$ is the kinetic energy, $Q_s = Q_s\left(t, q_k^{\sigma}(t), q_k^{\Delta}(t)\right)$ are the generalized forces, and q_s $(s = 1, 2, \ldots, n)$ are the generalized coordinates. The time scales calculus and its basic properties involved here and later can be found in [17].

Let the system be subjected to g ideal two-sided nonholonomic constraints of non-Chetaev type

$$f_\beta = f_\beta\left(t, q_s^{\sigma}(t), q_s^{\Delta}(t)\right) = 0, \quad (\beta = 1, 2, \ldots, g; s = 1, 2, \ldots, n). \tag{2}$$

The restriction conditions imposed on the virtual displacements by the nonholonomic constraints (2) are

$$f_{\beta s}\left(t, q_k^{\sigma}(t), q_k^{\Delta}(t)\right)\delta q_s^{\sigma} = 0, \quad (\beta = 1, 2, \ldots, g; s, k = 1, 2, \ldots, n), \tag{3}$$

Generally speaking, there is no relation between $f_{\beta s}$ and $\partial f_\beta / \partial q_s^{\Delta}$. If we take $f_{\beta s} = \left(\partial f_\beta / \partial q_s^{\Delta}\right)^{\sigma}$, then non-Chetaev constraints become Chetaev constraints [31].

According to the principle (1) and the restriction conditions (3), we can easily obtain

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^{\Delta}} - \frac{\partial L}{\partial q_s^{\sigma}} = Q_s'' + \lambda_\beta f_{\beta s}, \quad (s = 1, 2, \ldots, n) \tag{4}$$

by using the Lagrange multiplier method. Equations (4) are called the differential equations of motion for nonholonomic systems of non-Chetaev type on time scales. Here, $L = T - V$ is the Lagrangian, $Q_s'' = Q_s''\left(t, q_k^{\sigma}(t), q_k^{\Delta}(t)\right)$ the non-potential generalized forces, λ_β are the constraint multipliers. Suppose that the system is non-singular, i.e., $D = \det\left(A_{sk}\right) = \det\left(\partial^2 L / \left(\partial q_s^{\Delta} \Delta q_k^{\Delta}\right)\right) \neq 0$, from Eqs. (2) and (4), we can find $\lambda_\beta = \lambda_\beta\left(t, q_k^{\sigma}(t), q_k^{\Delta}(t)\right)$. Thus, Eq. (4) can be expressed as

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^{\Delta}} - \frac{\partial L}{\partial q_s^{\sigma}} = Q_s'' + \Lambda_s, \quad (s = 1, 2, \ldots, n), \tag{5}$$

where $\Lambda_s = \Lambda_s\left(t, q_k^{\sigma}(t), q_k^{\Delta}(t)\right) = \lambda_\beta f_{\beta s}$. Equation (5) are called the equations of the corresponding holonomic system of the nonholonomic system (2) and (4) on time scales. If the initial values of generalized coordinates q_{s0} and generalized velocities q_{s0}^{Δ} on time scales satisfy the constraint equations (2), i.e.,

$$f_\beta\left(t_0, q_{s0}^{\sigma}, q_{s0}^{\Delta}\right) = 0, \quad (\beta = 1, 2, \ldots, g; s = 1, 2, \ldots, n), \tag{6}$$

then the solution of Eq. (5) gives the motion of the nonholonomic system (2) and (4). Expanding Eq. (5), we can find all the generalized accelerations, that is

$$q_s^{\Delta\Delta} = h_s\left(t, q_k^{\sigma}, q_k^{\Delta}\right), \quad (s = 1, 2, \ldots, n). \tag{7}$$

If the system is subjected to the small disturbance forces υF_s, then Eq. (5) becomes

$$\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^{\Delta}} - \frac{\partial L}{\partial q_s^{\sigma}} = Q_s'' + \Lambda_s + \upsilon F_s, \quad (s = 1, 2, \ldots, n), \tag{8}$$

where υ is a small parameter. Equation (8) is the differential equations of disturbed motion for the nonholonomic system of non-Chetaev type on time scales. Expanding Eq. (8), we get

$$q_s^{\Delta\Delta} = h_s\left(t, q_k^{\sigma}, q_k^{\Delta}\right) + \upsilon \frac{M_{sk}}{D} F_k, \tag{9}$$

where M_{sk} is the cofactor of the element A_{sk} of determinant D.

3 Lie symmetries of nonholonomic systems of non-Chetaev on time scales

Let us consider a one-parameter Lie group of infinitesimal transformations as follows:

$$\bar{t} = t + \varepsilon\tau\left(t, q_k\right), \bar{q}_s\left(\bar{t}\right) = q_s\left(t\right) + \varepsilon\xi_s\left(t, q_k\right), \quad \left(s = 1, 2, \ldots, n\right), \tag{10}$$

where ε is an infinitesimal parameter, τ and ξ_s are the infinitesimal generators.

The criterion equations of Lie symmetry of Eq. (7) are

$$\xi_s^{\Delta\Delta} - 2h_s\tau^{\Delta} - q_s^{\Delta\sigma}\tau^{\Delta\Delta} = X^{(1)}\left(h_s\right), \quad \left(s = 1, 2, \ldots, n\right), \tag{11}$$

where [30]

$$\begin{aligned} X^{(0)} &= \tau\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k}, \\ X^{(1)} &= \tau\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k} + \left(\xi_k^{\Delta} - q_k^{\Delta}\tau^{\Delta}\right)\frac{\partial}{\partial q_k^{\Delta}}. \end{aligned} \tag{12}$$

The invariance of nonholonomic constraints (2) under the infinitesimal transformations (10) is reduced to the restriction equations of Lie symmetry as follows:

$$X^{(1)}\left[f_\beta\left(t, q_s^{\sigma}, q_s^{\Delta}\right)\right] = 0, \quad \left(\beta = 1, 2, \ldots, g\right). \tag{13}$$

Substituting $\delta q_s^{\sigma} = \varepsilon\left(\xi_s^{\sigma} - q_s^{\Delta\sigma}\tau^{\sigma}\right)$ into conditions (3), we get

$$f_{\beta s}\left(\xi_s^{\sigma} - q_s^{\Delta\sigma}\tau^{\sigma}\right) = 0, \quad \left(\beta = 1, 2, \ldots, g\right). \tag{14}$$

Equation (14) are the additional restriction equations of non-Chetaev nonholonomic constraint (2) on infinitesimal generators. So, we have

Definition 1 If the infinitesimal generators τ and ξ_s satisfy the criterion equation (11) and the restriction equation (13), as well as the additional restriction equation (14), then the invariance is called the Lie symmetry of the nonholonomic mechanical system (2) and (4) of non-Chetaev type on time scales.

When the system is subjected to small disturbance forces υF_s, the original Lie symmetry will be changed. It is assumed that the infinitesimal generators of the disturbed system are the small perturbation on the basis of the generators of the undisturbed system. For convenience, the generators of the undisturbed system are denoted as τ^0 and ξ_s^0, and the disturbed generators are denoted as τ and ξ_s; then, we get

$$\tau = \tau^0 + \upsilon\tau^1 + \upsilon^2\tau^2 + \cdots, \quad \xi_s = \xi_s^0 + \upsilon\xi_s^1 + \upsilon^2\xi_s^2 + \cdots \tag{15}$$

The disturbed infinitesimal generator vector $X^{(0)}$ and its first expansion $X^{(1)}$ are

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k} = \upsilon^m\left(\tau^m\frac{\partial}{\partial t} + \xi_k^m\frac{\partial}{\partial q_k}\right) = \upsilon^m X_m^{(0)}, \tag{16}$$

$$\begin{aligned} X^{(1)} &= \tau\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k} + \left(\xi_k^{\Delta} - q_k^{\Delta}\tau^{\Delta}\right)\frac{\partial}{\partial q_k^{\Delta}} \\ &= \upsilon^m\left[\tau^m\frac{\partial}{\partial t} + \xi_k^m\frac{\partial}{\partial q_k} + \left(\xi_k^{m\Delta} - q_k^{\Delta}\tau^{m\Delta}\right)\frac{\partial}{\partial q_k^{\Delta}}\right] = \upsilon^m X_m^{(1)}. \end{aligned} \tag{17}$$

The criterion equations of Lie symmetry of Eq. (9) are

$$\xi_s^{\Delta\Delta} - 2h_s\tau^{\Delta} - 2\upsilon\frac{M_{sk}}{D}F_k\tau^{\Delta} - q_s^{\Delta\sigma}\tau^{\Delta\Delta} = X^{(1)}\left(h_s\right) + \upsilon X^{(1)}\left(\frac{M_{sk}}{D}F_k\right), \quad \left(s = 1, 2, \ldots, n\right). \tag{18}$$

Substituting Eqs. (15) and (17) into Eqs. (13), (14) and (18), and making the coefficients of υ^m equal to each other, we get

$$\xi_s^{m\Delta\Delta} - 2h_s\tau^{m\Delta} - 2\frac{M_{sk}}{D}F_k\left(\tau^{m-1}\right)^{\Delta} - q_s^{\Delta\sigma}\tau^{m\Delta\Delta} = X_m^{(1)}\left(h_s\right) + X_{m-1}^{(1)}\left(\frac{M_{sk}}{D}F_k\right),$$
$$(s = 1, 2, \ldots, n; \quad m = 0, 1, 2, \ldots), \tag{19}$$
$$X_m^{(1)}\left[f_\beta\left(t, q_s^\sigma, q_s^\Delta\right)\right] = 0, \quad (\beta = 1, 2, \ldots, g), \tag{20}$$
$$f_{\beta s}\left(\xi_s^{m\sigma} - q_s^{\Delta\sigma}\tau^{m\sigma}\right) = 0, \quad (\beta = 1, 2, \ldots, g). \tag{21}$$

When $m = 0$, we specify that $\tau^{-1} = \xi_s^{-1} = 0$. Equations (19–21) are the criterion equation and the restriction equation, and additional restriction equation, respectively, of the disturbed nonholonomic system of non-Chetaev type on time scales.

Definition 2 If the infinitesimal generators τ^m and ξ_s^m satisfy the criterion equation (19) and the restriction equation (20), as well as the additional restriction equation (21), then the invariance is called the Lie symmetry of the disturbed nonholonomic mechanical system (2) and (8) of non-Chetaev type on time scales.

4 Lie symmetries and exact invariants of nonholonomic systems of non-Chetaev type on time scales

Lie symmetries can lead to conserved quantities under certain conditions. The following theorem gives the condition under which the Lie symmetry of nonholonomic mechanical system of non-Chetaev type leads to a conserved quantity on time scales, and the form of the conserved quantity.

Theorem 1 *For the undisturbed nonholonomic system* (2) *and* (4) *of non-Chetaev type on time scales, if the infinitesimal transformation* (10) *corresponds to the Lie symmetry of the system, and there is a gauge function* $G = G\left(t, q_s^\sigma, q_s^\Delta\right)$ *that satisfies the following structural equation:*

$$L\tau^{\Delta} + X^{(1)}(L) + \mu(t)\frac{\partial L}{\partial q_k^\sigma}q_k^\Delta\tau^\Delta + \left(Q_k'' + \Lambda_k\right)\left(\xi_k - q_k^\Delta\tau\right)^\sigma + G^\Delta = 0, \tag{22}$$

then the Lie symmetry of the system results in the following conserved quantity:

$$I = \frac{\partial L}{\partial q_k^\Delta}\xi_k + \left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau - \mu(t)\frac{\partial L}{\partial t}\tau + G = \text{const.} \tag{23}$$

Proof Due to

$$\frac{\Delta}{\Delta t}I = \frac{\partial L}{\partial q_k^\Delta}\xi_k^\Delta + \frac{\Delta}{\Delta t}\left(\frac{\partial L}{\partial q_k^\Delta}\right)\xi_k^\sigma + \left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau^\Delta + \frac{\Delta}{\Delta t}\left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau^\sigma$$
$$-\mu(t)\frac{\partial L}{\partial t}\tau^\Delta - \frac{\Delta}{\Delta t}\left(\mu(t)\frac{\partial L}{\partial t}\right)\tau^\sigma + G^\Delta, \tag{24}$$

similar to the proof of the second Euler–Lagrange equations of the Lagrange systems on time scales in the reference [22], for Eq. (5) of the nonholonomic system on time scales, the following relation can be obtained easily:

$$\frac{\Delta}{\Delta t}\left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta - \mu(t)\frac{\partial L}{\partial t}\right) = \frac{\partial L}{\partial t} - \left(Q_k'' + \Lambda_k\right)q_k^{\Delta\sigma}. \tag{25}$$

Equation (25) is actually the energy equation of the nonholonomic mechanical system on time scales. □

According to the formula (12), we get

$$X^{(1)}(L) = X^{(1)}\left[L\left(t, q_k^\sigma, q_k^\Delta\right)\right] = \tau\frac{\partial L}{\partial t} + \xi_k\frac{\partial L}{\partial q_k^\sigma} + \left(\xi_k^\Delta - q_k^\Delta\tau^\Delta\right)\left(\frac{\partial L}{\partial q_k^\Delta} + \mu(t)\frac{\partial L}{\partial q_k^\sigma}\right). \tag{26}$$

Substituting Eq. (25) and the structural equation (22) into the formula (24), and considering the relation (26) and Eq. (5), we have

$$\begin{aligned}\frac{\Delta}{\Delta t}I &= \frac{\partial L}{\partial q_k^\Delta}\xi_k^\Delta + \frac{\Delta}{\Delta t}\left(\frac{\partial L}{\partial q_k^\Delta}\right)\xi_k^\sigma - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\tau^\Delta + \frac{\partial L}{\partial t}\tau^\sigma - \mu(t)\frac{\partial L}{\partial t}\tau^\Delta \\ &\quad - X^{(1)}(L) - \mu(t)\frac{\partial L}{\partial q_k^\sigma}q_k^\Delta\tau^\Delta - \left(Q_k'' + \Lambda_k\right)\xi_k^\sigma \\ &= \left[\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_k^\Delta} - \frac{\partial L}{\partial q_k^\sigma} - Q_k'' - \Lambda_k\right]\xi_k^\sigma = 0. \end{aligned} \tag{27}$$

Therefore, the formula (23) is the conserved quantity of the system, and the theorem is proved.

Theorem 1 can be called the Lie symmetry theorem of undisturbed nonholonomic systems of non-Chetaev type on time scales. Since the system is not disturbed, the conserved quantity (23) is an exact invariant. The theorem reveals the relationship between Lie symmetries and invariants when the system is undisturbed.

If we take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, and the criterion equation (11) and the restriction equation (13) and the additional restriction equation (14) of Lie symmetry on time scales become

$$\ddot{\xi}_s - 2h_s\dot{\tau} - \dot{q}_s\ddot{\tau} = X^{(1)}(h_s), \tag{28}$$

$$X^{(1)}\left(f_\beta(t, q_s, \dot{q}_s)\right) = 0, \tag{29}$$

$$f_{\beta s}(\xi_s - \dot{q}_s\tau) = 0. \tag{30}$$

In this case, Theorem 1 gives

Theorem 2 *For the undisturbed nonholonomic system of non-Chetaev type on time scales, if the generators of infinitesimal transformations satisfy the criterion equation (28) and the restriction equation (29), as well as the additional restriction equation (30), and there is a gauge function $G = G(t, q_s, \dot{q}_s)$ that satisfies the following structural equation:*

$$L\dot{\tau} + X^{(1)}(L) + \left(Q_k'' + \Lambda_k\right)(\xi_k - \dot{q}_k\tau) + \dot{G} = 0, \tag{31}$$

then the Lie symmetry of the system results in the following conserved quantity:

$$I = \frac{\partial L}{\partial \dot{q}_k}\xi_k + \left(L - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)\tau + G = \text{const}. \tag{32}$$

Theorem 2 is given in the reference [6].

If we take $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, and the criterion equation (11) and the restriction equation (13) and the additional restriction equation (14) of Lie symmetry on time scales become

$$\Delta^2\xi_s - 2h_s\Delta\tau - \Delta q_s(t+1)\Delta^2\tau = X^{(1)}(h_s), \tag{33}$$

$$X^{(1)}\left[f_\beta(t, q_s(t+1), \Delta q_s(t))\right] = 0, \tag{34}$$

$$f_{\beta s}\left[\xi_s(t+1) - \Delta q_s(t+1)\tau(t+1)\right] = 0. \tag{35}$$

In this case, Theorem 1 gives

Theorem 3 *For the undisturbed discrete nonholonomic system of non-Chetaev type on time scales, if the generators of infinitesimal transformations satisfy the criterion equation* (33) *and the restriction equation* (34)*, as well as the additional restriction equation* (35)*, and there is a gauge function* $G = G\left(t, q_s\left(t+1\right), \Delta q_s\left(t\right)\right)$ *that satisfies the following structural equation:*

$$L\Delta\tau + X^{(1)}\left(L\right) + \frac{\partial L}{\partial q_k\left(t+1\right)}\Delta q_k \Delta\tau + \left(Q_k''\Lambda_k\right)\left(\xi_k\left(t+1\right) - \Delta q_k\left(t+1\right)\tau\left(t+1\right)\right) + \Delta G = 0, \quad (36)$$

then the Lie symmetry of the system results in the following conserved quantity:

$$I = \frac{\partial L}{\partial \Delta q_k}\xi_k + \left(L - \frac{\partial L}{\partial \Delta q_k}\Delta q_k\right)\tau - \frac{\partial L}{\partial t}\tau + G = \text{const.} \quad (37)$$

In Theorem 3, the relation between Lie symmetries and exact invariants for the discrete nonholonomic system of non-Chetaev type is established.

5 Lie symmetries and adiabatic invariants of nonholonomic systems of non-Chetaev type on time scales

The classical adiabatic invariant refers to a physical quantity that changes more slowly relative to the slow change in a parameter of the system. So the adiabatic invariant is an approximate invariant. The following theorem gives the conditions for adiabatic invariants resulting from Lie symmetries of disturbed nonholonomic systems of non-Chetaev type on time scales, and the form of adiabatic invariants.

Theorem 4 *For the disturbed nonholonomic system* (2) *and* (8) *of non-Chetaev type on time scales, if the infinitesimal transformation* (10) *corresponds to the Lie symmetry of the system, and there are gauge functions* $G_m = G_m\left(t, q_s^\sigma, q_s^\Delta\right)$ *that satisfy the following structural equations:*

$$\begin{aligned} &L\tau^{m\Delta} + X_m^{(1)}\left(L\right) + \mu\left(t\right)\frac{\partial L}{\partial q_k^\sigma}q_k^\Delta\tau^{m\Delta} + \left(Q_k'' + \Lambda_k\right)\left(\xi_k^m - q_k^\Delta\tau^m\right)^\sigma \\ &+ F_k\left(\xi_k^{m-1} - q_k^\Delta\tau^{m-1}\right)^\sigma + G_m^\Delta = 0, \quad \left(m = 0, 1, 2, \ldots\right), \end{aligned} \quad (38)$$

then

$$I_z = \sum_{m=0}^{z}\upsilon^m\left[\frac{\partial L}{\partial q_k^\Delta}\xi_k^m + \left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau^m - \mu\left(t\right)\frac{\partial L}{\partial t}\tau^m + G_m\right] \quad (39)$$

is a z-th adiabatic invariant of the nonholonomic system on time scales.

Proof Take the delta derivative of formula (39) with respect to time t, and we get

$$\begin{aligned} \frac{\Delta}{\Delta t}I_z = \sum_{m=0}^{z}\upsilon^m\Bigg[&\frac{\partial L}{\partial q_k^\Delta}\xi_k^{m\Delta} + \frac{\Delta}{\Delta t}\left(\frac{\partial L}{\partial q_k^\Delta}\right)\xi_k^{m\sigma} + \left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau^{m\Delta} + \frac{\Delta}{\Delta t}\left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta\right)\tau^{m\sigma} \\ &- \mu\left(t\right)\frac{\partial L}{\partial t}\tau^{m\Delta} - \frac{\Delta}{\Delta t}\left(\mu\left(t\right)\frac{\partial L}{\partial t}\right)\tau^{m\sigma} + \frac{\Delta}{\Delta t}G_m\Bigg]. \end{aligned} \quad (40)$$

For the disturbed system (8), the energy equation (25) can be extended as

$$\frac{\Delta}{\Delta t}\left(L - \frac{\partial L}{\partial q_k^\Delta}q_k^\Delta - \mu\left(t\right)\frac{\partial L}{\partial t}\right) = \frac{\partial L}{\partial t} - \left(Q_k'' + \Lambda_k\right)q_k^{\Delta\sigma} - \upsilon F_k q_k^{\Delta\sigma}. \quad (41)$$

By substituting Eqs. (38) and (41) into the formula (40), and considering Eq. (8), we get

$$\begin{aligned} \frac{\Delta}{\Delta t}I_z &= \sum_{m=0}^{z}\upsilon^m\left[\upsilon F_k\left(\xi_k^{m\sigma} - q_k^{\Delta\sigma}\tau^{m\sigma}\right) - F_k\left(\xi_k^{m-1} - q_k^\Delta\tau^{m-1}\right)^\sigma\right] \\ &= \upsilon^{z+1}F_k\left(\xi_k^{z\sigma} - q_k^{\Delta\sigma}\tau^{z\sigma}\right). \end{aligned} \quad (42)$$

Therefore, I_z is a z-th adiabatic invariant of the nonholonomic system on time scales. The theorem is proved.

Theorem 4 can be called the Lie symmetry theorem of disturbed nonholonomic systems of non-Chetaev type on time scales, which reveals the relationship between Lie symmetries and adiabatic invariants when the system is subject to small disturbance. When undisturbed, Theorem 4 degenerates to Theorem 1. □

If we take $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, and the criterion equation (19) and the restriction equation (20) and the additional restriction equation (21) of Lie symmetry on time scales become

$$\ddot{\xi}_s^m - 2h_s\dot{\tau}^m - 2\frac{M_{sk}}{D}F_k\dot{\tau}^{m-1} - \dot{q}_s\ddot{\tau}^m = X_m^{(1)}(h_s) + X_{m-1}^{(1)}\left(\frac{M_{sk}}{D}F_k\right), \tag{43}$$

$$X_m^{(1)}\left(f_\beta(t, q_s, \dot{q}_s)\right) = 0, \tag{44}$$

$$f_{\beta s}\left(\xi_s^m - \dot{q}_s\tau^m\right) = 0. \tag{45}$$

Hence, Theorem 4 gives

Theorem 5 *For the disturbed nonholonomic system of non-Chetaev type, if the generators of infinitesimal transformations satisfy the criterion equation (43) and the restriction equation (44), as well as the additional restriction equation (45), and there are gauge functions $G_m = G_m(t, q_s, \dot{q}_s)$ that satisfy the following structural equations:*

$$L\dot{\tau}^m + X_m^{(1)}(L) + \left(Q_k'' + \Lambda_k\right)\left(\xi_k^m - \dot{q}_k\tau^m\right) + F_k\left(\xi_k^{m-1} - \dot{q}_k\tau^{m-1}\right) + \dot{G}_m = 0, \tag{46}$$

then the system has a z-th adiabatic invariant as follows:

$$I_z = \sum_{m=0}^{z} \upsilon^m\left[\frac{\partial L}{\partial \dot{q}_k}\xi_k^m + \left(L - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right)\tau^m + G_m\right]. \tag{47}$$

If we take $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, and the criterion equation (19) and the restriction equation (20) and the additional restriction equation (21) of Lie symmetry on time scales become

$$\Delta^2\xi_s^m - 2h_s\Delta\tau^m - 2\frac{M_{sk}}{D}F_k\Delta\tau^{m-1} - \Delta q_s(t+1)\Delta^2\tau^m = X_m^{(1)}(h_s) + X_{m-1}^{(1)}\left(\frac{M_{sk}}{D}F_k\right), \tag{48}$$

$$X_m^{(1)}\left[f_\beta(t, q_s(t+1), \Delta q_s(t))\right] = 0, \tag{49}$$

$$f_{\beta s}\left[\xi_s^m(t+1) - \Delta q_s(t+1)\tau^m(t+1)\right] = 0. \tag{50}$$

Hence, Theorem 4 gives

Theorem 6 *For the disturbed discrete nonholonomic system of non-Chetaev type on time scales, if the generators of infinitesimal transformations satisfy the criterion equation (48) and the restriction equation (49), as well as the additional restriction equation (50), and there are gauge functions $G_m = G_m(t, q_s(t+1), \Delta q_s(t))$ that satisfy the following structural equations:*

$$\begin{aligned} &L\Delta\tau^m + X_m^{(1)}(L) + \frac{\partial L}{\partial q_k(t+1)}\Delta q_k\Delta\tau^m + \left(Q_k'' + \Lambda_k\right)\left(\xi_k^m(t+1) - \Delta q_k(t+1)\tau^m(t+1)\right) \\ &+F_k\left(\xi_k^{m-1}(t+1) - \Delta q_k(t+1)\tau^{m-1}(t+1)\right) + \Delta G_m = 0, \quad (m = 0, 1, 2, \ldots), \end{aligned} \tag{51}$$

then the system has a z-th adiabatic invariant as follows:

$$I_z = \sum_{m=0}^{z} \upsilon^m\left[\frac{\partial L}{\partial \Delta q_k}\xi_k^m + \left(L - \frac{\partial L}{\partial \Delta q_k}\Delta q_k\right)\tau^m - \frac{\partial L}{\partial t}\tau^m + G_m\right]. \tag{52}$$

In Theorem 6, the relation between Lie symmetries and adiabatic invariants for the discrete nonholonomic system of non-Chetaev type is established.

Due to the arbitrariness of time scales, apart from the above two special cases, different time scales can be selected according to the needs, so as to obtain the corresponding results.

6 An example

Assume that the Lagrangian of a Q nonholonomic mechanical system of non-Chetaev type on time scales is

$$L = \frac{1}{2}\left[\left(q_1^{\Delta}\right)^2 + \left(q_2^{\Delta}\right)^2\right]. \tag{53}$$

The system is subjected to the following nonholonomic constraint:

$$f = q_1^{\Delta} + btq_2^{\Delta} - bq_2 + t = 0, \quad (b = \text{const.}). \tag{54}$$

The virtual displacements of the system satisfy the following equation:

$$\delta q_1^{\sigma} - b\sigma\,(t)\,\delta q_2^{\sigma} = 0. \tag{55}$$

First, we establish the differential equations of motion of the system. Equation (4) gives

$$q_1^{\Delta\Delta} = \lambda,\, q_2^{\Delta\Delta} = -\lambda b\sigma\,(t)\,. \tag{56}$$

According to Eqs. (54) and (56), we obtain

$$\lambda = \frac{1}{b^2\sigma^2\,(t) - 1}, \tag{57}$$

so we have

$$\Lambda_1 = \frac{1}{b^2\sigma^2\,(t) - 1}, \quad \Lambda_2 = -\frac{b\sigma\,(t)}{b^2\sigma^2\,(t) - 1} \tag{58}$$

and Eq. (56) becomes

$$q_1^{\Delta\Delta} = \frac{1}{b^2\sigma^2\,(t) - 1}, \quad q_2^{\Delta\Delta} = -\frac{b\sigma\,(t)}{b^2\sigma^2\,(t) - 1}. \tag{59}$$

Second, we calculate the Lie symmetry and exact invariants of the system. The criterion equations of Lie symmetry are

$$\begin{aligned} \xi_1^{\Delta\Delta} - \frac{2}{b^2\sigma^2\,(t) - 1}\tau^{\Delta} - q_1^{\Delta\sigma}\tau^{\Delta\Delta} &= X^{(1)}\left(\frac{1}{b^2\sigma^2\,(t) - 1}\right), \\ \xi_2^{\Delta\Delta} + \frac{2b\sigma\,(t)}{b^2\sigma^2\,(t) - 1}\tau^{\Delta} - q_2^{\Delta\sigma}\tau^{\Delta\Delta} &= -X^{(1)}\left(\frac{b\sigma\,(t)}{b^2\sigma^2\,(t) - 1}\right). \end{aligned} \tag{60}$$

Equation (60) has a solution

$$\tau^0 = 0, \quad \xi_1^0 = bt, \quad \xi_2^0 = 1. \tag{61}$$

The restriction equation and the additional restriction equation of the system, respectively, are

$$\xi_1^{\Delta} - q_1^{\Delta}\tau^{\Delta} + bt\left(\xi_2^{\Delta} - q_2^{\Delta}\tau^{\Delta}\right) + bq_2^{\Delta}\tau - b\xi_2 + \tau = 0, \tag{62}$$

$$\xi_1^{\sigma} - q_1^{\Delta\sigma}\tau^{\sigma} - b\sigma\,(t)\left(\xi_2^{\sigma} - q_2^{\Delta\sigma}\tau^{\sigma}\right) = 0. \tag{63}$$

Obviously, the generators (61) satisfy Eqs. (62) and (63), so the generators (61) correspond to the Lie symmetry of the system.

Substituting the generators (61) into the structural equation (22), we get

$$G_0 = -bq_1. \tag{64}$$

According to Theorem 1, from the generators (61) and the gauge function (64), we obtain the following conserved quantity:

$$I_0 = btq_1^{\Delta} + q_2^{\Delta} - bq_1 = \text{const.} \tag{65}$$

This is the exact invariant caused by the Lie symmetry (61) of the system.

The third step is to calculate the adiabatic invariants led by the perturbation of Lie symmetry under small disturbance. Suppose the system is subjected to small disturbance forces υF_s, i.e.,

$$\upsilon F_1 = \upsilon b, \quad \upsilon F_2 = \upsilon\left(b^2 t + q_2^{\Delta}\right), \tag{66}$$

then Eq. (59) becomes

$$q_1^{\Delta\Delta} = \frac{1}{b^2\sigma^2(t) - 1} + \upsilon b, \quad q_2^{\Delta\Delta} = -\frac{b\sigma(t)}{b^2\sigma^2(t) - 1} + \upsilon\left(b^2 t + q_2^{\Delta}\right). \tag{67}$$

The criterion equation (19) of Lie symmetry gives

$$\begin{aligned} &\xi_1^{1\Delta\Delta} - \frac{2}{b^2\sigma^2(t) - 1}\tau^{1\Delta} - 2b\tau^{0\Delta} - q_1^{\Delta\sigma}\tau^{1\Delta\Delta} = X_1^{(1)}\left(\frac{1}{b^2\sigma^2(t) - 1}\right) + X_0^{(1)}(b), \\ &\xi_2^{1\Delta\Delta} + \frac{2b\sigma(t)}{b^2\sigma^2(t) - 1}\tau^{1\Delta} - 2\left(b^2 t + q_2^{\Delta}\right)\tau^{0\Delta} - q_2^{\Delta\sigma}\tau^{1\Delta\Delta} \\ &\quad = -X_1^{(1)}\left(\frac{b\sigma(t)}{b^2\sigma^2(t) - 1}\right) + X_0^{(1)}\left(b^2 t + q_2^{\Delta}\right). \end{aligned} \tag{68}$$

Equation (68) has a solution

$$\tau^1 = 0, \quad \xi_1^1 = t, \quad \xi_2^1 = \frac{1}{b}. \tag{69}$$

The restriction equations and the additional restriction equations of the disturbed system are

$$\xi_1^{1\Delta} - q_1^{\Delta}\tau^{1\Delta} + bt\left(\xi_2^{1\Delta} - q_2^{\Delta}\tau^{1\Delta}\right) + bq_2^{\Delta}\tau^1 - b\xi_2^1 + \tau^1 = 0, \tag{70}$$

$$\xi_1^{1\sigma} - q_1^{\Delta\sigma}\tau^{1\sigma} - b\sigma(t)\left(\xi_2^{1\sigma} - q_2^{\Delta\sigma}\tau^{1\sigma}\right) = 0. \tag{71}$$

The generators (69) satisfy Eqs. (70) and (71), so it corresponds to the Lie symmetry of the disturbed system.

The structural equation of the disturbed system is

$$\begin{aligned} &L\tau^{1\Delta} + \left(\xi_1^{1\Delta} - q_1^{\Delta}\tau^{1\Delta}\right)q_1^{\Delta} + \left(\xi_2^{1\Delta} - q_2^{\Delta}\tau^{1\Delta}\right)q_2^{\Delta} + \frac{1}{b^2\sigma^2(t) - 1}\left(\xi_1^1 - q_1^{\Delta}\tau^1\right)^{\sigma} \\ &- \frac{b\sigma(t)}{b^2\sigma^2(t) - 1}\left(\xi_2^1 - q_2^{\Delta}\tau^1\right)^{\sigma} + b\left(\xi_1^0 - q_1^{\Delta}\tau^0\right)^{\sigma} + \left(b^2 t + q_2^{\Delta}\right)\left(\xi_2^0 - q_2^{\Delta}\tau^0\right)^{\sigma} + G_1^{\Delta} = 0. \end{aligned} \tag{72}$$

Substituting Eq. (69) into Eq. (72), we can get

$$G_1 = -b^2 t^2 - q_1 - q_2. \tag{73}$$

According to Theorem 4, we have

$$I_1 = btq_1^{\Delta} + q_2^{\Delta} - bq_1 + \upsilon\left(tq_1^{\Delta} + \frac{1}{b}q_2^{\Delta} - q_1 - q_2 - b^2 t^2\right). \tag{74}$$

This is the first-order adiabatic invariant caused by the Lie symmetry of the disturbed system. Similarly, we can find adiabatic invariants of the second and higher orders.

We now consider two special cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, and $q^{\Delta}(t) = \frac{\mathrm{d}q}{\mathrm{d}t}$, and the Lagrangian (53), nonholonomic constraint (54) and the virtual displacement equation (55) are reduced to those [6] in the classical continuous case, respectively:

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right), \quad f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0, \quad \delta q_1 - bt\delta q_2 = 0, \tag{75}$$

and Eq. (74) gives classical continuous version of adiabatic invariant as follows:

$$I_1 = bt\dot{q}_1 + \dot{q}_2 - bq_1 + \upsilon\left(t\dot{q}_1 + \frac{1}{b}\dot{q}_2 - q_1 - q_2 - b^2t^2\right). \tag{76}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, and $q^{\Delta}(t) = q(t+1) - q(t) = \Delta q(t)$, and the Lagrangian (53), nonholonomic constraint (54) and the virtual displacement equation (55) are reduced to those in the classical discrete case, respectively:

$$\begin{aligned} &L = \frac{1}{2}\left[(\Delta q_1(t))^2 + (\Delta q_2(t))^2\right], \quad f = \Delta q_1(t) + bt\Delta q_2(t) - bq_2(t) + t = 0, \\ &\delta q_1(t+1) - b(t+1)\,\delta q_2(t+1) = 0, \end{aligned} \tag{77}$$

and Eq. (74) gives the discrete version of the adiabatic invariant as follows:

$$I_1 = bt\Delta q_1(t) + \Delta q_2(t) - bq_1(t) + \upsilon\left[t\Delta q_1(t) + \frac{1}{b}\Delta q_2(t) - q_1(t) - q_2(t) - b^2t^2\right], \tag{78}$$

where Δ is the usual forward difference operator.

If we choose another time scale, for example, $\mathbb{T} = \left\{2^j : j \in \mathbb{N}_0\right\}$, we can get another discrete version of the adiabatic invariant (74).

7 Conclusions

In this paper, we proposed and studied Lie symmetries and exact invariants of nonholonomic systems of non-Chetaev type on time scales, and studied the perturbation of Lie symmetry and adiabatic invariants of nonholonomic systems of non-Chetaev type on time scales under small disturbance. The main contributions of this paper are as follows: The first is that we established the criterion equations of Lie symmetry for nonholonomic systems of non-Chetaev type on time scales, proposed and proved the Lie symmetry theorem, and derived the corresponding exact invariants caused by the Lie symmetry of the undisturbed system; the second is that we studied the perturbation of Lie symmetry of the system under small disturbance, and derived the adiabatic invariants of nonholonomic system of non-Chetaev type on time scales. Since the time scales calculus has the two features, i.e., unification and extension, the results of this paper are universal. The method and results can be further extended to various constrained mechanical systems on time scales.

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