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ORIGINAL PAPER

Sergey V. Ershkov · Dmytro Leshchenko

On a new type of solving procedure for Euler–Poisson equations (rigid body rotation over the fixed point)

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Abstract In this paper, we proceed to develop a new approach which was formulated first in Ershkov (Acta Mech 228(7):2719–2723, 2017) for solving Poisson equations: a new type of the solving procedure for Euler–Poisson equations (rigid body rotation over the fixed point) is suggested in the current research. Meanwhile, the Euler–Poisson system of equations has been successfully explored for the existence of analytical solutions. As the main result, a new ansatz is suggested for solving Euler–Poisson equations: the Euler–Poisson equations are reduced to a system of three nonlinear ordinary differential equations of first order in regard to three functions Ω_i ($i = 1, 2, 3$); the proper elegant approximate solution has been obtained as a set of quasi-periodic cycles via re-inversing the proper elliptical integral. So the system of Euler–Poisson equations is proved to have analytical solutions (in quadratures) only in classical simplifying cases: (1) Lagrange's case, or (2) Kovalevskaya's case or (3) Euler's case or other well-known but particular cases.

1 Introduction, equations of motion

Euler–Poisson equations, describing the dynamics of rigid body rotation, are known to be one of the famous problems in classical mechanics.

In accordance with [1–3], Euler equations describe the rotation of a rigid body in a frame of reference fixed in the rotating body for the case of rotation over the fixed point as below (at given initial conditions):

$$\begin{cases} I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3 = P\,(\gamma_2 c_0 - \gamma_3 b_0)\,, \\ I_2 \frac{d\Omega_2}{dt} + (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1 = P\,(\gamma_3 a_0 - \gamma_1 c_0)\,, \\ I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2 = P\,(\gamma_1 b_0 - \gamma_2 a_0)\,, \end{cases} \tag{1.1}$$

where $I_i \neq 0$ are the principal moments of inertia ($i = 1, 2, 3$) and Ω_i are the components of the angular velocity vector along the proper principal axis; γ_i are the components of the weight of mass P and a_0, b_0, c_0 are the appropriate coordinates of the center of masses in a frame of reference fixed in the rotating body (in regard to the absolute system of coordinates X, Y, Z).

S. V. Ershkov (✉)
Engineering Academy, Peoples' Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya Street, Moscow 117198, Russian Federation
E-mail: sergej-ershkov@yandex.ru

D. Leshchenko
Odessa State Academy of Civil Engineering and Architecture, Odessa, Ukraine
E-mail: leshchenko_d@ukr.net

The Poisson equations for the components of the weight in a frame of reference fixed in the rotating body (in regard to the absolute system of coordinates X, Y, Z) can be presented as below [4,5]:

$$\begin{cases} \frac{d\gamma_1}{dt} = \Omega_3\gamma_2 - \Omega_2\gamma_3, \\ \frac{d\gamma_2}{dt} = \Omega_1\gamma_3 - \Omega_3\gamma_1, \\ \frac{d\gamma_3}{dt} = \Omega_2\gamma_1 - \Omega_1\gamma_2, \end{cases} \tag{1.2}$$

besides, we should present the invariants (first integrals of motion) as

$$\begin{cases} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_1 \cdot \Omega_1 \cdot \gamma_1 + I_2 \cdot \Omega_2 \cdot \gamma_2 + I_3 \cdot \Omega_3 \cdot \gamma_3 = \text{const.} = C_0, \\ \frac{1}{2}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + P(a_0\gamma_1 + b_0\gamma_2 + c_0\gamma_3) = \text{const.} = C_1. \end{cases} \tag{1.3}$$

2 Derivation of the invariants (first integrals) of motion

According to [6], let us recall how to derive the invariants (1.3). From (1.1), (1.2) we obtain

$$\begin{aligned} &\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2}+I_2\frac{\Omega_2^2}{2}+I_3\frac{\Omega_3^2}{2}\right)}{P} = \Omega_1 \cdot (\gamma_2 c_0 - \gamma_3 b_0) + \Omega_2 \cdot (\gamma_3 a_0 - \gamma_1 c_0) + \Omega_3 \cdot (\gamma_1 b_0 - \gamma_2 a_0), \\ \Rightarrow \quad &-\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2}+I_2\frac{\Omega_2^2}{2}+I_3\frac{\Omega_3^2}{2}\right)}{P} = a_0 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3) + b_0 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1) + c_0 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2), \\ \Rightarrow \quad &\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2}+I_2\frac{\Omega_2^2}{2}+I_3\frac{\Omega_3^2}{2}\right) + a_0 \cdot P \cdot \frac{d\gamma_1}{dt} + b_0 \cdot P \cdot \frac{d\gamma_2}{dt} + c_0 \cdot P \cdot \frac{d\gamma_3}{dt} = 0. \end{aligned}$$

So we have obtained the third integral of (1.3) in [7]. For obtaining the second integral of (1.3) in [7], we multiply each of Eqs. (1.1) by γ_i accordingly, but also each of equations of (1.2) by $(I_i \cdot \Omega_i)$ accordingly; then we add as below:

$$\begin{cases} \left(I_1 \cdot \gamma_1 \cdot \frac{d\Omega_1}{dt} + \gamma_1 \cdot (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3\right) + \left(I_1 \cdot \Omega_1 \frac{d\gamma_1}{dt}\right) = \gamma_1 \cdot P\,(\gamma_2 c_0 - \gamma_3 b_0) + I_1 \cdot \Omega_1 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3), \\ \left(I_2 \cdot \gamma_2 \cdot \frac{d\Omega_2}{dt} + \gamma_2 \cdot (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1\right) + \left(I_2 \cdot \Omega_2 \frac{d\gamma_2}{dt}\right) = \gamma_2 \cdot P\,(\gamma_3 a_0 - \gamma_1 c_0) + I_2 \cdot \Omega_2 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1), \\ \left(I_3 \cdot \gamma_3 \cdot \frac{d\Omega_3}{dt} + \gamma_3 \cdot (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2\right) + \left(I_3 \cdot \Omega_3 \frac{d\gamma_3}{dt}\right) = \gamma_3 \cdot P\,(\gamma_1 b_0 - \gamma_2 a_0) + I_3 \cdot \Omega_3 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2), \end{cases}$$

Having done this, we add all the three equations above:

$$I_1 \cdot \frac{d}{dt}(\Omega_1 \cdot \gamma_1) + I_2 \cdot \frac{d}{dt}(\Omega_2 \cdot \gamma_2) + I_3 \cdot \frac{d}{dt}(\Omega_3 \cdot \gamma_3) = 0.$$

So we have obtained the second integral of (1.3) in [7].

The first integral of (1.3) is trivial, but belongs to Poisson equations only: to obtain it, we multiply each of equations of (1.2) on γ_i accordingly, then add them (the constant of integration is chosen equal to 1, due to trigonometric sense of the presenting solution in absolute system of coordinates via Euler angles):

$$\frac{1}{2}\frac{d}{dt}(\gamma_1^2) + \frac{1}{2}\frac{d}{dt}(\gamma_2^2) + \frac{1}{2}\frac{d}{dt}(\gamma_3^2) = 0,$$

As we can see, 2 of 3 proper additional invariants above are obtained by using of all the 6 EP-equations (including Poisson equations).

But aforesaid argument is not sufficient for solving the EP-equations: indeed, the system of equations (1.1)–(1.2) is supposed to not be equivalent to the system of equations (1.1) along with all the invariants (1.3) (Dr. Hamad H. Yehya, personal communications) for some particular cases, as was suggested earlier in [7,8]. The rather complex case, which describes the motion of the constrained rigid body around a fixed point, was considered in the comprehensive article [9].

So, for solving the system of equations (1.1)–(1.2), we first solve the Poisson equations (1.2).

3 Presentation of the solution of Poisson equations

The system of equations (1.2) has the analytical solution [6] (with respect to the time-parameter t; in fluid mechanics see Refs. [10–13]):

$$\begin{aligned} &\gamma_1 = -\sigma \cdot \left(\tfrac{2a}{1+(a^2+b^2)}\right), \quad \gamma_2 = -\sigma \cdot \left(\tfrac{2b}{1+(a^2+b^2)}\right), \\ &\gamma_3 = \sigma \cdot \left(\tfrac{1-(a^2+b^2)}{1+(a^2+b^2)}\right), \end{aligned} \tag{3.1}$$

where σ is some arbitrary (real) constant, given by the initial conditions; Ω_i are functions of the time-parameter t only—we consider here only the case for the first approximation, which means that $\Omega_i \neq \Omega_i(\{\gamma_i\}, t)$.

Besides, the real-valued coefficients $a(t)$, $b(t)$ (3.1) are solutions of the mutual system of 2 Riccati ordinary differential equations:

$$\begin{cases} a' = \frac{\Omega_2}{2} \cdot a^2 - (\Omega_1 \cdot b) \cdot a - \frac{\Omega_2}{2}(b^2 - 1) + \Omega_3 \cdot b, \\ b' = -\frac{\Omega_1}{2} \cdot b^2 + (\Omega_2 \cdot a) \cdot b + \frac{\Omega_1}{2} \cdot (a^2 - 1) - \Omega_3 \cdot a. \end{cases} \tag{3.2}$$

Just to confirm [6] the Riccati-type of equations (3.2): indeed, if we multiply the first of Eqs. (3.2) on Ω_2, the second by Ω_1, then summarize properly, we obtain

$$\begin{aligned} &\Omega_2 \cdot a' - \tfrac{1}{2}((\Omega_1)^2 + (\Omega_2)^2) \cdot a^2 + \Omega_1 \cdot \Omega_3 \cdot a + \tfrac{(\Omega_1)^2}{2} \\ &= -\Omega_1 \cdot b' - \tfrac{1}{2}((\Omega_1)^2 + (\Omega_2)^2) \cdot b^2 + \Omega_2 \cdot \Omega_3 \cdot b + \tfrac{(\Omega_2)^2}{2}. \end{aligned} \tag{3.3}$$

Equation (3.3) above is the classical Riccati ODE. It describes the evolution of function $a(t)$ in dependence on the function $b(t)$ along with the functions $\{\Omega_i\}$ in regard to the time-parameter t; such a *Riccati* ODE has no analytical solution in the general case [14] and could be presented as:

$$a' = A \cdot a^2 + B \cdot a + D. \tag{3.4}$$

$$A = \frac{1}{2}\frac{((\Omega_1)^2 + (\Omega_2)^2)}{\Omega_2}, \quad B = -\left(\frac{\Omega_1 \cdot \Omega_3}{\Omega_2}\right),$$

$$D = -\frac{\Omega_1}{\Omega_2} \cdot b' - \frac{1}{2}\frac{((\Omega_1)^2 + (\Omega_2)^2)}{\Omega_2} \cdot b^2 + \Omega_3 \cdot b + \frac{\Omega_2}{2} - \frac{(\Omega_1)^2}{2\Omega_2}. \tag{3.5}$$

A lot of important partial solutions of (3.4) have been considered properly in refs. [12,13], but according to their results in each case we should restrict the choosing of the appropriate functions $\{\Omega_i\}$ (this means that one of $\{\Omega_i\}$ depends on each other). In the current research, we wish to avoid such restricting dependences.

The mathematical procedure for checking of the solution (3.1)–(3.2) (which is to be valid for the Poisson equations (1.2)) has been moved to "Appendix A1", with only the resulting formulae left in the main text.

4 Presentation of the solution of Euler–Poisson equations

According to the results of the previous section, the Poisson system of equations (1.2) has *the analytical* way to present its general solution in the form (3.1) (in regard to the time-parameter t).

Meanwhile, as we can see from Section 2 above ("Derivation of the invariants (*first integrals*) of motion"), two of three proper additional invariants (1.3) are obtained by using all the 6 Euler–Poisson equations (1.1)–(1.2).

Thus, we can make a reasonable conclusion that the system of equations (1.1)–(1.2) is supposed to be equivalent to the system of Poisson equations (1.2) along with updated Euler equations (1.1): in any two of them the last two invariants of (1.3) could be substituted (for the theory of invariants of ODE-systems, see [14]).

So, for solving the Euler–Poisson system of equations (1.1)–(1.2), we should first solve the Poisson equations (1.2) in a form (3.4)–(3.5), which should be accomplished with the two aforementioned invariants along with any one of the 3 equations (1.1) (for example, let us choose the third equation from Eqs. (1.1) for definiteness):

$$\left\{\begin{array}{l} \left\{\begin{array}{l} I_1 \cdot \Omega_1 \cdot \gamma_1 + I_2 \cdot \Omega_2 \cdot \gamma_2 + I_3 \cdot \Omega_3 \cdot \gamma_3 = \text{const.} = C_0 \ , \\ \frac{1}{2}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + P(a_0\gamma_1 + b_0\gamma_2 + c_0\gamma_3) = \text{const.} = C_1, \\ I_3 \frac{\mathrm{d}\Omega_3}{\mathrm{d}t} + (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2 = P\left(\gamma_1 b_0 - \gamma_2 a_0\right), \end{array}\right. \\ \gamma_1 = -\sigma \cdot \left(\frac{2a}{1+(a^2+b^2)}\right), \quad \gamma_2 = -\sigma \cdot \left(\frac{2b}{1+(a^2+b^2)}\right), \quad \gamma_3 = \sigma \cdot \left(\frac{1-(a^2+b^2)}{1+(a^2+b^2)}\right), \\ \qquad\qquad \Updownarrow \\ \qquad\qquad \left\{\Leftrightarrow \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\right\} \\ \left\{\begin{array}{l} a' = \frac{\Omega_2}{2} \cdot a^2 - (\Omega_1 \cdot b) \cdot a - \frac{\Omega_2}{2}(b^2 - 1) + \Omega_3 \cdot b, \\ b' = -\frac{\Omega_1}{2} \cdot b^2 + (\Omega_2 \cdot a) \cdot b + \frac{\Omega_1}{2} \cdot (a^2 - 1) - \Omega_3 \cdot a. \end{array}\right. \end{array}\right. \tag{4.1}$$

Using the expressions for γ_i in (4.1), we obtain from the first and second equation of system (4.1):

$$\left\{\begin{array}{l} I_1 \cdot \Omega_1 \cdot \left(-\sigma \cdot \left(\frac{2a}{1+(a^2+b^2)}\right)\right) + I_2 \cdot \Omega_2 \cdot \left(-\sigma \cdot \left(\frac{2b}{1+(a^2+b^2)}\right)\right) + I_3 \cdot \Omega_3 \cdot \sigma \cdot \left(\frac{2}{1+(a^2+b^2)} - 1\right) = C_0, \\ \frac{1}{2P}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + a_0\left(-\sigma \cdot \left(\frac{2a}{1+(a^2+b^2)}\right)\right) + b_0\left(-\sigma \cdot \left(\frac{2b}{1+(a^2+b^2)}\right)\right) + c_0 \cdot \sigma \cdot \left(\frac{2}{1+(a^2+b^2)} - 1\right) = \frac{C_1}{P}, \end{array}\right.$$

or

$$\left\{\begin{array}{l} -2I_1 \cdot \Omega_1 \cdot \frac{a\cdot\sigma}{1+(a^2+b^2)} - 2I_2 \cdot \Omega_2 \cdot \frac{b\cdot\sigma}{1+(a^2+b^2)} + 2I_3 \cdot \Omega_3 \cdot \frac{\sigma}{1+(a^2+b^2)} = C_0 + I_3 \cdot \Omega_3 \cdot \sigma, \\ -2a_0 \frac{a\cdot\sigma}{1+(a^2+b^2)} - 2b_0 \frac{b\cdot\sigma}{1+(a^2+b^2)} + 2c_0 \cdot \frac{\sigma}{1+(a^2+b^2)} = \frac{C_1}{P} + c_0 \cdot \sigma - \frac{1}{2P}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right). \end{array}\right. \tag{4.2}$$

Let us divide the left part of the second of Eqs. (4.2) above by the left part of the first equation (accomplishing also the dividing with respect to the right parts for both of these equations)

$$\frac{a_0 \frac{a\cdot\sigma}{1+(a^2+b^2)} + b_0 \frac{b\cdot\sigma}{1+(a^2+b^2)} - c_0 \cdot \frac{\sigma}{1+(a^2+b^2)}}{I_1\cdot\Omega_1 \cdot \frac{a\cdot\sigma}{1+(a^2+b^2)} + I_2\cdot\Omega_2 \cdot \frac{b\cdot\sigma}{1+(a^2+b^2)} - I_3\cdot\Omega_3 \cdot \frac{\sigma}{1+(a^2+b^2)}} = \frac{\frac{C_1}{P} + c_0\cdot\sigma - \frac{1}{2P}\left(I_1\cdot\Omega_1^2 + I_2\cdot\Omega_2^2 + I_3\cdot\Omega_3^2\right)}{C_0 + I_3\cdot\Omega_3\cdot\sigma}$$

$$\Rightarrow \frac{a_0\cdot a + b_0\cdot b - c_0}{I_1\cdot\Omega_1\cdot a + I_2\cdot\Omega_2\cdot b - I_3\cdot\Omega_3} = \frac{\frac{C_1}{P} + c_0\cdot\sigma - \frac{1}{2P}\left(I_1\cdot\Omega_1^2 + I_2\cdot\Omega_2^2 + I_3\cdot\Omega_3^2\right)}{C_0 + I_3\cdot\Omega_3\cdot\sigma} \quad \Rightarrow$$

$$a = b \cdot \left(\frac{\Upsilon\cdot I_2\cdot\Omega_2 - b_0}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right) + \left(\frac{c_0 - \Upsilon\cdot I_3\cdot\Omega_3}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right), \quad \left\{\Upsilon = \frac{\frac{C_1}{P} + c_0\cdot\sigma - \frac{1}{2P}\left(I_1\cdot\Omega_1^2 + I_2\cdot\Omega_2^2 + I_3\cdot\Omega_3^2\right)}{C_0 + I_3\cdot\Omega_3\cdot\sigma}\right\}. \tag{4.3}$$

So the last of Eqs. (4.3) reveals the linear dependence of the function $a(t)$ with respect to the function $b(t)$ for solutions of the system (4.1). In addition to the aforesaid linear invariant (4.3), we obtain from the first of Eqs. (4.2)

$$F \cdot b^2 + 2G \cdot b + H = 0 \quad \Rightarrow \quad b = \frac{-G \pm \sqrt{G^2 - F\cdot H}}{F}, \tag{4.4}$$

$$F = \left(\frac{C_0}{\sigma} + I_3 \cdot \Omega_3\right) \cdot \left(\left(\frac{\Upsilon \cdot I_2 \cdot \Omega_2 - b_0}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right)^2 + 1\right),$$

$$G = \left(\left(\frac{C_0}{\sigma} + I_3 \cdot \Omega_3\right) \cdot \left(\frac{\Upsilon \cdot I_2 \cdot \Omega_2 - b_0}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right) \cdot \left(\frac{c_0 - \Upsilon \cdot I_3 \cdot \Omega_3}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right) \right.$$
$$\left. + I_1 \cdot \Omega_1 \cdot \left(\frac{\Upsilon \cdot I_2 \cdot \Omega_2 - b_0}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right) + I_2 \cdot \Omega_2\right),$$

$$H = \left(\frac{C_0}{\sigma} + I_3 \cdot \Omega_3\right) \cdot \left(\left(\frac{c_0 - \Upsilon \cdot I_3 \cdot \Omega_3}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right)^2 + 1\right) + 2I_1 \cdot \Omega_1 \cdot \left(\frac{c_0 - \Upsilon \cdot I_3 \cdot \Omega_3}{a_0 - \Upsilon \cdot I_1 \cdot \Omega_1}\right) - 2I_3 \cdot \Omega_3.$$

Last, but not least, we should especially note that the system of equations (4.1) is reduced to the system (4.4)–(4.5) of three nonlinear ordinary differential equations of first order with respect to three functions Ω_i $(i = 1, 2, 3)$:

$$\begin{cases} I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1)\cdot\Omega_1\cdot\Omega_2 = P\,(\gamma_1 b_0 - \gamma_2 a_0)\,, \\ a' = \frac{\Omega_2}{2}\cdot a^2 - (\Omega_1\cdot b)\cdot a - \frac{\Omega_2}{2}(b^2-1) + \Omega_3\cdot b, \\ b' = -\frac{\Omega_1}{2}\cdot b^2 + (\Omega_2\cdot a)\cdot b + \frac{\Omega_1}{2}\cdot(a^2-1) - \Omega_3\cdot a, \end{cases} \tag{4.5}$$

where

$$\gamma_1 = -\sigma\cdot\left(\frac{2a}{1+(a^2+b^2)}\right),\quad \gamma_2 = -\sigma\cdot\left(\frac{2b}{1+(a^2+b^2)}\right),\quad \gamma_3 = \sigma\cdot\left(\frac{1-(a^2+b^2)}{1+(a^2+b^2)}\right),$$

$$a = b\cdot\left(\frac{\Upsilon\cdot I_2\cdot\Omega_2 - b_0}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right) + \left(\frac{c_0 - \Upsilon\cdot I_3\cdot\Omega_3}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right),\quad \left\{\Upsilon = \frac{\frac{C_1}{P} + c_0\cdot\sigma - \frac{1}{2P}\left(I_1\cdot\Omega_1^2 + I_2\cdot\Omega_2^2 + I_3\cdot\Omega_3^2\right)}{C_0 + I_3\cdot\Omega_3\cdot\sigma}\right\},$$

$$b = \frac{-G\pm\sqrt{G^2 - F\cdot H}}{F},$$

$$\left\{\begin{aligned} F &= \left(\frac{C_0}{\sigma} + I_3\cdot\Omega_3\right)\cdot\left(\left(\frac{\Upsilon\cdot I_2\cdot\Omega_2 - b_0}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right)^2 + 1\right), \\ G &= \left(\left(\frac{C_0}{\sigma} + I_3\cdot\Omega_3\right)\cdot\left(\frac{\Upsilon\cdot I_2\cdot\Omega_2 - b_0}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right)\cdot\left(\frac{c_0 - \Upsilon\cdot I_3\cdot\Omega_3}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right) + I_1\cdot\Omega_1\cdot\left(\frac{\Upsilon\cdot I_2\cdot\Omega_2 - b_0}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right) + I_2\cdot\Omega_2\right), \\ H &= \left(\frac{C_0}{\sigma} + I_3\cdot\Omega_3\right)\cdot\left(\left(\frac{c_0 - \Upsilon\cdot I_3\cdot\Omega_3}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right)^2 + 1\right) + 2I_1\cdot\Omega_1\cdot\left(\frac{c_0 - \Upsilon\cdot I_3\cdot\Omega_3}{a_0 - \Upsilon\cdot I_1\cdot\Omega_1}\right) - 2I_3\cdot\Omega_3. \end{aligned}\right\}$$

5 The case of approximate solution (4.5) of Euler–Poisson equations

Let us choose a simplifying assumption for solving the system of equations (4.5) (for example, assumption of symmetric rigid rotor):

$$I_1 = I_2,\quad \{a_0, b_0\} = 0. \tag{5.1}$$

So let us consider the case of a symmetric rigid rotor (5.1) whose center of mass lies on the symmetry axis. It is obviously similar to the Lagrange case, but in the Lagrange case we additionally assume [1]:

(1) the angular momentum component along the symmetry axis,
(2) the angular momentum in the Z-direction, and
(3) the magnitude of the $\boldsymbol{\gamma}$-vector,

whereas all of the aforesaid invariants should be constant.

If we assume (5.1), such simplifications should transform the first equation of (4.5) to the simple equality $\Omega_3 = \text{const}$.

As we can see from the structure of the components for the second and third equations of the system (4.5), we need additional simplifying assumptions to solve it easily:

$$|I_1\cdot\Omega_1| >> |I_2\cdot\Omega_2| >> |I_3\cdot\Omega_3|\,. \tag{5.2}$$

However, the aforesaid assumption (5.1) should simplify these components of the system of equations (4.5) accordingly as below ($I_1 = I_2$, $\{a_0, b_0\} = 0$):

$$a = -b\cdot\left(\frac{\Omega_2}{\Omega_1}\right) - \left(\frac{c_0}{\Upsilon\cdot I_1\cdot\Omega_1}\right),\quad b = \frac{-G\pm\sqrt{G^2 - F\cdot H}}{F},$$

$$\left\{\begin{aligned} F &= \frac{C_0}{\sigma}\cdot\left(\left(\frac{\Omega_2}{\Omega_1}\right)^2 + 1\right), \\ G &= \frac{C_0}{\sigma}\cdot\left(\frac{\Omega_2}{\Omega_1}\right)\cdot\left(\frac{c_0}{\Upsilon\cdot I_1\cdot\Omega_1}\right), \\ H &= \frac{C_0}{\sigma}\cdot\left(\left(\frac{c_0}{\Upsilon\cdot I_1\cdot\Omega_1}\right)^2 + 1\right) - 2\left(\frac{c_0}{\Upsilon}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} &G^2 - F\cdot H \\ &= 2\left(\frac{C_0}{\sigma}\right)\cdot\left(\frac{c_0}{\Upsilon}\right) - \left(\frac{C_0}{\sigma}\right)^2\cdot\left(\left(\frac{c_0}{\Upsilon\cdot I_1\cdot\Omega_1}\right)^2 + 1\right) \\ &\quad - \left(\frac{\Omega_2}{\Omega_1}\right)^2\left(\frac{C_0}{\sigma}\right)^2 + 2\left(\frac{\Omega_2}{\Omega_1}\right)^2\cdot\left(\frac{C_0}{\sigma}\right)\cdot\left(\frac{c_0}{\Upsilon}\right), \end{aligned}$$

which (if we choose $\Omega_3 = \text{const} \cong 0$) can be presented as

$$a = -b \cdot \left(\frac{\Omega_2}{\Omega_1}\right) - \left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right), \quad b = \frac{-G \pm \sqrt{G^2 - F \cdot H}}{F}, \tag{5.3}$$

$$\left\{\begin{aligned} F &= \frac{C_0}{\sigma} \cdot \left(\left(\frac{\Omega_2}{\Omega_1}\right)^2 + 1\right), \\ G &= \frac{C_0}{\sigma} \cdot \left(\frac{\Omega_2}{\Omega_1}\right) \cdot \left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right), \\ H &= \frac{C_0}{\sigma} \cdot \left(\left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right)^2 + 1\right) - 2\left(\frac{c_0}{\Upsilon}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} &G^2 - F \cdot H \\ &= 2\left(\frac{C_0}{\sigma}\right) \cdot \left(\frac{c_0}{\Upsilon}\right) - \left(\frac{C_0}{\sigma}\right)^2 \cdot \left(\left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right)^2 + 1\right) \\ &- \left(\frac{\Omega_2}{\Omega_1}\right)^2 \left(\frac{C_0}{\sigma}\right)^2 + 2\left(\frac{\Omega_2}{\Omega_1}\right)^2 \cdot \left(\frac{C_0}{\sigma}\right) \cdot \left(\frac{c_0}{\Upsilon}\right). \end{aligned}$$

Condition (5.2) can be associated with evolution of spin of the rotating rigid body from initial-current spin state toward the rotation about maximal-inertia axis due to the process of nutation relaxation or to the proper spin state corresponding to minimal energy [15] (which is fluctuating near the given appropriate constant) with a fixed angular momentum.

Let us also consider the additional simplifying assumption as below ($\Omega_3 \cong 0$):

$$c_0 \to 0. \tag{5.4}$$

If we take into account the additional assumptions (5.2) and (5.4) above, this should simplify expressions in (5.3) to the form below:

$$a = -b \cdot \left(\frac{\Omega_2}{\Omega_1}\right) - \left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right) \cong -b \cdot \left(\frac{\Omega_2}{\Omega_1}\right), \quad b = \frac{-G \pm \sqrt{G^2 - F \cdot H}}{F}, \tag{5.5}$$

$$\left\{\begin{aligned} F &= \frac{C_0}{\sigma}, \\ G &= \frac{C_0}{\sigma} \cdot \left(\frac{\Omega_2}{\Omega_1}\right) \cdot \left(\frac{c_0}{\Upsilon \cdot I_1 \cdot \Omega_1}\right) \cong 0, \\ H &= \frac{C_0}{\sigma} - 2\left(\frac{c_0}{\Upsilon}\right), \end{aligned}\right\} \Rightarrow \begin{aligned} b &= \frac{-G \pm \sqrt{G^2 - F \cdot H}}{F} \\ &= \pm\sqrt{2\left(\frac{\sigma}{C_0}\right) \cdot \left(\frac{c_0}{\Upsilon}\right) - 1}, \end{aligned}$$

$$\left\{ \Upsilon = \frac{\frac{C_1}{P} + c_0 \cdot \sigma - \frac{1}{2P}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right)}{C_0 + I_3 \cdot \Omega_3 \cdot \sigma} \right\}.$$

Conditions (5.5) above imply that functions $\{a(t), b(t)\}$ depend on both the components Ω_1 and Ω_2. However, assumptions (5.2) and (5.4) help us in simplifying the second and the third equation of system (4.5) (let us choose the sign "+" from "±" in the expression for $b(t)$ in formulae (5.5)):

$$\left\{\begin{aligned} &\frac{d}{dt}\left(b \cdot \left(\frac{\Omega_2}{\Omega_1}\right)\right) = -\frac{\Omega_2}{2} b^2 \cdot \left(1 + \left(\frac{\Omega_2}{\Omega_1}\right)^2\right) - \frac{\Omega_2}{2} - \Omega_2 \cdot \left(\frac{\Omega_3}{\Omega_2}\right) \cdot b, \\ &\frac{d}{dt} b = -\frac{\Omega_1}{2} \cdot b^2 \cdot \left(1 + \left(\frac{\Omega_2}{\Omega_1}\right)^2\right) - \frac{\Omega_1}{2} + \Omega_1 \cdot \left(\frac{\Omega_3}{\Omega_1}\right) \cdot \left(\frac{\Omega_2}{\Omega_1}\right) \cdot b, \end{aligned}\right. \tag{5.6}$$

or

$$\left\{\begin{aligned} &\frac{d}{dt}\left(b \cdot \left(\frac{\Omega_2}{\Omega_1}\right)\right) \cong -\frac{\Omega_2}{2} \cdot b^2 - \frac{\Omega_2}{2}, \\ &\frac{d}{dt} b \cong -\frac{\Omega_1}{2} \cdot b^2 - \frac{\Omega_1}{2}, \end{aligned}\right. \tag{5.7}$$

from which we obtain the invariant of the last system (5.7) (reduced version of (5.6))

$$\frac{\frac{d}{dt}\left(\left(\frac{\Omega_2}{\Omega_1}\right) \cdot b\right)}{\frac{d}{dt} b} \cong \frac{\Omega_2}{\Omega_1} \quad \Rightarrow \quad \frac{\Omega_2}{\Omega_1} = \text{const.} = \varepsilon \to 0. \tag{5.8}$$

Using (5.8), we should then solve, e.g., the first of equations of system (5.7) where we should use the expression for $b(t)$ and (approximated) expression for $\Upsilon(t)$ from (5.5):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\cdot\sqrt{2\left(\frac{\sigma}{C_0}\right)\cdot\left(\frac{c_0}{\Upsilon}\right)-1}\right)\cong-\frac{\varepsilon\cdot\Omega_1}{2}\cdot\left(2\left(\frac{\sigma}{C_0}\right)\cdot\left(\frac{c_0}{\Upsilon}\right)-1\right)-\frac{\varepsilon\cdot\Omega_1}{2}\Rightarrow\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{2\left(\frac{\sigma}{C_0}\right)\cdot\left(\frac{c_0}{\Upsilon}\right)-1}\right)=-\frac{\Omega_1}{2}\cdot\left(2\left(\frac{\sigma}{C_0}\right)\cdot\left(\frac{c_0}{\Upsilon}\right)-1\right)-\frac{\Omega_1}{2},\\
&\left\{\Upsilon\cong\frac{\frac{C_1}{P}+c_0\cdot\sigma-\frac{1}{2P}(I_1\cdot\Omega_1^2)}{C_0}\right\}.
\end{aligned}\tag{5.9}$$

Thus, we obtain from (5.9) the approximate solution for the component Ω_1 (the mathematical procedure of obtaining the solution (5.10) has been moved to "Appendix A2", with only the resulting formulae left in the main text):

$$C_1\pm c_0\cdot\sigma\cdot P\geq 0,\quad K\geq y>L>M>N,$$

$$\int\limits_K^y\frac{\mathrm{d}\Omega_1}{\sqrt{(K-\Omega_1)\cdot(\Omega_1-L)\cdot(\Omega_1-M)\cdot(\Omega_1-N)}}=gF(\phi,k)=-\frac{1}{2}(t_y-t_K),\tag{5.10}$$

$$K=\sqrt{2\left(\frac{C_1+c_0\cdot\sigma\cdot P}{I_1}\right)},\quad L=\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)},\quad M=-\sqrt{2\left(\frac{C_1+c_0\cdot\sigma\cdot P}{I_1}\right)},$$

$$N=-\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)},$$

$$g=\frac{2}{\sqrt{(K-M)\cdot(L-N)}},\quad \phi=\sin^{-1}\left(\sqrt{\frac{(K-M)\cdot(y-L)}{(K-L)\cdot(y-M)}}\right),\quad k=\sqrt{\frac{(K-L)\cdot(M-N)}{(K-M)\cdot(L-N)}}.$$

The left side of Eq. (5.10) is transformed to the proper elliptical integral in regard to the function Ω_1 (see "Appendix A2"). But the elliptical integral is known to be a generalization of a class of inverse periodic functions (e.g., see [16,17]). Then, by the obtaining the re-inverse dependence of function Ω_1 in (5.10) with respect to the time-parameter t, we could present the solution as a set of quasi-periodic cycles: a quasi-periodic character of the evolution of the components Ω_1, Ω_2 of the angular velocity of rigid body rotation.

6 Discussion

The system of Euler–Poisson equations (which governs by the dynamics of rigid body rotation over fixed point) is known to be very hard to solve analytically: the aforementioned system is proved to have analytical solutions (in quadratures) only in classical simplifying cases [18]: (1) Lagrange's case, or (2) Kovalevskaya's case or (3) Euler's case or other well-known but particular cases (where the existence of particular solutions depends on the choosing of the appropriate initial conditions).

A new approach has been developed previously in [6] for solving the Poisson equations, for the case that the components of the angular velocity of the rigid body rotation could be considered as functions of the time-parameter t only. A fundamental solution was presented by analytical formulae in dependence on two time-dependent, real-valued coefficients. These coefficients were proved to be the solutions of a mutual system of 2 Riccati ordinary differential equations (which has no analytical solution in the general case).

We proceed with this ansatz, which was formulated in [6] at first for solving of Poisson equations: the Euler–Poisson system of equations was explored here for the existence of an analytical solution.

As we can see from Section 2 above ("Derivation of the invariants (first integrals) of motion"), two of three of the proper additional invariants (1.3) of the EP-system of equations are obtained by using all 6 Euler–Poisson equations (1.1)–(1.2).

Thus, we can make a reasonable conclusion that the system of equations (1.1)–(1.2) is supposed to be equivalent to the system of Poisson equations (1.2) along with updated Euler equations (1.1): to any two of them the last two invariants of (1.3) could be substituted by.

So, for solving the Euler–Poisson system of equations (1.1)–(1.2), we should first solve the Poisson equations (1.2) in a form (3.1)–(3.2), which should be accomplished with the two aforementioned invariants along with any one of the 3 equations (1.1) (we chose the third equation from Eqs. (1.1) for definiteness).

Having solved them according to the aforementioned ansatz, we should especially note that the system of equations (4.1) is reduced to the system (4.5) of 3 nonlinear ordinary differential equations of first order with respect to three functions Ω_i $(i = 1, 2, 3)$.

In our derivation, the main motivation was the transforming of the presented system of equations (4.5) into a convenient form, in which the minimum numerical calculations are required to obtain the final solutions. Preferably, it should be the analytical or semi-analytical solutions; we have presented here the elegant approximate solution as a set of quasi-periodic cycles via re-inversing of the proper elliptical integral.

7 Conclusion, the short survey on recent issues in the literature on rigid body rotation

A new approach was first formulated in [6] for solving the Poisson equations: the Euler–Poisson system of equations was explored there for the existence of an analytical solution. The main motivation of the aforementioned research was to correct the previous issue: indeed, the system of equations (1.1)–(1.2) is supposed not to be equivalent to the system of equations (1.1) along with all the invariants (1.3) (Dr. Hamad H. Yehya, personal communications) for some particular cases, as it was suggested earlier in [7,8]. If you solve the dynamical equations (1.1) using only the integrals (1.3) without the Poisson equations (1.2), some untrue solutions of Euler–Poisson equations may appear [7,8].

As a result, a new ansatz was suggested for solving Euler–Poisson equations in [6], which has been successfully developed in the current research: the Euler–Poisson equations are reduced to the system (4.5) of three nonlinear ODE (ordinary differential equations of first order) with respect to to three functions Ω_i $(i = 1, 2, 3)$; the proper elegant approximate solution has been obtained as a set of quasi-periodic cycles via re-inversing the proper elliptical integral.

Also, some remarkable articles should be cited, which concern the problem under consideration, [19–23].

A review of the obtained results in dynamics of the rigid bodies, their classification, and detailed bibliography can be found in the books by Leimanis [24] and Gashenenko et al. [5].

The papers [25–28] deal with the averaging procedure for a system of equations, governing the motion of a dynamically symmetric rigid body with a fixed point acted upon by the gravitational and Lorentz forces.

The expressions (3.1) [25–27] and (4) [28] for the restoring torque, corresponding to the interaction of the electric charge with the magnetic field, are incorrect. These papers are based from the beginning on a wrong equation (3.1) or (4).

In [25] the resultant value of the restoring moment $\boldsymbol{K}$, taking into account equations (2.3) and (2.4), can be written in the form (3.1):

$$K = mgl + eHl'\left[r - \frac{1}{2}r^{-1}(p\sin\phi\sin\theta + q\cos\phi\sin\theta + r\cos\theta)^2\right], \tag{7.1}$$

where l' is the distance of the position of the point charge e from a fixed point, $\boldsymbol{H}$ is the strength of the electromagnetic field; the restoring moment depends on the angles θ and ϕ. The Lorentz force is $e\cdot[\,\boldsymbol{\omega}\times \boldsymbol{H}]$, where $\boldsymbol{\omega}$ is the angular velocity of a gyrostat.

In [26,27] the restoring moment (3.1), (4) has the form

$$K = mgl + eH(l')^2\cos\theta\left[p^2 + q^2 + (q\cos\phi + p\sin\phi)^2(\tan\theta)^2\right]^{1/2}. \tag{7.2}$$

It is known that the torque of the Lorentz force has the form [29]

$$\vec{M} = e(\vec{H}\cdot\vec{l}')\left[\vec{\omega}\times\vec{l}'\right] = eH(l')^2\cos\theta\,(q, -p, 0)^{\mathrm{T}}. \tag{7.3}$$

The gyrostat rotates in presence of a uniform electromagnetic field of strength $\boldsymbol{H}$ and a point charge e, located on its axis of symmetry. Thus, the gyrostat rotates under the action of Lorentz force $\boldsymbol{F} = e\cdot[\boldsymbol{v}\times\boldsymbol{H}]$, where $\boldsymbol{v} = [\boldsymbol{\omega}\times\boldsymbol{l}']$ and

$$\vec{l}' = \left(0, 0, l'\right)^{\mathrm{T}},$$

where $\boldsymbol{\omega}$ is the angular velocity vector and l' is the distance of the position of the point charge e to 0.

The torque of the Lorentz force (7.3) has another structure than the restoring torque (3.1) in [25–27] and (4) in [28]. In these papers [25–28], all suggestions, definitions, methods, examples and parts of text have been rewritten from the works [30,31]. The results of [30,31] were also discussed in Chapters 4, 11 of book [32]. The technique of averaging of received mechanical systems in [25–28] is plagiarism from papers [30,31].

The methodological treatment of the averaging method was not presented in papers [25–28]. The solutions, obtained in [25–28] represent a slight generalization of the results of previous papers [30,31].

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Author contributions

In this research, Sergey Ershkov is responsible for the general ansatz and the solving procedure, simple algebraic manipulations, calculations, results of the article in Sects. 1–6, and also for the search of approximate solutions (as well as self-critical remarks in Sect. 7). Dr. Dmytro Leshchenko is responsible for the short survey on recent issues in the literature on rigid body rotation in Sect. 7. Both authors agreed with the results and conclusions in Sects. 1–7.

Compliance with ethical standards

Conflict of interest
The authors declare that there is no conflict of interest regarding the publication of this article.

Appendix A1 (Checking of the solution (3.4)–(3.5) for the Poisson equations (1.2))

Let us check the solution (3.1)–(3.2), which is to be valid for the Poisson equations (1.2). Namely, let us substitute formulae (3.1) for the appropriate functions in (1.2):

$$\gamma_1 = -\sigma \cdot \left(\frac{2a}{1+(a^2+b^2)}\right), \quad \gamma_2 = -\sigma \cdot \left(\frac{2b}{1+(a^2+b^2)}\right),$$
$$\gamma_3 = \sigma \cdot \left(\frac{1-(a^2+b^2)}{1+(a^2+b^2)}\right),$$

but, in addition to this, we also substitute the expressions for derivatives (with respect to time) of the functions a and b from (3.2):

$$\begin{cases} a' = \frac{\Omega_2}{2} \cdot a^2 - (\Omega_1 \cdot b) \cdot a - \frac{\Omega_2}{2}(b^2 - 1) + \Omega_3 \cdot b, \\ b' = -\frac{\Omega_1}{2} \cdot b^2 + (\Omega_2 \cdot a) \cdot b + \frac{\Omega_1}{2} \cdot (a^2 - 1) - \Omega_3 \cdot a. \end{cases}$$

Let us begin checking with the first of Eqs. (1.2):

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = \Omega_3\gamma_2 - \Omega_2\gamma_3,$$

the other equations should be checked in the same manner. Let us begin:

$$\begin{aligned}
&\tfrac{\mathrm{d}\gamma_1}{\mathrm{d}t} = \Omega_3\gamma_2 - \Omega_2\gamma_3 \quad \Rightarrow \\
&-\sigma \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{2a}{1+(a^2+b^2)}\right) = \left(-\sigma \cdot \left(\tfrac{2b}{1+(a^2+b^2)}\right)\right) \cdot \Omega_3 - \left(\sigma \cdot \left(\tfrac{1-(a^2+b^2)}{1+(a^2+b^2)}\right)\right) \cdot \Omega_2 \quad \Rightarrow \\
&\frac{2\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)\cdot(1+(a^2+b^2)) - 4a\left(a\cdot\frac{\mathrm{d}a}{\mathrm{d}t} + b\cdot\frac{\mathrm{d}b}{\mathrm{d}t}\right)}{(1+(a^2+b^2))^2} = \frac{2b\cdot\Omega_3 + (1-(a^2+b^2))\cdot\Omega_2}{1+(a^2+b^2)} \quad \Rightarrow \\
&2\left(\tfrac{\Omega_2}{2} \cdot a^2 - (\Omega_1 \cdot b) \cdot a - \tfrac{\Omega_2}{2}(b^2 - 1) + \Omega_3 \cdot b\right) \cdot (1 + (a^2 + b^2)) - \\
&-4a\left(a \cdot \left(\tfrac{\Omega_2}{2} \cdot a^2 - (\Omega_1 \cdot b) \cdot a - \tfrac{\Omega_2}{2}(b^2 - 1) + \Omega_3 \cdot b\right)\right. \\
&\left.+ b \cdot \left(-\tfrac{\Omega_1}{2} \cdot b^2 + (\Omega_2 \cdot a) \cdot b + \tfrac{\Omega_1}{2} \cdot (a^2 - 1) - \Omega_3 \cdot a\right)\right) \\
&= (2b \cdot \Omega_3 + (1 - (a^2 + b^2)) \cdot \Omega_2) \cdot \left(1 + (a^2 + b^2)\right),
\end{aligned}$$

where the proper simplifying of the last equation yields:

$$
\begin{aligned}
&2\left(\tfrac{\Omega_2}{2}\cdot a^2-(\Omega_1\cdot b)\cdot a-\tfrac{\Omega_2}{2}(b^2-1)+\Omega_3\cdot b\right)\cdot(1+(a^2+b^2))-\\
&-4a\left(a\cdot\left(\tfrac{\Omega_2}{2}\cdot a^2-(\Omega_1\cdot b)\cdot a-\tfrac{\Omega_2}{2}(b^2-1)+\Omega_3\cdot b\right)\right.\\
&\quad\left.+b\cdot\left(-\tfrac{\Omega_1}{2}\cdot b^2+(\Omega_2\cdot a)\cdot b+\tfrac{\Omega_1}{2}\cdot(a^2-1)-\Omega_3\cdot a\right)\right)\\
&=(2b\cdot\Omega_3+(1-(a^2+b^2))\cdot\Omega_2)\cdot\left(1+(a^2+b^2)\right)\quad\Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Omega_2\cdot a^2-2\Omega_1\cdot b\cdot a-\Omega_2\cdot b^2+\Omega_2+2\Omega_3\cdot b\\
&\quad+\Omega_2\cdot a^4-2\Omega_1\cdot b\cdot a^3-\Omega_2\cdot b^2\cdot a^2+\Omega_2\cdot a^2+2\Omega_3\cdot b\cdot a^2+\Omega_2\cdot a^2\cdot b^2-2\Omega_1\cdot b^3\cdot a\\
&\quad-\Omega_2\cdot b^4+\Omega_2\cdot b^2+2\Omega_3\cdot b^3\\
&\quad-2\Omega_2\cdot a^4+4\Omega_1\cdot b\cdot a^3+2\Omega_2\cdot b^2\cdot a^2-2\Omega_2\cdot a^2-4\Omega_3\cdot b\cdot a^2+2\Omega_1\cdot a\cdot b^3-4\Omega_2\cdot a^2\cdot b^2\\
&\quad-2\Omega_1\cdot a^3\cdot b+2\Omega_1\cdot a\cdot b+4\Omega_3\cdot a^2\cdot b\\
&\quad=2b\cdot\Omega_3+\Omega_2-a^2\cdot\Omega_2-b^2\cdot\Omega_2+2a^2\cdot b\cdot\Omega_3+a^2\cdot\Omega_2-\Omega_2\cdot a^4\\
&\quad-\Omega_2\cdot a^2\cdot b^2+2b^3\cdot\Omega_3+b^2\cdot\Omega_2-\Omega_2\cdot a^2\cdot b^2-\Omega_2\cdot b^4,\quad\Rightarrow\\
&\quad-\Omega_2\cdot a^4-\Omega_2\cdot b^4+2\Omega_3\cdot b^3-2\Omega_2\cdot a^2\cdot b^2\\
&\quad+2\Omega_3\cdot a^2\cdot b+\Omega_2+2\Omega_3\cdot b\\
&\quad=-\Omega_2\cdot a^4-\Omega_2\cdot b^4+2b^3\cdot\Omega_3-2\Omega_2\cdot a^2\cdot b^2+2a^2\cdot b\cdot\Omega_3+\Omega_2+2b\cdot\Omega_3.
\end{aligned}
$$

Thus, we have obtained the valid equality (which means that the checking of the first of Eqs. (1.2) was successful finished).

Let us also perform the checking of the second of Eqs. (1.2):

$$\frac{\mathrm{d}\gamma_2}{\mathrm{d}t}=\Omega_1\gamma_3-\Omega_3\gamma_1,$$

in a similar way:

$$
\begin{aligned}
&\frac{\mathrm{d}\gamma_2}{\mathrm{d}t}=\Omega_1\gamma_3-\Omega_3\gamma_1\quad\Rightarrow\\
&\quad-\sigma\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2b}{1+(a^2+b^2)}\right)=\left(\sigma\cdot\left(\frac{1-(a^2+b^2)}{1+(a^2+b^2)}\right)\right)\cdot\Omega_1+\left(\sigma\cdot\left(\frac{2a}{1+(a^2+b^2)}\right)\right)\cdot\Omega_3\quad\Rightarrow\\
&\quad-\frac{\left(2\left(\frac{\mathrm{d}b}{\mathrm{d}t}\right)\cdot(1+(a^2+b^2))-4b\left(a\cdot\frac{\mathrm{d}a}{\mathrm{d}t}+b\cdot\frac{\mathrm{d}b}{\mathrm{d}t}\right)\right)}{\left(1+(a^2+b^2)\right)^2}=\frac{2a\cdot\Omega_3+(1-(a^2+b^2))\cdot\Omega_1}{1+(a^2+b^2)}\quad\Rightarrow\\
&\quad 2\left(-\frac{\Omega_1}{2}\cdot b^2+(\Omega_2\cdot a)\cdot b+\frac{\Omega_1}{2}\cdot(a^2-1)-\Omega_3\cdot a\right)\cdot(1+(a^2+b^2))-\\
&\quad-4b\left(a\cdot\left(\frac{\Omega_2}{2}\cdot a^2-(\Omega_1\cdot b)\cdot a-\frac{\Omega_2}{2}(b^2-1)+\Omega_3\cdot b\right)\right.\\
&\quad\left.+b\cdot\left(-\frac{\Omega_1}{2}\cdot b^2+(\Omega_2\cdot a)\cdot b+\frac{\Omega_1}{2}\cdot(a^2-1)-\Omega_3\cdot a\right)\right)\\
&\quad=-\left(2a\cdot\Omega_3+(1-(a^2+b^2))\cdot\Omega_1\right)\cdot\left(1+(a^2+b^2)\right),\\
&-\Omega_1\cdot b^2+2\Omega_2\cdot a\cdot b+\Omega_1\cdot a^2-\Omega_1-2\Omega_3\cdot a-\Omega_1\cdot a^2\cdot b^2+2\Omega_2\cdot a^3\cdot b+\Omega_1 a^4-\Omega_1 a^2\\
&\quad-2\Omega_3\cdot a^3+-\Omega_1\cdot b^4+2\Omega_2\cdot a\cdot b^3+\Omega_1\cdot b^2\cdot a^2-\Omega_1\cdot b^2-2\Omega_3\cdot b^2\cdot a\\
&\quad-2\Omega_2\cdot a^3 b+4\Omega_1\cdot b^2\cdot a^2+2\Omega_2 ab^3-2\Omega_2 ab-4\Omega_3\cdot ab^2+2\Omega_1\cdot b^4-4\Omega_2\cdot a\cdot b^3\\
&\quad-2\Omega_1\cdot a^2b^2+2\Omega_1\cdot b^2+4\Omega_3\cdot ab^2
\end{aligned}
$$

$$= -2a \cdot \Omega_3 - \Omega_1 + a^2 \cdot \Omega_1 + b^2 \cdot \Omega_1 - 2a^3 \cdot \Omega_3 - a^2\Omega_1 + a^4\Omega_1 + b^2 \cdot a^2\Omega_1$$
$$-2ab^2 \cdot \Omega_3 - b^2\Omega_1 + a^2b^2\Omega_1 + b^4\Omega_1, \quad \Rightarrow$$
$$\Omega_1 \cdot a^4 + \Omega_1 \cdot b^4 + 2\Omega_1 \cdot (b^2 \cdot a^2) - 2\Omega_3 \cdot a^3 - 2\Omega_3 \cdot (b^2 \cdot a) - 2\Omega_3 \cdot a - \Omega_1$$
$$= a^4 \cdot \Omega_1 + b^4 \cdot \Omega_1 + 2(a^2 \cdot b^2) \cdot \Omega_1 - 2a^3 \cdot \Omega_3 - 2(a \cdot b^2) \cdot \Omega_3 - 2a \cdot \Omega_3 - \Omega_1.$$

Thus, we have obtained the valid equality again (meaning that the checking of the second of Eqs. (1.2) is successfully finished).

Last, but not least, let us perform also the checking of the third of Eqs. (1.2):

$$\frac{\mathrm{d}\gamma_3}{\mathrm{d}t} = \Omega_2\gamma_1 - \Omega_1\gamma_2,$$

in the same way as above:

$$\frac{\mathrm{d}\gamma_3}{\mathrm{d}t} = \Omega_2\gamma_1 - \Omega_1\gamma_2 \quad \Rightarrow$$
$$\sigma \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1-(a^2+b^2)}{1+(a^2+b^2)}\right) = \left(-\sigma \cdot \left(\frac{2a}{1+(a^2+b^2)}\right)\right) \cdot \Omega_2 + \left(\sigma \cdot \left(\frac{2b}{1+(a^2+b^2)}\right)\right) \cdot \Omega_1 \quad \Rightarrow$$
$$\frac{\left(-2\left(a \cdot \frac{\mathrm{d}a}{\mathrm{d}t} + b \cdot \frac{\mathrm{d}b}{\mathrm{d}t}\right) \cdot (1+(a^2+b^2)) - 2(1-(a^2+b^2)) \cdot \left(a \cdot \frac{\mathrm{d}a}{\mathrm{d}t} + b \cdot \frac{\mathrm{d}b}{\mathrm{d}t}\right)\right)}{(1+(a^2+b^2))^2} = \frac{2b \cdot \Omega_1 - 2a \cdot \Omega_2}{1+(a^2+b^2)} \quad \Rightarrow$$
$$-4 \cdot \left(a \cdot \frac{\mathrm{d}a}{\mathrm{d}t} + b \cdot \frac{\mathrm{d}b}{\mathrm{d}t}\right) = (2b \cdot \Omega_1 - 2a \cdot \Omega_2) \cdot \left(1 + (a^2 + b^2)\right),$$

where proper simplifying of the last equation yields:

$$-4 \cdot \left(a \cdot \frac{\mathrm{d}a}{\mathrm{d}t} + b \cdot \frac{\mathrm{d}b}{\mathrm{d}t}\right) = (2b \cdot \Omega_1 - 2a \cdot \Omega_2) \cdot \left(1 + (a^2 + b^2)\right)$$
$$2\Omega_2 \cdot a^3 - 4\Omega_1 \cdot b \cdot a^2 - 2\Omega_2ab^2 + 2\Omega_2a + 4\Omega_3 \cdot ab - 2\Omega_1 \cdot b^3 + 4\Omega_2 \cdot a \cdot b^2$$
$$+2\Omega_1 \cdot ba^2 - 2\Omega_1 \cdot b - 4\Omega_3 \cdot ab$$
$$= -2b \cdot \Omega_1 + 2a \cdot \Omega_2 - 2a^2 \cdot b \cdot \Omega_1 + 2a^3 \cdot \Omega_2 - 2b^3 \cdot \Omega_1 + 2a \cdot b^2 \cdot \Omega_2 \quad \Rightarrow$$
$$2\Omega_2 \cdot a^3 - 2\Omega_1 \cdot b^3 + 2\Omega_2 \cdot (a \cdot b^2) - 2\Omega_1 \cdot (b \cdot a^2) + 2\Omega_2 \cdot a - 2\Omega_1 \cdot b$$
$$= 2a^3 \cdot \Omega_2 - 2b^3 \cdot \Omega_1 + 2(a \cdot b^2) \cdot \Omega_2 - 2(a^2 \cdot b) \cdot \Omega_1 + 2a \cdot \Omega_2 - 2b \cdot \Omega_1,$$

so we have obtained the valid equality once again (meaning that the checking of the third of Eqs. (1.2) is also successfully finished).

Appendix A2 (Obtaining solution (5.10))

Let us obtain the solution of Eq. (5.9) in the form (5.10) as below:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{2\left(\frac{\sigma}{C_0}\right) \cdot \left(\frac{c_0}{\mathrm{Y}}\right) - 1}\right) = \frac{c_0 \cdot \sigma}{\sqrt{\left(\frac{c_0 \cdot \sigma - \frac{C_1}{P} + \frac{1}{2P}\left(I_1 \cdot \Omega_1^2\right)}{c_0 \cdot \sigma + \frac{C_1}{P} - \frac{1}{2P}\left(I_1 \cdot \Omega_1^2\right)}\right)}} \cdot \frac{1}{\left(\frac{C_1}{P} + c_0 \cdot \sigma - \frac{1}{2P}\left(I_1 \cdot \Omega_1^2\right)\right)^2} \cdot \frac{1}{P}\left(I_1 \cdot \Omega_1\right) \cdot \frac{\mathrm{d}\Omega_1}{\mathrm{d}t}$$
$$\Rightarrow$$
$$\int \frac{\mathrm{d}\Omega_1}{\sqrt{\left(c_0 \cdot \sigma \cdot P + \left(C_1 - \frac{1}{2}\left(I_1 \cdot \Omega_1^2\right)\right)\right) \cdot \left(c_0 \cdot \sigma \cdot P - \left(C_1 - \frac{1}{2}\left(I_1 \cdot \Omega_1^2\right)\right)\right)}} = -\left(\frac{1}{I_1}\right) \int \mathrm{d}t,$$

which yields

$$\int \frac{\mathrm{d}\Omega_1}{\sqrt{\left(2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)-(\Omega_1^2)\right)\cdot\left((\Omega_1^2)-2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)\right)}} = -\frac{1}{2}\int \mathrm{d}t$$

$$\Rightarrow$$

$$\left\{\begin{array}{c} \left(2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)-(\Omega_1^2)\right)=\left(\sqrt{2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)}-(\Omega_1)\right)\cdot\left(\sqrt{2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)}+(\Omega_1)\right) \\ \left((\Omega_1^2)-2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)\right)=\left((\Omega_1)-\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)}\right)\cdot\left((\Omega_1)+\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)}\right) \end{array}\right\}$$

$$\Rightarrow$$

$$\int \frac{\mathrm{d}\Omega_1}{\sqrt{\left(\sqrt{2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)}-\Omega_1\right)\cdot\left(\Omega_1+\sqrt{2\left(\frac{c_0\cdot\sigma\cdot P+C_1}{I_1}\right)}\right)\cdot\left(\Omega_1-\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)}\right)\cdot\left(\Omega_1+\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)}\right)}} = -\frac{1}{2}\int \mathrm{d}t$$

$$C_1 \pm c_0\cdot\sigma\cdot P \geq 0.$$

For proper solution of the left part of the above equation via elliptic functions (integrals), see example 256.00 in [33]:

$$C_1 \pm c_0\cdot\sigma\cdot P \geq 0, \quad K \geq y > L > M > N,$$

$$\int_K^y \frac{\mathrm{d}\Omega_1}{\sqrt{(K-\Omega_1)\cdot(\Omega_1-L)\cdot(\Omega_1-M)\cdot(\Omega_1-N)}} = gF(\phi,k) = -\frac{1}{2}(t_y - t_K), \tag{5.10}$$

$$K=\sqrt{2\left(\frac{C_1+c_0\cdot\sigma\cdot P}{I_1}\right)}, \quad L=\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)}, \quad M=-\sqrt{2\left(\frac{C_1+c_0\cdot\sigma\cdot P}{I_1}\right)},$$

$$N=-\sqrt{2\left(\frac{C_1-c_0\cdot\sigma\cdot P}{I_1}\right)},$$

$$g=\frac{2}{\sqrt{(K-M)\cdot(L-N)}}, \quad \phi=\sin^{-1}\left(\sqrt{\frac{(K-M)\cdot(y-L)}{(K-L)\cdot(y-M)}}\right), \quad k=\sqrt{\frac{(K-L)\cdot(M-N)}{(K-M)\cdot(L-N)}}.$$

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