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Mei symmetry and new conserved quantities for non-material volumes

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Abstract This paper is focused on the Mei symmetry of non-material volumes. The Mei symmetrical definition and criterion are given. The criterion equations of the system are presented by introducing the invariance of dynamical functions of the system under infinitesimal transformations of Lie groups. The structure equations and the form of new conserved quantities are proposed. An example is given to illustrate the application of the method; a new conserved quantity is obtained under the Mei symmetrical transformations.

1 Introduction

Non-material volumes have received widespread attention and show important theoretical significance and engineering background. They are also widely used in rocketry, mechanical engineering and civil engineering [1,2]. Recently, there are many researchers who have addressed the fundamental principles of non-material volumes. Irschik and Holl [3] derived the Lagrange's equation of a non-material volume which instantaneously coincides with some part of a continuous and possibly deformable body. Casetta and Pesce [4] established the generalized Hamilton's principle for a non-material volume by introducing Reynolds' transport theorem. They [5] also investigated the inverse problem of Lagrangian mechanics connected to Meshchersky's equation and stated a variational formulation and a Hamiltonian formulation. Casetta [6] reported the inverse problem of Lagrangian mechanics for a non-material volume via the method of Darboux and proposed the Hamiltonian formalism and a conservation law. Irschik and Holl [7] proposed a formulation of Lagrange's equations for non-material volumes and calculated local forms and global form of Lagrange's equations in the framework of the Lagrange description of continuum mechanics. Casetta et al. [8] developed the generalization of Noether's theorem for a non-material volume and proposed a Noether conserved quantity and the corresponding Killing equations. Jiang and Xia [9] presented the Lie symmetry which is an invariance of differential equations of motion under infinitesimal transformations of Lie group and obtained four kinds of conserved quantities of non-material volumes. However, to the authors' knowledge, the Mei symmetry of non-material volumes has not been investigated yet.

Symmetry is a promising approach to seek conserved quantities for dynamical systems, which has profound theoretical significance. In 2000, Mei [10] pioneered a new symmetry, i.e., Mei symmetry which is an

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invariance of dynamical functions under infinitesimal transformations of Lie group. Subsequently, Mei and his coauthors extended the Mei symmetry to Appell equations [11], Nielsen equations [12] and generalized classical mechanics [13]. Liu et al. [14] investigated the Mei symmetry of non-conservative systems. Zheng et al. [15] reported Mei symmetry for Tzénoff equations. Jiang et al. [16] derived Mei conserved quantities for higher-order non-holonomic systems. Xia and Chen [17,18] promoted the Mei symmetry to different difference systems. Zhang et al. [19] described Mei conserved quantities of generalized Hamilton systems with additional terms. Wang and Xue [20] considered Mei conserved quantities of thin elastic rod. So far, to the authors' best knowledge, there is no Mei symmetry analysis on non-material volumes. To address the lack of research in this aspect, the present work develops the Mei symmetry technique to seek the conserved quantities of non-material volumes.

The paper is organized as follows. Section 2 reviews the differential equations of the non-material volumes. Section 3 considers Mei symmetrical definition and criterion of non-material volumes and obtains the criterion equations of Mei symmetry. Section 4 presents a new types of conserved quantities of non-material volumes under Mei symmetrical transformations. Section 5 gives an example to illustrate the application of the method. Section 6 contains the concluding remarks.

2 The differential equations of the non-material volumes

The Lagrange's equation of non-material volumes was pioneered by Irschik and Holl [3, p.243, Eq.(5.6)], and it can be given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k} - \frac{\partial T_u}{\partial q_k} - \int\limits_{\partial V_u} \frac{1}{2}\rho \upsilon^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right) \cdot \boldsymbol{n}\mathrm{d}\partial V_u + \int\limits_{\partial V_u} \rho \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{n}\mathrm{d}\partial V_u = Q_k, \tag{1}$$

where $T_u = T_u(\dot{q}_k, q_k, t)$ is the total kinetic energy of the material particles which come with non-material volume V_u , q_k represents generalized coordinates, $\boldsymbol{v}$ is the velocity of the material particles, $\boldsymbol{u}$ is the velocity of the fictitious particles, ρ is the volumetric mass density, Q_k is the generalized force applied to the material body, $\boldsymbol{n}$ is the outer normal unit vector at the surface of V_u, and ∂V_u depicts the bounding surface of V_u. In addition, the velocity of the material particles can be written as

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial \boldsymbol{p}}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{p}}{\partial t}, \tag{2}$$

and the velocity of fictitious particles is recognized as

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\partial \boldsymbol{r}}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r}}{\partial t}, \tag{3}$$

where $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{P}, q_k, t)$ denotes the position vector of the material particle, and $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R}, q_k, t)$ represents the position vector of the fictitious particle. In addition, the dummy index k in Eqs. (2) and (3) indicates summation.

Suppose that the system is non-singular, i.e.,

$$\Lambda = \det\left(\frac{\partial^2 T_u}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{4}$$

then the differential equations of the non-material volumes have the form

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\ (s = 1, 2, \ldots, k), \tag{5}$$

where

$$\alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{\Lambda_{ks}}{\Lambda}\left(\frac{\partial T_u}{\partial q_k} + Q_k + \Upsilon_k - \frac{\partial^2 T_u}{\partial \dot{q}_k \partial q_r}\dot{q}_r - \frac{\partial^2 T_u}{\partial \dot{q}_k \partial t}\right),$$
$$\Upsilon_k = \int\limits_{\partial V_u} \frac{1}{2}\rho \upsilon^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot \boldsymbol{n}\mathrm{d}\partial V_u - \int\limits_{\partial V_u} \rho \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{n}\mathrm{d}\partial V_u.$$

3 Mei symmetry of the non-material volumes

Choose the infinitesimal transformations t and q_i of Lie group as follows:

$$t^* = t + \Delta t, q_i^*(t^*) = q_i(t) + \Delta q_i, \quad (i = 1, 2, \ldots, k), \tag{6}$$

and their expanding forms are

$$t^* = t + \varepsilon\tau(t, q_j),\ q_i^*(t^*) = q_i(t) + \varepsilon\xi_i(t, q_j), \tag{7}$$

where ε is an infinitesimal parameter, and τ and ξ_i represent the infinitesimal generators. The infinitesimal generator vector is taken as

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \tag{8}$$

and its first extension vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s}. \tag{9}$$

By using the definition of Mei symmetries [10], the dynamical functions of the non-material volumes (1) under the infinitesimal transformations (6) are invariant, i.e., the total kinetic energy T_u of the material particles, generalized force Q_k, the position vectors $\boldsymbol{p}$ and $\boldsymbol{r}$ under the infinitesimal transformations (6) become ${T_u}^*$, ${Q_k}^*$, $\boldsymbol{p}^*$ and $\boldsymbol{r}^*$, respectively, and then the definition is given as follows:

Definition Under the infinitesimal transformations (6), the form of Eq. (1) preserves an invariance, i.e., if the following equations hold:

$$\frac{d}{dt}\frac{\partial {T_u}^*}{\partial \dot{q}_k} - \frac{\partial {T_u}^*}{\partial q_k} - \int\limits_{\partial V_u}\frac{1}{2}\rho v^{*2}\left(\frac{\partial \boldsymbol{v}^*}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}^*}{\partial \dot{q}_k}\right)\cdot\boldsymbol{n}\mathrm{d}\partial V_u + \int\limits_{\partial V_u}\rho v^*\frac{\partial \boldsymbol{v}^*}{\partial \dot{q}_k}\left(\boldsymbol{v}^* - \boldsymbol{u}^*\right)\cdot\boldsymbol{n}\mathrm{d}\partial V_u = {Q_k}^*, \tag{10}$$

then the invariance is called Mei symmetry of non-material volumes.

Expanding ${T_u}^*$, ${Q_k}^*$, $\boldsymbol{p}^*$ and $\boldsymbol{r}^*$, respectively, we have

$$T_u^* = T_u + \varepsilon X^{(1)}(T_u),\ Q_k^* = Q_k + \varepsilon X^{(1)}(Q_k),\ \boldsymbol{r}^* = \boldsymbol{r} + \varepsilon X^{(0)}(\boldsymbol{r}),\ \boldsymbol{p}^* = \boldsymbol{p} + \varepsilon X^{(0)}(\boldsymbol{p}). \tag{11}$$

Substituting Eq. (11) into Eq. (10), utilizing Eqs. (2) and (3), neglecting ε^2 and the higher-order infinitesimal terms in the resulting equation, yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_k} - \frac{\partial X^{(1)}(T_u)}{\partial q_k} - \int\limits_{\partial V_u}\frac{1}{2}\rho v^2\left(\frac{\partial \Xi}{\partial \dot{q}_k} - \frac{\partial \Psi}{\partial \dot{q}_k}\right)\cdot n\mathrm{d}\partial V_u - \int\limits_{\partial V_u}\rho\boldsymbol{v}\Xi\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot n\mathrm{d}\partial V_u\\
&+\int\limits_{\partial V_u}\rho\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\Xi - \Psi)\cdot n\mathrm{d}\partial V_u + \int\limits_{\partial V_u}\rho\boldsymbol{v}\frac{\partial \Xi}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot n\mathrm{d}\partial V_u + \int\limits_{\partial V_u}\rho\Xi\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot n\mathrm{d}\partial V_u = X^{(1)}(Q_k),
\end{aligned} \tag{12}$$

where

$$\Xi = \frac{\partial X^{(0)}(\boldsymbol{p})}{\partial q_s}\dot{q}_s + \frac{\partial X^{(0)}(\boldsymbol{p})}{\partial t}, \Psi = \frac{\partial X^{(0)}(\boldsymbol{r})}{\partial q_s}\dot{q}_s + \frac{\partial X^{(0)}(\boldsymbol{r})}{\partial t}.$$

Equation (12) is called the criterion equations of Mei symmetry of the non-material volumes. The symmetry of a non-material volume is Mei symmetry if its generators τ and ξ_i satisfy Eq.(12).

Note that we only considered the infinitesimal transformations of the total kinetic energy $T_u(q_k, \dot{q}_k, t)$ which is a function after the integral, but did not consider the transformations of the volume $V_u(q_k, t)$.

4 Mei symmetry and a new conserved quantity of the non-material volumes

Theorem *For the non-material volumes* (1)*, if the generators* τ *and* ξ_i *of infinitesimal transformations* (6) *satisfy the Mei symmetrical criterion equations* (12)*, and there exists a function* $G_M(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying the following condition:*

$$X^{(1)}\left\{X^{(1)}(T_u)\right\} + X^{(1)}(T_u)\frac{\bar{\mathrm{d}}\tau}{\mathrm{d}t} + [X^{(1)}(Q_k) + Z_k](\xi_s - \dot{q}_s\tau) + \frac{\bar{\mathrm{d}}G_M}{\mathrm{d}t} = 0, \tag{13}$$

where

$$\begin{aligned} Z_k = & \int\limits_{\partial V_u} \frac{1}{2}\rho\upsilon^2\left(\frac{\partial \varXi}{\partial \dot{q}_k} - \frac{\partial \varPsi}{\partial \dot{q}_k}\right)\cdot n\mathrm{d}\partial V_u + \int\limits_{\partial V_u} \rho\boldsymbol{v}\varXi\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot n\mathrm{d}\partial V_u \\ & - \int\limits_{\partial V_u} \rho\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\varXi - \varPsi)\cdot n\mathrm{d}\partial V_u - \int\limits_{\partial V_u} \rho\boldsymbol{v}\frac{\partial \varXi}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot n\mathrm{d}\partial V_u - \int\limits_{\partial V_u} \rho\varXi\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot n\mathrm{d}\partial V_u, \end{aligned}$$

then the system (1) *produces the following Mei conserved quantity:*

$$I_M = X^{(1)}(T_u)\tau + \frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\tau) + G_M = const. \tag{14}$$

Proof Differentiating Eq. (14) with respect to time, and using Eqs. (12) and (13) yields

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{M}} = & \left[\frac{\partial X^{(1)}(T_u)}{\partial t} + \dot{q}_s\frac{\partial X^{(1)}(T_u)}{\partial q_s} + \alpha_s\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\right]\tau + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\right)(\xi_s - \dot{q}_s\tau) \\ & + X^{(1)}(T_u)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau + \frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \alpha_s\tau - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau\right) - X^{(1)}(T_u)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau \\ & - X^{(1)}\left\{X^{(1)}(T_u)\right\} - [X^{(1)}(Q_k) + Z_k](\xi_s - \dot{q}_s\tau) \\ = & \left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s} - \frac{\partial X^{(1)}(T_u)}{\partial q_s} - [X^{(1)}(Q_k) + Z_k]\right\}(\xi_s - \dot{q}_s\tau) \\ = & 0, \end{aligned} \tag{15}$$

where the total time derivative is

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_k\frac{\partial}{\partial q_k} + \alpha_k\frac{\partial}{\partial \dot{q}_k},$$

then $I_{\mathrm{M}} = \mathrm{const}$. □

It is worth noting that the equation of motion and conserved quantities for non-material volumes have universal significance, and they can be reduced to other conventional dynamical systems [10,14].

5 Example

To illustrate the applicability of the Mei symmetry, an ideal two-dimensional problem of a rotating drum which has been derived by Casetta et al. [8, p.705, Eq.(67)] is considered; the governing equation of motion for the system can be described as

$$\frac{1}{2}\rho l\pi\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4\ddot{\phi} = \varPi, \tag{16}$$

where $\varPi = \varPi(\phi)$ is the torque acting upon the control volume, and the corresponding kinetic energy is

$$T_u = \frac{1}{4}\rho l\pi\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4\dot{\phi}^2, \tag{17}$$

the flux of kinetic energy is

$$\int_{\partial V_u} \frac{1}{2}\rho \upsilon^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u = \frac{1}{2}\rho\varepsilon l \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3 \dot{\phi}^2, \tag{18}$$

and the flux of linear momentum of the systems is

$$\int_{\partial V_u} \rho \boldsymbol{v} \frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} (\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}\partial V_u = \rho\varepsilon l \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3 \dot{\phi}^2 \tag{19}$$

It is noted that ε represents thickness of a thin strip, l is width, ρ describes mass density, $\phi = \phi(t)$ is the rotation angle, R_0 is the original radius, V_u describes the control volume, ∂V_u is the control surface, υ is the velocity of the material particles instantaneously included in V_u, and $\partial V_u = \varepsilon l$.

Using the definition of Mei symmetry, one has

$$\begin{aligned} X^{(1)}(T_u) &= \xi_1 \frac{\partial T_u}{\partial \phi} + (\dot{\xi}_1 - \dot{\phi}\dot{\tau}) \frac{\partial T_u}{\partial \dot{\phi}} \\ X^{(1)}(\varPi(\phi) - \varXi) &= \xi_1 \frac{\partial (\varPi(\phi) - \varXi)}{\partial \phi} + (\dot{\xi}_1 - \dot{\phi}\dot{\tau}) \frac{\partial (\varPi(\phi) - \varXi)}{\partial \dot{\phi}}, \end{aligned} \tag{20}$$

where $\varXi = \frac{1}{2}\varepsilon\rho l \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3 \dot{\phi}^2$.

Selecting the infinitesimal generators as

$$\tau = 1,\ \xi_1 = 0, \tag{21}$$

Eq. (20) turns out to be

$$X^{(1)}(T_u) = 0,\ X^{(1)}(\varPi(\phi) - \varXi) = 0. \tag{22}$$

Inserting Eqs. (20) and (21) into Eq. (13), we find

$$G_M = \frac{1}{4}\rho l \pi \dot{\phi}^2 - \int \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4 \varPi(\phi)\dot{\phi}\mathrm{d}t, \tag{23}$$

and then the corresponding Mei conserved quantity will also be given as follows:

$$I_M = \frac{1}{4}\rho l \pi \dot{\phi}^2 - \int \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4 \varPi(\phi)\dot{\phi}\mathrm{d}t = \text{const.} \tag{24}$$

In addition, selecting another infinitesimal generators as

$$\xi_1 = \frac{\varPi(\phi)}{\partial \varPi \big/ \partial\phi - \varepsilon^2 \rho l \dot{\phi}^2 \left(R_0 - 0.5\varepsilon\phi/\pi\right)^2 \big/ \pi}, \tau = \int \left[\frac{\xi_1}{\dot{\phi}} - \frac{\varepsilon}{\pi}\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-1} \xi_1 - 1\right] \mathrm{d}t \tag{25}$$

yields

$$X^{(1)}(T_u) = T_u,\ X^{(1)}(\varPi(\phi) - \varXi) = \varPi(\phi) - \varXi. \tag{26}$$

Then another Mei conserved quantity is found

$$I_M = T_u \tau + \rho\varepsilon l \left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3 \dot{\phi}\left(\xi_1 - \dot{\phi}\tau\right) + G_M = \text{const.}, \tag{27}$$

where the gauge function is

$$G_M = -\int \left[T_u + T_u \dot{\tau} + (\varPi(\phi) - \varXi)\left(\xi_1 - \dot{\phi}\tau\right)\right] \mathrm{d}t.$$

6 Conclusions

This paper is devoted to the Mei symmetry and conserved quantity of the non-material volumes under Lie groups' infinitesimal transformations. On the basis of the form invariance of dynamical functions under general infinitesimal transformations, the definition and the criterion equations of Mei symmetries for the non-material volumes are constructed. New types of conserved quantities of are obtained. Finally, a single-degree-of-freedom non-material volume is given to illustrate the application of the method.

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