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ORIGINAL PAPER

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Infinitesimal deformations and stability of rods made of nonlocal elastic materials

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Abstract Aim of the paper is the formulation of a criterion of infinitesimal stability for a class of rods made of nonlocal elastic materials. To that end, the nonlinear equilibrium equations of naturally straight, inextensible rods subject to terminal loads are written, and the constitutive equation assumed to represent the material response in rods of finite length is discussed. Then, the equations describing the infinitesimal deformations superimposed upon a finite one are deduced. The expression of the work done by the increments of the external loads associated with an infinitesimal deformation is employed to formulate, for the considered rods, the criterion of infinitesimal stability which does not require the existence of a stored-energy function. The criterion is applied to study the stability of simply supported rods subject to axial forces, of rods with one end clamped and the other one constrained to have the tangent parallel to the undeformed rod axis, subject to axial forces and twisting couples, and of annular rods formed from naturally straight rods with the addition of twist. The results show that rods made of nonlocal elastic materials exhibit a reduction in rigidity with respect to rods having the same geometry and made of usual elastic materials with the same tensile and shear moduli.

Mathematics Subject Classification 74B20 · 74A60 · 74K10 · 74H55

1 Introduction

The development of nanotechnology has induced a great interest in the study of models for the description of the mechanical behavior of nanostructures. In particular, one-dimensional nanostructures have become the object of intensive researches in consideration of their use in nanoscale devices, such as switchers, resonators, sensors, and oscillators (cf., e.g.,: [1–3]). The application of continuum models to nanoscale mechanical systems appears particularly appealing in the prospective of extending the advantages of a continuum approach to the study of nanosized bodies, avoiding the difficulties of atomic and molecular models. The theory of nonlocal materials [4] that describes the mechanical response of bodies in which an internal characteristic length (as lattice parameter, granular size, molecular diameter) is comparable with an external characteristic length (as wavelength, crack length, thickness) appeared to be appropriate to extend the continuum approach to nanoscale bodies. In the pioneering paper of Peddieson et al. [5], the flexure of small actuators in the form of simply supported beams and cantilevers was studied by employing, in the Euler–Bernoulli equilibrium equations, an approximate differential version of the integral constitutive equation of Eringen's theory of nonlocal elastic materials [6]. After this work, the same approach has been largely applied to the study of deformations, oscillations, buckling, and wave-propagation in nanobeams. The literature on the subject is so extensive that it is impossible to give here an account of the contributions to the matter, and we refer the interested readers to the review papers, such as [7–10].

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Most of the researches on the mechanics of one-dimensional nanobodies concern the linear behavior, but large deformations are important for nanorods, in particular for carbon nanotubes that possess a large flexibility and can undergo large reversible deformations: The remarkable flexibility of carbon nanotubes and their capability of suffering reversible strong bending have been evidenced in high resolution electron microscope observations [11]; large elastic deflections followed by elastic buckling processes have been observed in experiments for determining the mechanical properties of carbon nanotubes [12]; experiments on large deformations of carbon nanotubes have shown that they can be bent on themselves alternatively in opposite directions without exhibiting damage [13]; carbon nanotubes can undergo reversible buckling deformations that do not involve bond-breaking or atomic rearrangements [14].

Linear beam theories, in which the usual strains and constitutive equations have been replaced by the von-Kármán's strains and constitutive equations of nonlocal materials, have been applied to study buckling and approximate shapes of buckled configurations of nanorods (cf., e.g., [15,16]), while numerical and perturbation methods have been employed to study the post-buckling behavior of nanorods (cf., e.g., [17–21]).

A theory of nonlinear deformations of nanorods has been proposed in [22] by employing a differential form of the constitutive equation of Eringen's nonlocal materials in the nonlinear equilibrium equations of Kirchhoff's theory of rods. In that paper, the proposed model has been applied to the study of planar deformations of nonlocal rods subject to end loads: Exact equilibrium solutions in terms of elliptic functions have been deduced and utilized to discuss the inflexional and noninflexional elastica. In subsequent papers, the theory has been employed to obtain the exact solutions for planar post-buckling configurations of nanorods with various end constraints under the action of axial forces [23], and for spatial post-buckling configurations of nanorods under the action of axial forces and couples [24]. In the present study, the model is employed to determine the equations of infinitesimal deformations superimposed on a large one and to derive the expression of the work done in such deformations that is used to assess the infinitesimal stability of equilibrium configurations of nonlocal rods.

After a presentation of Kirchhoff's theory and in view of its extension to rods made of nonlocal materials, the present paper contains a discussion on the constitutive equation employed to describe the response of a rod of finite length made of a nonlocal material. In Kirchhoff's theory, the stresses acting on the section of the rod at a certain point of the axis produce their effects by a resultant force and a resultant moment. While the resultant force is a reactive variable not constitutively determined, the resultant moment is given by a linear constitutive equation as function of the value, at the considered point, of a strain measure which depends on the changes in curvature and twist with respect to a natural configuration. In a rod made of a nonlocal material of Eringen's type, the resultant moment at a section depends on the values of the strain measure at all the points of the axis, with an influence decreasing with the distance. The constitutive equation of an Eringen's nonlocal material has an integral form, and its introduction into the equilibrium equations produces an integro-differential problem, in general difficult to solve. For this reason, it is customary in application of Eringen's theory to employ nonlocal constitutive equations in forms that produce differential equilibrium problems, although in general such constitutive equations are only approximate for bodies having finite extension. The discussion in the present paper above mentioned has the purpose of investigating the relationship between the employed constitutive equation and the constitutive equation in integral form.

Then, the form that the criterion of infinitesimal stability takes for rods made of nonlocal materials is studied. Starting from the nonlinear equilibrium equations of the present theory, the equations that describe infinitesimal deformations superimposed on a finite deformation are deduced. The criterion of infinitesimal stability, which does not require that the body be hyperelastic, can be formulated as the requirement that the work done by the increments of the stress, with respect to their values in the configuration whose stability is under consideration, be nonnegative in any infinitesimal deformation. To make use of the criterion in the problems of interest in the present paper, we write the expression of the work done by the additional external loads associated with a superimposed infinitesimal deformation of the rod and examine its sign using as test functions the eigenfunctions of the linearized equilibrium problem. We apply the criterion to examine the stability of rods in three states of equilibrium: (i) simply supported rods subject to axial compressive forces; (ii) rods with one end clamped and the other one constrained to have the tangent parallel to the undeformed axial curve, subject to axial forces and twisting moments; (iii) naturally straight rods that have been deformed into annular configurations with the addition of twist. The results show that the considered equilibrium configurations are stable for values of the loads nonexceeding the critical ones, that is, the first loads that determine bifurcation from the trivial solutions; in particular, in the third example, the annular configurations are stable for values of the twisting moment nonexceeding the critical ones. Since the critical values of the external loads of rods made of nonlocal materials are lower than those of rods having the same geometry and made of usual materials with the same elastic moduli, the stability analysis confirms the results, obtained in the papers [22–24] on nonlinear deformations of nanorods, that the presence of a nonlocal material has effects analogous to those due to a reduction in the rigidity of the rods.

In the paper, use is made of the convention of the sum on repeated indices; the symbols $\cdot$, $\times$, and $\otimes$ denote, respectively, the scalar, vector, and tensor products of vectors. The derivative with respect to a variable is denoted by a comma followed by the symbol of the variable, e.g., $\frac{\mathrm{d}(\,\cdot\,)}{\mathrm{d}s} = (\,\cdot\,)_{,s}$. For the sake of conciseness, a rod made of a nonlocal elastic material is referred to as a "nonlocal rod" and a rod made of a usual elastic material as a "classic rod."

2 Extension of Kirchhoff's theory to rods made of nonlocal materials

We model a nonlocal rod as a body whose deformations are described by the nonlinear kinematics of Kirchhoff's theory [25], and that is made of a homogeneous elastic material whose response is given by a constitutive equation of Eringen's type [6]. In the following parts of this Section, we recall the principal features of Kirchhoff's theory, discuss the form of the constitutive equation that is employed for the material response of a rod, and write the nonlinear equilibrium equations of a nonlocal rod that will be linearized in the next Section.

2.1 Kirchhoff's theory

In Kirchhoff's theory, a rod $\mathcal{R}$ is viewed as a slender three-dimensional body that can undergo deformations in which the rotations may be large and the strains with respect to an undistorted configuration remain small. The theory is complete to within an error of order two in a dimensionless measure of thickness, curvature, twist, and extension. Expositions of the theory are given in [25], and [26], and in a modern notation in [27], and [28].

We here consider rods that are *naturally straight* and *kinetically symmetric* in the sense that they are cylindrical in an undistorted, stress free configuration $\mathfrak{C}^u$ and, in that configuration, their (congruent) cross sections have equal the two principal moments of inertia. In $\mathfrak{C}^u$, the centroids of the cross sections of a rod $\mathcal{R}$ are on a straight segment $\mathcal{C}^u$, of length L, whose points have the positions $\boldsymbol{x}^u = \boldsymbol{x}^u(s)$, where $s \in (0, L)$ is an arc-length parameter. The *axis* of the rod is the set of material points that lie on $\mathcal{C}^u$. With each point $\boldsymbol{x}^u(s)$ of $\mathcal{C}^u$ is associated a triad of orthonormal vectors $(\boldsymbol{d}_1^u, \boldsymbol{d}_2^u, \boldsymbol{d}_3^u)$, with $\boldsymbol{d}_3^u$ tangent to $\mathcal{C}^u$, and $\boldsymbol{d}_1^u$ and $\boldsymbol{d}_2^u$ along the directions of the principal moments of inertia of the cross section. Since $\mathfrak{C}^u$ is naturally straight, the vectors $\boldsymbol{d}_i^u$, $i = 1, 2, 3$ can be assumed to be independent of s.

A general configuration $\mathfrak{C}$ of a rod is described by giving: (i) the curve $\mathcal{C}$, of equation $\boldsymbol{x} = \boldsymbol{x}(s)$, that is formed by the material points of the axis and is called the *axial curve* of $\mathcal{R}$ in the configuration $\mathfrak{C}$ and (ii) for each point $\boldsymbol{x}(s)$ of $\mathcal{C}$, a triad of unit vectors $(\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s))$ such that $\boldsymbol{d}_3(s)$ coincides with the tangent $\boldsymbol{t}(s)$ to $\mathcal{C}$, $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ are tangent, at $\boldsymbol{x}(s)$, to the curves on which lie the material points that in the configuration $\mathfrak{C}^u$ are along the directions of the principal axes of inertia of the cross section at $\boldsymbol{x}^u(s)$. The triad $(\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s))$ is called the *torsion-flexure triad* of the rod at $\boldsymbol{x}(s)$, and the axes through $\boldsymbol{x}(s)$ having the directions of the vectors of the triad are the *principal torsion-flexure axes* (cf. [26, Sect. 252]).

To within the order of magnitude at which the theory holds, the vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ can be considered perpendicular to each other and to the axial curve, so that the torsion-flexure triad $(\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s))$ can be considered orthonormal and thought of as obtained from the triad $(\boldsymbol{d}_1^u, \boldsymbol{d}_2^u, \boldsymbol{d}_3^u)$ by means of a rotation $\boldsymbol{R}(s) = \boldsymbol{d}_i(s) \otimes \boldsymbol{d}_i^u$. To within the same order of approximation, the extensions of the axial curve can be disregarded, and the deformations of the rod can be assumed to leave invariant the distance between points of the axis measured along the axial curve. Hence, in each configuration of $\mathcal{R}$, at each point s, the following *inextensibility condition* holds:

$$\boldsymbol{d}_3(s) = \boldsymbol{t}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s) = \boldsymbol{x}_{,s}(s), \tag{1}$$

whose content is equivalently expressed by the equation

$$\boldsymbol{x}_{,s} \cdot \boldsymbol{x}_{,s} = 1.$$

The strain in a configuration $\mathfrak{C}$, with respect to the naturally straight configuration $\mathfrak{C}^u$, is measured by the *curvature vector* $\boldsymbol{\kappa}$,

$$\boldsymbol{\kappa} = \boldsymbol{t} \times \boldsymbol{t}_{,s} + \kappa \boldsymbol{t} = k\boldsymbol{b} + \kappa \boldsymbol{t}.$$

Here k is the *geometric curvature* of the axial curve $\mathcal{C}$, $\kappa = \boldsymbol{\kappa} \cdot \boldsymbol{t}$ is the *twist* of the rod, and $\boldsymbol{b}$ is the *binormal* of $\mathcal{C}$. The vector $\boldsymbol{\kappa}$ can be interpreted as the angular velocity of the torsion-flexure triad regarded as a frame moving along the axial curve with unit velocity; accordingly,

$$\boldsymbol{\kappa} = \frac{1}{2}\boldsymbol{d}_i \times \boldsymbol{d}_{i,s}, \qquad \boldsymbol{d}_{i,s} = \boldsymbol{\kappa} \times \boldsymbol{d}_i, \quad i = 1, 2, 3. \tag{2.1,2}$$

At each point s, the pair $(\boldsymbol{d}_1(s), \boldsymbol{d}_2(s))$ lies in the normal plane of the axial curve, spanned by the *principal normal* $\boldsymbol{n}(s)$ and the binormal $\boldsymbol{b}(s)$ of $\mathcal{C}$, but need not coincide with the pair $(\boldsymbol{n}(s), \boldsymbol{b}(s))$; thus, in general, the twist κ of a rod $\mathcal{R}$ does not coincide with the geometric torsion τ of its axial curve $\mathcal{C}$. Denoting by $\varphi(s)$ the angle from $\boldsymbol{n}(s)$ to $\boldsymbol{d}_1(s)$, the relation between the twist κ of $\mathcal{R}$ and the torsion τ of $\mathcal{C}$ is $\kappa(s) = \tau(s) + \varphi_{,s}(s)$.

In Kirchhoff's theory, the equilibrium of a rod subject only to terminal forces and couples is governed by the equations

$$\boldsymbol{F}_{,s} = \boldsymbol{0}, \qquad \boldsymbol{M}_{,s} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0} \tag{3.1,2}$$

where the vector $\boldsymbol{F}(s)$, called the *resultant force*, is the resultant of the Piola stresses that the material on the side of increasing s exerts on the surface formed in the configuration $\mathfrak{C}$ by the material points that in $\mathfrak{C}^u$ are on the cross section at s, and the vector $\boldsymbol{M}(s)$, called the *resultant moment*, is the moment of those stresses about $\boldsymbol{x}(s)$. The resultant force has a reactive nature and is not constitutively determined; the resultant moment is given by Kirchhoff's constitutive equation, that for a naturally straight, kinetically symmetric, materially homogeneous, isotropic, elastic rod has the form

$$\boldsymbol{M} = \boldsymbol{C}\boldsymbol{\kappa} = EI\boldsymbol{t} \times \boldsymbol{t}_{,s} + GJ\kappa \boldsymbol{t} \tag{4}$$

where $\boldsymbol{C}$ is the tensor

$$\boldsymbol{C} = EI(\boldsymbol{I} - \boldsymbol{t} \otimes \boldsymbol{t}) + GJ\boldsymbol{t} \otimes \boldsymbol{t},$$

with $\boldsymbol{I}$ the identity tensor, and EI and GJ the flexural and torsional rigidities of the cross sections in the configuration $\mathfrak{C}^u$. The components of $\boldsymbol{M}$ along the directions of the unit vectors $\boldsymbol{b}$ and $\boldsymbol{t}$,

$$\boldsymbol{M} \cdot \boldsymbol{b} = EI\boldsymbol{t} \times \boldsymbol{t}_{,s} \cdot \boldsymbol{b} = EIk, \qquad \boldsymbol{M} \cdot \boldsymbol{t} = GJ\kappa$$

are the *bending moment* and the *twisting moment*, respectively; as proved by Poisson [29], in a rod subject only to terminal loads, the twisting moment is constant along the axial curve. Namely, from the scalar product of Eq. (3.2) by $\boldsymbol{t}$, making use of the constitutive equation (4) and taking into account that $\boldsymbol{t}_{,s} \cdot \boldsymbol{t} = 0$, one has

$$0 = (\boldsymbol{M}_{,s} + \boldsymbol{t} \times \boldsymbol{F}) \cdot \boldsymbol{t} = \boldsymbol{M}_{,s} \cdot \boldsymbol{t} = (\boldsymbol{M} \cdot \boldsymbol{t})_{,s} - (EI\boldsymbol{t} \times \boldsymbol{t}_{,s} + GJ\kappa\boldsymbol{t}) \cdot \boldsymbol{t}_{,s} = (\boldsymbol{M} \cdot \boldsymbol{t})_{,s}.$$

In view of Eq. (3.1), the force vector $\boldsymbol{F}$ is constant; denoting by $\boldsymbol{e} = \boldsymbol{F}/|\boldsymbol{F}|$ the unit vector having its oriented direction, the scalar product of Eq. (3.2) by $\boldsymbol{e}$,

$$0 = (\boldsymbol{M}_{,s} + \boldsymbol{t} \times \boldsymbol{F}) \cdot \boldsymbol{e} = \boldsymbol{M}_{,s} \cdot \boldsymbol{e} = (\boldsymbol{M} \cdot \boldsymbol{e})_{,s},$$

shows that also the component of $\boldsymbol{M}$ along the direction of $\boldsymbol{F}$ is independent of s.

2.2 Nonlocal constitutive equation

The equilibrium of a nonlocal rod is governed by Eqs. (3.1,2) in which the usual resultant moment $\boldsymbol{M}$ is replaced by the *nonlocal moment* $\widetilde{\boldsymbol{M}}$,

$$\boldsymbol{F}_{,s} = \boldsymbol{0}, \qquad \widetilde{\boldsymbol{M}}_{,s} + \boldsymbol{t} \times \boldsymbol{F} = \boldsymbol{0}. \tag{5.1,2}$$

According to Eringen's theory, in a rod made of a nonlocal material the resultant moment at a point s of the axial curve $\mathcal{C}$ depends on the strain at each point ς of $\mathcal{C}$, with an influence that decreases with the distance of ς from s. Hence, the nonlocal moment $\widetilde{\boldsymbol{M}}$ is given by an integral equation of the type

$$\widetilde{\boldsymbol{M}}(s) = \int_{\mathcal{C}} \lambda(s, \varsigma)\boldsymbol{M}(\varsigma)\mathrm{d}\varsigma \tag{6}$$

where $\lambda = \lambda(s, \varsigma)$ is the *nonlocal kernel* which accounts for the reduction with the distance of the influence of the strain at ς on the stress at s, and the integral is extended to the axial curve $\mathcal{C}$ of the rod. Introduction of the constitutive law (6) into the equilibrium equation (5.2) produces an integro-differential problem. To avoid the difficulties of solving a problem of such a type, as it is customary in applications of Eringen's theory, the nonlocal kernel λ is taken to be the Green's function of a linear differential operator $\mathcal{L}$. In that case (cf., e.g., [30, Chapter V]), the kernel λ, considered as a function of s, except at the point $s = \varsigma$ is a solution of the differential equation

$$\mathcal{L}[\lambda(s)] = 0, \tag{7}$$

the nonlocal moment $\widetilde{\boldsymbol{M}}(s)$ satisfies the equation

$$\mathcal{L}[\widetilde{\boldsymbol{M}}(s)] = \boldsymbol{M}(s), \tag{8}$$

and, conversely, if the function $\widetilde{\boldsymbol{M}}(s)$ satisfies Eq. (8) and appropriate boundary conditions, $\widetilde{\boldsymbol{M}}(s)$ can be represented in the form (6).

The kernel function λ can be obtained by matching the dispersion curves of plane waves with those of atomic lattice dynamics [31]; in one-dimensional problems possible choices for λ and the corresponding linear differential operator $\mathcal{L}$ are

$$\lambda(s, \varsigma) = \frac{1}{2\varepsilon} e^{-|s-\varsigma|/\varepsilon}, \qquad \mathcal{L} = 1 - \varepsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2}. \tag{9.1,2}$$

The parameter ε depends on the material comprising the rod and governs nonlocality in the sense that, when ε goes to zero, the nonlocal effects disappear and the behavior of the material is described by a classic constitutive equation; it is usually written in the form

$$\varepsilon = \hat{\tau} l_{\mathrm{e}} = \left(e_{\mathrm{o}} \frac{l_{\mathrm{i}}}{l_{\mathrm{e}}} \right) l_{\mathrm{e}} \tag{10}$$

where e_{o} is a material modulus, l_{i} is an internal characteristic length (e.g., a lattice parameter), and l_{e} is an external characteristic length (e.g., a wavelength). It follows from the definition (9.1) that

$$\begin{aligned} \lambda &= \frac{1}{2\varepsilon} e^{-(\varsigma - s)/\varepsilon}, \quad \lambda_{,s} = +\frac{1}{\varepsilon}\lambda, \quad \lambda_{,ss} = \frac{1}{\varepsilon^2}\lambda, \quad \text{for } s \leq \varsigma, \\ \lambda &= \frac{1}{2\varepsilon} e^{-(s - \varsigma)/\varepsilon}, \quad \lambda_{,s} = -\frac{1}{\varepsilon}\lambda, \quad \lambda_{,ss} = \frac{1}{\varepsilon^2}\lambda, \quad \text{for } s \geq \varsigma. \end{aligned} \tag{11}$$

Thus, the kernel (9.1) satisfies Eq. (7) for $\mathcal{L}$ given by Eq. (9.2), and considered as a function of s, its first derivative has in ς the jump

$$\lim_{t \to 0} \lambda_{,s}(\varsigma + t, \varsigma) - \lim_{t \to 0} \lambda_{,s}(\varsigma - t, \varsigma) = -\frac{1}{\varepsilon^2} \tag{12}$$

where t approaches 0 taking positive values. Using this equation and the rule of differentiation of an integral with respect to a parameter, one can show that the nonlocal moment $\widetilde{\boldsymbol{M}}$ defined by Eq. (6) satisfies the differential equation (8), and this fact is exploited to transform the integro-differential equilibrium problem into a differential problem.

As remarked by Eringen [31], an equation of the type (6) holds for a body of infinite extension and, since the kernel goes rapidly to zero, in a finite body is valid for points that are not too close to the boundary. With reference to the constitutive equation for nonlocal rods, the resultant moment given by Eq. (6) with the kernel (9.1) is approximate for a rod of finite length L, because λ and $\widetilde{\boldsymbol{M}}$ do not satisfy homogeneous conditions at the ends of the rod[1]. Namely, for a rod of infinite length, the nonlocal resultant moment (6) with the kernel (9.1),

$$\widetilde{\boldsymbol{M}}(s) = \frac{1}{2\varepsilon} \int_{\mathcal{C}} e^{-|s-\varsigma|/\varepsilon} \boldsymbol{M}(\varsigma) \mathrm{d}\varsigma, \tag{13}$$

satisfies Eq. (8), which takes the form

$$\widetilde{\boldsymbol{M}}_{,ss}(s) - \varepsilon^2 \widetilde{\boldsymbol{M}}(s) = \boldsymbol{M}(s), \tag{14}$$

[1] According to the definitions of Courant and Hilbert [30, Sect. 14.1] in this case the kernel λ is a "fundamental solution" of the differential equation $\mathcal{L}[\lambda(s)] = 0$.

and, conversely, the solution $\widetilde{\boldsymbol{M}}(s)$ of this equation is the function (13). Moreover, differentiation of the Eq. (5.2), taking (5.1) into account, gives

$$\widetilde{\boldsymbol{M}}_{,ss}(s) = -\boldsymbol{t}_{,s}(s) \times \boldsymbol{F},$$

then, substitution into Eq. (14), putting $\mu = \varepsilon^2$, furnishes

$$\widetilde{\boldsymbol{M}}(s) = \boldsymbol{M}(s) - \mu \boldsymbol{t}_{,s}(s) \times \boldsymbol{F}. \tag{15}$$

Thus, *for a rod of infinite length in equilibrium under terminal loads, the constitutive equation* (15) *is equivalent to the integral constitutive equation* (13). In general, such result does not hold for a rod of finite length for which Eq. (15) is an approximation of the integral equation (13).

To see to which equation the constitutive equation (15) is equivalent in rods of finite length, we write Eq. (14) with ς in place of s, multiply both its members by the kernel function, and integrate with respect to ς on the intervals $(0, s)$ and (s, L); making use of Eqs. (7), (11), and (12), we obtain (cf. [23])

$$\widetilde{\boldsymbol{M}}(s) = \frac{1}{2\varepsilon}\int_{\mathcal{C}} e^{-|s-\varsigma|/\varepsilon}\boldsymbol{M}(\varsigma)\mathrm{d}\varsigma + \frac{1}{2}\left[e^{-|s-\varsigma|/\varepsilon}(\varepsilon\widetilde{\boldsymbol{M}}_{,\varsigma}(\varsigma) - \widetilde{\boldsymbol{M}}(\varsigma))\right]_{\varsigma=0}^{\varsigma=L}; \tag{16}$$

by use of Eq. (5.1,2), this equation can be put in the form

$$\widetilde{\boldsymbol{M}}(s) = \frac{1}{2\varepsilon}\int_{\mathcal{C}} e^{-|s-\varsigma|/\varepsilon}\boldsymbol{M}(\varsigma)\mathrm{d}\varsigma - \frac{1}{2}\left[e^{-|s-\varsigma|/\varepsilon}(\varepsilon\boldsymbol{t}(\varsigma) \times \boldsymbol{F} + \widetilde{\boldsymbol{M}}(\varsigma))\right]_{\varsigma=0}^{\varsigma=L}, \tag{17}$$

which gives the solution of the differential equation (14) with the boundary term expressed by means of the resultant forces and moments at the rod ends. Since the boundary terms in the last two equations depend on the kernel (9.1), they have an influence on the nonlocal moment that becomes negligible in sections that are not close to the rod ends[2]. We can check that, conversely, the function $\widetilde{\boldsymbol{M}}$ given by Eq. (16) satisfies Eq. (14). By splitting the interval of integration $(0, L)$ into the two intervals $(0, s)$ and (s, L), using the rule of differentiation of an integral with respect to a parameter, and taking Eq. (12) into account, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^L \lambda(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma = \int_0^s \lambda_{,s}(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma + \int_s^L \lambda_{,s}(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\int_0^L \lambda(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma = \int_0^s \lambda_{,ss}(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma + \int_s^L \lambda_{,ss}(s,\varsigma)\boldsymbol{M}(\sigma)\mathrm{d}\sigma - \frac{1}{\varepsilon^2}\boldsymbol{M}(s)$$

where λ is given by (9.1); the last two equations together with Eq. (7) imply the desired result. For a rod of finite length in equilibrium under terminal loads, Eqs. (14) and (16) can be put in the forms (15) and (17), respectively. Hence, for such a rod, the constitutive equations (15) and (17) are equivalent.

In what follows we will make use of the constitutive equation (15). We observe that it can be expressed as the sum of two terms having the directions of the tangent and the binormal to the axial curve, as the constitutive equation (4) for classic rods. Since $\boldsymbol{t}_{,s} = k\boldsymbol{n}$, the vector $\mu\boldsymbol{t}_{,s} \times \boldsymbol{F}$ has a null component along the direction of the principal normal of $\mathcal{C}$. Writing the resultant force $\boldsymbol{F}$ in the form $F\boldsymbol{e}$, we have

$$\begin{aligned}\mu\boldsymbol{t}_{,s} \times \boldsymbol{F} = \mu F\boldsymbol{t}_{,s} \times \boldsymbol{e} &= \mu F(\boldsymbol{t}_{,s} \times \boldsymbol{e} \cdot \boldsymbol{t})\boldsymbol{t} + \mu F(\boldsymbol{t}_{,s} \times \boldsymbol{e} \cdot \boldsymbol{b})\boldsymbol{b}\\ &= \mu F(\boldsymbol{t} \times \boldsymbol{t}_{,s} \cdot \boldsymbol{e})\boldsymbol{t} + \mu F(\boldsymbol{b} \times k\boldsymbol{n} \cdot \boldsymbol{e})\boldsymbol{b}\\ &= \mu F(\boldsymbol{t} \times \boldsymbol{t}_{,s} \cdot \boldsymbol{e})\boldsymbol{t} - \mu F(\boldsymbol{t} \cdot \boldsymbol{e})\boldsymbol{t} \times \boldsymbol{t}_{,s}.\end{aligned}$$

In view of this result and Eq. (4), the constitutive equation (15) becomes

$$\widetilde{\boldsymbol{M}} = (EI + \mu F(\boldsymbol{t} \cdot \boldsymbol{e}))\,\boldsymbol{t} \times \boldsymbol{t}_{,s} + \big(GJ\kappa - \mu F\boldsymbol{t} \times \boldsymbol{t}_{,s} \cdot \boldsymbol{e}\big)\,\boldsymbol{t}. \tag{18}$$

The terms on the right-hand member of the equation are the expressions of the bending moment and the twisting moment in a nonlocal rod in equilibrium; when ε (and thus μ) goes to zero they tend to the expressions (4) of the moments in a classic rod.

[2] The boundary conditions that the nonlocal moment should satisfy in order that its integral expression (13) hold in a beam of finite length, and alternative forms of the kernel function have been discussed in [32] and [33].

2.3 Equilibrium of a nonlocal rod

Substitution of the constitutive equation (15) into Eq. (5.2) yields the following form of the equilibrium equations of a nonlocal rod subject to loads applied at its ends:

$$\boldsymbol{F}_{,s} = \boldsymbol{0}, \qquad \boldsymbol{M}_{,s} + (\boldsymbol{t} - \mu \boldsymbol{t}_{,ss}) \times \boldsymbol{F} = \boldsymbol{0}. \tag{19.1,2}$$

These equations and the inextensibility condition (1) imply that

$$\boldsymbol{M} + (\boldsymbol{x} - \mu \boldsymbol{t}_{,s}) \times \boldsymbol{F} = \boldsymbol{c}_\mathrm{o} \tag{20}$$

where $\boldsymbol{c}_\mathrm{o}$ is a vector constant that, by changing the origin of the position vectors, can be made parallel to $\boldsymbol{F}$, or null if $\boldsymbol{M} \cdot \boldsymbol{F} = 0$. Precisely (cf. [22]), the new origin must be placed on the straight line whose equation is

$$\boldsymbol{y} = \frac{\boldsymbol{F} \times (\boldsymbol{M}(\overline{s}) + (\boldsymbol{x}(\overline{s}) - \mu \boldsymbol{x}_{,ss}(\overline{s})) \times \boldsymbol{F})}{\boldsymbol{F} \cdot \boldsymbol{F}} + t\boldsymbol{F}, \qquad t \in \mathbb{R},$$

where $\overline{s}$ is any point of the axial curve.

The components of the nonlocal resultant moment along the directions of $\boldsymbol{t}$ and $\boldsymbol{e} = \boldsymbol{F}/|\boldsymbol{F}|$ are independent of s, as for the usual resultant moment in a classic rod. Namely, taking the scalar product of Eq. (19.2) by $\boldsymbol{t}$ one has

$$\begin{aligned} 0 &= (\boldsymbol{M}_{,s} + (\boldsymbol{t} - \mu \boldsymbol{t}_{,ss}) \times \boldsymbol{F}) \cdot \boldsymbol{t} = (\boldsymbol{M}_{,s} - \mu \boldsymbol{t}_{,ss} \times \boldsymbol{F}) \cdot \boldsymbol{t} \\ &= ((\boldsymbol{M} - \mu \boldsymbol{t}_{,s} \times \boldsymbol{F}) \cdot \boldsymbol{t})_{,s} - (\boldsymbol{M} - \mu \boldsymbol{t}_{,s} \times \boldsymbol{F}) \cdot \boldsymbol{t}_{,s} = ((\boldsymbol{M} - \mu \boldsymbol{t}_{,s} \times \boldsymbol{F}) \cdot \boldsymbol{t})_{,s}; \end{aligned}$$

thus, the nonlocal twisting moment

$$a_\mathrm{o} = (\boldsymbol{M} - \mu \boldsymbol{t}_{,s} \times \boldsymbol{F}) \cdot \boldsymbol{t} = GJ\kappa - \mu F \boldsymbol{t} \times \boldsymbol{t}_{,s} \cdot \boldsymbol{e},$$

is constant along the axial curve. Moreover, by taking the scalar product of Eq. (20) by $\boldsymbol{e}$, one finds that the component of the nonlocal resultant moment along the direction of the resultant force $\boldsymbol{F}$ is equal to the constant $c_\mathrm{o} = \boldsymbol{c}_\mathrm{o} \cdot \boldsymbol{e}$,

$$c_\mathrm{o} = \big(\boldsymbol{M} - \mu \boldsymbol{t}_{,s} \times \boldsymbol{F}\big) \cdot \boldsymbol{e} = \boldsymbol{M} \cdot \boldsymbol{e},$$

and coincides with the corresponding component of the usual resultant moment. For future use, we introduce the quantities

$$\alpha = \frac{a_\mathrm{o}}{EI}, \qquad \gamma = \frac{c_\mathrm{o}}{EI},$$

and observe that, since α and γ are independent of s, if there is a section of the rod in which $\boldsymbol{t} = \pm \boldsymbol{e}$, then it is $\alpha = \pm \gamma$.

From this point on, we adopt dimensionless variables that are obtained by taking $\sqrt{I/A}$ as unit of length and EA as unit of force; accordingly we make the replacements

$$\begin{aligned} &\{s, \boldsymbol{x}\} \to \{s, \boldsymbol{x}\}\sqrt{I/A}, \qquad \{k, \kappa\} \to \{k, \kappa\}\sqrt{A/I}, \\ &\mu \to \mu\, I/A, \qquad F \to F(EA), \qquad \boldsymbol{M} \to \boldsymbol{M}(E\sqrt{AI}). \end{aligned}$$

In terms of the dimensionless variables, the constitutive equation (4) of a classic rod becomes

$$\boldsymbol{M} = \boldsymbol{C}\boldsymbol{\kappa} = \boldsymbol{t} \times \boldsymbol{t}_{,s} + \Omega \kappa \boldsymbol{t}, \tag{21}$$

where $\Omega = \dfrac{GJ}{EI}$, and the dimensionless tensor $\boldsymbol{C}$ is

$$\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{t} \otimes \boldsymbol{t}) + \Omega \boldsymbol{t} \otimes \boldsymbol{t};$$

the equilibrium equations (3.1,2) and (5.1,2) of classic and nonlocal rods, and the nonlocal constitutive equations (13), (15), and (17) keep unaltered their form but, in them, the moment $\boldsymbol{M}$ has the expression (21); the dimensionless form of the constitutive equation (18) of a nonlocal material is

$$\widetilde{\boldsymbol{M}} = (1 + \mu F \chi)\, \boldsymbol{t} \times \boldsymbol{t}_{,s} + \alpha \boldsymbol{t}$$

where $\chi = \boldsymbol{t} \cdot \boldsymbol{e}$ is the component of the tangent to $\mathcal{C}$ along the direction of $\boldsymbol{F}$.

Fig. 1 Equilibrium configuration of a nonlocal rod with the end at $s = 0$ clamped and the end at $s = L$ constrained to have the tangent parallel to the undeformed axial curve $\mathcal{C}^u$, subject at the end $s = L$ to a force parallel to $\mathcal{C}^u$ and a couple whose component parallel to $\mathcal{C}^u$ is assigned. In dimensionless units the length of the rod is $L = 360$, its circular cross section has diameter $d = 4$, the force is $F = 6.853 \times 10^{-4}$, and the twisting moment is $\alpha = -3.75 \times 10^{-2}$

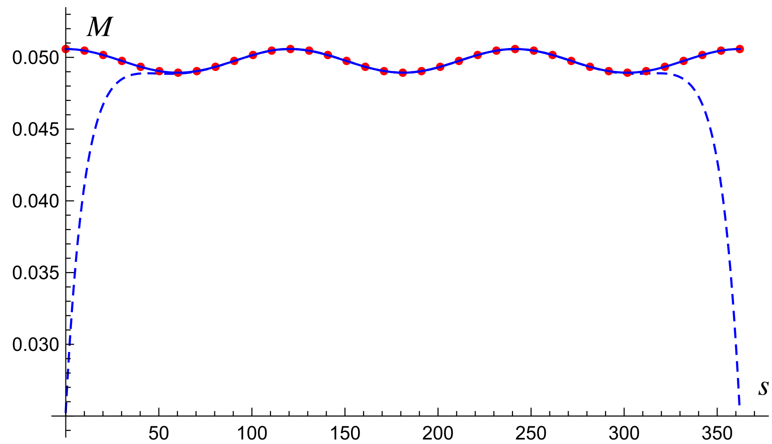

Fig. 2 Comparison of the moduli of the resultant moments expressed, for the nonlocal rod of Fig. 1, by the constitutive equation (15) (red dots) employed in the equilibrium equations, and by the integral constitutive equations (17) (blue curve) and (13) (dashed blue curve). The modulus of the moments given by the constitutive equation (17) coincides with that of the equilibrium moments given by Eq. (15), while the modulus of the moments given by the constitutive equation (13) practically coincides with that of the equilibrium moments except for the parts of the axial curve close to the rod ends

2.4 An illustrative example

We have observed that, for a nonlocal rod in equilibrium under the action of end loads, the constitutive equation (15) is equivalent to the integral constitutive equation (13) if the rod has infinite length, and to the constitutive equation (17) if the rod has finite length. We conclude this Section with an example in which, with reference to a nonlocal rod in equilibrium in a spatial configuration, we compare the values of the resultant moment along the axial curve given by the constitutive equations (13), (15), and (17).

We consider the equilibrium, in a post-buckling configuration, of a nonlocal rod which has one end clamped (i.e., constrained to have null displacements and rotations) and the other one constrained to have the tangent parallel to the direction of the undeformed straight axial curve $\mathcal{C}^u$ but otherwise free to move, and which is subject to a force parallel to the direction of $\mathcal{C}^u$ and a couple whose component along the direction of $\mathcal{C}^u$ is assigned (Fig. 1). We refer to a rod that in dimensionless units has length $L = 360$, with a circular cross section whose diameter is $d = 4$, and utilize the results obtained in [24], where the exact solution in terms of Weierstrass elliptic functions of the nonlinear equations governing the problem is deduced. From that solution, it is possible to compute the curvature and twist of the deformed axial curve that, by means of the constitutive equation (21), give the usual resultant moment; then, this moment is introduced into the integral constitutive equations (13) and (17) to obtain the corresponding expressions of the nonlocal moment[3]. According to Eringen's remark recalled in Sect. 2.2, we expect that, in regions close to the rod ends, the moment given by Eq. (13) will depart from the equilibrium values of the moment given by Eq. (15).

The plots of Fig. 2 compare: (i) the modulus of the resultant moment given by the constitutive equation (15) employed in the equilibrium equations (red dots), (ii) the modulus of the moment given by the integral constitutive equation (13) which holds for a rod of infinite length (dashed blue curve), and (iii) the modulus of the moment given by the integral constitutive equation (17) (blue curve), which holds for a rod of finite length. The plots show that the moduli of the moments given by the constitutive equations (17) and (15) coincide

[3] Integration of Eq. (5.2) yields $\widetilde{\boldsymbol{M}} - \boldsymbol{x} \times \boldsymbol{F} = \boldsymbol{c}_o$, from which the values of the resultant moment in the boundary term of Eq. (17) can be deduced.

on the entire axial curve while, as it was expected, the modulus of the moment given by Eq. (13) practically coincides with that of the equilibrium moment except in regions that are close to the rod ends.

Remark An interpretation of the effects of the boundary terms in the constitutive equation (17) can be obtained as follows. One can imagine of extracting a central part from the rod of Fig. 1 and keeping it in equilibrium by applying at its ends the stress resultants due to the actions of the removed parts of the rod. Let $\mathcal{R}$ denote the original rod, $\mathcal{R}^\star$ its extracted part, and S a section of $\mathcal{R}^\star$. In the original equilibrium state of $\mathcal{R}$, the resultant moment at S was influenced by the strain at points of $\mathcal{R}$ that have been removed and do not belong to $\mathcal{R}^\star$, but the resultant moment at S in $\mathcal{R}^\star$ given by Eq. (17) is unaltered with respect to that of the original state because the stress resultants applied at ends of $\mathcal{R}^\star$ influence the resultant moment in the interior of the rod through the boundary terms of Eq. (17).

3 Infinitesimal deformations superimposed on a finite deformation

We now deduce the equations governing the infinitesimal deformations of a nonlocal rod from an equilibrium configuration $\mathfrak{C}$ to an equilibrium configuration $\mathfrak{C}'$. We denote by a prime the quantities pertaining to the configuration $\mathfrak{C}'$; accordingly, in this configuration $\mathcal{C}'$ is the axial curve, $\boldsymbol{x}'$ are the positions of the points of $\mathcal{C}'$, $\boldsymbol{t}'$ is the tangent to $\mathcal{C}'$, $\boldsymbol{\kappa}'$ is the curvature vector, $\boldsymbol{F}'$, $\boldsymbol{M}'$, and $\widetilde{\boldsymbol{M}}'$ are the resultant force, the classic, and nonlocal resultant moments.

In the configuration $\mathfrak{C}$, that is a state of equilibrium for the nonlocal rod, Eqs. (5) hold, and the resultant moment is given by the constitutive equation (15). Let $\boldsymbol{u} = \boldsymbol{u}(s)$ be the infinitesimal displacement that transforms the points $\boldsymbol{x}$ of $\mathcal{C}$ into the points $\boldsymbol{x}'$ of $\mathcal{C}'$,

$$\boldsymbol{u}(s) = \boldsymbol{x}'(s) - \boldsymbol{x}(s), \quad s \in (0, L). \tag{22}$$

The orthonormal vectors $\boldsymbol{d}_i'$, $i = 1, 2, 3$, forming the torsion-flexure triad in the configuration $\mathfrak{C}'$ are obtained from the corresponding vectors $\boldsymbol{d}_i$ of $\mathfrak{C}$ by means of an infinitesimal rotation that can be represented as $\boldsymbol{I}+\boldsymbol{W}(s)$, where $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{W} = \boldsymbol{W}(s)$ is a skew tensor whose axial vector we denote by $\boldsymbol{w} = \boldsymbol{w}(s)$,

$$\boldsymbol{d}_i'(s) = (\boldsymbol{I} + \boldsymbol{W}(s))\,\boldsymbol{d}_i(s) = \boldsymbol{d}_i(s) + \boldsymbol{w}(s) \times \boldsymbol{d}_i(s), \quad i = 1, 2, 3. \tag{23.1–3}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ determine the infinitesimal deformation. We assume that the norm of $\boldsymbol{w}$, those of its first two derivatives, and that of the increment $\Delta\boldsymbol{F} = \boldsymbol{F}' - \boldsymbol{F}$ of the force vector, are "small" and we neglect their products in the derivations that follow. Substitution of the expressions (23) of the vectors $\boldsymbol{d}_i'$ into the formula (2.1), written for the curvature vector $\boldsymbol{\kappa}'$ of the configuration $\mathfrak{C}'$, at the first order furnishes

$$\boldsymbol{\kappa}' = \frac{1}{2}\left(\boldsymbol{d}_i \times \boldsymbol{d}_{i,s} + (\boldsymbol{w} \times \boldsymbol{d}_i) \times \boldsymbol{d}_{i,s} + \boldsymbol{d}_i \times \left(\boldsymbol{w} \times \boldsymbol{d}_{i,s}\right)\right) + \boldsymbol{w}_{,s}. \tag{24}$$

Taking into account that $\boldsymbol{d}_i \times \boldsymbol{d}_{i,s}/2$ is the curvature vector $\boldsymbol{\kappa}$, and that, given three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, the properties of the vector product imply the identity

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \times \boldsymbol{a}_3 = \boldsymbol{a}_1 \times (\boldsymbol{a}_2 \times \boldsymbol{a}_3) - \boldsymbol{a}_2 \times (\boldsymbol{a}_1 \times \boldsymbol{a}_3), \tag{25}$$

Eq. (24) can be written as

$$\boldsymbol{\kappa}' = \boldsymbol{\kappa} + \boldsymbol{w} \times \boldsymbol{\kappa} + \boldsymbol{w}_{,s}. \tag{26}$$

In view of the condition of inextensibility, differentiation of Eq. (22) gives

$$\boldsymbol{u}_{,s} = \boldsymbol{t}' - \boldsymbol{t},$$

which, together with Eq. (23.3), yields

$$\boldsymbol{u}_{,s} = \boldsymbol{w} \times \boldsymbol{t}. \tag{27}$$

This equation implies that, when the condition of inextensibility holds, the infinitesimal rotation vector $\boldsymbol{w}$ can be represented as

$$\boldsymbol{w} = \boldsymbol{t} \times \boldsymbol{u}_{,s} + \beta\boldsymbol{t} \tag{28}$$

where $\beta = \beta(s)$ is the component along $\boldsymbol{t}$ of the infinitesimal rotation. Equation (27) implies also that the component of $\boldsymbol{u}_{,s}$ along the direction of the tangent to the axial curve vanishes,

$$\boldsymbol{u}_{,s} \cdot \boldsymbol{t} = 0. \tag{29}$$

The stress resultants in the configuration $\mathfrak{C}'$ are written as sum of those in $\mathfrak{C}$ and the increments associated with the superimposed infinitesimal deformation,

$$\boldsymbol{F}' = \boldsymbol{F} + \Delta\boldsymbol{F}, \qquad \widetilde{\boldsymbol{M}}' = \widetilde{\boldsymbol{M}} + \Delta\boldsymbol{M}. \tag{30.1,2}$$

As the force $\boldsymbol{F}$, also its infinitesimal increment $\Delta\boldsymbol{F}$ is not constitutively determined. The resultant moment $\widetilde{\boldsymbol{M}}'$ in $\mathfrak{C}'$ is given by the constitutive equation (15),

$$\widetilde{\boldsymbol{M}}' = \boldsymbol{M}' - \mu\boldsymbol{t}'_{,s} \times \boldsymbol{F}' = \boldsymbol{t}' \times \boldsymbol{t}'_{,s} + \Omega(\boldsymbol{\kappa}' \cdot \boldsymbol{t}')\boldsymbol{t}' - \mu\boldsymbol{t}'_{,s} \times \boldsymbol{F}';$$

substituting the expressions (23.3), (26), and (30.1) of $\boldsymbol{t}' = \boldsymbol{d}'_3$, $\boldsymbol{\kappa}'$, and $\boldsymbol{F}'$, and making use of the identity (25), at the first order we find that the additional resultant moment is

$$\Delta\boldsymbol{M} = \boldsymbol{w} \times \widetilde{\boldsymbol{M}} + \boldsymbol{C}\boldsymbol{w}_{,s} - \mu\boldsymbol{t}_{,s} \times (\Delta\boldsymbol{F} - \boldsymbol{w} \times \boldsymbol{F}) - \mu\left(\boldsymbol{w}_{,s} \times \boldsymbol{t}\right) \times \boldsymbol{F}. \tag{31}$$

By use of Eqs. (23) and (30), at the first order, the equilibrium equations (5) for the configuration $\mathfrak{C}'$ are

$$(\boldsymbol{F} + \Delta\boldsymbol{F})_{,s} = \boldsymbol{0}, \quad (\widetilde{\boldsymbol{M}} + \Delta\boldsymbol{M})_{,s} + \boldsymbol{t} \times (\boldsymbol{F} + \Delta\boldsymbol{F}) + (\boldsymbol{w} \times \boldsymbol{t}) \times \boldsymbol{F} = \boldsymbol{0}; \tag{32}$$

taking into account that $\mathfrak{C}$ is a state of equilibrium, they become

$$\Delta\boldsymbol{F}_{,s} = \boldsymbol{0}, \qquad \Delta\boldsymbol{M}_{,s} + (\boldsymbol{w} \times \boldsymbol{t}) \times \boldsymbol{F} + \boldsymbol{t} \times \Delta\boldsymbol{F} = \boldsymbol{0}. \tag{33.1,2}$$

These equations describe the infinitesimal deformations from $\mathfrak{C}$ to an equilibrium configuration $\mathfrak{C}'$. When $\Delta\boldsymbol{M}$ has the expression (31) and $\boldsymbol{w}$ is given by Eq. (28), Eqs. (33) and (29) form a system of seven scalar equations for seven unknowns: the components of $\boldsymbol{u}$ and $\Delta\boldsymbol{F}$, and the infinitesimal angle β.

Since, by Eq. (15), the nonlocal resultant moment is $\widetilde{\boldsymbol{M}} = \boldsymbol{M} - \mu\boldsymbol{t}_{,s} \times \boldsymbol{F}$, making use of the identity (25), Eq. (31) can be written in the form

$$\Delta\boldsymbol{M} = \boldsymbol{w} \times \boldsymbol{M} + \boldsymbol{C}\boldsymbol{w}_{,s} - \mu\boldsymbol{t}_{,s} \times \Delta\boldsymbol{F} - \mu\,(\boldsymbol{w} \times \boldsymbol{t})_{,s} \times \boldsymbol{F},$$

which, for $\mu \to 0$, shows that in an infinitesimal deformation of a rod made of usual elastic material the additional resultant moment is

$$\Delta\boldsymbol{M} = \boldsymbol{w} \times \boldsymbol{M} + \boldsymbol{C}\boldsymbol{w}_{,s}. \tag{34}$$

It follows that the equations of infinitesimal deformations from an equilibrium configuration $\mathfrak{C}$ of a classic rod have the form (33) with $\Delta\boldsymbol{M}$ given by Eq. (34).

4 Stability of equilibrium configurations

The stability of equilibrium configurations of nonlocal rods will be evaluated on the basis of the criterion of infinitesimal stability that Truesdell and Noll [34, Sect.68 bis], [35, Sect.III.8B], state in the following words: "A static deformation of a body subject to boundary conditions of place and traction is said to be *infinitesimally stable* if the work done in any further infinitesimal deformation compatible with the boundary conditions is not less than that required to effect the same infinitesimal deformation subject to dead loading, i.e., at the same state of stress as in the ground state of strain."

The criterion does not require the assumption of hyperelasticity [34, Sects.68 bis and 89]; hence, it can be employed in the present theory of nonlocal rods which are assumed to be made of materials that do not admit a stored-energy function. We recall that another criterion of stability that does not presuppose the hyperelasticity of a body was formulated by Gurtin and Spector [36], who defined stable a configuration of a body if the incremental power required to move the body from that configuration is always positive[4].

[4] The criterion has been applied by Gurtin and Spector [36], and by Spector [37,38] to the study of uniqueness problems in finite elasticity.

In order to find the form that the criterion of Truesdell and Noll assumes in the context of the present theory of nonlocal rods, we observe that integration over $(0, L)$ of the quantity $\widetilde{\boldsymbol{M}} \cdot \boldsymbol{w}_{,s} + (\boldsymbol{w} \times \boldsymbol{t}) \cdot (\boldsymbol{w} \times \boldsymbol{F})$, making use of integration by parts and the relation $(\boldsymbol{w} \times \boldsymbol{t} - \boldsymbol{u}_{,s}) \cdot \Delta \boldsymbol{F} = 0$ following from Eq. (27), yields

$$\int_0^L \left(\Delta \boldsymbol{M} \cdot \boldsymbol{w}_{,s} + (\boldsymbol{w} \times \boldsymbol{t}) \cdot (\boldsymbol{w} \times \boldsymbol{F})\right) \mathrm{d}s$$

$$= [\Delta \boldsymbol{F} \cdot \boldsymbol{u} + \Delta \boldsymbol{M} \cdot \boldsymbol{w}]_0^L + \int_0^L (\Delta \boldsymbol{m} \cdot \boldsymbol{w} + \Delta \boldsymbol{f} \cdot \boldsymbol{u}) \, \mathrm{d}s$$

where we have put $\Delta \boldsymbol{f} = -\Delta \boldsymbol{F}_{,s}$, and $\Delta \boldsymbol{m} = -(\Delta \boldsymbol{M}_{,s} + (\boldsymbol{w} \times \boldsymbol{t}) \times \boldsymbol{F} + \boldsymbol{t} \times \Delta \boldsymbol{F})$. The vectors $\Delta \boldsymbol{f}$ and $\Delta \boldsymbol{m}$ can be regarded as the incremental force and couple per unit length of $\mathcal{C}$ which together with the incremental terminal force and couple $\Delta \boldsymbol{F}$ and $\Delta \boldsymbol{M}$ are the external loads associated with the considered infinitesimal deformation[5]. Then, the last equation shows that the integral on the left-hand member is equal to the work that the incremental external loads should perform to support the infinitesimal deformation characterized by the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

The criterion of infinitesimal stability requires that the incremental external work be nonnegative in all infinitesimal deformations; hence, an equilibrium configuration $\mathfrak{C}$ of a nonlocal rod is infinitesimally stable if

$$\mathcal{W} = \int_0^L \left(\Delta \boldsymbol{M} \cdot \boldsymbol{w}_{,s} + (\boldsymbol{w} \times \boldsymbol{t}) \cdot (\boldsymbol{w} \times \boldsymbol{F})\right) \mathrm{d}s \geq 0, \tag{35}$$

for all infinitesimal deformations consistent with the boundary conditions. In the examples of the following section the criterion will be applied by assuming that the deformations from $\mathfrak{C}$, on which the inequality (35) is tested, are generated by the eigenfunctions of the equilibrium problem linearized about $\mathfrak{C}$. For a rod made of classic elastic material the criterion of infinitesimal stability requires that Eq. (35), with $\Delta \boldsymbol{M}$ given by Eq. (34), holds for all deformations consistent with the boundary conditions.

Truesdell and Noll [34, Sect.68 bis] remark that, although the criterion of infinitesimal stability does not require the existence of a stored-energy function, for hyperelastic bodies it coincides with the criterion derived by Hadamard from a general definition of finite stability based on the minimum of an energy functional. In fact, Hadamard ([39, Sect.269], cf. also [40, Sect.31]) made the assumption that, for the stability of the equilibrium of an elastic body, it is necessary that its energy be a minimum and obtained the criterion from the condition that the second variation of the energy functional were nonnegative. When referred to classic rods (for which $\boldsymbol{M} \cdot \boldsymbol{\kappa}/2$ is the strain energy per unit length along the axial curve), also the inequality (35) can be derived from a minimum condition of the energy functional $\mathcal{E}$ associated with the deformation of hyperelastic rods. Namely, let $\delta \boldsymbol{Q}$ be a variation of the rotation $\boldsymbol{Q} = \boldsymbol{d}_i' \otimes \boldsymbol{d}_i$ that transforms the torsion-flexure triad of the equilibrium configuration $\mathfrak{C}$ in the corresponding triad of the configuration $\mathfrak{C}'$, and let $\delta \boldsymbol{q}$ be the axial vector of the skew tensor $\boldsymbol{Q}^{\mathsf{T}} \delta \boldsymbol{Q}$. Then, the condition that the second variation of the energy functional $\mathcal{E}$ be nonnegative is (cf. [41])

$$\delta^2 \mathcal{E} = \int_0^L \left(\delta \boldsymbol{M} \cdot \delta \boldsymbol{q}_{,s} + (\delta \boldsymbol{q} \times \boldsymbol{t}) \cdot (\delta \boldsymbol{q} \times \boldsymbol{F})\right) \mathrm{d}s \geq 0,$$

for all $\delta \boldsymbol{q}$. Except for the presence of the vector $\delta \boldsymbol{q}$ in place of the infinitesimal rotation $\boldsymbol{w}$, and of the variation $\delta \boldsymbol{M}$ of the classic moment in place of the additional nonlocal moment $\Delta \boldsymbol{M}$, this condition coincides with the criterion (35) and can be interpreted as the requirement that the work done by the variations of the stress resultants be nonnegative.

[5] In view of Eq. (33.1,2), the vectors $\Delta \boldsymbol{f}$ and $\Delta \boldsymbol{m}$ vanish when the infinitesimal deformation brings the rod from $\mathfrak{C}$ into another equilibrium configuration $\mathfrak{C}'$, as it is the case for deformations described by the eigenfunctions of the linearized equilibrium problem.

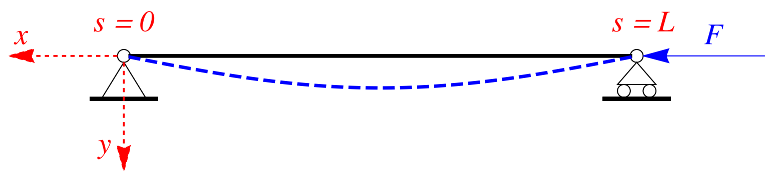

Fig. 3 The coordinate system for the study of a simply supported rod subject to a compressive force and (dashed line) the shape of the eigenfunction corresponding to the first eigenvalue

5 Examples

As examples of application of the results of the previous sections, we examine the infinitesimal stability of equilibrium configurations of naturally straight nonlocal rods that: (i) are simple supported and are subject to a compressive axial force; (ii) have one end clamped and the other one subject to the constraint of having the tangent parallel to the undeformed straight axial curve, and are acted upon by an axial force and a twisting couple; (iii) have been bent to form annular equilibrium configurations with the addition of a given twist.

In each example, we write the equilibrium equations (19) for the configuration $\mathfrak{C}$ whose stability is examined and specialize Eqs. (33) of the superimposed infinitesimal deformations with reference to that configuration. To determine the conditions under which the inequality (35) assuring the infinitesimal stability of $\mathfrak{C}$ is verified, we find the eigenfunctions of the equilibrium problem linearized about $\mathfrak{C}$ and, by assuming that the infinitesimal deformations are generated by the eigenfunctions, test on them the sign of the work made in the additional deformation. In the process of derivation of the eigenfunctions, the critical values of the loads that can determine the buckling are obtained[6]; the subsequent application of the criterion of stability shows that the loss of stability of the straight configurations of the rods occurs when the external loads reach their critical values.

5.1 Simply supported rod subject to a compressive axial force

As first example, we consider the stability of a simply supported nonlocal rod subject to a compressive axial force (Fig. 3). In the equilibrium configuration $\mathfrak{C}$, whose stability we examine, the rod is straight and, due to the condition of inextensibility, undeformed; the resultant moment vanishes, and the resultant force is equal to the compressive axial force $\boldsymbol{F}$. We employ a Cartesian coordinate system (x, y, z), whose base vectors are $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ with $\boldsymbol{e}_x = \boldsymbol{e} = \boldsymbol{F}/|\boldsymbol{F}|$. We take the origin of the coordinates at the rod end where $s = 0$ and choose the positive orientation of the x-axis so that $\boldsymbol{e}_x = -\boldsymbol{t}$; accordingly, writing $\boldsymbol{F} = F\boldsymbol{e}_x$, it is $F > 0$ for a compressive force. The vectors $\boldsymbol{u}$ and $\boldsymbol{w} = \boldsymbol{t} \times \boldsymbol{u}_{,s} + \beta\boldsymbol{t}$, characterizing an infinitesimal deformation, in components are

$$\begin{aligned}\boldsymbol{u} &= u_x\boldsymbol{e}_x, + u_y\boldsymbol{e}_y + u_z\boldsymbol{e}_z,\\ \boldsymbol{w} &= w_x\boldsymbol{e}_x + w_y\boldsymbol{e}_y + w_z\boldsymbol{e}_z = -\beta\boldsymbol{e}_x + u_{z,s}\boldsymbol{e}_y - u_{y,s}\boldsymbol{e}_z;\end{aligned} \tag{36}$$

the condition (29) becomes

$$\boldsymbol{u}_{,s} \cdot \boldsymbol{t} = -\boldsymbol{u}_{,s} \cdot \boldsymbol{e}_x = -u_{x,s} = 0. \tag{37}$$

In an infinitesimal deformation, the rod axis forms a planar curve that we assume to be in the (x, y) plane; accordingly, the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ reduce to

$$\boldsymbol{u} = u_y\boldsymbol{e}_y, \qquad \boldsymbol{w} = -u_{y,s}\boldsymbol{e}_z.$$

The vector $\Delta\boldsymbol{F}$ vanishes because the equilibrium equation (33.1) requires that $\Delta\boldsymbol{F}$ be constant and the increment of the external forces applied at the rod ends is zero. The components of the Eq. (33.2) along the x and y directions are identically null; taking into account that $\Delta\boldsymbol{F} = \boldsymbol{0}$, the z-component can be put in the form

$$u_{y,sss} + \nu^2 u_{y,s} = 0, \tag{38}$$

with

$$\nu^2 = \frac{F}{1 - \mu F}. \tag{39}$$

[6] Except for the third example for which the buckling problem was not studied before, the critical loads here determined coincide with those given in the literature on the subject (cf., e.g.,: [15,17,19,23,24,42,43]).

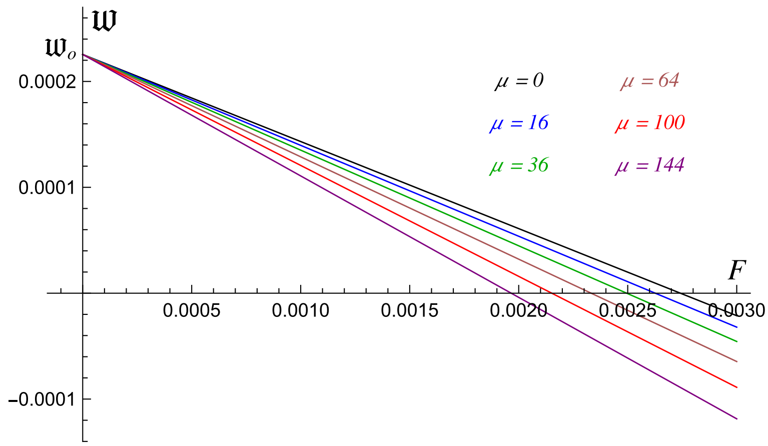

Fig. 4 The work $\mathcal{W}$ of the stress resultant increments as function of F, for a simply supported rod that in dimensionless units has length $L = 60$ and circular cross section of diameter $d = 4$, for the values $\{0, 16, 36, 64, 100, 144\}$ of μ which correspond to the values $\{0, 1, 1.5, 2, 2.5, 3\}$ of the nonlocal material modulus e_o; it has been put $A_1 = 1$. The lines intersect the F-axis at the critical loads $F_{\rm cr}$ corresponding to the values of μ

The boundary conditions associated with Eq. (38) are

$$u_y(0) = u_y(L) = 0, \qquad \Delta M_z(0) = -(1-\mu F)u_{y,ss} = 0. \tag{40}$$

For $\nu = \nu_n = n\pi/L$, the problem defined by Eqs. (38) and (40) has the non-null solutions

$$u_y^{(n)}(s) = A_n \sin\frac{n\pi s}{L}, \qquad n = 1, 2, \ldots$$

where the A_n are constants. By Eq. (39), the dimensionless values of the axial force that determine bifurcation from the solution $u_y = 0$ are

$$F_n = \frac{n^2\pi^2}{L^2 + \mu n^2\pi^2}, \qquad n = 1, 2, \ldots;$$

the first of these values is the critical value $F_{cr} = F_1$.

The condition (35) of infinitesimal stability is

$$\mathcal{W} = \int_0^L \big((1-\mu F)u_{y,ss}u_{y,ss} - F u_{y,s}u_{y,s}\big)\,\mathrm{d}s \geq 0,$$

for all deformations consistent with the boundary conditions. When the rod undergoes the infinitesimal deformation in which the displacement of the axial curve is $u_y^{(n)}$, this condition becomes

$$\mathcal{W} = A_n^2\frac{n^2\pi^2}{2L^3}\big(n^2\pi^2 - F(L^2 + \mu n^2\pi^2)\big) \geq 0, \tag{41.1}$$

and is satisfied for

$$F \leq \frac{n^2\pi^2}{L^2 + \mu n^2\pi^2}; \tag{41.2}$$

the most restrictive of these inequalities occurs for $n = 1$ and requires that

$$F \leq F_{\rm cr} = \frac{\pi^2}{L^2 + \mu\pi^2}. \tag{42}$$

Thus, the straight configuration of the rod is infinitesimally stable for values of the compressive axial force F nonexceeding the critical value $F_{\rm cr}$. Since for a nonlocal rod $\mu > 0$ and for a classic rod $\mu = 0$, Eq. (42) shows that $F_{\rm cr}$ of a nonlocal rod is always less than that of a classical rod with the same geometry and made of a material with the same tensile modulus E and is a decreasing function of $\mu = \varepsilon^2$ and, thus, of the nonlocal

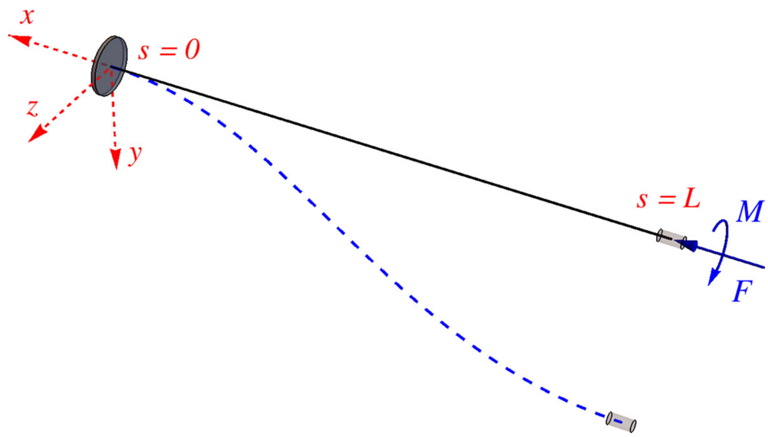

Fig. 5 The coordinate system employed to study the stability of a rod subject to an axial force and a twisting couple, with the end at $s = 0$ clamped (i.e., displacements and rotations are null) and the end at $s = L$ constrained to have the tangent parallel to the undeformed straight axial curve, but otherwise free to move; the dashed line represents the shape of the first eigenfunction

material modulus e_o that appears in the expression (10) of ε. With reference to a rod of given geometry and tensile modulus, for $n = 1$ and an assigned value of A_1, the work $\mathcal{W}$ defined by Eq. (41.1) can be viewed as a function $\mathcal{W}_\mu$ of F depending on the parameter μ. In the plane $(F, \mathcal{W})$, the curves $\mathcal{W} = \mathcal{W}_\mu(F)$ are straight lines through the point $(0, \mathcal{W}_o)$ with $\mathcal{W}_o = A_1^2\pi^4/(2L^3)$, whose negative slope decreases when μ increases (Fig. 4). The intersections of these lines with the F-axis are the critical values F_{cr}, for different values of μ, of the problem defined by Eqs. (38) and (40). The influence of the values of μ on the results confirms that nonlocal rods exhibit a reduction in rigidity with respect to classic rods having the same geometry and made of materials with the same tensile modulus and that such reduction increases with μ.

5.2 Rod subject to an axial force and a twisting couple

We examine the stability of a nonlocal rod, with one end clamped and the other one constrained to have the tangent parallel to the undeformed axial curve, that is in a straight equilibrium configuration $\mathfrak{C}$ under the action of an axial force and a twisting moment (Fig. 5). We employ a Cartesian coordinate system whose origin and axes are chosen as in the previous example. The resultant force and the twisting moment in $\mathfrak{C}$ are $F\boldsymbol{e}_x$ and $\alpha\boldsymbol{t} = \gamma\boldsymbol{e}_x$, respectively.

In an infinitesimal deformation from $\mathfrak{C}$, the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ have the expressions (36), and the constraint (37) holds. The increment $\Delta\boldsymbol{F}$ of the resultant force vanishes because $\Delta\boldsymbol{F}_{,s} = \boldsymbol{0}$ by the equilibrium equation (33.1), and $\Delta\boldsymbol{F} = \boldsymbol{0}$ at the end $s = L$ of the rod. By Eq. (31), the increment $\Delta\boldsymbol{M}$ of the nonlocal resultant moment is

$$\Delta\boldsymbol{M} = -\Omega\beta_{,s}\boldsymbol{e}_x + \big((1-\mu F)u_{z,ss} - \gamma u_{y,s}\big)\boldsymbol{e}_y - \big((1-\mu F)u_{y,ss} + \gamma u_{y,s}\big)\boldsymbol{e}_z.$$

The equilibrium equation (33.2) yields the three scalar equations

$$\Omega\beta_{,ss} = 0, \tag{43.1}$$

$$(1-\mu F)u_{z,sss} - \gamma u_{y,ss} + Fu_{z,s} = 0, \tag{43.2}$$

$$(1-\mu F)u_{y,sss} + \gamma u_{z,ss} + Fu_{y,s} = 0. \tag{43.3}$$

Equation (43.1), together with the boundary conditions $\beta(0) = 0$ and $\Delta M_x(L) = 0$, implies that β is identically zero. The boundary conditions associated with Eqs. (43.2, 3) are

$$u_y(0) = u_z(0) = 0, \qquad u_{y,s}(0) = u_{z,s}(0) = u_{y,s}(L) = u_{z,s}(L) = 0, \tag{44}$$

and the equilibrium problem has the solution $u_y = u_z = 0$ for all the values of F and γ. Equations (43.2, 3) have the general solution

$$\begin{aligned} u_y(s) &= C_1\cos(q_a s) + C_2\sin(q_a s) + C_3\cos(q_b s) + C_4\sin(q_b s) + C_5, \\ u_z(s) &= C_2\cos(q_a s) - C_1\sin(q_a s) + C_4\cos(q_b s) - C_3\sin(q_b s) + C_6 \end{aligned} \tag{45}$$

where $C_1, \ldots, C_6$ are integration constants, and

$$q_a = \frac{-\gamma - \sqrt{\gamma^2 + 4F(1-\mu F)}}{2(1-\mu F)}, \qquad q_b = \frac{-\gamma + \sqrt{\gamma^2 + 4F(1-\mu F)}}{2(1-\mu F)}. \tag{46}$$

Substitution of the functions (45) into the boundary conditions (44) yields a homogeneous system of six linear equations for the constants $C_1, \ldots, C_6$, which admits non-null solutions when q_a and q_b are such that

$$q_b^{(n)} = q_a^{(n)} + 2\nu_n, \qquad n = 1, 2, \ldots. \tag{47}$$

where $\nu_n = n\pi/L$. By squaring Eqs. (47) in which the expressions (46) of q_a and q_b have been substituted, one has

$$\left(1 + 2\nu_n^2\mu\right) 4F - \left(1 + \nu_n^2\mu\right) 4\mu F^2 + \gamma^2 = 4\nu_n^2, \tag{48}$$

which, for each n, give a family of conics in the plane (F, γ) depending on the parameter μ. By putting

$$b_n^2 = \frac{1}{\mu(1+\nu_n^2\mu)}, \qquad a_n^2 = \frac{b_n^4}{4}, \qquad c_n^2 = a_n^2 + b_n^2, \qquad F_{\mathrm{o}}^{(n)} = \frac{b_n^2(1+2\nu_n^2\mu)}{2},$$

Eq. (48), for $\mu > 0$, can be written in the form

$$\frac{(F - F_{\mathrm{o}})^2}{a_n^2} - \frac{\gamma^2}{b_n^2} = 1, \tag{49}$$

which shows that they represent hyperbolae having the transverse axis coinciding with the F-axis, centers at the points $(F_{\mathrm{o}}^{(n)}, 0)$, foci at the points $(F_{\mathrm{o}}^{(n)} \pm c_n, 0)$, and vertices at the points $(F_{\mathrm{o}}^{(n)} \pm a_n, 0)$.

The problem defined by Eqs. (43.2, 3) with the boundary conditions (44) has non-null solutions for values of the resultant force F and twisting moment $\alpha = -\gamma$ that satisfy Eq. (48). In that case, the matrix of the coefficient of the homogeneous system obtained from the boundary conditions has rank 4, the integration constants $C_1, \ldots, C_6$ depend on 2 parameters, and the functions u_y and u_z are

$$\begin{aligned} u_y^{(n)}(s) &= A_n \left(q_b^{(n)} \cos(q_a^{(n)}s) - q_a^{(n)} \cos(q_b^{(n)}s) - 2\nu_n\right) \\ &\quad + B_n \left(q_a^{(n)} \sin(q_b^{(n)}s) - q_b^{(n)} \sin(q_a^{(n)}s)\right), \\ u_z^{(n)}(s) &= A_n \left(q_a^{(n)} \sin(q_b^{(n)}s) - q_b^{(n)} \sin(q_a^{(n)}s)\right) \\ &\quad + B_n \left(q_a^{(n)} \cos(q_b^{(n)}s) - q_b^{(n)} \cos(q_a^{(n)}s) + 2\nu_n\right) \end{aligned}$$

where A_n and B_n are constants. For $\mu \to 0$, Eq. (48) tends to the equations

$$F + \frac{\gamma^2}{4} = \nu_n^2, \tag{50}$$

which, for each n, represent a parabola whose axis coincides with the F-axis and whose vertex is at the point $(\nu_n^2, 0)$. The parabolae (50) are the eigencurves of the equilibrium problem for the case in which the rod is made of a usual elastic material (cf. [44, Sect.IX.5] and [23]; for different boundary conditions cf. [45] and [46]).

In the case under consideration, the condition (35) for the infinitesimal stability of the rod is

$$\begin{aligned} \mathcal{W} = \int_0^L &\Big((1-\mu F)(u_{y,ss}u_{y,ss} + u_{z,ss}u_{z,ss}) \\ &- F(u_{y,s}u_{y,s} + u_{z,s}u_{z,s}) - \gamma(u_{y,s}u_{z,ss} - u_{y,ss}u_{z,s}) + \Omega\beta_{,s}\beta_{,s}\Big)\,\mathrm{d}s \geq 0, \end{aligned}$$

and, in a deformation in which β is zero and the displacement of the axial curve has components $u_y^{(n)}$ and $u_z^{(n)}$, becomes

$$\mathcal{W} = \left((A_n q_a^{(n)})^2 + (B_n q_b^{(n)})^2\right) 2L\nu_n \left(2\nu_n(1-\mu F) - \sqrt{\gamma^2 + 4F(1-\mu F)}\right) \geq 0, \tag{51}$$

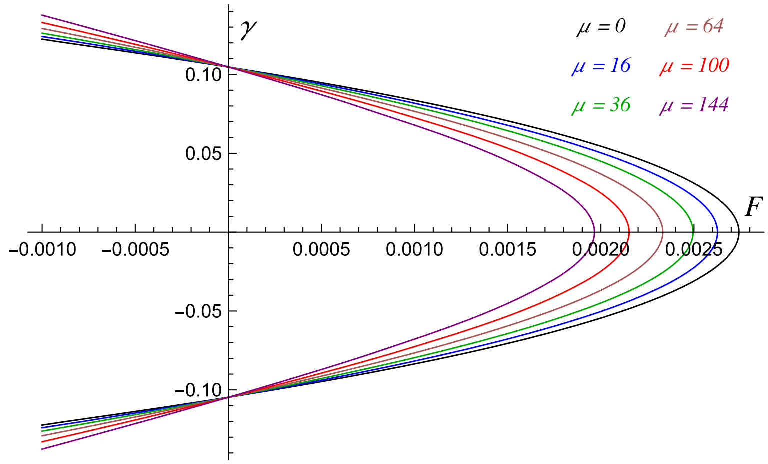

Fig. 6 Intersections of quadrics $\mathcal{W} = \mathcal{W}_\mu(F, \gamma)$ with the plane (F, γ), for a rod that in dimensionless units has length $L = 60$ and circular cross section of diameter $d = 4$, for the values $\{0, 16, 36, 64, 100, 144\}$ of μ which correspond to the values $\{0, 1, 1.5, 2, 2.5, 3\}$ of the nonlocal material modulus e_o. The intersections are hyperbolae for $\mu > 0$ and a parabola for $\mu = 0$ (only the significative branches of the hyperbolae are plotted). Each curve intersects the F-axis at the critical load F_{cr} of Eq. (42) corresponding to the value of μ of the curve, and all the curves intersect the γ-axis at $\gamma_{cr} = \pm 2\pi/L$, the critical values of the twisting moment for $F = 0$

and is satisfied for

$$\left(1 + 2\nu_n^2\mu\right) 4F - \left(1 + \nu_n^2\mu\right) 4\mu F^2 + \gamma^2 - 4\nu_n^2 \leq 0.$$

The most restrictive of these conditions occurs for $n = 1$; thus, the straight configuration of the nonlocal rod is infinitesimally stable if the axial force F and the twisting moment $\alpha = -\gamma$ satisfy the inequality

$$4F\left(1 - \mu(F + (\mu F - 2)\pi^2/L^2)\right) + \gamma^2 - 4\pi^2/L^2 \leq 0. \tag{52}$$

By equaling to zero the first member of this equation, one obtains the hyperbola given by Eq. (49) for $n = 1$; geometrically, the condition (52) requires that, in the plane (F, γ), the point corresponding to the axial force and the twisting moment $\alpha = -\gamma$ acting on the rod be not external to the region of the plane containing the origin and bounded by the closer branch of the hyperbola. The distance from the origin of points that are intersections of a straight line for the origin with hyperbolae corresponding to ($n = 1$ and) different values of μ, is a decreasing function of μ for $F > 0$ and an increasing function of μ for $F < 0$. Namely, the distance from the origin of a point of the straight line $\gamma = mF$ is $|F|\sqrt{1 + m^2}$; the coordinate F of the intersection of that line with the branch of the hyperbola that is closer to the origin is

$$F = \frac{2(1 - \sqrt{1 + \nu_1^2 m^2} + 2\nu_1^2\mu)}{4\mu(1 + \nu_1^2\mu) - m^2},$$

and is a decreasing function of μ. It follows that, for increasing values of μ, the branches of the hyperbolae that bound the regions of stability become closer to the origin for $F > 0$, that is for compressive forces, and become more distant from the origin for $F < 0$, that is for tensile forces. Hence, also for the problem under consideration, the influence of the values of μ on the results shows that nonlocality has effects analogous to those due to a reduction in rigidity of the rod.

For a rod of given geometry and made of a nonlocal material of given tensile and shear moduli, for $n = 1$ and assigned values of A_1 and B_1, the work $\mathcal{W}$ defined by Eq. (51) can be viewed as a function $\mathcal{W}_\mu$ of F and γ, depending on the parameter μ. In the space $(F, \gamma, \mathcal{W})$ the surfaces $\mathcal{W} = \mathcal{W}_\mu(F, \gamma)$ form a family of quadrics that, for $\mu > 0$, are hyperboloids of one sheet which intersect the plane $\mathcal{W} = 0$ along the hyperbolae defining the region of stability; for $\mu \to 0$, those hyperboloids tend to become a hyperbolic paraboloid, which intersects the plane $\mathcal{W} = 0$ along the parabola whose equation is given by Eq. (50) for $n = 1$ (Fig. 6).

5.3 Annular configurations of a naturally straight rod

We now examine the infinitesimal stability of annular equilibrium configurations that have been obtained by joining and sealing, with the addition of twist, the two ends of a naturally straight nonlocal rod. In an equilibrium configuration $\mathfrak{C}$ of that type, the axial curve $\mathcal{C}$ of the rod is a circle of radius $R = L/(2\pi)$, but there is a critical value of the twisting moment for which bifurcation occurs and the rod tends to buckle into equilibrium configurations in which the axial curve is not planar.

We use a cylindrical coordinate system (r, z, θ), whose base vectors are $(\boldsymbol{e}_r, \boldsymbol{e}_z, \boldsymbol{e}_\theta)$, with the origin at the center of the circle $\mathcal{C}$, the z-axis orthogonal to the plane of $\mathcal{C}$, and the radial vectors $\boldsymbol{e}_r$ pointing toward the z-axis; we take the origin of θ and the orientation of $\mathcal{C}$ so that $\theta = s/R$. Along the axial curve, whose curvature is $k = 1/R$, the basis $(\boldsymbol{e}_r, \boldsymbol{e}_z, \boldsymbol{e}_\theta)$ coincides with the Serret–Frenet basis $(\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t})$, and we have

$$\boldsymbol{e}_{r,s} = -\frac{1}{R}\boldsymbol{e}_\theta, \qquad \boldsymbol{e}_{\theta,s} = \frac{1}{R}\boldsymbol{e}_r.$$

In the annular configuration the equilibrium equations (19.1,2) become

$$\boldsymbol{F}_{,s} = \boldsymbol{0}, \qquad \boldsymbol{M}_{,s} + (1+\eta)\boldsymbol{e}_\theta \times \boldsymbol{F} = \boldsymbol{0} \tag{53}$$

where we have put

$$\eta = \frac{\mu}{R^2};$$

denoting by κ the uniform twist imposed to the rod, the classic moment is

$$\boldsymbol{M} = \frac{1}{R}\boldsymbol{e}_z + \Omega\kappa\boldsymbol{e}_\theta. \tag{54}$$

It follows from Eqs. (53) that the nonlocal moment and the resultant force in $\mathfrak{C}$ are

$$\widetilde{\boldsymbol{M}} = \widetilde{M}_z\boldsymbol{e}_z + \widetilde{M}_\theta\boldsymbol{e}_\theta = \frac{1}{R}\boldsymbol{e}_z + \frac{\Omega\kappa}{1+\eta}\boldsymbol{e}_\theta, \qquad \boldsymbol{F} = F_z\boldsymbol{e}_z = \frac{\widetilde{M}_\theta}{R}\boldsymbol{e}_z. \tag{55}$$

Equations (54) and (55) show that in a nonlocal rod the twisting moment is less than that in a classic rod having the same geometry and made of a material with the same tensile and shear moduli, subject to the same amount of twist. To describe the infinitesimal deformations from the annular configuration, we assume the angle θ as independent variable along $\mathcal{C}$. In the chosen cylindrical system, the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ have the expressions

$$\boldsymbol{u} = u_e\boldsymbol{e}_r + u_z\boldsymbol{e}_z + u_\theta\boldsymbol{e}_\theta, \quad \boldsymbol{w} = -\frac{1}{R}u_{z,\theta}\boldsymbol{e}_r + \frac{1}{R}(u_{r,\theta} + u_\theta)\boldsymbol{e}_z + \beta\boldsymbol{e}_\theta;$$

the consequence (29) of the inextensibility condition requires that

$$u_{\theta,\theta} - u_r = 0. \tag{56}$$

In components Eq. (33.1) is

$$\Delta F_{r,\theta} + \Delta F_\theta = 0, \quad \Delta F_{z,\theta} = 0, \quad \Delta F_{\theta,\theta} - \Delta F_r = 0, \tag{57}$$

with

$$\Delta F_r = \Delta\boldsymbol{F}\cdot\boldsymbol{e}_r, \quad \Delta F_z = \Delta\boldsymbol{F}\cdot\boldsymbol{e}_z, \quad \Delta F_\theta = \Delta\boldsymbol{F}\cdot\boldsymbol{e}_\theta.$$

The component form of Eq. (33.2) is

$$\begin{aligned} &u_{z,\theta\theta\theta} + (1-\Omega)u_{z,\theta} - \Omega R\beta_{,\theta} - \zeta(1-\eta)(u_{r,\theta\theta} + u_{\theta,\theta}) + (1+\eta)R^3\Delta F_z = 0,\\ &u_{r,\theta\theta\theta} + u_{\theta,\theta\theta} + \zeta(1+\eta)u_{z,\theta\theta} + (1+\eta)R^3\Delta F_r = 0,\\ &R\Omega\beta_{,\theta\theta} + \Omega u_{z,\theta\theta} - \zeta\eta(u_{r,\theta\theta\theta} + u_{\theta,\theta\theta}) = 0, \end{aligned} \tag{58}$$

with

$$\zeta = \frac{\widetilde{M}_\theta}{\widetilde{M}_z} = \widetilde{M}_\theta R. \tag{59}$$

Equations (56), (57), and (58) form a system for the unknown functions ΔF_r, ΔF_z, ΔF_θ, u_r, u_z, u_θ, and β; they can be transformed into the equivalent system[7]:

$$u_{z,\theta\theta\theta\theta\theta\theta} + (2+\zeta^2(1+\eta))u_{z,\theta\theta\theta\theta} + (1+\zeta^2(1+\eta))u_{z,\theta\theta} = 0, \tag{60.1}$$
$$u_{z,\theta\theta\theta\theta} + u_{z,\theta\theta} - \zeta(u_{\theta,\theta\theta\theta\theta} + u_{\theta,\theta\theta}) = 0, \tag{60.2}$$
$$R\Omega\beta_{\theta\theta} + \Omega u_{z,\theta\theta} - \eta(u_{z,\theta\theta\theta\theta} + u_{z,\theta\theta}) = 0, \tag{60.3}$$
$$u_{\theta,\theta} - u_r = 0, \tag{60.4}$$
$$u_{\theta,\theta\theta\theta\theta} + u_{\theta,\theta\theta} + \zeta(1+\eta)u_{z,\theta\theta} + (1+\eta)R^3\Delta F_r = 0, \tag{60.5}$$
$$u_{z,\theta\theta\theta} + (1-\Omega)u_{z,\theta} - \zeta(1-\eta)(u_{\theta,\theta\theta\theta} + u_{\theta,\theta}) - R\Omega\beta_{,\theta} + (1+\eta)R^3\Delta F_z = 0, \tag{60.6}$$
$$\Delta F_r - \Delta F_{r,\theta} + \Delta F_{r,\theta\theta} - \Delta F_\theta = 0, \tag{60.7}$$

in which the first equation contains only the unknown u_z, the second one contains only u_z and u_θ, the third one contains only u_z and β, and so on. Hence the solution of the system (60) can be obtained by solving in sequence equations that involve only one unknown function. Since in the annular equilibrium configuration the ends of the rod have been sealed, all the functions that appear in the equations must take the same values for $s = 0$ and $s = L$ (that is, for $\theta = 0$ and $\theta = 2\pi$). The general solution of Eq. (60.1) is

$$u_z(\theta) = c_1 + c_2\theta + c_3\cos(\theta + \widetilde{\delta}) + c_4\cos(\theta\sqrt{1+\zeta^2(1+\eta)} + \widehat{\delta})$$

where $c_1, \ldots, c_4$, $\widetilde{\delta}$, and $\widehat{\delta}$ are integration constants.

Excluding the constant solution, the periodic boundary conditions can be satisfied by the integral $c_3\cos(\theta + \widetilde{\delta})$, and by the integral $c_4\cos(\theta\sqrt{1+\zeta^2(1+\eta)} + \widehat{\delta})$ provided that $\sqrt{1+\zeta^2(1+\eta)}$ is equal to an integer larger than 1. Using Eqs. (60.2–4) it can be shown [41] that to the integral $c_3\cos(\theta + \widetilde{\delta})$ of u_z there correspond functions u_θ, u_z, and β such that the resulting deformation is rigid. Thus, the solutions of Eq. (60.1) that are of interest are possible when ζ^2 has the values

$$\zeta_n^2 = \frac{n^2-1}{1+\eta}, \qquad n = 2, 3, 4, \ldots, \tag{61}$$

and are of the type

$$u_z^{(n)}(\theta) = U_n\cos(n\theta + \delta_n) \tag{62}$$

where U_n and δ_n are constants. It follows from Eqs. (60.2–7) that when u_z is given by Eq. (62), the other unknown functions are

$$\begin{aligned} u_\theta^{(n)}(\theta) &= \frac{U_n}{\zeta}\cos(n\theta + \delta_n), \\ u_r^{(n)}(\theta) &= -\frac{nU_n}{\zeta}\sin(n\theta + \delta_n), \\ \beta^{(n)}(\theta) &= -\frac{U_n((n^2-1)\eta + \Omega)}{R\Omega}\cos(n\theta + \delta_n), \\ \Delta F_r^{(n)} &= \Delta F_z^{(n)} = \Delta F_\theta^{(n)} = 0. \end{aligned} \tag{63}$$

The value $\zeta_{cr} = \zeta_2$ of ζ, obtained from Eq. (61) for $n = 2$ furnishes, by Eq. (59), the critical value $\widetilde{M}_{\theta\mathrm{cr}}$ of the twisting moment in the annular equilibrium configuration for which bifurcation occurs and the rod tends to buckle into a configuration in which the axial curve is not planar:

$$\zeta_{\mathrm{cr}} = \pm\sqrt{\frac{3}{1+\eta}}, \qquad \widetilde{M}_{\theta\mathrm{cr}} = \frac{\zeta_{\mathrm{cr}}}{R} = \pm\frac{1}{R}\sqrt{\frac{3}{1+\eta}}.$$

Since the critical value of the twisting moment for a classical annular rod is $M_{\theta\mathrm{cr}} = \pm\sqrt{3}/R$ (cf., e.g., [41]), the above result shows that in a nonlocal rod the bifurcation is reached for values of the twisting moment lower than those of a classic rod.

[7] That is, the two systems have the same solutions (cf., e.g., [47, Sect.6.41]).

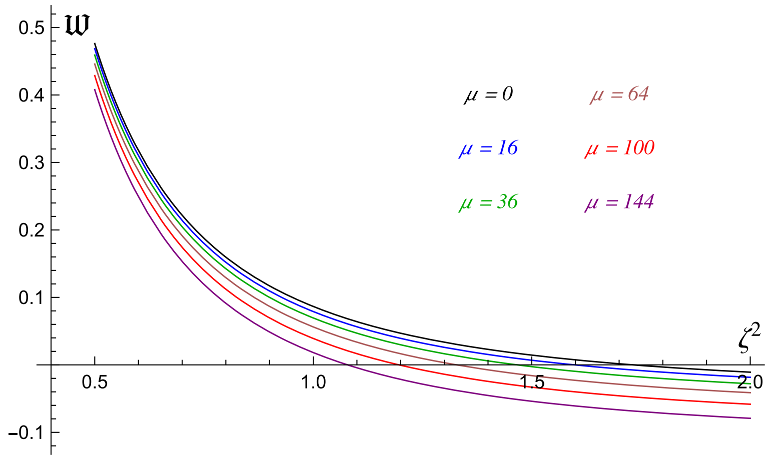

Fig. 7 The significative branches of the hyperbolae $\mathcal{W}_\mu = 0$ for different values of μ intersect the ζ^2 axis at the critical values $\zeta^2_{\rm cr}$, that decrease for μ increasing. The curves refer to a rod that in dimensionless units has length $L = 60$ and circular cross section of diameter $d = 4$, and to the values {0, 16, 36, 64, 100, 144} of μ which correspond to the values {0, 1, 1.5, 2, 2.5, 3} of the nonlocal material modulus $e_{\rm o}$; it has been assumed $U_2 = 1$

In a deformation of the rod in which the displacement is given by the eigenfunctions (62), (63), the condition of infinitesimal stability (35) becomes

$$\mathcal{W} = U_n^2 \frac{\pi}{R^3} \frac{n^2(n^2-1)(n^2-1-\zeta^2(1+\eta))}{\zeta^2} \geq 0. \tag{64}$$

For $n = 2$, we have the most restrictive of these conditions that is satisfied for

$$\zeta^2 \leq \zeta^2_{\rm cr} = \frac{3}{1+\eta} = \frac{3}{1+\mu/R^2}. \tag{65}$$

Thus, the annular equilibrium configuration of a nonlocal rod is infinitesimally stable when, in absolute value, the twisting moment $\widetilde{M}_\theta$ does not exceed the critical twisting moment $\zeta_{\rm cr}/R$. The form of Eq. (65) shows that the value of the maximum moment for which the annular configuration is stable decreases with μ.

For a rod of given geometry and given tensile and shear moduli, for $n = 2$ and an assigned value of U_2, the work $\mathcal{W}$ defined by Eq. (64) can be viewed as a function $\mathcal{W}_\mu$ of ζ^2 depending on the parameter $\mu = R^2\eta$.

In the plane $(\zeta^2, \mathcal{W})$, the curves $\mathcal{W} = \mathcal{W}_\mu(\zeta^2)$ are equilateral hyperbolae with centers at the points $(0, -12(1+\eta)U_2^2\pi^2/R^3)$, asymptotes parallel to ζ^2 and $\mathcal{W}$, and axes that form angles $\pm\pi/4$ with ζ^2 and $\mathcal{W}$. The positive intersections of the hyperbolae with the ζ^2-axis are the critical values $\zeta^2_{\rm cr}$ for the values of μ characterizing the curves (Fig. 7). The curves furnish a graphical representation of the property that the critical values $\zeta^2_{\rm cr}$ decrease for increasing values of μ.

6 Conclusions

The paper has presented the deduction of the criterion of infinitesimally stability for equilibrium configurations of nonlocal rods. To attain this result, the nonlinear equilibrium equations of nonlocal rods have been written by employing the kinematics of Kirchhoff theory and a constitutive equation of nonlocal materials. The adopted constitutive equation (15) has been discussed, showing that it is equivalent to an integral constitutive equation of Eringen's type for rods of infinite length and, in general, is an approximation of that equation for rods of finite length. The equilibrium equations have been linearized obtaining the equations describing the infinitesimal deformations that are superimposed on a finite deformation of a rod. Starting from these equations, the expression of the work done, in an infinitesimal deformation from a configuration $\mathfrak{C}$, by the increments of the external loads with respect to their values in $\mathfrak{C}$ has been deduced and employed to establish the criterion of infinitesimal stability for the equilibrium of nonlocal rods. The criterion has been applied to the study of the stability of rods in three equilibrium states: (i) simply supported rods subject to axial forces; (ii) rods with a clamped end and the other one constrained to have the tangent parallel to the undeformed rod axis acted upon by axial forces and twisting moments; (iii) annular rods obtained bending naturally straight rods with the addition of twist. The work done in infinitesimal additional deformations by the increments of the stress resultants has been tested on the eigenfunctions of the equilibrium problem linearized about the configuration whose stability was under consideration. The obtained results are in agreement with those of classic rods in the sense that, when the nonlocal parameter μ tends to zero, they reduce to those of classic rods. Moreover, the results show that, for nonlocal rods, the loss of stability of the considered equilibrium configurations occurs for values of the external actions that are lower than those that are necessary for classical rods having the same geometry and made of materials with the same elastic moduli. Hence, also under the aspect examined in the paper, the presence of a nonlocal material has effects analogous to those due to a reduction in rigidity of the rods.

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