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ORIGINAL PAPER

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Algebraic structure and Poisson brackets of single degree of freedom non-material volumes

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Abstract This paper investigates an algebraic structure and Poisson theory of single degree of freedom non-material volumes. The equations of motion are proposed in a contravariant algebraic form, and an algebraic product is determined. A consistent algebraic structure and a Lie algebra structure are proposed, and a proposition is obtained. The Poisson theory of the non-material volume is established, and five theorems are derived. Three examples are given to illustrate the application of the method.

1 Introduction

Non-material volumes present some particular characteristics which are not known in the dynamics of material volumes. Theoretical investigations of non-material volumes have flourished in recent years, providing important theoretical significance and engineering background. They are also widely used in rocketry [1] and mechanical engineering [2,3]. Recently, there are many researchers who have addressed the fundamental principles of the non-material volumes. Irschik and Holl [1] first derived the Lagrange's equation of a non-material volume by employing the method of fictitious particles. Subsequently, they [3] reviewed the development and the situation of variable-mass systems. Casetta [4] reported the inverse problem of Lagrangian mechanics for a non-material volume via the method of Darboux, and proposed a Hamiltonian formalism and a conservation law. Casetta and Pesce [5] established the generalized Hamilton's principle for a non-material volume, rewriting this theorem properly in the context of non-material volumes and fictitious particles artifacts. They [6] also investigated the inverse problem of Lagrangian mechanics connected to Meshchersky's equation, stated a variational formulation and a Hamiltonian formulation. Irschik and Holl [7] proposed a formulation of Lagrange's equations for non-material volumes, calculated local forms and global form of Lagrange's equations in the framework of the Lagrange description of Continuum Mechanics. Casetta et al. [8] developed the generalization of Noether's theorem for a non-material volume and proposed a Noether conserved quantity and the corresponding Killing equations. Casetta [9] established the Poisson brackets definition for the dynamics of a position-dependent mass particle. However, to the authors' knowledge, the algebraic structure of non-material volumes has not been investigated yet.

Algebraic structure and Poisson's theory is an important integration method in classical mechanics, which has been explored to find invariants of dynamical and physical systems. There are many researchers who carried

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out interesting research on algebraic structure and Poisson's theory. Mei investigated the algebraic structure and Poisson's theory of a Birkhoffian system [10] and non-holonomic systems [11]. Subsequently, algebraic structure and Poisson's theory were extended to rotational relativistic systems [12], rotational relativistic Birkhoffian system [13], mechanico-electrical system [14], controllable mechanical system [15], in the event space [16], dynamical system with relative motion [17], and $f(R)$ cosmology [18]. Recently, Colarusso and Lau [19] addressed the limit of Lie algebra by using the Lie-Poisson theory. Benini and Schenkel [20] constructed Poisson algebras of field theories by introducing differential geometry. Ershkov [21] proposed a method to solve the Poisson equations where the angular velocity is dependent on time, obtaining a Riccati-type solution. So far, to the authors' best knowledge, there is no integral theory on non-material volumes. To address the lack of research in this aspect, the present work develops the Poisson's theory to determine the conserved quantities of non-material volumes.

The paper is organized as follows. Section 2 reviews the differential equations of the non-material volumes. Section 3 considers the algebraic structure of non-material volumes, presents a contravariant algebraic form and an algebraic product and obtains a Lie algebra proposition of the non-material volumes. Section 4 constructs the Poisson theory for non-material volumes, giving five theorems. Section 5 gives three examples to illustrate the application of the method. Section 6 contains the concluding remarks.

2 The differential equations of the non-material volumes

The Lagrange's equations for non-material volumes are pioneered by Irschik and Holl [1, p. 243, Eq. (5.6)], given as

$$\frac{d}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k}-\frac{\partial T_u}{\partial q_k}-\int\limits_{\partial V_u}\frac{1}{2}\rho \upsilon^2\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}-\frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot \boldsymbol{n}\mathrm{d}\partial V_u+\int\limits_{\partial V_u}\rho \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{n}\mathrm{d}\partial V_u=Q_k \tag{1}$$

where $T_u = T_u(\dot{q}_k, q_k, t)$ is the total kinetic energy of the material particles enclosed by the non-material volume V_u, q_k represents generalized coordinates, $\boldsymbol{v}$ is the velocity of the material particles, $\boldsymbol{u}$ is the velocity of fictitious particles, ρ is the volumetric mass density, Q_k is a generalized force applied to the material body, $\boldsymbol{n}$ is the outer normal unit vector at the surface of V_u, and ∂V_u depicts the bounding surface of V_u.

The Lagrange's Eq. (1) for a single degree of freedom non-material volume can be reformatted within a Hamiltonian formalism by Casetta [4, p. 7, Eqs. (37–38)], namely

$$\begin{aligned}\dot{q}&=\frac{\partial \tilde{H}_u}{\partial \tilde{p}_u},\\ \dot{\tilde{p}}_u&=-\frac{\partial \tilde{H}_u}{\partial q},\end{aligned} \tag{2}$$

where

$$\tilde{p}_u=\tilde{L}_u\big/\dot{q}=\int\limits_0^{\dot{q}}\exp\int\left(\frac{B_u+2\omega C_u-\partial Q(q,\omega,t)/\partial\omega}{A_u}\right)\mathrm{d}t\mathrm{d}\omega,$$

$$\tilde{L}_u=\int\limits_0^{\dot{q}}(\dot{q}-\omega)\exp\int\left(\frac{B_u+2\omega C_u-\partial Q(q,\omega,t)/\partial\omega}{A_u}\right)\mathrm{d}t\mathrm{d}\omega$$

$$-\int\limits_0^{\dot{q}}\left(\frac{D_u(\xi,t)-Q(\xi,0,t)}{A_u(\xi,t)}\right)\exp\int\left(\frac{B_u(\xi,t)-[\partial Q(\xi,\dot{q},t)/\partial\dot{q}]_{\dot{q}=0}}{A_u(\xi,t)}\right)\mathrm{d}t\mathrm{d}\xi,$$

$$\tilde{H}_u = \tilde{p}_u \dot{q} - \tilde{L}_u = \int_0^{\dot{q}} \omega \exp \int \left(\frac{B_u + 2\omega C_u - \partial Q(q,\omega,t)/\partial\omega}{A_u} \right) \mathrm{d}t\mathrm{d}\omega$$

$$+ \int_0^{\dot{q}} \left(\frac{D_u(\xi,t) - Q(\xi,0,t)}{A_u(\xi,t)} \right) \exp \int \left(\frac{B_u(\xi,t) - [\partial Q(\xi,\dot{q},t)/\partial\dot{q}]_{\dot{q}=0}}{A_u(\xi,t)} \right) \mathrm{d}t\mathrm{d}\xi,$$

$$A_u(q,t) = \int_{V_u} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial \upsilon}{\partial \dot{q}} dV_u,\ B_u(q,t) = \frac{\partial}{\partial t} \int_{V_u} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial \upsilon}{\partial \dot{q}} dV_u + \int_{\partial V_u} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial \upsilon}{\partial \dot{q}} \left(\frac{\partial p}{\partial t} - \frac{\partial r}{\partial t} \right) \cdot n \mathrm{d}\partial V_u,$$

$$C_u(q,t) = \frac{\partial}{\partial q} \int_{V_u} \frac{1}{2} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial \upsilon}{\partial \dot{q}} dV_u + \int_{\partial V_u} \frac{1}{2} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial \upsilon}{\partial \dot{q}} \left(\frac{\partial \upsilon}{\partial \dot{q}} - \frac{\partial u}{\partial \dot{q}} \right) \cdot n \mathrm{d}\partial V_u,$$

$$D_u(q,t) = \frac{\partial}{\partial t} \int_{V_u} \rho \frac{\partial \upsilon}{\partial \dot{q}} \frac{\partial p}{\partial t} dV_u - \frac{\partial}{\partial q} \int_{V_u} \frac{1}{2} \rho \frac{\partial p}{\partial t} \frac{\partial p}{\partial t} dV_u - \int_{\partial V_u} \frac{1}{2} \rho \frac{\partial p}{\partial t} \frac{\partial p}{\partial t} \left(\frac{\partial \upsilon}{\partial \dot{q}} - \frac{\partial u}{\partial \dot{q}} \right) \cdot n \mathrm{d}\partial V_u$$

$$+ \int_{\partial V_u} \rho \frac{\partial p}{\partial t} \frac{\partial \upsilon}{\partial \dot{q}} \left(\frac{\partial p}{\partial t} - \frac{\partial r}{\partial t} \right) \cdot n \mathrm{d}\partial V_u.$$

The derivation of A_u, B_u, C_u, D_u can be found in [4, p. 5, Eqs. (24–27)], and $\tilde{L}_u$, $\tilde{p}_u$ and $\tilde{H}_u$ can be found in [4, p. 6, Eqs. (32), (34), (36)].

3 Algebraic structure of non-material volumes

This section is devoted to the algebraic structure of single degree of freedom non-material volumes. Let

$$a^\mu = \begin{cases} q, & (\mu = 1) \\ \tilde{p}_u, & (\mu = 2) \end{cases}. \tag{3}$$

The Hamiltonian Eq. (2) can be rewritten as

$$\dot{a}^\mu - \Omega^{\mu\nu} \frac{\partial \tilde{H}_u}{\partial a^\nu} = 0, (\mu, \nu = 1, 2) \tag{4}$$

where the contravariant is given by

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

According to Eq. (4) define an algebraic product, namely

$$\dot{A} = \frac{\partial \tilde{H}_u}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial \tilde{H}_u}{\partial a^\nu} \stackrel{\text{def}}{=} A \circ \tilde{H}_u. \tag{5}$$

The product (5) accords with the right distributive law

$$A \circ (B + C) = A \circ B + A \circ C, \tag{6}$$

and the left distributive law

$$(A + B) \circ C = A \circ C + B \circ C, \tag{7}$$

as well as the scalar law

$$(\alpha A) \circ B = A \circ (\alpha B) = \alpha (A \circ B). \tag{8}$$

If an algebraic product conforms to the relationship Eqs. (6–8), it is a consistent algebra. Thus, Eq. (2) satisfies a consistent algebra structure.

The conditions (6–8) under the product (5) also possess the antisymmetry properties

$$A \circ B + B \circ A = 0, \tag{9}$$

and the Jacobi identical relation

$$A \circ (B \circ C) + B \circ (C \circ A) + C \circ (A \circ B) = 0. \tag{10}$$

Hence, the Lagrange equation and Hamilton equation of a single degree of freedom non-material volume satisfy a Lie algebra structure.

Actually, expanding the product (5), we have

$$\frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial \tilde{H}_u}{\partial a^{\nu}} = \frac{\partial A}{\partial q} \frac{\partial \tilde{H}_u}{\partial \tilde{p}_u} - \frac{\partial A}{\partial \tilde{p}_u} \frac{\partial \tilde{H}_u}{\partial q}. \tag{11}$$

Obviously, Eq. (11) is the Poisson bracket which is denoted by the symbol $\left(A, \tilde{H}_u\right)$. It is well known that the Poisson brackets have the following properties:

$$(A, B) + (B, A) = 0, \tag{12}$$

$$(A, (B, C)) + (B, (C, A)) + (C, (A, B)) = 0. \tag{13}$$

They are equivalent to the Lie algebra structure (9) and (10).

Proposition 1 *Both, the Lagrange equation and Hamilton equations of a single degree of freedom non-material volume, not only define a consistent algebra structure, but also a Lie algebra structure.*

4 Poisson theory of a single degree of freedom non-material volume

Since the dynamical equations for a non-material volume accord with a Lie algebra structure, the Poisson's theory associated with first integrals can be applied to the system. Therefore, the following theorems are given as

Theorem 1 *$I(a^{\mu}, t) = const.$ is the first integral of Eq. (2) if and only if*

$$\frac{\partial I}{\partial t} + \left(I, \tilde{H}_u\right) = 0, \tag{14}$$

where $\tilde{H}_u$ is the Hamiltonian of a single degree of freedom non-material volume.

Proof Consider the function $I(a^{\mu}, t)$. From the definition of total derivative, one has that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial \tilde{H}_u}{\partial a^{\nu}} = \frac{\partial I}{\partial t} + \left(I, \tilde{H}_u\right). \tag{15}$$

Given that (14) holds true, then $I = \text{const}$. □

Theorem 2 *If the Hamiltonian $\tilde{H}_u$ of a non-material volume does not explicitly contain time t, then $\tilde{H}_u$ is the first integral of Eq. (2).*

Proof Using Theorem 1, it holds obviously. □

Theorem 3 *If Eq. (2) has two first integrals $I_1(a^{\mu}, t)$ and $I_2(a^{\mu}, t)$ having not involution, then the Poisson bracket (I_1, I_2) is also the first integral of system (2).*

Proof Since $I_1(a^\mu, t)$ and $I_2(a^\mu, t)$ are first integrals, then

$$\frac{\partial I_1}{\partial t} + \left(I_1, \tilde{H}_u\right) = 0, \ \frac{\partial I_2}{\partial t} + \left(I_2, \tilde{H}_u\right) = 0. \tag{16}$$

According to Eq. (4), it follows that

$$\frac{\partial}{\partial t}(I_1, I_2) = \left(\frac{\partial}{\partial t} I_1, I_2\right) + \left(I_1, \frac{\partial}{\partial t} I_2\right). \tag{17}$$

Using the Lie algebraic axiom Eq. (13) and the left distributive law Eq. (7), combining Eqs. (16, 17) leads to

$$\begin{aligned}
&\frac{\partial}{\partial t}(I_1, I_2) + \left((I_1, I_2), \tilde{H}_u\right) \\
&= \left(\frac{\partial}{\partial t} I_1, I_2\right) + \left(I_1, \frac{\partial}{\partial t} I_2\right) + \left(\left(I_1, \tilde{H}_u\right), I_2\right) + \left(I_1, \left(I_2, \tilde{H}_u\right)\right) \\
&= \left(\frac{\partial}{\partial t} I_1 + \left(I_1, \tilde{H}_u\right), I_2\right) + \left(I_1, \frac{\partial}{\partial t} I_2 + \left(I_2, \tilde{H}_u\right)\right) \\
&= 0;
\end{aligned} \tag{18}$$

thus, (I_1, I_2) is the first integral of system (2). □

It is worth noting that Theorem 3 has universal significance, and it can be reduced to the traditional Poisson's theorem of classical analytical mechanics when $\tilde{L}_u$, $\tilde{p}_u$, and $\tilde{H}_u$ of system (2) reduce to the classical Lagrangian, generalized momentum and Hamiltonian.

Theorem 4 *If Eq.* (2) *possesses a first integral* $I(a^\mu, t)$ *including time* t*, and the Hamiltonian* $\tilde{H}_u$ *does not include time* t*, then* $\frac{\partial I}{\partial t}, \frac{\partial^2 I}{\partial t^2}, \ldots$ *are the first integrals of system* (2).

Proof By differentiating Eq. (14) with respect to time t, it is found that

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, \tilde{H}_u\right) + \left(I, \frac{\partial \tilde{H}_u}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, \tilde{H}_u\right) = 0;
\end{aligned} \tag{19}$$

hence, $\frac{\partial I}{\partial t}$ is the first integral of system (2). Analogously, one can immediately prove that $\frac{\partial^2 I}{\partial t^2}, \frac{\partial^3 I}{\partial t^3}, \ldots$ are also the first integrals of the system. □

Theorem 5 *If Eq.* (2) *has a first integral* $I(a^\mu, t)$=*const. involving* a^ρ*, but the Hamiltonian* $\tilde{H}_u$ *does not explicitly depend on* a^ρ*, then* $\frac{\partial I}{\partial a^\rho}, \frac{\partial^2 I}{\partial a^{\rho 2}}, \ldots$ *are the first integrals of systems* (2).

Proof By differentiating Eq. (14) with respect to a^ρ, it is found that

$$\begin{aligned}
&\frac{\partial}{\partial a^\rho}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial a^\rho}, \tilde{H}_u\right) + \left(I, \frac{\partial \tilde{H}_u}{\partial a^\rho}\right) \\
&= \frac{\partial}{\partial t}\frac{\partial I}{\partial a^\rho} + \left(\frac{\partial I}{\partial a^\rho}, \tilde{H}_u\right) = 0,
\end{aligned} \tag{20}$$

and so, $\frac{\partial I}{\partial a^\rho}$ is the first integral of system (2). Similarly, one can demonstrate that $\frac{\partial^2 I}{\partial a^{\rho 2}}, \frac{\partial^3 I}{\partial a^{\rho 3}}, \ldots$ are also the first integrals of the system. □

5 Applications

In order to illustrate the applicability of the proposed scheme, three examples will be investigated in the following.

5.1 Flow of liquid from an open tube

Casetta [4, p. 10, Eqs. (73–74)] derived the Hamiltonian canonical equations of flow of liquid from an open tube which can be described as

$$\begin{aligned} \dot{q} &= \tilde{p}_u, \\ \dot{\tilde{p}}_u &= -g \end{aligned} \tag{21}$$

where q represents the height of the amount of liquid inside the tube, g is the acceleration of gravity, $\tilde{p}_u$ describes the canonical momentum, and instantaneously the Hamiltonian is reported by Casetta [4, p. 10, Eq. (72)] as

$$\tilde{H}_u = \frac{1}{2}\tilde{p}_u^2 + gq. \tag{22}$$

Thereby, the system (21) has a Lie algebra structure.

According to Theorem 1, the Hamiltonian of system (21) is the first integral which can be found in [4, p. 10, Eq. (75)] as

$$\frac{1}{2}\tilde{p}_u^2 + gq = \text{const.} \tag{23}$$

By employing Theorem 1 of Poisson theory, we have

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial q}\frac{\partial \tilde{H}_u}{\partial \tilde{p}_u} - \frac{\partial I}{\partial \tilde{p}_u}\frac{\partial \tilde{H}_u}{\partial q} = 0, \tag{24}$$

and the characteristic equation of Eq. (24) is

$$\mathrm{d}t = \frac{\mathrm{d}q}{\partial \tilde{H}_u \big/ \partial \tilde{p}_u} = -\frac{d\tilde{p}_u}{\partial \tilde{H}_u \big/ \partial q}. \tag{25}$$

The first integral of Eq. (25) can be obtained as

$$q - \int \tilde{p}_u \mathrm{d}t = C^1, \; \tilde{p}_u + gt = C^2, \tag{26}$$

then a particular solution of (25) is

$$I = q + \tilde{p}_u - \int \tilde{p}_u \mathrm{d}t + gt. \tag{27}$$

5.2 The rocket motion

Casetta [4, p. 12, Eq. (100)] considered the Hamiltonian canonical equations of rocket motion which can be expressed as

$$\begin{aligned} \dot{q} &= \tilde{p}_u, \\ \dot{\tilde{p}}_u &= -\left(\frac{\dot{m}_u(t)v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)}\right) \end{aligned} \tag{28}$$

where q represents the displacement of the rocket, $m_u(t)$ is the total mass, $v_{\mathrm{rel}}(t)$ shows the relative velocity of the rocket with respect to the expelled propellant, $Q(t)$ is the external force, $\tilde{p}_u$ describes the canonical momentum, and analogously the Hamiltonian is

$$\tilde{H}_u = \frac{1}{2}\tilde{p}_u^2 + \left(\frac{\dot{m}_u(t)v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)}\right)q. \tag{29}$$

Then, the system (28) possesses a Lie algebra structure.

By using Theorem 1, we find that

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial q}\frac{\partial \tilde{H}_u}{\partial \tilde{p}_u} - \frac{\partial I}{\partial \tilde{p}_u}\frac{\partial \tilde{H}_u}{\partial q} = 0, \tag{30}$$

and the characteristic equation of Eq. (30) is

$$\mathrm{d}t = \frac{\mathrm{d}q}{\partial \tilde{H}_u \big/ \partial \tilde{p}_u} = -\frac{d\tilde{p}_u}{\partial \tilde{H}_u \big/ \partial q}. \tag{31}$$

The first integral of Eq. (31) can be written as

$$q - \int \tilde{p}_u \mathrm{d}t = C^1,\ \tilde{p} + \int \left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right) \mathrm{d}t = C^2, \tag{32}$$

then a particular solution of (31) is

$$I = q + \tilde{p}_u - \int \tilde{p}_u \mathrm{d}t + \int \left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right) \mathrm{d}t. \tag{33}$$

5.3 The rotating drum uncoiling a thin strip

Casetta et al. [8, p. 705, Eq. (67)] derived the Lagrangian equation of a rotating drum uncoiling a thin strip which can be expressed as

$$\frac{1}{2} \rho l \pi \left(R_0 - \frac{\varepsilon \phi}{2\pi} \right)^4 \ddot{\phi} = \Pi(\phi) \tag{34}$$

where ε represents the thickness of a thin strip, l is width, ρ describes mass density, $\phi = \phi(t)$ is the rotation angle, and R_0 is the original radius.

The Hamiltonian canonical equation can be reformatted as the following form:

$$\begin{aligned} \dot{\phi} &= \tilde{p}_u, \\ \dot{\tilde{p}}_u &= \Lambda(\phi), \end{aligned} \tag{35}$$

and the Hamiltonian is

$$\tilde{H} = \frac{1}{2} \tilde{p}_u^2 - \int \Lambda(\phi) d\phi \tag{36}$$

where

$$\Lambda(\phi) = \frac{2\Pi(\phi)}{\rho l \pi} \left(R_0 - \frac{\varepsilon \phi}{2\pi} \right)^{-4};$$

then, the system (35) proposes a Lie algebra structure.

Since the Hamiltonian (36) does not explicitly depend on time, consequently, the system (35) has the first integral which was reported by Casetta et al. [8, p. 705, Eq. (69)] as

$$\frac{1}{2} \tilde{p}_u^2 - \int \Lambda(\phi) d\phi = \mathrm{const.} \tag{37}$$

By recalling Theorem 1, one has

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial \phi} \frac{\partial \tilde{H}_u}{\partial \tilde{p}_u} - \frac{\partial I}{\partial \tilde{p}_u} \frac{\partial \tilde{H}_u}{\partial \phi} = 0, \tag{38}$$

and the characteristic equation of Eq. (38) is

$$\mathrm{d}t = \frac{\mathrm{d}\phi}{\partial \tilde{H}_u \big/ \partial \tilde{p}_u} = -\frac{d\tilde{p}_u}{\partial \tilde{H}_u \big/ \partial \phi}. \tag{39}$$

The first integral of Eq. (39) can be calculated as

$$\phi - \int \tilde{p}_u \mathrm{d}t = C^1,\ \tilde{p} - \Lambda(\phi) t = C^2, \tag{40}$$

then a particular solution of (39) is

$$I = \phi + \tilde{p}_u - \int \tilde{p}_u \mathrm{d}t - \Lambda(\phi) t. \tag{41}$$

6 Conclusions

This paper focuses on the algebraic structure and Poisson theory of non-material volumes. The contravariant algebraic form of the equations of motion for a single degree of freedom non-material volume is proposed, an algebraic product is defined, and a proposition is employed. In this context, Poisson theory is constructed, and five theorems are established. Three examples are proposed to illustrate the application of the method, and the corresponding first integrals are solved.

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References

1. Irschik, H., Holl, H.J.: The equations of Lagrange written for a non-material volume. Acta Mech. **153**, 231–248 (2002)
2. Cveticanin, L.: Dynamics of Bodies with Time-Variable Mass. Springer, Seelisberg (2016)
3. Irschik, H., Holl, H.J.: Mechanics of variable-mass systems—part 1: balance of mass and linear momentum. Appl. Mech. Rev. **57**, 145–160 (2004)
4. Casetta, L.: The inverse problem of Lagrangian mechanics for a non-material volume. Acta Mech. **226**, 1–15 (2015)
5. Casetta, L., Pesce, C.P.: The generalized Hamilton's principle for a non-material volume. Acta Mech. **224**, 919–924 (2013)
6. Casetta, L., Pesce, C.P.: The inverse problem of Lagrangian mechanics for Meshchersky's equation. Acta Mech. **225**, 1607–1623 (2014)
7. Irschik, H., Holl, H.J.: Lagrange's equations for open systems, derived via the method of fictitious particles, and written in the Lagrange description of continuum mechanics. Acta Mech. **226**, 63–79 (2015)
8. Casetta, L., Irschik, H., Pesce, C.P.: A generalization of Noether's theorem for a non-material volume. Z. Angew. Math. Mech. **96**, 696–706 (2016)
9. Casetta, L.: Poisson brackets formulation for the dynamics of a position-dependent mass particle. Acta Mech. **228**, 4491–4496 (2017)
10. Mei, F.X.: Poisson's theory of Birkhoffian system. Chin. Sci. Bull. **41**, 641–645 (1996)
11. Mei, F.X.: The algebraic structure and Poisson's theory for the equations of motion of non-holonomic systems. J. Appl. Math. Mech. **62**, 155–158 (1998)
12. Fu, J.L., Chen, X.W., Luo, S.K.: Algebraic structures and Poisson integrals of relativistic dynamical equations for rotational systems. Appl. Math. Mech. **20**, 1266–1275 (1999)
13. Luo, S.K., Chen, X.W., Guo, Y.X.: Algebraic structure and Poisson integrals of rotational relativistic Birkhoff system. Chin. Phys. **51**, 523–528 (2002)
14. Liu, H.J., Tang, Y.F., Fu, J.L.: Algebraic structure and Poisson's theory of mechanico-electrical systems. Chin. Phys. **15**, 1653–1662 (2006)
15. Xia, L.L.: Poisson theory and inverse problem in a controllable mechanical system. Chin. Phys. Lett. **28**, 120202 (2011)
16. Zhang, Y.: Poisson theory and integration method of Birkhoffian systems in the event space. Chin. Phys. B **19**, 080301 (2010)
17. Zhang, Y., Shang, M.: Poisson theory and integration method for a dynamical system of relative motion. Chin. Phys. B **20**, 024501 (2011)
18. Fu, J.L., Xie, F.P., Guo, Y.X.: Algebraic structure and Poissons integral theory of $f(R)$ cosmology. Int. J. Theor. Phys. **51**, 35–48 (2012)
19. Colarusso, M., Lau, M.: Lie–Poisson theory for direct limit Lie algebras. J. Pure Appl. Algebra **220**, 1489–1516 (2016)
20. Benini, M., Schenkel, A.: Poisson algebras for non-linear field theories in the cahiers topos. Ann. Henri Poincaré **18**, 1435–1464 (2017)
21. Ershkov, S.V.: A Riccati-type solution of Euler–Poisson equations of rigid body rotation over the fixed point. Acta Mech. **228**, 2719–2723 (2017)