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ORIGINAL PAPER

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Ordered rate constitutive theories for thermoviscoelastic solids without memory incorporating internal and Cosserat rotations

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Abstract The work in this paper is based on a non-classical continuum theory in the Lagrangian description for thermoviscoelastic solids without memory in which the conservation and balance laws are derived by incorporating internal rotations (${}_i\boldsymbol{\Theta}$) arising from the Jacobian of deformation ($\boldsymbol{J}$), as well as Cosserat rotations (${}_e\boldsymbol{\Theta}$) at a material point. Such non-classical solids have additional energy storage due to rotations and additional dissipation due to rotation rates compared to classical continuum theories. Rotations ${}_i\boldsymbol{\Theta}$ are completely defined by $\boldsymbol{J}$, whereas displacements $\boldsymbol{u}$ and Cosserat rotations ${}_e\boldsymbol{\Theta}$ are degrees of freedom at each material point. When ${}_i\boldsymbol{\Theta}$ and ${}_e\boldsymbol{\Theta}$ are resisted by the deforming matter, conjugate moments arise, which together with (${}_i\boldsymbol{\Theta}$, ${}_e\boldsymbol{\Theta}$) and (${}_i\dot{\boldsymbol{\Theta}}$, ${}_e\dot{\boldsymbol{\Theta}}$) result in additional work and rate of work. This paper utilizes thermodynamic framework for non-classical solids derived based on internal as well as Cosserat rotations and presents a thermodynamically consistent derivation of the constitutive theories that incorporate the aforementioned deformation physics. The constitutive theories are derived using the conditions resulting from the entropy inequality in conjunction with the representation theorem.

1 Literature review and scope of work

The Cosserat theories for continuous media consider rotations as independent degrees of freedom in addition to the usual three displacements. Beam, plate, and shell theories are classical examples in which rotational degrees of freedom at a material point are often used to incorporate desired physics. The stress-couple theories due to Voigt [1,2] are perhaps the first presentation of incorporating the idea of rotations in the deformation physics through the assumption of the existence of the moment tensor. Cosserat and Cosserat [3] presented a unified treatment of deformable bodies using rotations at a material point as additional three degrees of freedom based on what they called *Euclidean action* and the variation of internal energy density. This theory is referred to as Cosserat theory based on rotations as independent degrees of freedom at a material point (Cosserat rotations). The idea of Cosserat theory was revived some 50 years later in Refs. [4–8].

Aero and Kuvshinskii [6] in 1960 pointed out that classical mechanics is inadequate for short acoustic waves in crystals and for the laws of piezoelectric phenomena. The authors presented what they call “a phenomenological theory of continuous media by taking into account rotational interaction among the particles.” By assuming the existence of the moment tensor on the oblique plane of the tetrahedron, use of Cauchy principle, and the balance of angular momenta, the derivation shows that the Cauchy stress tensor must be non-symmetric and that the gradients of the Cauchy moment tensors are balanced by the antisymmetric components of the

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Cauchy stress tensor. In the rate of work, the antisymmetric part of the velocity gradient tensor is assumed to be conjugate to the Cauchy moment tensor; hence, the rates of rotation are due to the internal deformation rate of the deforming matter. Mindlin and Tiersten [9] assumed the existence of a moment on the oblique plane of the tetrahedron and gave a similar derivation to Ref. [6], but included a discussion of the constitutive theories presented in Refs. [5,6]. Examples of thickness-shear vibrations of a plate, torsional vibrations of a circular cylinder, spherical cavity in a field of simple tension, and cylindrical cavity in a field of simple tension were presented. Theories similar to those in Ref. [4–9] can also found in the works of Toupin [10] and Koiter [11]. Eringen [12] presented a special form of Cosserat theory in which rotational degrees of freedom are considered. This theory is similar to that in Refs. [6,9–11]. Eringen and Suhubi [13,14] and Eringen [15,16] presented a general theory of nonlinear microelastic continua in which the intrinsic motion of the microelements in a macrovolume are taken into consideration. The authors showed that the presently known couple stress theory and Cosserat theory are a subset of the theories presented by them. A microstructure theory of elasticity by Mindlin [17] and a micropolar theory by Green and Rivlin [18], both in 1964, have similarities with theories in [13–16] for special situations.

In another paper [19], Mindlin presented stress functions for a Cosserat continuum. In this paper, both internal rotations and Cosserat rotations were considered. Constitutive theories were derived using potential energy density which was assumed to be a function of strains, rotations, and rotation gradients. The energy equation and the entropy inequality were not considered in this work; consequently, there was no discussion of conjugate pairs (i.e., dual variables associated with the internal and Cosserat rotations were not discussed). The constitutive theories are derived based on assumed physics for the strain energy density function. Following the works cited above, there has been a large number of publications on stress-couple theories as well as Cosserat theories (some containing both), alternate formulations, diverse applications, computational methods, finite element formulations, and so on (see, e.g., Refs. [20–38]). It is not essential to discuss these in the context of the work presented in this paper; however, they provide some background for the present work as well as some help to the readers in terms of specific applications of this theory.

For any deforming solid matter, the displacements and the Jacobian of deformation $\boldsymbol{J}$ defining gradients of deformed coordinates with respect to undeformed coordinates are the fundamental measures of deformation physics. Thus, the conservation and balance laws must consider complete $\boldsymbol{J}$ in their derivations. For small deformation, small strain, ${}^{d}\boldsymbol{J}$, the gradients of displacements, replaces $\boldsymbol{J}$. Decomposition of ${}^{d}\boldsymbol{J}$ into a symmetric tensor ${}^{d}_{s}\boldsymbol{J}$ and an antisymmetric tensor ${}^{d}_{a}\boldsymbol{J}$ suggests that consideration of complete ${}^{d}\boldsymbol{J}$ requires consideration of ${}^{d}_{s}\boldsymbol{J}$ and ${}^{d}_{a}\boldsymbol{J}$. ${}^{d}_{s}\boldsymbol{J}$ defines strain measures whereas ${}^{d}_{a}\boldsymbol{J}$ defines rotations at a material point. The rotations at a material point are defined using a triad whose axes are parallel to the x-frame.

Classical continuum theories are derived using displacements as degrees of freedom and ${}^{d}_{s}\boldsymbol{J}$ at a material point. ${}^{d}_{a}\boldsymbol{J}$ is not considered at all. Any continuum theory that deviates from the consideration of displacements $\boldsymbol{u}$ and ${}^{d}_{s}\boldsymbol{J}$ is a non-classical continuum theory. Rotations in ${}^{d}_{a}\boldsymbol{J}$ are due to ${}^{d}\boldsymbol{J}$ that are always intrinsically present in all deformations; hence, we refer to these rotations as internal rotations. A continuum theory that considers $\boldsymbol{u}$, ${}^{d}_{s}\boldsymbol{J}$, and ${}^{d}_{a}\boldsymbol{J}$ is referred to as an internal polar continuum theory, or a non-classical continuum theory with internal rotations. Obviously, the latter is a more precise description of the theory. A continuum theory that incorporates only Cosserat rotations is a non-classical Cosserat continuum theory. The theories that incorporate both rotations are simply called non-classical continuum theories.

Confusion regarding the assumption of a material point with mass but no dimension and considerations of rotations is rather natural. We go back to the definitions of strains and stresses in solids. We consider a tetrahedron in the undeformed configuration (reference configuration) such that its oblique plane is subjected to external loads. Upon deformation, the edges of the tetrahedron representing material lines become curved. Tangent vectors to these deformed material lines are the edges of the deformed tetrahedron (covariant basis). Their extensions and change in the 90° angle (in the undeformed tetrahedron) between them define strain measures. The contravariant stress tensor is defined using planes of the deformed tetrahedron whose components, when converted to the x-frame using covariant base vectors, define the Cauchy stress tensor $\boldsymbol{\sigma}$. The concept of stress at a material point is in fact the limiting case of the deformed tetrahedron visualized as a material point.

Since ${}^{d}\boldsymbol{J}$ may vary between a material point and its neighbors, and if the material points stay connected (as they do in the absence of damage), then these varying rotations between the material points are resisted by the deforming matter, creating internal moments (in the same manner as the varying displacement gradients, when resisted, result in stresses). The concepts of rotations and moments at a material point are easily understood by considering the deformed tetrahedron. Existence of the internal moments in the deforming matter results in the Cauchy moment tensor that can be derived in a similar manner as the Cauchy stress tensor and is related to the average moment on the oblique plane of the deformed tetrahedron by the Cauchy principle.

In short, existence of the Cauchy stress tensor and the Cauchy moment tensor are established using the deformed tetrahedron. Its oblique plane contains average stress and average moment and its planes contain contravariant stress and moment tensors that yield Cauchy stress and moment tensors. The Cauchy principle relates the average stress and moment on the oblique plane of the tetrahedron to the Cauchy stress and moment tensors derived using the planes of the deformed tetrahedron.

We remark that rotations at a material point exist but due not result in rotational inertia as the limiting case of a material point has no dimensions. This fact is honored in the derivation of the law of balance of moment of moments presented in this paper.

In the present work, we consider the internal rotations due to ${}^d_a\boldsymbol{J}$ as well as Cosserat rotations that are additional three degrees of freedom about the axes of the same triad about which internal rotations are defined. The continuum theory presented here considers $\boldsymbol{u}$, ${}^d_s\boldsymbol{J}$, ${}^d_a\boldsymbol{J}$, as well as Cosserat rotations; hence, this is a non-classical continuum theory. A theory that considers only $\boldsymbol{u}$, ${}^d_s\boldsymbol{J}$, and ${}^d_a\boldsymbol{J}$ is also a non-classical continuum theory, but more specifically it is a non-classical internal polar continuum theory or a non-classical continuum theory with internal rotations.

The work presented in this paper considers rotations ${}_i\boldsymbol{\Theta}$ due to the antisymmetric part of the Jacobian of deformation (or displacement gradient tensor) as well as Cosserat rotations ${}_e\boldsymbol{\Theta}$, both acting about the axes of a triad parallel to the fixed x-frame at each material point. The rotations ${}_i\boldsymbol{\Theta}$ are due to $\boldsymbol{J}$, hence are referred to as internal rotations. These are always present in deforming solids and are completely defined by the antisymmetric part of $\boldsymbol{J}$, hence do not constitute unknown degrees of freedom at a material point in addition to the three displacements.

Surana et al. [39–48] proposed that if $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$ (for small deformation) for solid matter and the velocity gradient tensor $\bar{\boldsymbol{L}}$ for fluent continua are complete measures of the deformation physics, then the corresponding thermodynamic frameworks must incorporate ${}^d\boldsymbol{J}$ and $\bar{\boldsymbol{L}}$ in their entirety. The resulting theories are called internal polar continuum theories or non-classical continuum theories with internal rotations and rotation rates. This work is obviously independent of Cosserat theories. The works by Surana et al. are motivated by incorporating the complete physics of deformation in ${}^d\boldsymbol{J}$ and $\bar{\boldsymbol{L}}$ in the thermodynamic framework as opposed to the assumption of the existence of the moment tensor in the stress-couple theories. The works by Surana et al. [39,40,43,45,46] discuss internal polar theories for solids and the constitutive theories. In Refs. [41,42,44], the authors present internal polar theories for fluent continua including constitutive theories. In their more recent works [47,48], non-classical continuum theories are presented for (i) thermoelastic solids incorporating internal and Cosserat rotations and (ii) thermoviscous fluent continua (without memory) incorporating internal and Cosserat rotation rates.

This paper presents a non-classical continuum theory for thermoviscoelastic solid continua (without memory) in which internal rotations and their rates as well as Cosserat rotations and their rates are considered in the conservation and balance laws. Constitutive theories are derived using the conditions resulting from the entropy inequality in conjunction with the representation theorem [49–68]. Material coefficients are derived and discussed. For simplicity, only small deformation, small strain physics is considered in this paper.

2 Continuum theories with internal rotations due to $\boldsymbol{J}$ or ${}^d\boldsymbol{J}$

This paper uses the same notations as Ref. [69]. Quantities with an over-bar are quantities in the current (deformed) configuration (i.e., all quantities with over-bar are functions of coordinates $\bar{x}_i$ and time t—the Eulerian description). Quantities without an over-bar are quantities referred to the reference configuration (i.e., these are functions of undeformed coordinates x_i and time t—Lagrangian description). The configuration at time $t = t_0 = 0$, commencement of the evolution, is considered as the reference configuration. Thus, x_i and $\bar{x}_i$ are coordinates of the same material point in reference and current configurations, respectively, both measured in a fixed Cartesian x-frame. This paper only considers the Lagrangian description.

If the Jacobian of deformation $[J]$ (finite deformation) or the displacement gradient tensor $[{}^dJ]$ (small deformation) and velocity gradient tensor $[\bar{L}]$ are measures of deformation in solid and fluent continua, then the thermodynamic frameworks for solid and fluent continua must incorporate $[J]$, $[{}^dJ]$, and $[\bar{L}]$ in their entirety. First, consider solid continua (using $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ as coordinates in deformed and undeformed configurations, respectively). The Jacobian of deformation (or deformation gradient) $[J]$ and displacement gradient tensor $[{}^dJ]$ are given by

$$[J] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[\frac{\partial\{u\}}{\partial\{x\}}\right] + [I] = [{}^dJ] + [I]. \tag{1}$$

Polar decomposition of $[J]$ gives

$$[J] = [R][S_r] = [S_l][R] \tag{2}$$

where $[S_r]$ and $[S_l]$ are left and right symmetric and positive-definite stretch tensors and $[R]$ is an orthogonal rotation tensor. Decomposition of $[J]$ and $[{}^dJ]$ into symmetric ($[{}_sJ]$ and $[{}^d_sJ]$) and antisymmetric ($[{}_aJ]$ and $[{}^d_aJ]$) tensors gives

$$[J] = [{}_sJ] + [{}_aJ], \tag{3}$$

$$[{}_sJ] = \frac{1}{2}\left([J] + [J]^T\right), \tag{4}$$

$$[{}_aJ] = \frac{1}{2}\left([J] - [J]^T\right) \tag{5}$$

and

$$[{}^dJ] = \left[{}^d_sJ\right] + \left[{}^d_aJ\right], \tag{6}$$

$$\left[{}^d_sJ\right] = \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right), \tag{7}$$

$$\left[{}^d_aJ\right] = \frac{1}{2}\left([{}^dJ] - [{}^dJ]^T\right). \tag{8}$$

It is clear that

$$[{}_aJ] = \left[{}^d_aJ\right]. \tag{9}$$

Note that $\left[{}^d_sJ\right]$ is a measure of infinitesimal strain (based on the linearization of the Green strain tensor), and $[{}_aJ] = \left[{}^d_aJ\right]$ defines rotation angles at a material point about the axes of a triad with its axes parallel to the x-frame whereas $[R]$ is a rotation matrix.

For infinitesimal deformation consider $[{}^dJ]$; hence, the strain measure $\left[{}^d_sJ\right]$ and the rotation measure in $\left[{}^d_aJ\right]$ must be considered in the thermodynamic framework. Thus, consideration of $[J]$ in its entirety implies incorporating $[{}_aJ]$ in the presently used classical theories, as $\left[{}^d_sJ\right]$ is already considered in the form of strain measures. When $[J]$ varies between neighboring material points (so does $[{}_aJ]$), and if it is resisted by deforming solid continua, conjugate moments are created. Varying rotations and their rates and the conjugate moments clearly result in additional resistance to motion (i.e., additional energy storage and/or dissipation as well as possible additional physics of relaxation or memory).

The physics of internal rotations and rotation rates that vary between neighboring material points and the resulting conjugate moments has been incorporated in the derivation of conservation and balance laws as well as associated constitutive theories for simple solids and fluids by Surana et al. [39–44]. The resulting theories are referred to as internal polar non-classical continuum theories for solid and fluent continua.

The consequence of the internal polar non-classical continuum theory is that the Cauchy stress tensor becomes non-symmetric and the moment vector on the oblique plane of the tetrahedron and the Cauchy principle for it in conjunction with the balance of angular momenta establishes that the gradients of the Cauchy moment tensor are balanced by the antisymmetric components of the Cauchy stress tensor. The balance of moment of moments [70,71] establishes that the Cauchy moment tensor is symmetric. The energy equation and the entropy inequality establish that the symmetric part of the Cauchy stress tensor is conjugate to ${}^d_s\dot{\boldsymbol{J}}$ (small deformation case), the rates of linear strains, and the symmetric moment tensor is conjugate to the symmetric part of the rates of rotation gradient tensor. Surana et al. [43] provided constitutive theories for the symmetric part of the Cauchy stress tensor and the symmetric Cauchy moment tensor for thermoelastic solid continua. In the non-classical continuum theories, antisymmetric components of the Cauchy stress tensor are balanced or defined by the gradients of the Cauchy moment tensor, hence remain as dependent variables in the mathematical model and are completely defined when the Cauchy moment tensor is known. Some important points to note in this approach are listed in the following.

(i) The work is motivated by the necessity of incorporating the complete ${}^{d}\boldsymbol{J}$ in the conservation and balance laws as it represents the complete deformation physics.
(ii) Non-symmetry of Cauchy stress tensor and existence of conjugate moment tensor and its symmetry are consequences of the new physics in ${}^{d}\boldsymbol{J}$, i.e., ${}_{a}\boldsymbol{J}$ or ${}^{d}_{a}\boldsymbol{J}$.
(iii) The conservation and balance laws along with the constitutive equations for the symmetric part of the Cauchy stress tensor, symmetric moment tensor, and heat vector provide closure to the mathematical model.
(iv) Unlike stress-couple theories, existence of the moment is not assumed, but rather necessitated due to varying rotations and rotation rates between the material points that are intrinsic in ${}^{d}\boldsymbol{J}$ and ${}^{d}\dot{\boldsymbol{J}}$ and are completely defined by displacement and velocity gradients.
(v) This theory is non-classical due to the consideration of internal rotations but is not a Cosserat theory that requires consideration of Cosserat rotations as additional unknown degrees of freedom at a material point. In the internal polar non-classical theories, rotations are internal, vary between material points, and are completely defined by ${}_{a}\boldsymbol{J}$ or ${}^{d}_{a}\boldsymbol{J}$.
(vi) When the balance of moment of moments is not used as a balance law but the Cauchy moment tensor is assumed to be symmetric, it amounts to neglecting the antisymmetric part of the moment tensor (for whatever reason).

3 Considerations of internal and Cosserat rotations and their gradients, $\mathbb{J}$, stress, moment, and strain tensors

Since the work presented in this paper only considers small strain and small deformation, the distinction between covariant and contravariant measures disappears as $\bar{x}_i \approx x_i$ (i.e., the deformed configuration is not substantially different from the undeformed configuration). For such deformation, $\det[J] = \det[\bar{J}] \approx 1$, hence in the development of the theory there is a need to separate displacements from the deformed coordinates. The displacement gradient $[{}^{d}J]$ in (1) is defined as

$$[{}^{d}J] = \frac{\partial\{u\}}{\partial\{x\}} = \left[\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{10}$$

The Cauchy stress tensor can be used as a measure of stress because the deformed and undeformed tetrahedron can be treated the same for small deformation. Hence, the conservation and balance laws must be based entirely on $[{}^{d}J]$ (i.e., $[{}^{d}_{s}J]$ and $[{}^{d}_{a}J]$ both must be considered in the conservation and balance laws).

The displacement gradient tensor $[{}^{d}J]$ can be written in component form as

$$ {}^{d}J_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = {}^{d}_{s}J_{ij} + {}^{d}_{a}J_{ij} \tag{11}$$

in which

$$\left[{}^{d}_{a}J\right] = \begin{bmatrix} 0 & {}_{i}\Theta_{x_3} & -{}_{i}\Theta_{x_2} \\ -{}_{i}\Theta_{x_3} & 0 & {}_{i}\Theta_{x_1} \\ {}_{i}\Theta_{x_2} & -{}_{i}\Theta_{x_1} & 0 \end{bmatrix}, \tag{12}$$

$${}_{i}\Theta_{x_1} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right); \quad {}_{i}\Theta_{x_2} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right); \quad {}_{i}\Theta_{x_3} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right). \tag{13}$$

Alternatively (13) can be derived as

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk}\boldsymbol{e}_k \frac{\partial u_j}{\partial x_i}, \tag{14}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \boldsymbol{e}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \boldsymbol{e}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), \tag{15}$$

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_1\left(-2{}_{i}\Theta_{x_1}\right) + \boldsymbol{e}_2\left(-2{}_{i}\Theta_{x_2}\right) + \boldsymbol{e}_3\left(-2{}_{i}\Theta_{x_3}\right). \tag{16}$$

The sign difference in (13) and (16) is due to the fact that rotations in (13) are clockwise, whereas quantities in (15) are twice the magnitude compared to those in (13) and are positive when counterclockwise. In this paper, (13) is considered as the definition of rotations (i.e., clockwise). The rotations defined in (13) can exist at every material point in the deforming solid.

The right and left stretch tensors $[S_r]$ and $[S_l]$ are symmetric and positive-definite, and $[R]$ is an orthogonal rotation tensor, a rotation matrix corresponding to the rotation angles defined in (13). $[_a J]$ contains rotation angles while $[R]$ is the corresponding rotation matrix or tensor. Both their forms given here can be used in derivations as needed. However, deriving $[R]$ from $[_a J]$ or vice versa in general in $\mathbb{R}^3$ may not be possible or unique [72–74]. Fortunately, there is no need for this here.

Incorporating $[^d J]$ in its entirety in the derivation of conservation and balance laws implies incorporating $\left[{}^d_s J\right]$ and $\left[{}^d_a J\right]$ (i.e., rotations ${}_i\Theta_{x_1}$, ${}_i\Theta_{x_2}$, and ${}_i\Theta_{x_3}$ about the axes of a triad located at each material point). Rotations in $\left[{}^d_a J\right]$ are internal and are completely defined by the skew-symmetric part of $[^d J]$.

Let ${}_e\Theta_{x_1}$, ${}_e\Theta_{x_2}$, and ${}_e\Theta_{x_3}$ be the additional Cosserat rotations (unknown) about the same triad as used for internal rotations ${}_i\boldsymbol{\Theta}$, assumed positive counterclockwise. Let $\left[{}^e_a\gamma\right]$ be the antisymmetric matrix of rotation angles defined using rotations ${}_e\boldsymbol{\Theta}$, then

$$[{}^e_a\gamma] = \begin{bmatrix} 0 & {}_e\Theta_{x_3} & -{}_e\Theta_{x_2} \\ -{}_e\Theta_{x_3} & 0 & {}_e\Theta_{x_1} \\ {}_e\Theta_{x_2} & -{}_e\Theta_{x_1} & 0 \end{bmatrix}. \tag{17}$$

Angles ${}_e\Theta_j$ in (17) are positive when counterclockwise. Let

$$[\mathbb{J}] = \left[{}^d_s J\right] + \left[{}^d_a J\right] - \left[{}^e_a\gamma\right], \tag{18}$$

$$[\mathbb{J}] = \left[{}^d_s J\right] + \left[{}^t_a r\right], \tag{19}$$

$$\left[{}^t_a r\right] = \left[{}^d_a J\right] - \left[{}^e_a\gamma\right] = \begin{bmatrix} 0 & {}_t\Theta_{x_3} & -{}_t\Theta_{x_2} \\ -{}_t\Theta_{x_3} & 0 & {}_t\Theta_{x_1} \\ {}_t\Theta_{x_2} & -{}_t\Theta_{x_1} & 0 \end{bmatrix} \tag{20}$$

in which $[{}^t_a r]$ is the antisymmetric matrix containing total rotations ${}_t\Theta_{x_1}$, ${}_t\Theta_{x_2}$, and ${}_t\Theta_{x_3}$ about the axes of the triad at a material point, considered positive in the clockwise sense. Obviously,

$$\begin{aligned} {}_t\Theta_{x_1} &= {}_i\Theta_{x_1} - {}_e\Theta_{x_1}, \\ {}_t\Theta_{x_2} &= {}_i\Theta_{x_2} - {}_e\Theta_{x_2}, \\ {}_t\Theta_{x_3} &= {}_i\Theta_{x_3} - {}_e\Theta_{x_3}. \end{aligned} \tag{21}$$

Due to varying $[J]$ between material points, total rotations ${}_t\boldsymbol{\Theta}$ vary between neighboring material points. When these are resisted by the deforming matter, conjugate moments are generated which, together with ${}_t\boldsymbol{\Theta}$ and their rates, result in additional energy storage and/or dissipation as well as additional rheology.

Remarks

(i) $\left[{}^d_s J\right]$ represents the usual infinitesimal strain tensor as in the linear theory of elasticity.
(ii) $\left[{}^d_a J\right]$, $\left[{}^e_a\gamma\right]$, and $\left[{}^t_a r\right]$ are antisymmetric tensors containing rotation angles, hence are not measures of strain.
(iii) Based on (i) and (ii), $[\mathbb{J}]$ is not a strain tensor, but rather addition of strain tensor $\left[{}^d_s J\right]$ and the internal and Cosserat rotation angle tensors $\left[{}^d_a J\right]$ and $\left[{}^e_a\gamma\right]$.

When the gradients of displacements vary between neighboring material points, so do the internal rotations ${}^d_a\boldsymbol{J}$, and likewise the Cosserat rotations ${}^e_a\boldsymbol{\gamma}$ may also vary between the neighboring material points. Hence, the total rotation tensor ${}^t_a\boldsymbol{r}$ can vary between the material points. When rotations ${}^t_a\boldsymbol{r}$ are resisted by the deforming matter conjugate moments are created. ${}^t_a\boldsymbol{r}$ and their rates and conjugate moments can result in additional energy storage, dissipation, and rheology, i.e., in addition to those which are already present due to the Cauchy stress tensor, strain, and strain rate tensors. Thus, in the deforming matter, the total rotations ${}^t_a\boldsymbol{r}$ are conjugate to the moment tensor which necessitates that on the boundary of the deformed volume there must exist a resultant moment tensor.

Consider a volume of matter $\underset{\sim}{V}$ in the reference configuration with closed boundary $\partial\underset{\sim}{V}$. The volume V is isolated from $\underset{\sim}{V}$ by a hypothetical surface ∂V as in the cut principle of Cauchy. Consider a tetrahedron T_1 such that its oblique plane is part of ∂V and its other three planes are orthogonal to each other and parallel to the planes of the x-frame. Upon deformation, $\underset{\sim}{V}$ and $\partial\underset{\sim}{V}$ occupy $\bar{\underset{\sim}{V}}$ and $\partial\bar{\underset{\sim}{V}}$, and likewise V and ∂V deform into $\bar{V}$ and $\partial\bar{V}$. The tetrahedron T_1 deforms into $\bar{T}_1$ whose edges (under finite deformation) are non-orthogonal covariant base vectors $\tilde{g}_i$. The planes of the tetrahedron formed by the covariant base vectors are flat but obviously non-orthogonal to each other. We assume the tetrahedron to be the small neighborhood of material point $\bar{o}$ so that the assumption of the oblique plane $\bar{A}\bar{B}\bar{C}$ being flat but still part of $\partial\bar{V}$ is valid. When the deformed tetrahedron is isolated from volume $\bar{V}$ it must be in equilibrium under the action of disturbance on surface $\bar{A}\bar{B}\bar{C}$ from the volume surrounding $\bar{V}$ and the internal fields that act on the flat faces which equilibrate with the mating faces in volume $\bar{V}$ when the tetrahedron T_2 is placed back in the volume $\bar{V}$.

Consider the deformed tetrahedron $\bar{T}_1$. Let $\bar{\boldsymbol{P}}$ be the average stress per unit area on plane $\bar{A}\bar{B}\bar{C}$, $\bar{\boldsymbol{M}}$ be the average moment per unit area on plane $\bar{A}\bar{B}\bar{C}$ henceforth referred to as moment for short, and $\bar{\boldsymbol{n}}$ be the normal to the face $\bar{A}\bar{B}\bar{C}$. $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{M}}$, and $\bar{\boldsymbol{n}}$ all have different directions when the deformation is finite. Based on the small deformation assumption, the deformed coordinates $\bar{x}_i$ are approximately the same as the undeformed coordinates x_i; thus, the deformed tetrahedron $\bar{T}_1$ in the current configuration is close to its map T_1 in the reference configuration. With this assumption, all stress measures (first and second Piola-Kirchhoff stress tensors, Cauchy stress tensor) are approximately the same. The same holds for the moment tensors. Thus with the assumption $\bar{\boldsymbol{x}} \approx \boldsymbol{x}$ we can write

$$\bar{\boldsymbol{P}} = \boldsymbol{P}, \quad \bar{\boldsymbol{M}} = \boldsymbol{M}. \tag{22}$$

The Cauchy principle for $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{M}}$ gives (hence for $\boldsymbol{P}$ and $\boldsymbol{M}$)

$$\boldsymbol{P} = \boldsymbol{\sigma}^T \cdot \boldsymbol{n}, \quad \boldsymbol{M} = \boldsymbol{m}^T \cdot \boldsymbol{n} \tag{23}$$

in which $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{m}$ is the Cauchy moment tensor (per unit area). Let

$$\{{}_t\Theta\}^T = \left[{}_t\Theta_{x_1},\ {}_t\Theta_{x_2},\ {}_t\Theta_{x_3}\right]. \tag{24}$$

Gradients of total rotations in (24) ($[{}^{\Theta}J^t]$) can be defined using

$$\left[{}^{\Theta}J^t\right] = \left[\frac{\partial\{{}_t\Theta\}}{\partial\{x\}}\right] \quad \text{or} \quad {}^{\Theta}J^t_{ij} = \frac{\partial\, {}_t\Theta_i}{\partial x_j}. \tag{25}$$

The gradient tensor of total rotations $[{}^{\Theta}J^t]$ in (25) can be decomposed into symmetric and antisymmetric parts $[{}^{\Theta}_{s}J^t]$ and $[{}^{\Theta}_{a}J^t]$,

$$\left[{}^{\Theta}J^t\right] = \left[{}^{\Theta}_{s}J^t\right] + \left[{}^{\Theta}_{a}J^t\right], \tag{26}$$

$$\begin{aligned} \left[{}^{\Theta}_{s}J^t\right] &= \frac{1}{2}\left(\left[{}^{\Theta}J^t\right] + \left[{}^{\Theta}J^t\right]^T\right), \\ \left[{}^{\Theta}_{a}J^t\right] &= \frac{1}{2}\left(\left[{}^{\Theta}J^t\right] - \left[{}^{\Theta}J^t\right]^T\right). \end{aligned} \tag{27}$$

Both $\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are non-symmetric tensors.

4 Conservation and balance laws

The conservation and balance laws for a non-classical solid continuum with internal and Cosserat rotations have been derived by Surana et al. [47,48] (see also Eringen et al. [75–83]). In the following, the standard conservation and balance laws used in classical continuum theories that are also applicable to non-classical continuum theories considered in this paper are presented. Conservation of mass, balance of linear momenta, balance of angular momenta, and the first and second laws of thermodynamics yield the following set of equations:

$$\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x}, t), \tag{28}$$

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0, \tag{29}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0, \tag{30}$$

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - \text{tr}\left([\sigma]\left[\frac{\partial\{v\}}{\partial\{x\}}\right]\right) - \text{tr}\left([m]\left[\frac{\partial\{{}_t\dot{\Theta}\}}{\partial\{x\}}\right]\right) - {}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) = 0, \tag{31}$$

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q} \cdot \boldsymbol{g}}{\theta} - \text{tr}\left([\sigma]\left[\frac{\partial\{v\}}{\partial\{x\}}\right]\right) - \text{tr}\left([m]\left[\frac{\partial\{{}_t\dot{\Theta}\}}{\partial\{x\}}\right]\right) - {}_t\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{m}) \leq 0 \tag{32}$$

in which ρ_0 and ρ are the density in the reference and current configurations, $\boldsymbol{v}$ are the velocities, $\boldsymbol{F}^b$ are the body forces per unit mass, e is the specific internal energy, $\boldsymbol{q}$ is the heat vector, $\boldsymbol{g}$ are the temperature gradients, Φ is the Helmholtz free energy density, η is the entropy density, and θ is the absolute temperature.

4.1 Why the law of balance of moment of moments is necessary in non-classical continuum theories for solid continua

Yang et al. [84] presented the following reasoning for the consideration of additional requirements advocated to be necessary for equilibrium of deforming solid matter in couple stress non-classical continuum theories.

When a system of forces is applied to a system of multiple particles, the equilibrium relations are derived from a resultant force and a resultant couple of forces applied to an arbitrary point. The moment of a couple of forces is a free vector in classical mechanics, which means that the effect of the couple applied at an arbitrary point in the space of the system of materials particles is independent of the position of the point. In other words, the couple can translate to any point in the space freely and the resulting effects are unchanged. As a result, only the conventional force equilibrium and the moment equilibrium (balance of linear and angular momenta) are involved in the equilibrium relations [9–11]. Equivalence of a couple resulting from the rotations ${}_i\Theta$ that is not a free vector but a driving force that rotates the material particles requires considerations (see [84] for details) that eventually result in balance of moment of moments or couples for static equilibrium.

In this reasoning at the onset of the derivation, the moment of the moments due to the antisymmetric part of the Cauchy stress tensor and the moment of the moment $\bar{\boldsymbol{M}}$ acting on the oblique plane of the deformed tetrahedron are assumed to equilibrate. This is termed by the authors as *balance of moment of moments balance law*. In this approach, there are two major points to be clarified: (i) Is this a balance law, and (ii) if what Yang et al. [84] presented is based on static considerations, then does it ensure dynamic equilibrium of the deforming non-classical solid continua during evolution? Based on the definition of balance laws in continuum mechanics (balance of linear and angular momenta, for example), a balance law must be based on rate considerations. The work of Yang et al. [84] as presented by them is static equilibrium; hence, it is perhaps more appropriate to label this as an equilibrium consideration at this stage rather than a balance law.

First, we consider inductive reasoning to demonstrate and establish why there is a need for an additional law in non-classical continuum theories to ensure dynamic equilibrium of the deforming solid continua in the presence of internal rotations and conjugate moments. In classical continuum theories for solid continua that consider displacements as the only observable quantities at the material points and their conjugate forces (or stresses), it is well known that the balance of linear momenta and the balance of angular momenta must hold for the dynamic equilibrium of the deforming solid continua. That is, the rate of change of linear momenta must be balanced by body forces and the average stress $\bar{\boldsymbol{P}}$ on the oblique plane of the tetrahedron for any arbitrary volume of matter (balance of linear momenta). The rate of change of the moment of linear momenta must be balanced by the moment of the body forces and the moment of average stress $\bar{\boldsymbol{P}}$ on the oblique plane of the tetrahedron (balance of angular momenta balance law). These two balance laws ensure stable dynamic equilibrium of the deforming volume of solid continua in classical continuum mechanics at any instant of time. Thus, we note that when the displacements are the only kinematic variables, two balance laws are required. The first balance law is the dynamic balance of the quantities conjugate to displacements that are forces, the balance of linear momenta, and the second one is the dynamic balance of the moments of the quantities conjugate to the displacements, i.e., moments of forces, the balance of angular momenta.

Remarks.

(i) We note that the balances of linear and angular momenta contain physics purely related to the forces and the moments due to the forces.

(ii) When rotations and their conjugate moments are introduced, the balance of linear momenta (purely related to the forces) remains unchanged as the non-symmetry of the Cauchy stress tensor is also present in classical theories until the balance of angular momenta establishes it to be symmetric.

(iii) Additional moments introduced by the consideration of internal rotations must now by considered to modify the dynamic balance of moments, i.e., the balance of angular momenta used in classical theories. The end result is the relationship between additional conjugate quantities introduced due to non-classical theories, i.e., the antisymmetric components of the Cauchy stress tensor and the Cauchy moment tensor. We note that neither of these exists in the classical continuum theory. Thus, due to internal rotations and the conjugate moments, the first balance law needed is the balance of angular momenta. This already exists due to the classical theory; hence, it is modified due to the presence of additional moments conjugate to the internal rotations and also due to the antisymmetric components of the Cauchy stress tensor.

(iv) A new balance law, *the law of balance of moment of moments* (parallel to the balance of angular momenta in classical theories), is required for dynamic equilibrium in the presence of internal rotations, their rates, and conjugate moments. This balance law must be a rate law just like all other balance laws, and must only contain the physics related to the non-classical behavior, i.e., possibly rotations and their rates, the conjugate Cauchy moment tensor, and the antisymmetric part of the Cauchy stress tensor. Thus, in the derivation of this balance law, we must consider the rate of the moment of angular momenta only due to internal rotation rates to balance with: (i) the moment of moments of those components associated with $\bar{\boldsymbol{P}}$ that are only related to non-classical physics, i.e., the antisymmetric components of the Cauchy stress tensor, and (ii) the moment of $\bar{\boldsymbol{M}}$ which is only due to non-classical physics.

(v) We remark that consideration of the following as a balance law in the non-classical theory considered here is invalid,

$$\frac{D}{Dt}\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{v}})\, d\bar{V} = \int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}} - \bar{\boldsymbol{M}})\, d\bar{\boldsymbol{A}} + \int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b)\, d\bar{V}. \tag{33}$$

(a) The left-hand side is purely due to the classical continuum physics, hence cannot be part of this balance law. (b) $\int_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{P}})\, d\bar{\boldsymbol{A}}$ is invalid as it contains the symmetric part of the Cauchy stress tensor (after application of the Cauchy principle) which is also part of the classical continuum theory. In this expression, only antisymmetric components of the Cauchy stress tensor should be considered as these are the only components related to non-classical behavior. (c) $\int_{\bar{V}(t)} \bar{\boldsymbol{x}} \times (\bar{\boldsymbol{x}} \times \bar{\rho}\bar{\boldsymbol{F}}^b)\, d\bar{V}$ is also purely due to classical continuum physics, hence cannot be considered in this balance law. Presence of $\int_{\partial\bar{V}(t)} (\bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}})\, d\bar{\boldsymbol{A}}$ is valid as $\bar{\boldsymbol{M}}$ is purely due to internal rotation physics.

(vi) The derivation using (33) leads to erroneous results, as expected due to the fact that (33) mostly contains physics that is purely related to the classical continuum theory (except $\bar{\boldsymbol{M}}$) that should be eliminated from this balance law as this balance law is only necessitated due to new physics related to internal rotations.

(vii) We note that a kinematic variable requires two balance laws: (i) the first is related to the dynamic balance of the quantity conjugate to the kinematic variable and (ii) the second one is related to the dynamic balance of the moment of the quantities conjugate to the kinematic variable. Displacements as kinematic variables need the balance of linear and angular momenta, which are dynamic balances of forces and their moments. Introduction of internal rotations and their rates require dynamic balance of moments (which already exists as the balance of angular momenta) and dynamic balance of moment of moments, a new balance law. Thus, for each new kinematic variable we need to consider (i) the dynamic balance of its conjugate quantity that already exists from the previous kinematic variable, hence can be modified to accommodate the influence of new physics, and (ii) dynamic balance of the moment of its conjugate quantity related only to the physics associated with the new kinematic variable, which is a new balance law that needs to be derived using rate considerations. This inductive reasoning holds for the introduction of each new kinematic variable.

(viii) In the next Section, we present the derivation of the *law of balance of moment of moments* based on rate considerations.

4.2 The law of balance of moment of moments

In this balance law, we must consider the rate of moment of angular momenta due to rotation rates to balance with the moment of moments of the antisymmetric components of the Cauchy stress tensor and the moments of $\bar{\boldsymbol{M}}$, all of which are only related to the non-classical physics due to internal rotations and the associated conjugate moments. We can write

$$\begin{pmatrix} \text{rate of moment of the} \\ \text{angular momenta due to} \\ \text{internal rotation rates} \\ \text{over } \bar{V}(t) \end{pmatrix} = \begin{pmatrix} \text{moment of moments due} \\ \text{to antisymmetric components} \\ \text{of the Cauchy stress} \\ \text{tensor over } \bar{V}(t) \end{pmatrix} - \begin{pmatrix} \text{moment of } \bar{\boldsymbol{M}} \\ \text{over } \partial\bar{V}(t) \end{pmatrix} \tag{34}$$

The negative sign is due to the assumption of clockwise internal rotations to be positive; hence, the corresponding moment tensor must be positive in the same sense. We note that in continuum theories for continuous media, we assume that the material points or particles have mass but no dimensions; thus, the angular momenta associated with the material particles due to rotation rates are zero. Thus, (34) reduces to

$$\begin{pmatrix} \text{moment of moments due} \\ \text{to antisymmetric components} \\ \text{of the Cauchy stress} \\ \text{tensor over } \bar{V}(t) \end{pmatrix} - \begin{pmatrix} \text{moment of } \bar{\boldsymbol{M}} \\ \text{over } \partial\bar{V}(t) \end{pmatrix} = 0 \tag{35}$$

If we consider the current configuration at time t, then in the Eulerian description we can write (35) as

$$\int\limits_{\bar{V}(t)} \bar{\boldsymbol{x}} \times (\boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}^{(0)})\, d\bar{V} - \int\limits_{\partial\bar{V}(t)} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} = 0. \tag{36}$$

Remarks.

(i) In the absence of (34), which is a rate statement, if we consider (35) and (36) directly, then we could mistakenly view (35) and (36) as equilibrium of moment of moments, a static consideration. This is obviously incorrect. As stated by Yang et al. [84], (36) indeed is a balance law, even though its derivation based on a statement like (34) is not reported in Ref. [84]. Due to the fact that the left-hand side of (34) is zero, the balance law (34) results in (36), which unfortunately has the appearance of an equilibrium statement.

(ii) Henceforth, in this paper, we refer to (36) as *the law of balance of moment of moments*. For the non-classical physics considered in this paper, the balance law (34) reduces to (36), which can also be labeled as the equilibrium of moment of moments due to the absence of rate terms (as they are zero). Thus, we can also say *the law of balance/equilibrium of moment of moments*.

We expand the second term in (36) and then convert the integral over $\partial\bar{V}$ to the integral over $\bar{V}$ using the divergence theorem:

$$\begin{aligned} \int\limits_{\partial\bar{V}} \bar{\boldsymbol{x}} \times \bar{\boldsymbol{M}}\, d\bar{A} &= \int\limits_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} x_i \bar{M}_j\, d\bar{A} \\ &= \int\limits_{\partial\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{x}_i \bar{m}_{mj} \bar{n}_m\, d\bar{A} \\ &= \int\limits_{\bar{V}} \boldsymbol{e}_k (\epsilon_{ijk} \bar{x}_i \bar{m}_{mj})_{,m}\, d\bar{V} \\ &= \int\limits_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} (\bar{m}_{ij} + \bar{x}_i \bar{m}_{mj,m})\, d\bar{V} \\ &= \int\limits_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij}\, d\bar{V} + \int\limits_{\bar{V}} \bar{\boldsymbol{x}} \times \left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}}\right) d\bar{V}. \end{aligned} \tag{37}$$

Using Eq. (37) in (36) and collecting terms yields:

$$\int_{\bar{V}} \bar{\boldsymbol{x}} \times \left(-\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{m}} + \boldsymbol{\epsilon} : \bar{\boldsymbol{\sigma}}\right) d\bar{V} - \int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0. \tag{38}$$

The first term in (38) vanishes due to the balance of angular momenta, giving the condition

$$\int_{\bar{V}} \boldsymbol{e}_k \epsilon_{ijk} \bar{m}_{ij} \, d\bar{V} = 0 \tag{39}$$

which, because $\bar{V}$ is arbitrary, yields

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \quad \text{and} \quad \epsilon_{ijk} m_{ij} = 0. \tag{40}$$

Equation (40) implies that $m_{ij} = m_{ji}$, i.e., the Cauchy moment tensor is symmetric. On the other hand, in the absence of this balance law, symmetry of the Cauchy moment tensor is not established; hence, the Cauchy moment tensor $\boldsymbol{m}$ will be non-symmetric. In the derivation of the constitutive theory for $\boldsymbol{m}$ in this paper, we assume $\boldsymbol{m}$ to be non-symmetric, implying that the balance of moment of moments is not considered as a necessary balance law. This is a more general case. The constitutive theories when $\boldsymbol{m}$ is symmetric are a subset of the more general case in which $\boldsymbol{m}$ is non-symmetric.

5 Conjugate pairs in the entropy inequality, constitutive variables, and their argument tensors

From the entropy inequality, note that in each of the two trace terms both tensors are non-symmetric, thus based on the works of Spencer, Wang, Zheng etc. [49–68], the pair of tensors in each trace term does not constitute a conjugate pair. That is, either of the tensors in each pair cannot be expressed in terms of the other due to the lack of existence of integrity for non-symmetric tensors. Recall

$$\left[\frac{\partial\{v\}}{\partial\{x\}}\right] = [L] = [\dot{J}] = [{}^{d}\dot{J}] = [\dot{\mathbb{J}}] + \left[{}^{e}_{a}\dot{\gamma}\right], \tag{41}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{m} = \boldsymbol{\epsilon} : \boldsymbol{\sigma}, \tag{42}$$

$${}_{t}\dot{\boldsymbol{\Theta}} = {}_{i}\dot{\boldsymbol{\Theta}} - {}_{e}\dot{\boldsymbol{\Theta}}. \tag{43}$$

Substituting (41)–(43) in the entropy inequality (32) and using

$$\operatorname{tr}\left([\sigma]\left[{}^{e}_{a}\dot{\gamma}\right]\right) = {}_{e}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \tag{44}$$

leads to the following:

$$\rho_0\left(\frac{D\Phi}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - \operatorname{tr}\left([\sigma][\dot{\mathbb{J}}]\right) - \operatorname{tr}\left([m][{}^{\Theta}\dot{J}^{t}]\right) - {}_{i}\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0. \tag{45}$$

Note that in (45) both terms in each trace are non-symmetric tensors. Consider

$$[\sigma] = [{}_{s}\sigma] + [{}_{a}\sigma], \tag{46}$$

$$[m] = [{}_{s}m] + [{}_{a}m], \tag{47}$$

$$[\dot{\mathbb{J}}] = \left[{}^{d}_{s}\dot{J}\right] + \left[{}^{d}_{a}\dot{J}\right] - \left[{}^{e}_{a}\dot{\gamma}\right] = \left[{}^{d}_{s}\dot{J}\right] + \left[{}^{t}_{a}\dot{r}\right] = [\dot{\varepsilon}] + \left[{}^{t}_{a}\dot{r}\right], \tag{48}$$

$$\begin{aligned} &\operatorname{tr}\left([{}_{s}\sigma]\left[{}^{t}_{a}\dot{r}\right]\right) = 0, \\ &\operatorname{tr}\left([{}_{a}\sigma][\dot{\varepsilon}]\right) = 0, \\ &\operatorname{tr}\left([{}_{s}m]\left[{}^{\Theta}_{a}\dot{J}^{t}\right]\right) = 0, \\ &\operatorname{tr}\left([{}_{a}m]\left[{}^{\Theta}_{s}\dot{J}^{t}\right]\right) = 0. \end{aligned} \tag{49}$$

Substituting (46)–(48) and then using (49),

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-({}_s\sigma_{ki})(\dot{\varepsilon}_{ki})-({}_a\sigma_{ki})\left({}^t_a\dot{r}_{ki}\right)-({}_sm_{ki})\left({}^\Theta_s\dot{J}^t_{ki}\right)$$
$$-({}_am_{ki})({}^\Theta_a\dot{J}^t_{ki})-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})\leq 0. \tag{50}$$

The energy equation can accordingly be modified as

$$\rho_0\frac{De}{Dt}+\nabla\cdot\boldsymbol{q}-({}_s\sigma_{ki})(\dot{\varepsilon}_{ki})-({}_a\sigma_{ki})\left({}^t_a\dot{r}_{ki}\right)-({}_sm_{ki})\left({}^\Theta_s\dot{J}^t_{ki}\right)-({}_am_{ki})\left({}^\Theta_a\dot{J}^t_{ki}\right)-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})=0. \tag{51}$$

Note that in (51) in the trace terms both tensors are either symmetric or antisymmetric; hence, all trace terms in (51) could be used to determine conjugate pairs in the constitutive theories. First, from (51), it is easy to infer that Φ, η, $\boldsymbol{q}$, ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, and ${}_a\boldsymbol{m}$ are possible choices of constitutive variables. The argument tensors of these constitutive variables are decided using the conjugate pairs as well as the desired physics these are to represent. Since the deforming matter has elasticity as well as dissipation, $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ (or $\boldsymbol{\varepsilon}_{[1]}$), both need to be argument tensors of ${}_s\boldsymbol{\sigma}$. If we assume that higher-order time derivatives of $\boldsymbol{\varepsilon}$ also contribute to dissipation, then $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_{[k]}$; $k=1,2,\ldots,n_\varepsilon$ need to be argument tensors of ${}_s\boldsymbol{\sigma}$. The resistance to deformation results in ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$. We also assume that the dissipation mechanism is also due to ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, and ${}_a\boldsymbol{\sigma}$ and the time derivatives of the corresponding conjugate tensors. Temperature θ is surely an argument tensor of all constitutive variables. Based on (51), the choice of $\boldsymbol{g}$ and θ as argument tensors of $\boldsymbol{q}$ is valid. Since the deformation and strains are small, i.e., $\bar{\boldsymbol{x}}\approx\boldsymbol{x}$, $\rho\approx\rho_0$ (the matter is assumed incompressible), hence $|J|=1$. Thus, we have the following for the argument tensors of the constitutive variables:

$$\begin{aligned}
{}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n_\varepsilon,\ \theta\right),\\
{}_a\boldsymbol{\sigma} &= {}_a\boldsymbol{\sigma}\left({}^t_a\boldsymbol{r},{}^t_a\boldsymbol{r}_{[j]};\ j=1,2,\ldots,n_{({}^t_a r)},\ \theta\right),\\
{}_s\boldsymbol{m} &= {}_s\boldsymbol{m}\left({}^\Theta_s\boldsymbol{J}^t,{}^\Theta_s\boldsymbol{J}^t_{[k]};\ k=1,2,\ldots,n_{({}^\Theta_s J^t)},\ \theta\right),\\
{}_a\boldsymbol{m} &= {}_a\boldsymbol{m}\left({}^\Theta_a\boldsymbol{J}^t,{}^\Theta_a\boldsymbol{J}^t_{[l]};\ l=1,2,\ldots,n_{({}^\Theta_a J^t)},\ \theta\right),\\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g},\theta),\\
\Phi &= \Phi\Big(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n_\varepsilon,\ {}^t_a\boldsymbol{r},{}^t_a\boldsymbol{r}_{[j]};\ j=1,2,\ldots,n_{({}^t_a r)},\\
&\qquad {}^\Theta_s\boldsymbol{J}^t,{}^\Theta_s\boldsymbol{J}^t_{[k]};\ k=1,2,\ldots,n_{({}^\Theta_s J^t)},\ {}^\Theta_a\boldsymbol{J}^t,{}^\Theta_a\boldsymbol{J}^t_{[l]};\ l=1,2,\ldots,n_{({}^\Theta_a J^t)},\ \boldsymbol{g},\theta\Big),\\
\eta &= \eta\Big(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n_\varepsilon,\ {}^t_a\boldsymbol{r},{}^t_a\boldsymbol{r}_{[j]};\ j=1,2,\ldots,n_{({}^t_a r)},\\
&\qquad {}^\Theta_s\boldsymbol{J}^t,{}^\Theta_s\boldsymbol{J}^t_{[k]};\ k=1,2,\ldots,n_{({}^\Theta_s J^t)},\ {}^\Theta_a\boldsymbol{J}^t,{}^\Theta_a\boldsymbol{J}^t_{[l]};\ l=1,2,\ldots,n_{({}^\Theta_a J^t)},\ \boldsymbol{g},\theta\Big).
\end{aligned} \tag{52}$$

Remarks.

(i) In defining the argument tensors of ${}_s\boldsymbol{\sigma}$, ${}_a\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, and $\boldsymbol{q}$ in (52), we could have used the principle of equipresence, in which case the argument tensors of these constitutive variables would be the same as those of Φ and η in (52). We have intentionally not done so based on the discussion in the paragraph immediately preceding Eq. (52). We elaborate on this in the following.

(ii) The conjugate pairs in (50) dictate rate of work conjugate pairs, hence facilitate the choices of the argument tensors of the constitutive variables.

(iii) As an example consider the constitutive variable ${}_s\boldsymbol{\sigma}$. From (50), $\boldsymbol{\varepsilon}$ needs to be its argument tensor as $\dot{\boldsymbol{\varepsilon}}$ is the rate of work conjugate to ${}_s\boldsymbol{\sigma}$. The dissipation mechanism necessitates that $\boldsymbol{\varepsilon}_{[i]}$; $i=1,2,\ldots,n$ be argument tensors of ${}_s\boldsymbol{\sigma}$ as well, and the choice of θ as an argument tensor is obvious as we are considering thermoviscoelastic solids. The choices of $\{g\}$, ${}^\Theta_s\boldsymbol{J}^t$, ${}^\Theta_s\boldsymbol{J}^t_{[k]}$; $k=1,2,\ldots,n_{({}^\Theta_s J^t)}$, etc. are ruled out as their rates are not rate of work conjugate with ${}_s\boldsymbol{\sigma}$.

(iv) From the $\mathrm{tr}\big([{}_s\sigma][\dot{\varepsilon}]\big)$ term in the entropy inequality (50), one could argue whether $\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n$ should be considered as argument tensors of ${}_s\boldsymbol{\sigma}$. A closer examination of the derivation of the energy equation reveals that the rates of work due to $\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n$ have not been included in its derivation; hence, the lack of appearance of $\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n$ conjugate to ${}_s\boldsymbol{\sigma}$ in the entropy inequality is a rather natural consequence. This work, currently in progress, shows that the choice of $\boldsymbol{\varepsilon}_{[i]};\ i=1,2,\ldots,n$ as argument tensors of ${}_s\boldsymbol{\sigma}$ is supported by a more complete derivation of the energy equation in which higher strain rates are included.

(v) Based on remarks (i)–(iv), the choice of argument tensors of the constitutive variables in (52) is appropriate. Starting with the principle of equipresence and ruling out those argument tensors that are not supported by the rate of work conjugate argument will result in (52).

Note that the argument tensors of Φ and η are the totality of all of the argument tensors of all of the constitutive variables. At this stage (52) is the most general choice. During the derivation of constitutive theories, some arguments of some constitutive variables may be ruled out due to some other considerations.

Using $\Phi(\cdot)$ in (52), $\dot{\Phi}$ can be obtained,

$$
\begin{aligned}
\frac{D\Phi}{Dt}=\dot{\Phi}=&\frac{\partial\Phi}{\partial\varepsilon_{ki}}\dot{\varepsilon}_{ki}+\sum_{j=1}^{n_\varepsilon}\frac{\partial\Phi}{\partial(\varepsilon_{[j]})_{ki}}(\dot{\varepsilon}_{[j]})_{ki}+\frac{\partial\Phi}{\partial\left({}^t_ar_{ki}\right)}\left({}^t_a\dot{r}_{ki}\right)+\sum_{j=1}^{n_{({}^t_ar)}}\frac{\partial\Phi}{\partial\left({}^t_ar_{[j]}\right)_{ki}}\left({}^t_a\dot{r}_{[j]}\right)_{ki}\\
&+\frac{\partial\Phi}{\partial\left({}^\Theta_sJ^t_{ki}\right)}\left({}^\Theta_s\dot{J}^t_{ki}\right)+\sum_{j=1}^{n_{{}^\Theta_sJ^t}}\frac{\partial\Phi}{\partial\left({}^\Theta_sJ^t_{[j]}\right)_{ki}}\left({}^\Theta_s\dot{J}^t_{[j]}\right)_{ki}+\frac{\partial\Phi}{\partial\left({}^\Theta_aJ^t_{ki}\right)}\left({}^\Theta_a\dot{J}^t_{ki}\right)+\sum_{j=1}^{n_{{}^\Theta_aJ^t}}\frac{\partial\Phi}{\partial\left({}^\Theta_aJ^t_{[j]}\right)_{ki}}\left({}^\Theta_a\dot{J}^t_{[j]}\right)_{ki}\\
&+\frac{\partial\Phi}{\partial g_i}\dot{g}_i+\frac{\partial\Phi}{\partial\theta}\dot{\theta}.
\end{aligned}
\tag{53}
$$

Substituting (53) into the entropy inequality (50) and collecting terms yields

$$
\begin{aligned}
&\left(\rho_0\frac{\partial\Phi}{\partial\varepsilon_{ki}}-{}_s\sigma_{ki}\right)\dot{\varepsilon}_{ki}+\left(\rho_0\frac{\partial\Phi}{\partial\left({}^t_ar_{ki}\right)}-{}_a\sigma_{ki}\right){}^t_a\dot{r}_{ki}+\left(\rho_0\frac{\partial\Phi}{\partial\left({}^\Theta_sJ^t_{ki}\right)}-{}_sm_{ki}\right){}^\Theta_s\dot{J}^t_{ki}\\
&+\left(\rho_0\frac{\partial\Phi}{\partial\left({}^\Theta_aJ^t_{ki}\right)}-{}_am_{ki}\right){}^\Theta_a\dot{J}^t_{ki}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}+\rho_0\left(\eta+\frac{\partial\Phi}{\partial\theta}\right)\dot{\theta}+\frac{\partial\Phi}{\partial g_i}\dot{g}_i+\sum_{j=1}^{n_\varepsilon}\frac{\partial\Phi}{\partial\left(\varepsilon_{[j]}\right)_{ki}}\left(\dot{\varepsilon}_{[j]}\right)_{ki}\\
&+\sum_{j=1}^{n_{({}^t_ar)}}\frac{\partial\Phi}{\partial\left({}^t_ar_{[j]}\right)_{ki}}\left({}^t_a\dot{r}_{[j]}\right)_{ki}+\sum_{j=1}^{n_{{}^\Theta_sJ^t}}\frac{\partial\Phi}{\partial\left({}^\Theta_sJ^t_{[j]}\right)_{ki}}\left({}^\Theta_s\dot{J}^t_{[j]}\right)_{ki}+\sum_{j=1}^{n_{{}^\Theta_aJ^t}}\frac{\partial\Phi}{\partial\left({}^\Theta_aJ^t_{[j]}\right)_{ki}}\left({}^\Theta_a\dot{J}^t_{[j]}\right)_{ki}\\
&-{}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma})\leq 0.
\end{aligned}
\tag{54}
$$

For arbitrary but admissible $\dot{\theta}$, $\dot{\boldsymbol{g}}$, $\dot{\boldsymbol{\varepsilon}}_{[j]};\ j=1,2,\ldots,n_\varepsilon$, ${}^t_a\dot{\boldsymbol{r}}_{[j]};\ j=1,2,\ldots,n_{{}^t_ar}$, ${}^\Theta_s\dot{\boldsymbol{J}}^t_{[j]};\ j=1,2,\ldots,n_{{}^\Theta_sJ^t}$, and ${}^\Theta_a\dot{\boldsymbol{J}}^t_{[j]};\ j=1,2,\ldots,n_{{}^\Theta_aJ^t}$, the entropy inequality will hold if their coefficients are zero. Hence, we obtain the following:

$$
\rho_0\left(\eta+\frac{\partial\Phi}{\partial\theta}\right)=0\implies\eta=-\frac{\partial\Phi}{\partial\theta},
\tag{55}
$$

$$
\begin{aligned}
&\frac{\partial\Phi}{\partial g_i}=0\implies\Phi\neq\Phi(\boldsymbol{g}),\\
&\frac{\partial\Phi}{\partial\boldsymbol{\varepsilon}_{[j]}}=0;\quad j=1,2,\ldots,n_\varepsilon\implies\Phi\neq\Phi\left(\boldsymbol{\varepsilon}_{[j]};\ j=1,2,\ldots,n_\varepsilon\right),\\
&\frac{\partial\Phi}{\partial{}^t_a\boldsymbol{r}_{[j]}}=0;\quad j=1,2,\ldots,n_{{}^t_ar}\implies\Phi\neq\Phi\left({}^t_a\boldsymbol{r}_{[j]};\ j=1,2,\ldots,n_{{}^t_ar}\right),\\
&\frac{\partial\Phi}{\partial{}^\Theta_s\boldsymbol{J}^t_{[j]}}=0;\quad j=1,2,\ldots,n_{{}^\Theta_sJ^t}\implies\Phi\neq\Phi\left({}^\Theta_s\boldsymbol{J}^t_{[j]};\ j=1,2,\ldots,n_{{}^\Theta_sJ^t}\right),\\
&\frac{\partial\Phi}{\partial{}^\Theta_a\boldsymbol{J}^t_{[j]}}=0;\quad j=1,2,\ldots,n_{{}^\Theta_aJ^t}\implies\Phi\neq\Phi\left({}^\Theta_a\boldsymbol{J}^t_{[j]};\ j=1,2,\ldots,n_{{}^\Theta_aJ^t}\right).
\end{aligned}
\tag{56}
$$

Condition (55) implies that η is not a constitutive variable. Using (56), the entropy inequality reduces to

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki}\right)\dot{\varepsilon}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial \left({}^t_a r_{ki}\right)} - {}_a\sigma_{ki}\right){}^t_a\dot{r}_{ki} + \left(\rho_0 \frac{\partial \Phi}{\partial \left({}^\Theta_s J^t_{ki}\right)} - {}_s m_{ki}\right){}^\Theta_s\dot{J}^t_{ki}$$
$$+ \left(\rho_0 \frac{\partial \Phi}{\partial \left({}^\Theta_a J^t_{ki}\right)} - {}_a m_{ki}\right){}^\Theta_a\dot{J}^t_{ki} + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0, \tag{57}$$

and the argument tensors of Φ are modified as well,

$$\Phi = \Phi\left(\boldsymbol{\varepsilon}, {}^t_a\boldsymbol{r}, {}^\Theta_s\boldsymbol{J}^t, {}^\Theta_a\boldsymbol{J}^t, \theta\right), \tag{58}$$

The argument tensors of the remaining constitutive variables remain the same as in (52).

The entropy inequality in the form given by (57) is essential. For example, for arbitrary but admissible choices of $\dot{\boldsymbol{\varepsilon}}$, ${}^t_a\dot{\boldsymbol{r}}$, ${}^\Theta_s\dot{\boldsymbol{J}}^t$, and ${}^\Theta_a\dot{\boldsymbol{J}}^t$, if it is assumed that their coefficients in (57) are zero, then

$$\begin{aligned}
{}_s\boldsymbol{\sigma} &= \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} \implies {}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta),\\
{}_a\boldsymbol{\sigma} &= \rho_0 \frac{\partial \Phi}{\partial {}^t_a\boldsymbol{r}} \implies {}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}({}^t_a\boldsymbol{r}, \theta),\\
{}_s\boldsymbol{m} &= \rho_0 \frac{\partial \Phi}{\partial {}^\Theta_s\boldsymbol{J}^t} \implies {}_s\boldsymbol{m} = {}_s\boldsymbol{m}({}^\Theta_s\boldsymbol{J}^t, \theta),\\
{}_a\boldsymbol{m} &= \rho_0 \frac{\partial \Phi}{\partial {}^\Theta_a\boldsymbol{J}^t} \implies {}_a\boldsymbol{m} = {}_a\boldsymbol{m}({}^\Theta_a\boldsymbol{J}^t, \theta).
\end{aligned} \tag{59}$$

We note that (59) are invalid based on (52), hence in the entropy inequality the following must hold (leaving the first term as is):

$$\frac{\partial \Phi}{\partial {}^t_a\boldsymbol{r}} = 0; \quad \frac{\partial \Phi}{\partial {}^\Theta_s\boldsymbol{J}^t} = 0; \quad \frac{\partial \Phi}{\partial {}^\Theta_a\boldsymbol{J}^t} = 0, \tag{60}$$

Using (60) in (57), the entropy inequality reduces to

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki}\right)\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}\left({}^t_a\dot{r}_{ki}\right) - {}_s m_{ki}\left({}^\Theta_s\dot{J}^t_{ki}\right) - {}_a m_{ki}\left({}^\Theta_a\dot{J}^t_{ki}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0, \tag{61}$$

and the argument tensors of Φ are modified,

$$\Phi = \Phi(\boldsymbol{\varepsilon}, \theta), \tag{62}$$

In order to proceed further, consider the decomposition of the symmetric Cauchy stress tensor, ${}_s\boldsymbol{\sigma}$ into equilibrium ${}_e({}_s\boldsymbol{\sigma})$ and deviatoric ${}_d({}_s\boldsymbol{\sigma})$ stress tensors,

$${}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma}). \tag{63}$$

Substituting (63) in (61) gives

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_e({}_s\sigma)_{ki}\right)\dot{\varepsilon}_{ki} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}\left({}^t_a\dot{r}_{ki}\right) - {}_s m_{ki}\left({}^\Theta_s\dot{J}^t_{ki}\right) - {}_a m_{ki}\left({}^\Theta_a\dot{J}^t_{ki}\right)$$
$$+ \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon} : \boldsymbol{\sigma}) \leq 0. \tag{64}$$

For small deformation and small strain the matter is incompressible ($|J| = 1$), hence

$$\frac{\partial \Phi}{\partial \varepsilon_{ki}} = \frac{\partial \Phi}{\partial |J|}\frac{\partial |J|}{\partial \varepsilon_{ki}} = 0 \quad \text{as} \quad \frac{\partial \Phi}{\partial |J|} = 0. \tag{65}$$

Thus, the first term in (64) cannot be used to derive the constitutive theory for ${}_e({}_s\boldsymbol{\sigma})$. Note that ${}_e({}_s\boldsymbol{\sigma})$ in (64) is in fact only valid for compressible matter if the coefficient of $\dot{\varepsilon}_{kl}$ is set to zero. The incompressibility condition must be introduced in (64),

$$\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = \mathrm{tr}[\bar{D}] = \mathrm{tr}[\bar{L}] = \mathrm{tr}(\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}) = \dot{J}_{kl}(J^{-1})_{lk} = \dot{J}_{kl}\delta_{lk} = 0. \tag{66}$$

Also

$$\mathrm{tr}([\bar{L}]^T) = \mathrm{tr}\big((\boldsymbol{J}^{-1})^T \dot{\boldsymbol{J}}^T\big) = (J^{-1})_{lk}\dot{J}_{lk} = \dot{J}_{lk}\delta_{lk} = 0. \tag{67}$$

Since

$$\mathrm{tr}[\bar{L}] = \mathrm{tr}([\bar{L}]^T) \tag{68}$$

we can write

$$\frac{1}{2}\big(\mathrm{tr}[\bar{L}] + \mathrm{tr}([\bar{L}]^T)\big) = \frac{1}{2}(\dot{J}_{kl}\delta_{lk} + \dot{J}_{lk}\delta_{lk}) = \dot{\varepsilon}_{kl}\delta_{kl} = 0, \tag{69}$$

Let $p(\theta)$ be an arbitrary Lagrange multiplier. Then, the incompressibility condition based on (69) becomes

$$p(\theta)\dot{\varepsilon}_{ki}\delta_{ki} = 0. \tag{70}$$

Adding (70) to the left side of (64) and using $\partial\Phi/\partial\boldsymbol{\varepsilon} = 0$,

$$\begin{aligned}&\big(p(\theta)\delta_{ki} - {}_e({}_s\sigma)_{ki}\big)\dot{\varepsilon}_{ki} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}\left({}^t_a\dot{r}_{ki}\right) - {}_sm_{ki}\left({}^\Theta_s\dot{J}^t_{ki}\right) - {}_am_{ki}\left({}^\Theta_a\dot{J}^t_{ki}\right)\\ &+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma}) \le 0.\end{aligned} \tag{71}$$

For arbitrary but admissible $\dot{\boldsymbol{\varepsilon}}$, (71) holds if

$$p(\theta)\delta_{ki} - {}_e({}_s\sigma)_{ki} = 0. \tag{72}$$

or

$${}_e({}_s\boldsymbol{\sigma}) = p(\theta)\boldsymbol{I}. \tag{73}$$

This is the constitutive theory for the equilibrium stress for an incompressible solid. $p(\theta)$ is called the mechanical pressure. If compressive pressure is assumed to be positive, then $p(\theta)$ in (73) can be replaced by $-p(\theta)$. The entropy inequality now reduces to

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_d({}_s\sigma)_{ki}\dot{\varepsilon}_{ki} - {}_a\sigma_{ki}\left({}^t_a\dot{r}_{ki}\right) - {}_sm_{ki}\left({}^\Theta_s\dot{J}^t_{ki}\right) - {}_am_{ki}\left({}^\Theta_a\dot{J}^t_{ki}\right) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma}) \le 0. \tag{74}$$

The corresponding energy equation becomes

$$\rho_0\frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \mathrm{tr}\big([{}_d({}_s\sigma)][\dot{\varepsilon}]\big) - \mathrm{tr}\big([{}_a\sigma][{}^t_a\dot{r}]\big) - \mathrm{tr}\big([{}_sm][{}^\Theta_s\dot{J}^t]\big) - \mathrm{tr}\big([{}_am][{}^\Theta_a\dot{J}^t]\big) = 0. \tag{75}$$

The entropy inequality (74) is satisfied if

$$\begin{aligned}{}^{s\sigma}\Psi &= \mathrm{tr}\big([{}_d({}_s\sigma)][\dot{\varepsilon}]\big) > 0,\\ {}^{a\sigma}\Psi &= \mathrm{tr}\big([{}_a\sigma][{}^t_a\dot{r}]\big) > 0,\\ {}^{sm}\Psi &= \mathrm{tr}\big([{}_sm][{}^\Theta_s\dot{J}^t]\big) > 0,\\ {}^{am}\Psi &= \mathrm{tr}\big([{}_am][{}^\Theta_a\dot{J}^t]\big) > 0,\end{aligned} \tag{76}$$

$$\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{77}$$

and

$${}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma}) = 0. \tag{78}$$

The conditions (76) imply that the rate of work due to ${}_d({}_s\boldsymbol{\sigma})$, ${}_a\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, and ${}_a\boldsymbol{m}$ must be positive. Inequality (77) can be used to derive the constitutive theory for $\boldsymbol{q}$. Equation (78) serves as a constraint (compatibility condition) on ${}_i\boldsymbol{\Theta}$ and the antisymmetric components of the Cauchy stress tensor $\boldsymbol{\sigma}$. The rate of work or the work conjugate pairs in (74) are in conformity with (52). The argument tensors of the constitutive variables in (52) can now be revised,

$$\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}) &= {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n_\varepsilon,\ \theta),\\
{}_a\boldsymbol{\sigma} &= {}_a\boldsymbol{\sigma}\left({}^t_a\boldsymbol{r}, {}^t_a\boldsymbol{r}_{[j]};\ j = 1, 2, \dots, n_{({}^t_d r)},\ \theta\right),\\
{}_s\boldsymbol{m} &= {}_s\boldsymbol{m}\left({}^{\Theta}_s\boldsymbol{J}^t, {}^{\Theta}_s\boldsymbol{J}^t_{[k]};\ k = 1, 2, \dots, n_{({}^{\Theta}_s J^t)},\ \theta\right),\\
{}_a\boldsymbol{m} &= {}_a\boldsymbol{m}\left({}^{\Theta}_a\boldsymbol{J}^t, {}^{\Theta}_a\boldsymbol{J}^t_{[l]};\ l = 1, 2, \dots, n_{({}^{\Theta}_a J^t)},\ \theta\right),\\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{g}, \theta),\\
{}_e({}_s\boldsymbol{\sigma}) &= p(\theta)\boldsymbol{I},\\
\Phi &= \Phi(\theta)
\end{aligned} \tag{79}$$

and

$${}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma}). \tag{80}$$

6 Constitutive theories

Note that in (79), ${}_d({}_s\boldsymbol{\sigma})$ and ${}_s\boldsymbol{m}$ are symmetric tensors of rank two, and their argument tensors are also symmetric tensors of rank two (except temperature θ). ${}_a\boldsymbol{\sigma}$ and ${}_a\boldsymbol{m}$ are antisymmetric tensors of rank two, and their argument tensors are also antisymmetric tensors of rank two (except θ). $\boldsymbol{q}$ is a tensor of rank one, and its argument tensors $\boldsymbol{g}$ and θ are tensors of rank one and zero. Hence, the representation theorem (theory of generators and invariants) of Spencer, Wang, Zheng, and others [49–68] can be used to derive constitutive theories for these constitutive variables.

6.1 Representation theorem

The following Sections present derivations of the constitutive theories for ${}_d({}_s\boldsymbol{\sigma})$, ${}_a\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, ${}_a\boldsymbol{m}$, and $\boldsymbol{q}$ using the theory of generators and invariants (representation theorem) based on pioneering works of Spencer, Wang, Zheng, etc. [49–68]. To illustrate the basic concepts of representation theorem, consider a symmetric tensor $\boldsymbol{T}(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_k)$ of rank two with $\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_k$ as its arguments that could be a mix of tensors of rank two or lower. If tensor $\boldsymbol{T}$ belongs to a space then the space must have a basis, referred to as integrity. Spencer, Wang, Zheng, etc. [49–68] have shown that for a symmetric tensor $\boldsymbol{T}$ of rank two, the basis consists of all possible symmetric tensors of rank two that are derived using its arguments $\boldsymbol{A}_i,\ i = 1, 2, \dots, k$, referred to as combined generators of the argument tensors. If $\boldsymbol{I}, \boldsymbol{\mathcal{G}}_i,\ i = 1, 2, \dots, N$ are the combined generators constituting the basis (symmetric tensors of rank two) of the of space of tensor $\boldsymbol{T}$, then $\boldsymbol{T}$ can be represented by a linear combination of $\boldsymbol{I}, \boldsymbol{\mathcal{G}}_i,\ i = 1, 2, \dots, N$, i.e.,

$$\boldsymbol{T} = \alpha^0\boldsymbol{I} + \sum_{i=1}^{N}\alpha^i\boldsymbol{\mathcal{G}}_i \tag{81}$$

$$\alpha^i = \alpha^i(\mathcal{I}^j;\ j = 1, 2, \dots, M);\quad i = 0, 1, \dots, N \tag{82}$$

in which $\mathcal{I}^j;\ j = 1, 2, \dots, M$ are the combined invariants of the argument tensors of $\boldsymbol{T}(\cdot)$.
Remarks.

(i) When $\boldsymbol{T}$ is an antisymmetric tensor of rank two, then the same representation theorem concept applies except that in this case the combined generators $\boldsymbol{\mathcal{G}}_i$ will all be antisymmetric tensors of rank two.
(ii) If $\boldsymbol{T}$ is a tensor of rank one with its arguments as tensors of rank two, one, or zero, then the combined generators of these argument tensors are tensors of rank one and the representation theorem holds.

(iii) It has not been shown in Ref. [49–68] or elsewhere, to our knowledge, that if $\boldsymbol{T}$ is a non-symmetric tensor of some rank with non-symmetric tensors as its arguments, then the representation theorem holds.
(iv) Material coefficients are derived from $\alpha^i(\bullet)$ $i = 0, 1, \ldots, N$ using Taylor series expansion in the invariants and others (like temperature θ).
(v) We remark that the representation theorem holds even if $\boldsymbol{A}_i$ are a mix of symmetric and antisymmetric tensors (chosen based on $\boldsymbol{T}$ and the desired physics), keeping in mind that a non-symmetric tensor can be decomposed into symmetric and antisymmetric tensors. If we wish to represent a symmetric tensor using the basis of a space in which it resides, then the basis must consist of symmetric tensors only. The same argument holds for antisymmetric tensors. In view of the representation theorem, non-symmetric tensors must be decomposed into symmetric and antisymmetric tensors as done in the present work in the derivation of the entropy inequality (50).

6.2 Constitutive theory for ${}_d({}_s\boldsymbol{\sigma})$

Consider the argument tensors of ${}_d({}_s\boldsymbol{\sigma})$ in (79). Let ${}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i$; $i = 1, 2, \ldots, N_{s\sigma}$ be the combined generators of the argument tensors of ${}_d({}_s\boldsymbol{\sigma})$ that are symmetric tensors of rank two and let ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_{s\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration:

$$
{}_d({}_s\boldsymbol{\sigma}) = {}^{s\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{\alpha}^i \left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) \tag{83}
$$

in which

$$
{}^{s\sigma}\underset{\sim}{\alpha}^i = {}^{s\sigma}\underset{\sim}{\alpha}^i\left({}^{s\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \ldots, M_{s\sigma},\ \theta\right); \quad i = 1, 2, \ldots, N_{s\sigma}. \tag{84}
$$

To determine the material coefficients in (83), expand each ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in a Taylor series in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_{s\sigma}$ and θ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{s\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M_{s\sigma}$, and θ, and then substitute these ${}^{s\sigma}\underset{\sim}{\alpha}^i$ in (83). After collecting coefficients of those terms that are defined in the current configuration, we obtain the following:

$$
\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}) = {}^0_s\sigma\big|_{\underline{\Omega}} \boldsymbol{I} + \sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underline{a}_j \left({}^{s\sigma}\underset{\sim}{I}^j\right)\boldsymbol{I} - {}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}}\left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{I} \\
+ \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underline{b}_i \left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{s\sigma}}\sum_{j=1}^{M_{s\sigma}} {}^{s\sigma}\underset{\sim}{c}_{ij}\left({}^{s\sigma}\underset{\sim}{I}^j\right)\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N_{s\sigma}} {}^{s\sigma}\underset{\sim}{d}_i\left(\theta - \theta_{\underline{\Omega}}\right)\left({}^{s\sigma}\underset{\sim}{\boldsymbol{G}}^i\right)
\end{aligned} \tag{85}
$$

in which

$$
\begin{aligned}
{}^0_s\sigma\big|_{\underline{\Omega}} &= {}^{s\sigma}\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}, \\
{}^{s\sigma}\underline{a}_j &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\Bigg|_{\underline{\Omega}}; \quad j = 1, 2, \ldots, M_{s\sigma}, \\
{}^{s\sigma}\underline{b}_i &= {}^{s\sigma}\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{M_{s\sigma}} \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\Bigg|_{\underline{\Omega}} \left({}^{s\sigma}\underset{\sim}{I}^j\right)_{\underline{\Omega}}; \quad i = 1, 2, \ldots, N_{s\sigma}, \\
{}^{s\sigma}\underset{\sim}{c}_{ij} &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial({}^{s\sigma}\underset{\sim}{I}^j)}\Bigg|_{\underline{\Omega}}; \quad \begin{array}{l} i = 1, 2, \ldots, N_{s\sigma} \\ j = 1, 2, \ldots, M_{s\sigma}, \end{array} \\
{}^{s\sigma}\underset{\sim}{\alpha}_{\text{tm}} &= -\frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^0)}{\partial\theta}\Bigg|_{\underline{\Omega}}, \\
{}^{s\sigma}\underset{\sim}{d}_i &= \frac{\partial({}^{s\sigma}\underset{\sim}{\alpha}^i)}{\partial\theta}\Bigg|_{\underline{\Omega}}; \quad i = 1, 2, \ldots, N_{s\sigma}.
\end{aligned} \tag{86}
$$

${}^{s\sigma}\underline{a}_j, {}^{s\sigma}\underline{b}_i, {}^{s\sigma}\underline{c}_{ij}, {}^{s\sigma}\underline{d}_i$, and ${}^{s\sigma}\underline{\alpha}_{\text{tm}}$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{s\sigma} + N_{s\sigma} + M_{s\sigma} N_{s\sigma} + N_{s\sigma} + 1)$ material coefficients. The material coefficients defined in (85) are functions of $({}^{s\sigma}\mathcal{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{s\sigma}\alpha^i;\ i = 0, 1, \dots, N_{s\sigma}$.

6.2.1 *Simplified constitutive theory for* ${}_d({}_s\boldsymbol{\sigma})$

The constitutive theory (85) obviously requires the determination of too many material coefficients. If n_ε is limited to 1, then

$$ {}_d({}_s\boldsymbol{\sigma}) = {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{[1]}, \theta) = {}_d({}_s\boldsymbol{\sigma})(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, \theta). \tag{87} $$

If the constitutive theory is further limited to be linear in $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ and product terms containing $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ are neglected, then (85) simplifies to (after neglecting initial stress and temperature terms without loss of generality)

$$ {}_d({}_s\boldsymbol{\sigma}) = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_1\dot{\boldsymbol{\varepsilon}} + \lambda_1 \operatorname{tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I}. \tag{88} $$

This constitutive theory requires only four material coefficients. μ and λ are Lamé's constants for the strain terms. μ_1 and λ_1 are corresponding material coefficients: related to strain rates.

6.3 Constitutive theory for ${}_a\boldsymbol{\sigma}$

Consider the argument tensors of ${}_a\boldsymbol{\sigma}$ in (79). Let ${}^{a\sigma}\boldsymbol{G}^i;\ i = 1, 2, \dots, N_{a\sigma}$ be the combined generators of the argument tensors of ${}_a\boldsymbol{\sigma}$ that are antisymmetric tensors of rank two and let ${}^{a\sigma}\mathcal{I}^j;\ j = 1, 2, \dots, M_{a\sigma}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration:

$$ {}_a\boldsymbol{\sigma} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\alpha^i\, ({}^{a\sigma}\boldsymbol{G}^i) \tag{89} $$

in which

$$ {}^{a\sigma}\alpha^i = {}^{a\sigma}\alpha^i({}^{a\sigma}\mathcal{I}^j;\ j = 1, 2, \dots, M_{a\sigma},\ \theta); \quad i = 1, 2, \dots, N_{a\sigma}. \tag{90} $$

To determine the material coefficients in (89), expand each ${}^{a\sigma}\alpha^i$ in a Taylor series in ${}^{a\sigma}\mathcal{I}^j;\ j = 1, 2, \dots, M_{a\sigma}$ and θ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{a\sigma}\mathcal{I}^j;\ j = 1, 2, \dots, M_{a\sigma}$ and θ, and then substitute these ${}^{a\sigma}\alpha^i$ in (89). After collecting coefficients of those terms that are defined in the current configuration, the following is obtained:

$$ {}_a\boldsymbol{\sigma} = \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underline{b}_i\left({}^{a\sigma}\boldsymbol{G}^i\right) + \sum_{i=1}^{N_{a\sigma}}\sum_{j=1}^{M_{a\sigma}} {}^{a\sigma}\underline{c}_{ij}\left({}^{a\sigma}\mathcal{I}^j\right)\left({}^{a\sigma}\boldsymbol{G}^i\right) + \sum_{i=1}^{N_{a\sigma}} {}^{a\sigma}\underline{d}_i\left(\theta - \theta_{\underline{\Omega}}\right)\left({}^{a\sigma}\boldsymbol{G}^i\right). \tag{91} $$

${}^{a\sigma}\underline{a}_j, {}^{a\sigma}\underline{b}_i, {}^{a\sigma}\underline{c}_{ij}, {}^{a\sigma}\underline{d}_i$, and ${}^{a\sigma}\underline{\alpha}_{\text{tm}}$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{a\sigma} + N_{a\sigma} + M_{a\sigma} N_{a\sigma} + N_{a\sigma} + 1)$ material coefficients. The material coefficients are functions of $({}^{a\sigma}\mathcal{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{a\sigma}\alpha^i;\ i = 0, 1, \dots, N_{a\sigma}$. Explicit forms of the material coefficients can be obtained from (86) by simply replacing the back superscript s with a.

6.3.1 Simplified constitutive theory for ${}_a\boldsymbol{\sigma}$

The constitutive theory (91) based on integrity can be simplified by choosing $n_{{}^t_d r} = 1$,

$$ {}_a\boldsymbol{\sigma} = {}_a\boldsymbol{\sigma}\left({}^t_a\boldsymbol{r}, {}^t_a\dot{\boldsymbol{r}}, \theta\right). \tag{92} $$

In this case ${}^{a\sigma}\mathcal{G}^1 = {}^t_a\boldsymbol{r}$, ${}^{a\sigma}\mathcal{G}^2 = {}^t_a\dot{\boldsymbol{r}}$, and ${}^{\sigma}\mathcal{G}^3 = \left[{}^t_a r\right]\left[{}^t_a\dot{r}\right] - \left[{}^t_a\dot{r}\right]\left[{}^t_a r\right]$ are the only combined generators and ${}^{a\sigma}\mathcal{I}^1 = \mathrm{tr}\left(\left({}^t_a\boldsymbol{r}\right)^2\right)$, ${}^{a\sigma}\mathcal{I}^2 = \mathrm{tr}\left(\left({}^t_a\dot{\boldsymbol{r}}\right)^2\right)$, and ${}^{a\sigma}\mathcal{I}^3 = \mathrm{tr}\left(\left[{}^t_a r\right]\left[{}^t_a\dot{r}\right]\right)$ are the only invariants, giving rise to a constitutive theory with 19 material coefficients ($N_{a\sigma} = 3$, $M_{a\sigma} = 3$). A linear constitutive theory for ${}_a\boldsymbol{\sigma}$ (neglecting initial stress and θ terms and excluding products of ${}^t_a\boldsymbol{r}$ and ${}^t_a\dot{\boldsymbol{r}}$) will be

$$ {}_a\boldsymbol{\sigma} = {}^{a\sigma}\beta_1\left({}^t_a\boldsymbol{r}\right) + {}^{a\sigma}\beta_2\left({}^t_a\dot{\boldsymbol{r}}\right). \tag{93} $$

The material coefficients ${}^{a\sigma}\beta_1$ and ${}^{a\sigma}\beta_2$ can be functions of the invariants and the temperature θ.

6.4 Constitutive theory for ${}_s\boldsymbol{m}$

Consider the argument tensors of ${}_s\boldsymbol{m}$ in (79). Let ${}^{sm}\mathcal{G}^i$; $i = 1, 2, \ldots, N_{sm}$ be the combined generators of the argument tensors of ${}_s\boldsymbol{m}$ that are symmetric tensors of rank two and let ${}^{sm}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{sm}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration:

$$ {}_s\boldsymbol{m} = {}^{sm}\alpha^0\boldsymbol{I} + \sum_{i=1}^{N_{sm}} {}^{sm}\alpha^i\left({}^{sm}\mathcal{G}^i\right) \tag{94} $$

in which

$$ {}^{sm}\alpha^i = {}^{sm}\alpha^i\left({}^{sm}\mathcal{I}^j;\ j = 1, 2, \ldots, M_{sm},\ \theta\right);\quad i = 1, 2, \ldots, N_{sm}. \tag{95} $$

To determine the material coefficients in (94), expand each ${}^{sm}\alpha^i$ in a Taylor series in ${}^{sm}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{sm}$ and θ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{sm}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{sm}$ and θ, and then substitute these ${}^{sm}\alpha^i$ in (94). After collecting coefficients of those terms that are defined in the current configuration, the following is obtained:

$$ \begin{aligned} {}_s\boldsymbol{m} = {}^0_s m|_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M_{sm}} {}^{sm}\underline{a}_j\left({}^{sm}\mathcal{I}^j\right)\boldsymbol{I} - {}^{sm}\underline{\alpha}_{\mathrm{tm}}\left(\theta - \theta_{\underline{\Omega}}\right)\boldsymbol{I} \\ + \sum_{i=1}^{N_{sm}} {}^{sm}\underline{b}_i\left({}^{sm}\mathcal{G}^i\right) + \sum_{i=1}^{N_{sm}}\sum_{j=1}^{M_{sm}} {}^{sm}\underline{c}_{ij}\left({}^{sm}\mathcal{I}^j\right)\left({}^{sm}\mathcal{G}^i\right) + \sum_{i=1}^{N_{sm}} {}^{sm}\underline{d}_i\left(\theta - \theta_{\underline{\Omega}}\right)\left({}^{sm}\mathcal{G}^i\right). \end{aligned} \tag{96} $$

${}^{sm}\underline{a}_j$, ${}^{sm}\underline{b}_i$, ${}^{sm}\underline{c}_{ij}$, ${}^{sm}\underline{d}_i$, and ${}^{sm}\underline{\alpha}_{\mathrm{tm}}$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires ($M_{sm} + N_{sm} + M_{sm}N_{sm} + N_{sm} + 1$) material coefficients. The material coefficients are functions of $({}^{sm}\mathcal{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{sm}\alpha^i$; $i = 0, 1, \ldots, N_{sm}$. Explicit forms of the material coefficients can be obtained from (86) by simply replacing the back superscript ${}_s\sigma$ with ${}_s m$ and ${}^0_s\sigma|_{\underline{\Omega}}$ by ${}^0_s m|_{\underline{\Omega}}$.

6.4.1 Simplified constitutive theory for ${}_s\boldsymbol{m}$

A much simplified constitutive theory for ${}_s\boldsymbol{m}$ can be obtained if n_{sm} is limited to 1,

$$ {}_{s}\boldsymbol{m} = {}_{s}\boldsymbol{m}\left({}^{\Theta}_{s}\boldsymbol{J}^t, {}^{\Theta}_{s}\dot{\boldsymbol{J}}^t, \theta \right), \tag{97} $$

If the constitutive theory is further limited to be linear in ${}^{\Theta}_{s}\boldsymbol{J}^t$ and ${}^{\Theta}_{s}\dot{\boldsymbol{J}}^t$ and product terms containing ${}^{\Theta}_{s}\boldsymbol{J}^t$ and ${}^{\Theta}_{s}\dot{\boldsymbol{J}}^t$ are neglected, then (96) simplifies to (after neglecting initial moment and temperature terms without loss of generality)

$$ {}_{s}\boldsymbol{m} = {}^{sm}\beta_1 \left({}^{\Theta}_{s}\boldsymbol{J}^t \right) + {}^{sm}\beta_2 \left({}^{\Theta}_{s}\dot{\boldsymbol{J}}^t \right). \tag{98} $$

The material coefficients ${}^{sm}\beta_1$ and ${}^{sm}\beta_2$ can be functions of the invariants and the temperature θ.

6.5 Constitutive theory for ${}_{a}\boldsymbol{m}$

Consider the argument tensors of ${}_{a}\boldsymbol{m}$ in (79). Let ${}^{am}\boldsymbol{G}^i$; $i = 1, 2, \ldots, N_{am}$ be the combined generators of the argument tensors of ${}_{a}\boldsymbol{m}$ that are antisymmetric tensors of rank two and let ${}^{am}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{am}$ be the combined invariants of the same argument tensors. Then, the following holds in the current configuration:

$$ {}_{a}\boldsymbol{m} = \sum_{i=1}^{N_{am}} {}^{am}\alpha^i \left({}^{am}\boldsymbol{G}^i \right) \tag{99} $$

in which

$$ {}^{am}\alpha^i = {}^{am}\alpha^i \left({}^{am}\mathcal{I}^j;\ j = 1, 2, \ldots, M_{am},\ \theta \right); \quad i = 1, 2, \ldots, N_{am}. \tag{100} $$

To determine the material coefficients in (99), expand each ${}^{am}\alpha^i$ in a Taylor series in ${}^{am}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{am}$ and θ about a known configuration $\underline{\Omega}$, retaining only up to linear terms in ${}^{am}\mathcal{I}^j$; $j = 1, 2, \ldots, M_{am}$, and θ, and then substitute these ${}^{am}\alpha^i$ in (99). After collecting coefficients of those terms that are defined in the current configuration, the following is obtained:

$$ {}_{a}\boldsymbol{m} = \sum_{i=1}^{N_{am}} {}^{am}\underline{b}_i \left({}^{am}\boldsymbol{G}^i \right) + \sum_{i=1}^{N_{am}} \sum_{j=1}^{M_{am}} {}^{am}\underline{c}_{ij} \left({}^{am}\mathcal{I}^j \right) \left({}^{am}\boldsymbol{G}^i \right) + \sum_{i=1}^{N_{am}} {}^{am}\underline{d}_i \left(\theta - \theta_{\underline{\Omega}} \right) \left({}^{am}\boldsymbol{G}^i \right). \tag{101} $$

${}^{am}\underline{a}_j$, ${}^{am}\underline{b}_i$, ${}^{am}\underline{c}_{ij}$, ${}^{am}\underline{d}_i$, and ${}^{am}\underline{\alpha}_{\text{tm}}$ are material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory requires $(M_{am} + N_{am} + M_{am}N_{am} + N_{am} + 1)$ material coefficients. The material coefficients are functions of $({}^{am}\mathcal{I}^j)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$. This constitutive theory is based on integrity, the only assumption being in the truncation of the Taylor series expansion of ${}^{am}\alpha^i$; $i = 0, 1, \ldots, N_{am}$. Explicit forms of the material coefficients can be obtained from (86) by simply replacing the back superscript ${}_{s}\sigma$ with ${}_{a}m$ and ${}^{0}_{a}\sigma|_{\underline{\Omega}}$ by ${}^{0}_{a}m|_{\underline{\Omega}}$.

6.5.1 Simplified constitutive theory for ${}_{a}\boldsymbol{m}$

The constitutive theory (101) based on integrity can be simplified by choosing $n_{{}^{\Theta}_{a}J^t} = 1$,

$$ {}_{a}\boldsymbol{m} = {}_{a}\boldsymbol{m}\left({}^{\Theta}_{a}\boldsymbol{J}^t, {}^{\Theta}_{a}\dot{\boldsymbol{J}}^t, \theta \right). \tag{102} $$

In this case ${}^{am}\boldsymbol{G}^1 = {}^{\Theta}_{a}\boldsymbol{J}^t$, ${}^{am}\boldsymbol{G}^2 = {}^{\Theta}_{a}\dot{\boldsymbol{J}}^t$, and ${}^{am}\boldsymbol{G}^3 = [{}^{\Theta}_{a}J^t][{}^{\Theta}_{a}\dot{J}^t] - [{}^{\Theta}_{a}\dot{J}^t][{}^{\Theta}_{a}J^t]$ are the only combined generators and ${}^{am}\mathcal{I}^1 = \text{tr}\big(({}^{\Theta}_{a}\boldsymbol{J}^t)^2\big)$, ${}^{am}\mathcal{I}^2 = \text{tr}\big(({}^{\Theta}_{a}\dot{\boldsymbol{J}}^t)^2\big)$, and ${}^{am}\mathcal{I}^3 = \text{tr}\big([{}^{\Theta}_{a}J^t][{}^{\Theta}_{a}\dot{J}^t]\big)$ are the only invariants, giving rise to a constitutive theory with 19 material coefficients ($N_{am} = 3$, $M_{am} = 3$). A linear constitutive theory for ${}_{a}\boldsymbol{m}$ (neglecting initial moment and θ terms and excluding products of ${}^{\Theta}_{a}\boldsymbol{J}^t$ and ${}^{\Theta}_{a}\dot{\boldsymbol{J}}^t$) will be

$$ {}_{a}\boldsymbol{m} = {}^{am}\beta_1 \left({}^{\Theta}_{a}\boldsymbol{J}^t \right) + {}^{am}\beta_2 \left({}^{\Theta}_{a}\dot{\boldsymbol{J}}^t \right). \tag{103} $$

The material coefficients ${}^{am}\beta_1$ and ${}^{am}\beta_2$ can be functions of the invariants and the temperature θ.

6.6 Constitutive theory for $\boldsymbol{q}$

Recall the inequality (77) resulting from the entropy inequality,

$$\boldsymbol{q} \cdot \boldsymbol{g} \leq 0 \quad (\text{as } \theta > 0). \tag{104}$$

In (104), $\boldsymbol{q}$ and $\boldsymbol{g}$ are conjugate. The simplest possible constitutive theory for $\boldsymbol{q}$ can be derived by assuming that $\boldsymbol{q}$ is proportional to $-\boldsymbol{g}$ which leads to the following for $\boldsymbol{q}$ [69,75]:

$$\boldsymbol{q} = -k(\theta)\boldsymbol{g}. \tag{105}$$

Alternatively if we assume

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{106}$$

then using the representation theorem (theory of generators and invariants), we can begin with (as $\boldsymbol{g}$ is the only combined generator of $\boldsymbol{g}$ and θ that is a tensor of rank one) in the current configuration

$$\boldsymbol{q} = -{}^{q}\alpha \boldsymbol{g} \tag{107}$$

in which

$${}^{q}\alpha = {}^{q}\alpha({}^{q}\underset{\sim}{I}, \theta); \quad {}^{q}\underset{\sim}{I} = \boldsymbol{g} \cdot \boldsymbol{g}, \tag{108}$$

${}^{q}\underset{\sim}{I}$ being the only invariant of the argument tensors $\boldsymbol{g}$ and θ. Expanding ${}^{q}\alpha$ in a Taylor series in ${}^{q}\underset{\sim}{I}$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^{q}\underset{\sim}{I}$ and θ, we can obtain the following [69]:

$$\boldsymbol{q} = -k|_{\underline{\Omega}}\boldsymbol{g} - k_1|_{\underline{\Omega}}\{g\}^T\{g\}\boldsymbol{g} - k_2|_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{109}$$

where

$$\begin{aligned} k|_{\underline{\Omega}} &= {}^{q}\alpha|_{\underline{\Omega}} + \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}} (\{g\}^T\{g\})_{\underline{\Omega}}, \\ k_1|_{\underline{\Omega}} &= \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{I}}\right|_{\underline{\Omega}}, \\ k_2|_{\underline{\Omega}} &= \left.\frac{\partial\, {}^{q}\alpha}{\partial \theta}\right|_{\underline{\Omega}}. \end{aligned} \tag{110}$$

The constitutive theory (109) is the simplest possible constitutive theory based on the representation theorem (using (106)). The only assumption in this constitutive theory is the truncation of Taylor series beyond linear terms in ${}^{q}\underset{\sim}{I}$ and θ. This constitutive theory is based on integrity, hence complete. Obviously (105), the Fourier heat conduction law is a subset of (109) when k is the only material coefficient and it only depends on temperature θ.

7 Complete mathematical model

This Section presents the complete mathematical model including the linear constitutive theories when (i) the balance of moment of moments is not a balance law and (ii) when the balance of moment of moments is a balance law. Only linear constitutive theories are included here for simplicity. The complete constitutive theories based on integrity given in this paper can be used to derive any desired subset suitable for the application at hand.

7.1 When the balance of moment of moments is not a balance law

For this case, we use the same conservation and balance laws that are used in classical continuum theories. In this case, the Cauchy moment tensor is not symmetric; hence, constitutive theories are required for both ${}_s\boldsymbol{m}$ and ${}_a\boldsymbol{m}$. Details of the complete mathematical model (including the final form of the entropy inequality) are given in the following (only for linear constitutive theories):

$$
\begin{aligned}
&\rho_0(\boldsymbol{x}) = |J|\rho(\boldsymbol{x},t)\ ; \quad |J| = 1,\\
&\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0,\\
&\boldsymbol{\nabla}\cdot\boldsymbol{m} - \boldsymbol{\epsilon} : \boldsymbol{\sigma} = 0,\\
&\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} - \operatorname{tr}\big([{}_d({}_s\sigma)][\dot{\varepsilon}]\big) - \operatorname{tr}\big([{}_a\sigma][{}^t_a\dot{r}]\big) - \operatorname{tr}\big([{}_s m][{}^\Theta_s\dot{J}^t]\big) - \operatorname{tr}\big([{}_a m][{}^\Theta_a\dot{J}^t]\big) = 0,\\
&\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} - {}_d({}_s\sigma)_{ki}\dot{\boldsymbol{\varepsilon}}_{ki} - {}_a\sigma_{ki}\big({}^t_a\dot{r}_{ki}\big) - {}_s m_{ki}\big({}^\Theta_s\dot{J}^t_{ki}\big) - {}_a m_{ki}\big({}^\Theta_a\dot{J}^t_{ki}\big) - {}_i\dot{\boldsymbol{\Theta}}\cdot(\boldsymbol{\epsilon}:\boldsymbol{\sigma}) \le 0,\\
&\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}\ ; \qquad {}_s\boldsymbol{\sigma} = {}_e({}_s\boldsymbol{\sigma}) + {}_d({}_s\boldsymbol{\sigma}),\\
&\boldsymbol{m} = {}_s\boldsymbol{m} + {}_a\boldsymbol{m}.
\end{aligned}
\tag{111}
$$

Constitutive Theories

$$
\begin{aligned}
{}_d({}_s\boldsymbol{\sigma}) &= 2\mu\boldsymbol{\varepsilon} + \lambda\operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu_1\dot{\boldsymbol{\varepsilon}} + \lambda_1\operatorname{tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I},\\
{}_a\boldsymbol{\sigma} &= {}^{a\sigma}\beta_1\big({}^t_a\boldsymbol{r}\big) + {}^{a\sigma}\beta_2\big({}^t_a\dot{\boldsymbol{r}}\big),\\
{}_s\boldsymbol{m} &= {}^{sm}\beta_1\big({}^\Theta_s\boldsymbol{J}^t\big) + {}^{sm}\beta_2\big({}^\Theta_s\dot{\boldsymbol{J}}^t\big),\\
{}_a\boldsymbol{m} &= {}^{am}\beta_1\big({}^\Theta_a\boldsymbol{J}^t\big) + {}^{am}\beta_2\big({}^\Theta_a\dot{\boldsymbol{J}}^t\big),\\
\boldsymbol{q} &= -k(\theta)\boldsymbol{g},\\
{}_e({}_s\boldsymbol{\sigma}) &= p(\theta)\boldsymbol{I}; \quad \text{mechanical pressure (mean normal stress).}
\end{aligned}
\tag{112}
$$

These are a system of 28 partial differential equations (linear momenta (3), angular momenta (3), energy (1), constitutive theories for: ${}_s\boldsymbol{\sigma}$ (6), ${}_a\boldsymbol{\sigma}$ (3), ${}_s\boldsymbol{m}$ (6), ${}_a\boldsymbol{m}$ (3), $\boldsymbol{q}$ (3)) in 28 variables: displacements $\boldsymbol{u}$ (3), Cosserat rotations ${}_e\boldsymbol{\Theta}$ (3), ${}_s\boldsymbol{\sigma}$ (6), ${}_a\boldsymbol{\sigma}$ (3), ${}_s\boldsymbol{m}$ (6), ${}_a\boldsymbol{m}$ (3), $\boldsymbol{q}$ (3), and temperature θ (1); hence, the mathematical model has closure. This mathematical model requires the minimum of eleven material coefficients for 2D and 3D.

7.2 When the balance of moment of moments is a balance law

In this case, the Cauchy moment tensor is symmetric,

$$
\boldsymbol{m} = {}_s\boldsymbol{m}; \quad {}_a\boldsymbol{m} = 0. \tag{113}
$$

Hence, the constitutive theory for ${}_a\boldsymbol{m}$ is not needed. This reduces the number of equations by three (constitutive equations for ${}_a\boldsymbol{m}$ are eliminated) and the number of variables by three (${}_a\boldsymbol{m}$ are eliminated as constitutive variables); thus, in this case there are 25 equations in 25 dependent variables. The two material coefficients associated with ${}_a\boldsymbol{m}$ are eliminated, leading to the minimum of nine material coefficients in 2D and 3D.

8 Summary and conclusions

In this paper, we have presented a non-classical continuum theory and associated constitutive theories for thermoviscoelastic solids without memory that incorporates internal rotations due to the Jacobian of deformation as well as the Cosserat rotations at a material point. Both rotations are defined about a triad at each material point with axes parallel to the fixed x-frame. The internal rotations are completely defined by the antisymmetric part of the Jacobian of deformation (or the antisymmetric part of the displacement gradient tensor), hence are known, whereas the Cosserat rotations are additional three unknown degrees of freedom at each material point, thus giving rise to a total of six degrees of freedom at each material point. Derivations of the conservation and balance laws are presented, followed by detailed derivations of the constitutive theories consistent with the conditions resulting from the second law of thermodynamics in conjunction with the representation theorem. In the following, we summarize significant aspects of the work presented in this paper.

1. The rate of work due to internal rotations resulting from the Jacobian of deformation as well as due to Cosserat rotations is considered in the conservation and balance laws as opposed to the rate of work only due to Cosserat rotations in [79].
2. As shown by Yang et al. [84] and Surana et al. [70,71], the *balance of moment of moments* balance law is required in non-classical continuum theories to ensure that the deforming volume of matter is in equilibrium. Due to this balance law, the Cauchy moment tensor becomes symmetric. In this paper, we have derived constitutive theories when the balance of moment of moments is not a balance law as well as when it is considered as a balance law.
3. For thermoviscoelastic solids, some part of the mechanical work results in dissipation (entropy generation). In this paper, we consider:
   (a) When balance of moment of moments is not a balance law, the dissipation mechanism is due to ${}_d({}_s\boldsymbol{\sigma})$, ${}_a\boldsymbol{\sigma}$, ${}_s\boldsymbol{m}$, as well as ${}_a\boldsymbol{m}$.
   (b) When the balance of moment of moments is a balance law, ${}_a\boldsymbol{m} = 0$, hence there is no entropy generation due to ${}_a\boldsymbol{m}$. All others remain the same as in (a).
4. Many shortcomings and inconsistencies in the balance laws in the works of Eringen [79] pointed out by Surana et al. [47] hold here as well, but are not repeated for the sake of brevity. An important point to note is that when the constitutive variable and its argument tensors (some or all) are non-symmetric tensors, then the constitutive theory for the constitutive variable can not be derived using the representation theorem. Decomposition of the dependent variables and their argument tensors into symmetric and antisymmetric tensors is necessary to establish conjugate pairs that either contain symmetric tensors or antisymmetric tensors so that the representation theorem can be used to derive the constitutive theories as done in the present work.
5. As shown in this paper, the constitutive theories based on integrity are almost always nonlinear in their argument tensors. Their linearizations are perfectly valid if limited physics is of interest, however the conclusions that may be drawn from the superposition of linear constitutive theories are obviously invalid. An example would be linear constitutive theories for ${}_d({}_s\boldsymbol{\sigma})$ and ${}_a\boldsymbol{\sigma}$, suggesting a constitutive theory for $({}_d({}_s\boldsymbol{\sigma}) + {}_a\boldsymbol{\sigma})$, a non-symmetric tensor in terms of non-symmetric argument tensors.
6. In the present work, we have assumed that since the internal rotations due to the Jacobian of deformation and Cosserat rotations are additive, a constitutive theory should be possible for the combined total rotations. This appears to be the approach used in published works, hence has been adopted here as well. However, if the two sets of rotations are associated with different physics, then obviously separate constitutive theories are warranted. This requires rederivation of the first and second laws of thermodynamics, new conjugate pairs, and associated constitutive theories. This work is currently in progress and will be the subject of upcoming papers.

In conclusion, the work presented in this paper is a consistent thermodynamic framework for non-classical thermoviscoelastic solids without memory incorporating internal and Cosserat rotations at a material point. The paper contains thermodynamically consistent derivations of constitutive theories in which all possible mechanisms of mechanical energy storage and dissipation that are supported by the second law of thermodynamics are considered. Modifications of the general constitutive theories presented here for specific applications are rather straightforward.

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