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ORIGINAL PAPER

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Analytical solutions for the thermal vibration of strain gradient beams with elastic boundary conditions

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Abstract A strain gradient Euler beam described by a sixth-order differential equation is used to investigate the thermal vibrations of beams made of strain gradient elastic materials. The sixth-order differential equation of motion and elastic boundary conditions are determined simultaneously by a variation formulation based on Hamilton's principle. Analytical solutions for the free vibration of the elastic constraint strain gradient beams subjected to axial thermal stress are obtained. The effects of the thermal stress, nonlocal effect parameter, and boundary spring stiffness on the vibration behaviors of the strain gradient beams are investigated. The results show that the natural frequencies obtained by the strain gradient Euler beam model with the thermal stress decrease while the temperature is rising. The thermal effects are sensitive to the boundary spring stiffness at a certain stiffness range. In addition, numerical results also show the importance of the nonlocal effect parameter on the vibration of the strain gradient beams.

1 Introduction

Structures, such as bars, beams, plates or shells, within extremely small scales have attracted considerable attention for widely potential applications in modern micro- and nanoelectromechanical systems [1–4]. Their characteristic scales are on the order of micrometer and nanometer. Experiments have shown that mechanical behaviors of microstructures display strong size effects [5–8]. Hence, it is essential to consider small-scale effects in the analysis of the mechanical behaviors of microstructures. Due to lacking intrinsic material length-scale parameters, the classical continuum elasticity theory is incapable of predicting size effects phenomena. To this end, some higher-order continuum theories, such as stress gradient theories [9–16], strain gradient theories [17–28], and nonlocal strain gradient theories [29–32], which incorporate size-dependent material length-scale parameters, have been successfully developed and employed to describe the mechanical behaviors of micro- and nanostructures.

Aifantis proposed a second-order strain gradient continuum elastic theory which contains only one material length-scale parameter [17,18]. Based on the strain gradient elasticity theory, several beam and plate models have been developed to capture size effects of the micro- and nanostructures [33–38]. For example, Papargyri-Beskou et al. [33] investigated the bending and stability of the gradient elastic beams. Two boundary value problems for bending and stability were solved analytically. The gradient elasticity effect on the bending and critical load of the beam was investigated for both cases. Later on, Papargyri-Beskou and Beskos [34] carried out the static, stability, and dynamic analysis for gradient elastic flexural Kirchhoff plates. Three boundary value problems for statics, stability, and dynamics of an all edges simply supported rectangular plate of gradient elasticity were solved analytically. The effects of the gradient coefficient on the static, and dynamic response,

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buckling load, and natural frequencies of the plate were studied. Askes and Aifantis [35] discussed various formats of gradient elasticity and their performance in static and dynamic applications. An overview of length-scale identification and quantification procedures were given. Some commonly used gradient elastic finite element methods with different boundary conditions were discussed.

In the above-mentioned studies, the boundary conditions of structures were all restricted to simply supported, clamped, or free cases. The boundary conditions of structures in engineering are far different from those classic cases in nature. Therefore, supports at edges should be approximated described by translational and rotational spring restraints at the boundaries. The vibration problems of elastically restrained beams and plates have drawn considerable attention in the past decades [39–52]. Based on the classic continuum models, the vibration analysis of elastically supported beams and plates was performed by some efficient numerical methods. Li [40,42] proposed a modified Fourier series method to analyze the free vibrations of Euler beams and Kirchhoff plates with general elastic supports. Xing and Wang [45] presented a general model for the vibration of beams restrained with two transversal and two rotational elastic springs subject to a constant axial load. The natural frequencies and the shape functions were derived analytically. Suddoung et al. [46] employed a differential transformation method to study the free vibration response of a stepped beam made of functionally graded materials with elastical end constraints. Wattanasakulpong and Mao [47] investigated the dynamic response of Timoshenko beams made of functionally graded materials with classical and non-classical boundary conditions using Chebyshev collocation method. Zhang et al. [48] used the element-free improved moving least-squares Ritz method to analyze the vibration of thick plates made of a functionally graded carbon nanotube reinforced composite with elastically restrained edges. Jiang et al. [49] used the modified Fourier series method to study the free vibration of single-walled carbon nanotubes with elastic boundary conditions based on Timoshenko beam models. Based on theory of stress gradient elasticity, Kiani [50,51] proposed a meshless approach to investigate the free vibration of the embedded single- and double-walled carbon nanotubes with elastic boundary conditions based on the nonlocal Euler, Timoshenko, and higher-order beam theory. Rosa and Lippiello [52] developed the differential quadrature method to study the free vibration of embedded single-walled carbon nanotubes using the nonlocal Euler beam models. Although numerous researches have been carried out on the elastic boundaries of the structures, no literature has been reported to consider the microstructures with elastic boundaries based on the strain gradient theories.

But all of the above-reviewed research works did not take thermal effects into consideration. Thermal stresses of structures are often induced by changes of temperature conditions, which can lead to the reduction in stiffness [53–72]. The vibrational behaviors of structures will change due to thermal stress. Thus, thermal effects must be considered in vibrational problems of the micro- and nanostructures. Recently, there are an increasing number of works on the thermal vibration of beam, plate, and shell structures based on classic elasticity theory [53,54], stress gradient theories [56–62], the modified couple stress theory [66,67], and the strain gradient theory [68–72]. Ansari et al. [68] used Mindlin's strain gradient theory to investigate the effects of temperature and length-scale parameters on the vibrational behaviors of the functionally graded nanoshells. Ebrahimi and Barati [69] studied thermal and surface effects on the vibration characteristics of viscoelastic functionally graded nanobeams embedded in a viscoelastic foundation based on nonlocal strain gradient elasticity theory and Euler beam model. Rahmani et al. [71] investigated the influences of uniform thermomechanical loading on buckling and free vibration of a curved functionally graded microbeam based on strain gradient theory and Timoshenko beam model. Nematollahi et al. [72] studied the vibration of thin rectangular nanoplates under different thermal conditions based on the higher-order nonlocal strain gradient theory. However, very few works on the vibration behaviors of strain gradient beams considering the effect of temperature changes are reported.

To the best knowledge of the authors, the thermal vibration analysis of strain gradient beams with elastic boundary conditions has not been reported in the available literature. The primary objective of this work is to propose an analytical solution for investigating the vibration of the elastically constrained strain gradient beams subjected to axial thermal stress. For this purpose, this paper is organized as follows. A sixth-order equation of motion and elastic boundary conditions are obtained simultaneously by employing Hamilton's principle in Sect. 2. Then, the boundary value problems for the free vibrational strain gradient beams are solved analytically in Sect. 3. Vibration analyses of the strain gradient beams with elastic boundary conditions are presented and discussed in Sect. 4. Finally, some concluding remarks are drawn in Sect. 5.

2 Strain gradient Euler beam model

One of the simplest constitutive laws between stress and strain of gradient elasticity theory in one dimension can be expressed as [18]

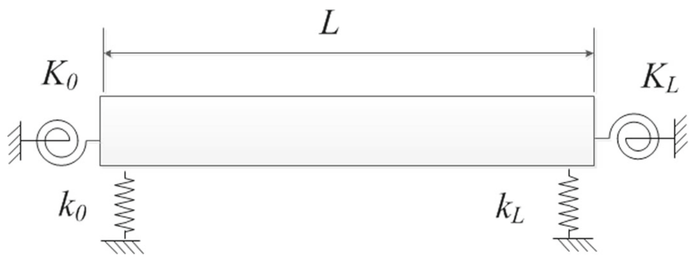

Fig. 1 An elastically constrained strain gradient beam

$$\sigma_x = E\left(\varepsilon_x - r^2\frac{\partial^2\varepsilon_x}{\partial x^2}\right) \tag{1}$$

where E is Young's modulus, σ_x is the axial stress, ε_x is the axial strain, and r is a material length-scale parameter related to the intrinsic microstructure.

The displacement components (u_x, u_y, u_z) of the Euler beam along the (x, y, z) coordinate directions can be written as

$$u_x = -z\frac{\partial w(x,t)}{\partial x}, \quad u_y = 0, \quad u_z = w(x,t) \tag{2}$$

where t denotes the time, and w is the transverse displacement of the middle line in z direction. The strain field can be expressed as

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}. \tag{3}$$

The strain energy of the strain gradient elastic beam in bending is [33]

$$U = \frac{1}{2}\int_0^L EI\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + r^2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right]\mathrm{d}x \tag{4}$$

where I is the inertia moment of the beam, and L is the length of the beam.

Considering boundary springs and additional thermal stress in the strain energy of the beam, one obtains the following expression:

$$\begin{aligned} U = {} & \frac{1}{2}\int_0^L \left\{EI\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + r^2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right] + N\left(\frac{\partial w}{\partial x}\right)^2\right\}\mathrm{d}x \\ & + \frac{1}{2}\left[k_0 w^2 + K_0\left(\frac{\partial w}{\partial x}\right)^2\right]\Bigg|_{x=0} + \frac{1}{2}\left[k_L w^2 + K_L\left(\frac{\partial w}{\partial x}\right)^2\right]\Bigg|_{x=L} \end{aligned} \tag{5}$$

where N denotes the axial force caused by the thermal stress, k_0 and k_L are the translational spring constants, and K_0 and K_L are the rotational spring constants at $x = 0$ and $x = L$, respectively, as shown in Fig. 1.

The kinetic energy of the beam can be expressed as

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w}{\partial t}\right)^2\mathrm{d}x \tag{6}$$

where ρ is the mass density, and A is the area of the cross section of the beam. Thus, Hamilton's principle for the strain gradient beam in the time interval $[0, t_0]$ has the form

$$\begin{aligned} 0 &= \delta\int_0^{t_0}[U - T]\mathrm{d}t \\ &= \delta\int_0^{t_0}\left\{\int_0^L \left\{EI\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + r^2\left(\frac{\partial^3 w}{\partial x^3}\right)^2\right] + N\left(\frac{\partial w}{\partial x}\right)^2 - \rho A\left(\frac{\partial w}{\partial t}\right)^2\right\}\mathrm{d}x\right. \end{aligned}$$

$$+\left[k_0 w^2+K_0\left(\frac{\partial w}{\partial x}\right)^2\right]\Bigg|_{x=0}+\left[k_L w^2+K_L\left(\frac{\partial w}{\partial x}\right)^2\right]\Bigg|_{x=L}\Bigg\}\mathrm{d}t$$
$$=\int_0^{t_0}\Bigg\{\int_0^L\left[EI\left(\frac{\partial^4 w}{\partial x^4}-r^2\frac{\partial^6 w}{\partial x^6}\right)-N\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}\right]\delta w\mathrm{d}x$$
$$+\left[N\frac{\partial w}{\partial x}-EI\left(\frac{\partial^3 w}{\partial x^3}-r^2\frac{\partial^5 w}{\partial x^5}\right)\right]\delta w\Bigg|_0^L$$
$$+EIr^2\frac{\partial^3 w}{\partial x^3}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\Bigg|_0^L+EI\left(\frac{\partial^2 w}{\partial x^2}-r^2\frac{\partial^4 w}{\partial x^4}\right)\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L$$
$$+\left[k_0 w\delta w+K_0\frac{\partial w}{\partial x}\delta\left(\frac{\partial w}{\partial x}\right)\right]\Bigg|_{x=0}+\left[k_L w\delta w+K_L\frac{\partial w}{\partial x}\delta\left(\frac{\partial w}{\partial x}\right)\right]\Bigg|_{x=L}\Bigg\}\mathrm{d}t. \tag{7}$$

The above variational equation implies that each term of Eq. (7) must be equal to zero. Hence, the equation of motion is given by

$$EI\left(\frac{\partial^4 w}{\partial x^4}-r^2\frac{\partial^6 w}{\partial x^6}\right)-N\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}=0, \tag{8.1}$$

and the boundary conditions satisfy these equations:

$$\left[k_0 w+EI\left(\frac{\partial^3 w}{\partial x^3}-r^2\frac{\partial^5 w}{\partial x^5}\right)-N\frac{\partial w}{\partial x}\right]\delta w\Bigg|_{x=0}=0, \tag{8.2}$$
$$\left[k_L w-EI\left(\frac{\partial^3 w}{\partial x^3}-r^2\frac{\partial^5 w}{\partial x^5}\right)+N\frac{\partial w}{\partial x}\right]\delta w\Bigg|_{x=L}=0, \tag{8.3}$$
$$\left[K_0\frac{\partial w}{\partial x}-EI\left(\frac{\partial^2 w}{\partial x^2}-r^2\frac{\partial^4 w}{\partial x^4}\right)\right]\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_{x=0}=0, \tag{8.4}$$
$$\left[K_L\frac{\partial w}{\partial x}+EI\left(\frac{\partial^2 w}{\partial x^2}-r^2\frac{\partial^4 w}{\partial x^4}\right)\right]\delta\left(\frac{\partial w}{\partial x}\right)\Bigg|_{x=L}=0, \tag{8.5}$$
$$EIr^2\left[\frac{\partial^3 w}{\partial x^3}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right]\Bigg|_{x=0}=0, \tag{8.6}$$
$$EIr^2\left[\frac{\partial^3 w}{\partial x^3}\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right]\Bigg|_{x=L}=0. \tag{8.7}$$

In particular, one can observe that Eq. (8.1) of the strain gradient elastic case can be reduced to the classical elastic case for $r=0$.

Further, the boundary conditions (8.2)–(8.5) for an elastically constrained beam are as follows:

$$k_0 w=-EI\left(\frac{\partial^3 w}{\partial x^3}-r^2\frac{\partial^5 w}{\partial x^5}\right)+N\frac{\partial w}{\partial x}, \tag{9.1}$$
$$K_0\frac{\partial w}{\partial x}=EI\left(\frac{\partial^2 w}{\partial x^2}-r^2\frac{\partial^4 w}{\partial x^4}\right) \tag{9.2}$$

at $x=0$, and

$$k_L w=EI\left(\frac{\partial^3 w}{\partial x^3}-r^2\frac{\partial^5 w}{\partial x^5}\right)-N\frac{\partial w}{\partial x}, \tag{9.3}$$
$$K_L\frac{\partial w}{\partial x}=-EI\left(\frac{\partial^2 w}{\partial x^2}-r^2\frac{\partial^4 w}{\partial x^4}\right) \tag{9.4}$$

at $x = L$. The higher-order boundary conditions at both ends may be assumed as the following four possible cases.

$$\text{Case 1}: \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} = 0, \quad \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = 0, \tag{10.1}$$

$$\text{Case 2}: \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0, \tag{10.2}$$

$$\text{Case 3}: \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0, \tag{10.3}$$

$$\text{Case 4}: \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = 0. \tag{10.4}$$

The solution $w(x, t)$ of the vibration of the beam can be expressed as follows:

$$w(x, t) = \bar{w}(x)e^{i\omega t} \tag{11}$$

where $\bar{w}(x)$ represents the amplitude of the deflection of the beam, ω is the natural frequency, and $i = \sqrt{-1}$.

Without loss of generality, the following dimensionless quantities are introduced:

$$\xi = \frac{x}{L}, \quad W(\xi) = \frac{\bar{w}(x)}{L}, \quad \mu = \frac{r}{L}, \quad \Omega^2 = \omega^2\frac{\rho A L^4}{EI}, \quad \bar{k}_0 = \frac{k_0 L^3}{EI},$$
$$\bar{k}_1 = \frac{k_L L^3}{EI}, \quad \bar{K}_0 = \frac{K_0 L}{EI}, \quad \bar{K}_1 = \frac{K_L L}{EI}, \quad \delta = -\frac{NL^2}{EI} \tag{12}$$

where ξ is the dimensionless x coordinate, $W(\xi)$ is the dimensionless amplitude of deflection of the beam, the nonlocal effect parameter μ is the dimensionless material length-scale parameter to account for the microstructural effect, Ω is the dimensionless natural frequency, the thermal effect parameter δ can reflect the effects of thermal stress, $\bar{k}_0$ and $\bar{k}_1$ are the dimensionless translational spring constants, and $\bar{K}_0$ and $\bar{K}_1$ are dimensionless rotational spring constants at $\xi = 0$ and $\xi = 1$, respectively.

Substitution of Eqs. (11) and (12) into Eq. (8.1) leads to the dimensionless governing equation:

$$\mu^2\frac{\partial^6 W}{\partial \xi^6} - \frac{\partial^4 W}{\partial \xi^4} - \delta\frac{\partial^2 W}{\partial \xi^2} + \Omega^2 W = 0. \tag{13}$$

Substituting of Eqs. (11) and (12) into Eqs. (9.1) and (10.1), the elastic boundary conditions can be expressed as follows:

BC1:

$$\bar{k}_0 W + \frac{\partial^3 W}{\partial \xi^3} - \mu^2\frac{\partial^5 W}{\partial \xi^5} + \delta\frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_0\frac{\partial W}{\partial \xi} - \frac{\partial^2 W}{\partial \xi^2} + \mu^2\frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^3 W}{\partial \xi^3} = 0, \tag{14.1}$$

at $\xi = 0$,

$$\bar{k}_1 W - \frac{\partial^3 W}{\partial \xi^3} + \mu^2\frac{\partial^5 W}{\partial \xi^5} - \delta\frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_1\frac{\partial W}{\partial \xi} + \frac{\partial^2 W}{\partial \xi^2} - \mu^2\frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^3 W}{\partial \xi^3} = 0, \tag{14.2}$$

at $\xi = 1$.

BC2:

$$\bar{k}_0 W + \frac{\partial^3 W}{\partial \xi^3} - \mu^2\frac{\partial^5 W}{\partial \xi^5} + \delta\frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_0\frac{\partial W}{\partial \xi} - \frac{\partial^2 W}{\partial \xi^2} + \mu^2\frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^2 W}{\partial \xi^2} = 0, \tag{15.1}$$

at $\xi = 0$,

$$\bar{k}_1 W - \frac{\partial^3 W}{\partial \xi^3} + \mu^2\frac{\partial^5 W}{\partial \xi^5} - \delta\frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_1\frac{\partial W}{\partial \xi} + \frac{\partial^2 W}{\partial \xi^2} - \mu^2\frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^2 W}{\partial \xi^2} = 0, \tag{15.2}$$

Table 1 The dimensionless natural frequencies Ω of the strain gradient beams with H–H boundary condition (the nonlocal effect parameter $\mu = 0.1$, the thermal effect parameter $\delta = 0$, $\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 0$, $\bar{K}_1 = 0$)

Boundary conditions	Mode					
H–H	1	2	3	4	5	6
BC1	10.1639	44.1696	112.0999	228.9215	412.3364	681.1025
[74]	10.1639	44.1696	112.0949	228.9214	412.3388	681.1816
BC2	10.3452	46.6244	122.0601	253.6045	459.4537	758.1477
[74]	10.3452	46.6244	122.0600	253.6043	459.4597	758.3004
Exact	10.3452	46.6244	122.0601	253.6045	459.4537	758.1477
BC3	10.2531	45.3550	116.8652	240.7486	435.0290	718.4084
BC4	10.2531	45.3550	116.8652	240.7486	435.0290	718.4084

Table 2 The dimensionless natural frequencies Ω of the strain gradient beams with C–C boundary condition (the nonlocal effect parameter $\mu = 0.1$, the thermal effect parameter $\delta = 0$, $\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 10^{10}$)

Boundary conditions	Mode					
C–C	1	2	3	4	5	6
BC1	26.6861	85.6811	195.6367	374.1180	639.4457	1010.1456
[74]	26.6861	85.6811	195.6366	374.1191	639.5018	998.9380
BC2	35.8935	108.9315	239.1782	444.2280	742.4001	1152.2046
[74]	35.8935	108.9315	239.1780	444.2329	742.5377	1153.9843
BC3	30.9154	96.6456	216.4654	407.9569	689.4383	1079.4256
BC4	30.9154	96.6456	216.4654	407.9569	689.4383	1079.4256

Table 3 The dimensionless natural frequencies Ω of the strain gradient beams with C–F boundary condition (the nonlocal effect parameter $\mu = 0.1$, the thermal effect parameter $\delta = 0$, $\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 0$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 0$)

Boundary conditions	Mode					
C–F	1	2	3	4	5	6
BC1	3.5843	24.7156	78.1543	176.1846	335.4597	574.0578
[74]	3.5843	24.7156	78.1543	176.185	335.460	574.085
BC2	4.3087	28.5909	89.2686	200.1563	379.2151	644.9123
[74]	4.3087	28.5909	89.2686	200.1562	379.2162	644.9710
BC3	3.5850	24.8100	79.3072	180.9587	347.6107	597.7278
BC4	4.3074	28.4554	87.8029	194.5273	365.5268	619.0029

Table 4 The dimensionless natural frequencies Ω of the strain gradient beams with C–H boundary condition (the nonlocal effect parameter $\mu = 0.1$, the thermal effect parameter $\delta = 0$, $\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 0$)

Boundary conditions	Mode					
C–H	1	2	3	4	5	6
BC1	16.9473	61.7966	148.3390	293.2959	514.9594	832.0661
[74]	16.9473	61.7966	148.3390	293.2958	514.9730	832.3242
BC2	19.9926	72.4153	172.8229	338.6914	588.3035	940.1784
[74]	19.9926	72.4153	172.8228	338.6917	588.3368	940.6942
BC3	17.1719	63.9029	155.5377	309.4787	544.0044	877.6480
BC4	19.6878	69.8236	164.5166	320.7252	556.8250	891.5515

at $\xi = 1$.

BC3:

$$\bar{k}_0 W + \frac{\partial^3 W}{\partial \xi^3} - \mu^2 \frac{\partial^5 W}{\partial \xi^5} + \delta \frac{\partial W}{\partial \xi} = 0 \quad \bar{K}_0 \frac{\partial W}{\partial \xi} - \frac{\partial^2 W}{\partial \xi^2} + \mu^2 \frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^3 W}{\partial \xi^3} = 0 \tag{16.1}$$

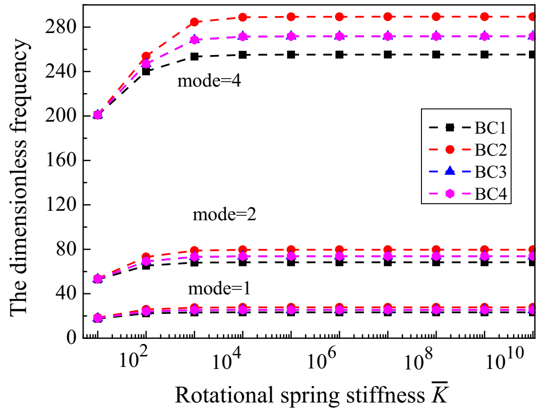

Fig. 2 Effect of the rotational spring stiffness on the dimensionless natural frequencies of the strain gradient beams (the nonlocal effect parameter $\mu = 0.05$, the translational spring stiffness $\bar{k} = 10^{10}$, the thermal effect parameter $\delta = 1$)

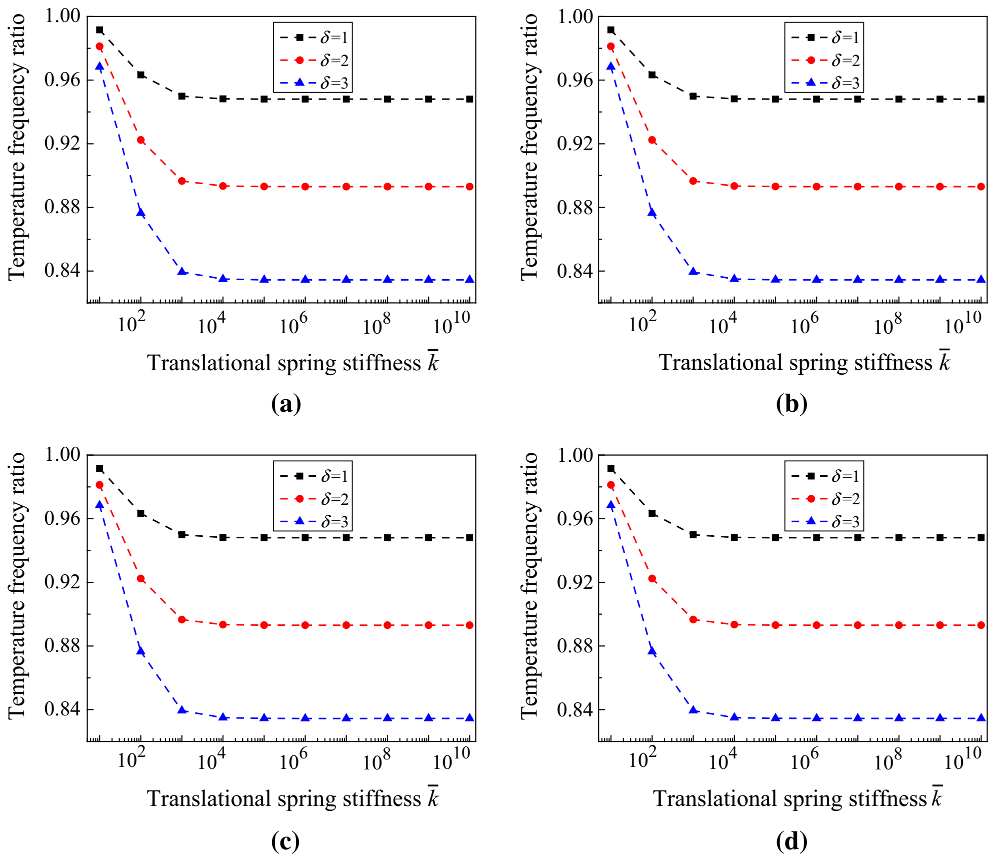

Fig. 3 Effect of the translational spring stiffness on the temperature frequency ratios of the strain gradient beams (the nonlocal effect parameter $\mu = 0.01$, rotational spring stiffness $\bar{K} = 0$). **a** BC1, **b** BC2, **c** BC3, and **d** BC4

at $\xi = 0$,

$$\bar{k}_1 W - \frac{\partial^3 W}{\partial \xi^3} + \mu^2 \frac{\partial^5 W}{\partial \xi^5} - \delta \frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_1 \frac{\partial W}{\partial \xi} + \frac{\partial^2 W}{\partial \xi^2} - \mu^2 \frac{\partial^4 W}{\partial \xi^4} = 0 \quad \frac{\partial^2 W}{\partial \xi^2} = 0 \tag{16.2}$$

at $\xi = 1$.

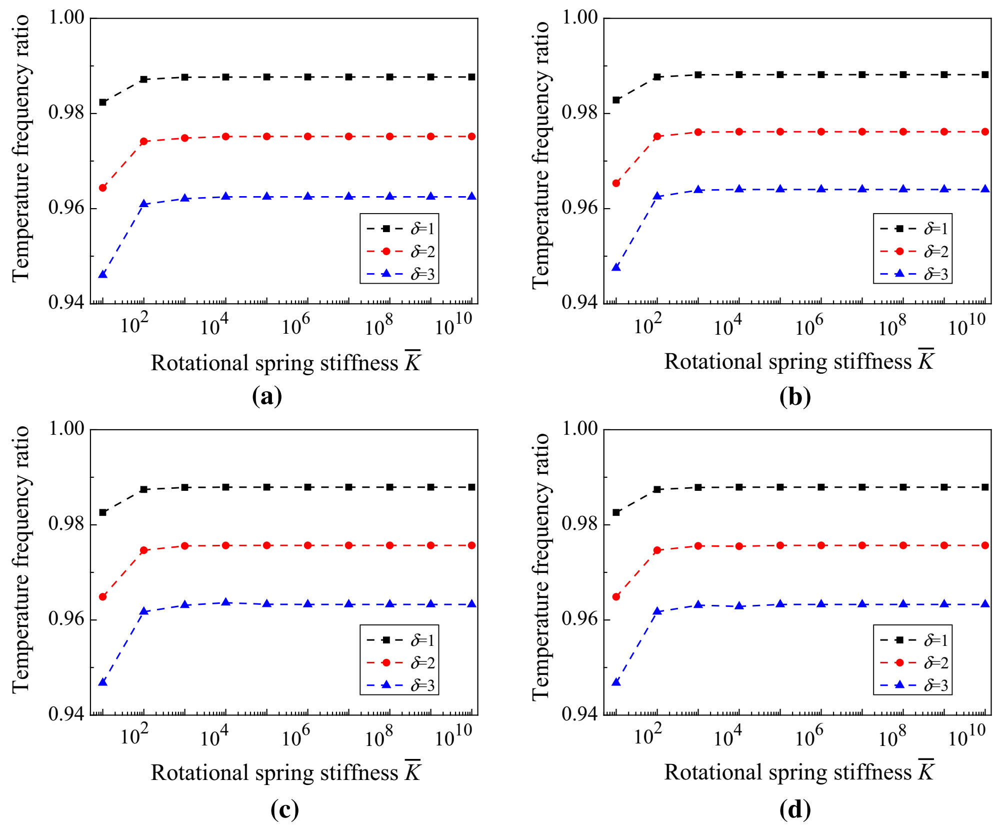

Fig. 4 Effect of the rotational spring stiffness on the temperature frequency ratios of the strain gradient beams (the nonlocal effect parameter $\mu = 0.01$, translational spring stiffness $\bar{k} = 10^{10}$). **a** BC1, **b** BC2, **c** BC3, and **d** BC4

BC4:

$$\bar{k}_0 W + \frac{\partial^3 W}{\partial \xi^3} - \mu^2 \frac{\partial^5 W}{\partial \xi^5} + \delta \frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_0 \frac{\partial W}{\partial \xi} - \frac{\partial^2 W}{\partial \xi^2} + \mu^2 \frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^2 W}{\partial \xi^2} = 0 \tag{17.1}$$

at $\xi = 0$,

$$\bar{k}_1 W - \frac{\partial^3 W}{\partial \xi^3} + \mu^2 \frac{\partial^5 W}{\partial \xi^5} - \delta \frac{\partial W}{\partial \xi} = 0, \quad \bar{K}_1 \frac{\partial W}{\partial \xi} + \frac{\partial^2 W}{\partial \xi^2} - \mu^2 \frac{\partial^4 W}{\partial \xi^4} = 0, \quad \frac{\partial^3 W}{\partial \xi^3} = 0 \tag{17.2}$$

at $\xi = 1$.

3 Analytical solution for the free vibration of the strain gradient beams

This section deals with the solution of the boundary value problem for the vibration of the strain gradient beams. Analytical solutions for the boundary value problems are usually a challenge except for some very simple case. In particular, the analytical solution for the high-order differential equation in vibrations will be more difficult. The analytical solutions for the vibration problems of the strain gradient beams are presented in the following. The characteristic equation corresponding to Eq. (13) can be expressed as

$$aX^3 + bX^2 + cX + d = 0 \tag{18}$$

where $a = \mu^2$, $b = -1$, $c = -\delta$, $d = \Omega^2$, and $X = \lambda^2$, here λ is the eigenvalue of Eq. (13). In fact, this is a cubic algebraic equation with respect to X. Let $B = b^2 - 3ac$, $C = bc - 9ad$, $F = c^2 - 3bd$, $\Delta = C^2 - 4BF$,

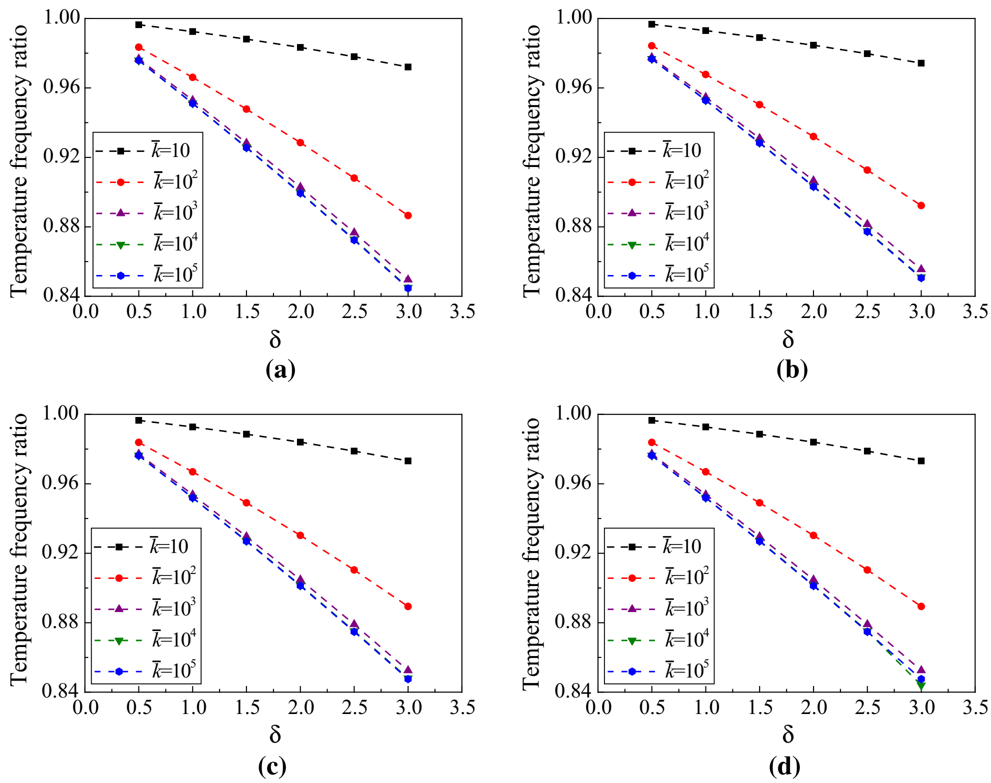

Fig. 5 Effect of the thermal effect parameter δ on the temperature frequency ratios for the strain gradient beams (the nonlocal effect parameter $\mu = 0.1$, the rotational spring stiffness $\bar{K} = 0$). **a** BC1, **b** BC2, **c** BC3, and **d** BC4

the roots of the characteristic cubic algebraic Eq. (18) are shown in "Appendix A" [73]. Here, the characteristic roots of Eq. (18) are complex in general. Thus, the general solution of Eq. (13) can be of the form

$$W(\xi) = \sum_{i=1}^{6} c_i t_i \tag{19}$$

where $t_i\,(i = 1, 2, \ldots, 6)$ are the fundamental solutions of Eq. (13) which can be obtained following the step described in "Appendix B". The constants of $c_i\,(i = 1, 2, \ldots, 6)$ in Eq. (19) can be determined by the boundary conditions. For the elastic boundary condition of BC1, substituting Eq. (19) into Eq. (14.1), one gets the linear equations in matrix form,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \vdots & \vdots & \vdots & \vdots \\ a_{61} & a_{62} & \cdots & a_{66} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_6 \end{pmatrix} = 0, \tag{20}$$

where $a_{ij}\,(i, j = 1, 2, \ldots, 6)$ are functions of the dimensionless natural frequency Ω. In order to have a nontrivial solution for Eq. (20), the dimensionless natural frequency equation is obtained as follows:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \vdots & \vdots & \vdots & \vdots \\ a_{61} & a_{62} & \cdots & a_{66} \end{vmatrix} = 0. \tag{21}$$

Hence, all the dimensionless natural frequencies can be obtained numerically by a direct iterative process. For BC2, BC3, and BC4, the dimensionless natural frequencies can also be obtained following the above procedure.

Once the dimensionless natural frequencies are obtained, the corresponding mode shapes can also be determined. At each dimensionless natural frequency, Eq. (20) has a nontrivial solution. Hence, at least one of the

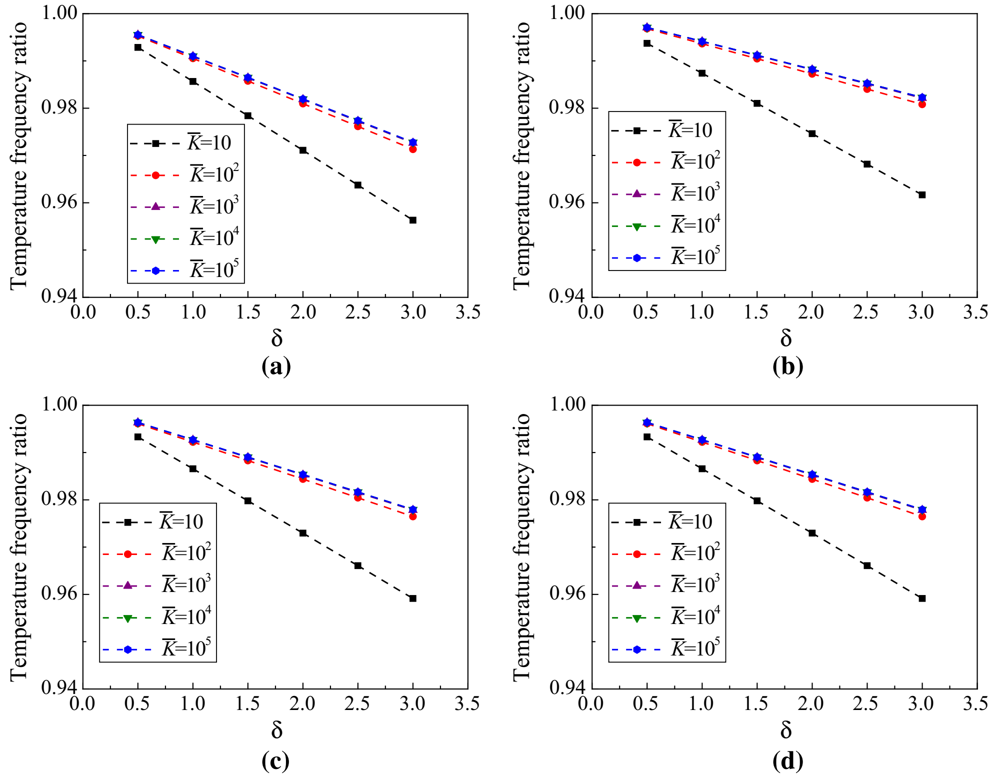

Fig. 6 Effect of the thermal effect parameter δ on the temperature frequency ratios for the strain gradient beams (the nonlocal effect parameter $\mu = 0.1$, the translational spring stiffness $\bar{k} = 10^{10}$). **a** BC1, **b** BC2, **c** BC3, and **d** BC4

constants $c_i(i = 1, 2, \ldots, 6)$ is nonzero. Without loss of generality, we assume that $c_6 \neq 0$. The constants $c_i(i = 1, 2, \ldots, 5)$ are determined as follows:

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_5 \end{pmatrix} = -c_6 \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ a_{21} & a_{22} & \cdots & a_{25} \\ \vdots & \vdots & \vdots & \vdots \\ a_{51} & a_{52} & \cdots & a_{55} \end{pmatrix}^{-1} \begin{pmatrix} a_{16} \\ a_{26} \\ \vdots \\ a_{56} \end{pmatrix}. \quad (22)$$

During the numerical calculation, the square matrix of fifth order in Eq. (22) is nonsingular. Then, the mode shapes of vibration of the strain gradient beam are obtained by using Eq. (19).

4 Numerical results and discussion

In this section, the vibrations of the strain gradient beams with elastic boundaries are investigated. The classical boundary conditions can be considered as special cases of elastic boundary conditions. For a hinged edge (H), the dimensionless stiffness value of the translational spring is infinitely large (set as 1.0×10^{10}), but the dimensionless stiffness value of the rotational spring is set to be 0. For a clamped supported edge (C), the dimensionless stiffness values of the translational and rotational boundary springs are infinitely large (set as 1.0×10^{10}). For a free edge (F), the dimensionless stiffness values of both the translational springs and the rotational boundary springs are set to be 0. To illustrate the efficiency and accuracy of the analytical solutions, for these four classical boundary cases of BC1 to BC4, the first six dimensionless natural frequencies of strain gradient beams with the nonlocal effect parameter $\mu = 0.1$ and the thermal effect parameter $\delta = 0$ are compared with the results reported by Artan and Batra [74] in Tables 1, 2, 3, and 4. In Table 1, the exact solution of the dimensionless natural frequencies of the hinged supported strain gradient beam for the case of BC2 where the boundary conditions are $W\big|_{\xi=0} = 0$, $\frac{\partial^2 W}{\partial \xi^2}\big|_{\xi=0} = 0$, $\frac{\partial^4 W}{\partial \xi^4}\big|_{\xi=0} = 0$, $W\big|_{\xi=1} = 0$, $\frac{\partial^2 W}{\partial \xi^2}\big|_{\xi=1} = 0$, and $\frac{\partial^4 W}{\partial \xi^4}\big|_{\xi=1} = 0$ can be

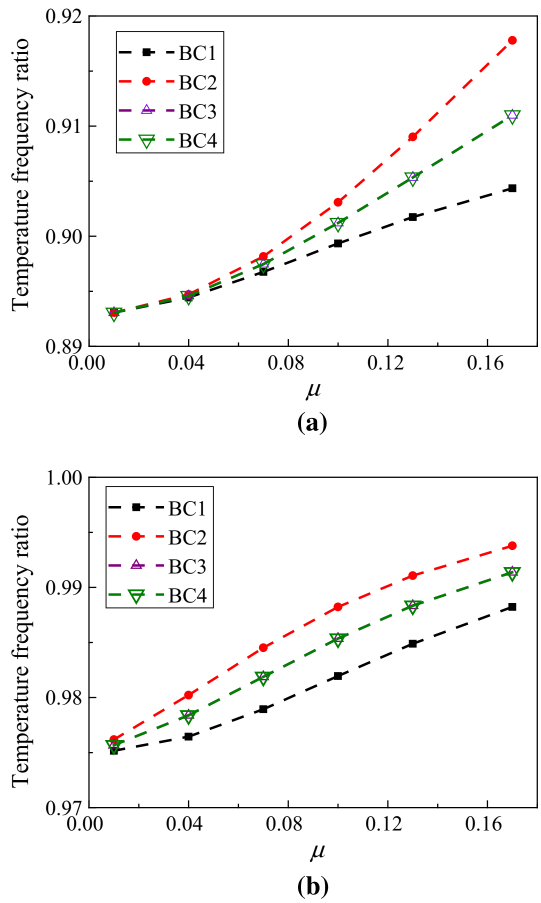

Fig. 7 Effect of the nonlocal effect parameters on the temperature frequency ratios of the strain gradient beams with the thermal effect parameter $\delta = 2$. **a** H–H and **b** C–C

easily obtained from Eq. (13), by assuming $W(\xi) = \sum_m^\infty A_m \sin m\pi\xi$. The dimensionless natural frequency is

$$\Omega = m\pi\sqrt{\mu^2(m\pi)^4 + (m\pi)^2 - \delta} \tag{23}$$

where m is the half-wave number. The present results are the same as the exact solution for the hinged supported case and agree well with available results obtained by the transfer matrix method [74]. It validates that the analytical solutions have very good accuracy. Meanwhile, it can also be observed that the dimensionless natural frequencies of H–H ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 0$, $\bar{K}_1 = 0$), C–C ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 10^{10}$), C–F ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 0$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 0$), and C–H ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 0$) are somewhat different for the four possible cases of BC1 to BC4, respectively. That is, the high-order boundary conditions have slight effect on the dimensionless natural frequencies of the strain gradient beams.

To evaluate the effects of the boundary conditions on the natural frequencies of beams, the effects of rotational spring stiffness on the dimensionless natural frequencies of the strain gradient beams with different high-order boundaries are shown in Fig. 2. For simplicity, it is assumed that the translational spring stiffness and rotational spring stiffness at both ends are the same, i.e., $\bar{k}_0 = \bar{k}_1 = \bar{k}$ and $\bar{K}_0 = \bar{K}_1 = \bar{K}$, in the following discussion. Here, the nonlocal effect parameter $\mu = 0.05$, the translational spring stiffness $\bar{k} = 10^{10}$, and the thermal effect parameter $\delta = 1$. It can be seen that the dimensionless natural frequencies increase with the increase of the rotational spring stiffness. Specially, the dimensionless natural frequencies increase rapidly in a certain range of 10–10^3. The dimensionless natural frequencies have little change, when the rotational spring stiffness is beyond 10^3. It is obvious that their natural frequencies are nearly the same for the boundary conditions of BC3 and BC4. The reason is that high-order boundaries of BC3 and BC4 are nearly equivalent when the boundary constraint spring stiffness at both ends keeps the same. Also, the dimensionless natural frequencies of BC3 and BC4 are larger than those of BC1, and less than those of BC2. It indicates that the

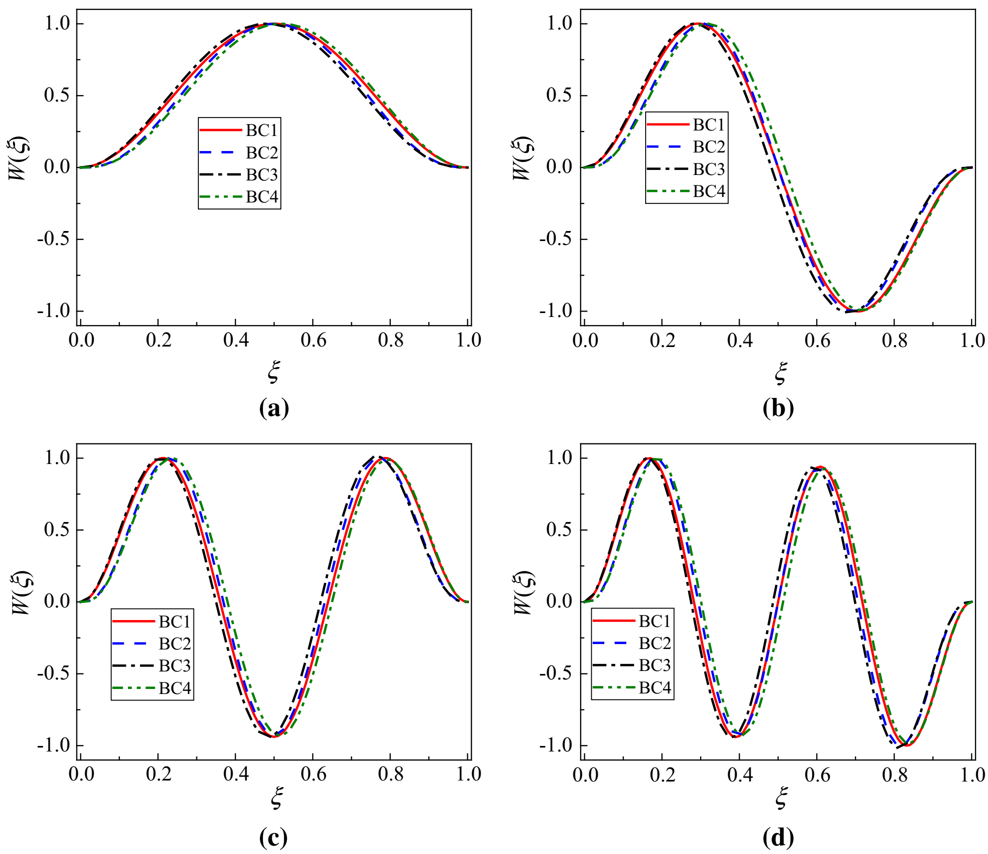

Fig. 8 The first four-order modal shapes of the strain gradient beams with C–C boundary condition (the nonlocal effect parameter $\mu = 0.05$, the thermal effect parameter $\delta = 1$). **a** mode = 1, **b** mode = 2, **c** mode = 3, and **d** mode = 4

boundary conditions BC3 and BC4 are the softer than BC2, and stiffer than BC1. Furthermore, the differences of these natural frequency values become more obvious for the higher-order mode.

To investigated the influence of the thermal effect on the vibrations of the strain gradient beams, the temperature frequency ratio is defined as $\Omega/\bar{\Omega}$. Here, Ω and $\bar{\Omega}$ are the dimensionless natural frequencies of the free vibrational beam including and excluding thermal stress, respectively. The effect of the translational spring stiffness on the temperature frequency ratios of the strain gradient beams with the rotational spring stiffness $\bar{K} = 0$ and the nonlocal effect parameter $\mu = 0.01$ is shown in Fig. 3. It is observed from Fig. 3 that the temperature frequency ratios of the first-order vibrational mode increase as the translational spring stiffness decreases. The temperature frequency ratios change rapidly as the dimensionless translational spring stiffness changes at the range of 10–10^3. Particularly, the temperature frequency ratios change considerably for larger thermal effect parameter δ.

Figure 4 illustrates effect of rotational spring stiffness on the temperature frequency ratios of the strain gradient beams for the translational spring stiffness $\bar{k} = 10^{10}$. The temperature frequency ratios of the first-order vibrational mode increase as the rotational spring stiffness increases. Opposite change tendencies of the temperature frequency ratios can be seen between Figs. 3 and 4. Additionally, the temperature frequency ratios change rapidly as the dimensionless rotational spring stiffness is at the range of 10–10^3. The changing is more considerable for larger thermal effect parameter δ.

The influences of the thermal effect parameter δ on the vibration of strain gradient beams with different boundary translational spring stiffnesses are shown in Fig. 5. The temperature frequency ratios of the first-order vibrational mode decrease with the increasing of δ for a nonlocal effect parameter $\mu = 0.1$. Figure 5 manifests that the translational spring stiffness $\bar{k}$ almost has no effect on the temperature frequency ratios for $\bar{k} > 10^3$. Figure 6 depicts the changes of the temperature frequency ratios of the first-order vibrational mode of strain gradient beams with different boundary rotational spring stiffness versus thermal effect parameters δ for the

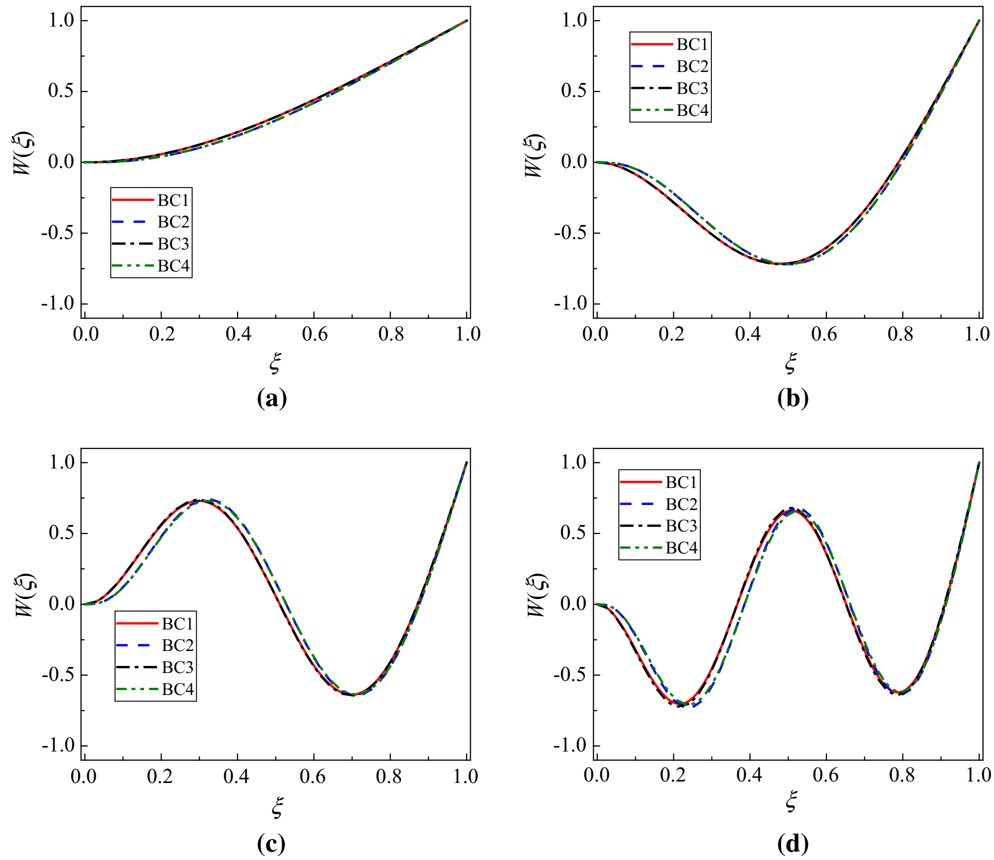

Fig. 9 The first four-order modal shapes of the strain gradient beams with C–F boundary condition (the nonlocal effect parameter $\mu = 0.05$, the thermal effect parameter $\delta = 1$). **a** mode = 1, **b** mode = 2, **c** mode = 3 and **d** mode = 4

nonlocal effect parameter $\mu = 0.1$. It is found that the rotational spring stiffness $\bar{K}$ almost has no effect on the temperature frequency ratios for $\bar{K} > 10^2$.

To investigate the influence of the nonlocal effect parameter μ on the vibrations of strain gradient beams, the temperature frequency ratios of the first-order vibrational mode for the strain gradient beams with H–H ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 0$, $\bar{K}_1 = 0$) and C–C ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 10^{10}$) boundary conditions (the thermal effect parameter $\delta = 2$) are illustrated in Fig. 7. It can be seen that the temperature frequency ratios increase as the nonlocal effect parameter increases. In addition, an obvious difference of the temperature frequency ratios is also observed among the BC1, BC3 and BC4, BC2.

In Figs. 8 and 9, the first four-order modal shapes of the strain gradient beams with C–C ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 10^{10}$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 10^{10}$) and C–F ($\bar{k}_0 = 10^{10}$, $\bar{k}_1 = 0$, $\bar{K}_0 = 10^{10}$, $\bar{K}_1 = 0$) boundary conditions (the nonlocal effect parameter $\mu = 0.05$, thermal effect parameter $\delta = 1$) are obtained, respectively. As can be seen from these figures, the high-order boundary conditions have slight effect on the modal shapes of the strain gradient beams.

5 Concluding remarks

In this paper, the thermal vibration of strain gradient beams with elastic boundaries is investigated based on the strain gradient Euler beam model. A sixth-order differential equation for the beam and elastic boundary condition are derived via Hamilton's principle. To predict accurate vibrational behaviors, the natural frequency equations and the modal shapes for free vibration analysis of the beams subjected to axial thermal stress are obtained analytically. The results show that natural frequencies including the thermal stress are lower than those without the thermal stress when the temperature is rising. The constraining stiffness has significant effects on the temperature ratios when the dimensionless translational spring stiffness or rotational spring stiffness is at the range from 10 to 10^3. The boundary conditions BC3 and BC4 are softer than BC2, and stiffer than BC1. In addition, the thermal effect parameter has an important influence on the vibration of the strain gradient beams.

Appendix A

The roots of the normal cubic algebraic equation:

$$\begin{cases} B=C=0, \quad X_1=X_2=X_3=\frac{-b}{3a}=\frac{-c}{b}=\frac{-3d}{c}, \\ B\neq 0 \; or \; C\neq 0 \begin{cases} \Delta>0, \quad \begin{cases} X_1=\frac{-b-\sqrt[3]{Y_1}-\sqrt[3]{Y_2}}{3a}, \\ X_{2,3}=\frac{-2b+\sqrt[3]{Y_1}+\sqrt[3]{Y_2}\pm\sqrt{3}\left(\sqrt[3]{Y_1}-\sqrt[3]{Y_2}\right)i}{6a}, \end{cases} \\ \Delta=0, \quad X_1=\frac{-b}{a}+\frac{C}{B},\; X_2=X_3=-\frac{1}{2}\frac{C}{B}, \\ \Delta<0, \quad X_1=\frac{-b-2\sqrt{B}\cos\frac{\theta}{3}}{3a},\; X_{2,3}=\frac{-b+\sqrt{B}\left(\cos\frac{\theta}{3}\pm\sqrt{3}\sin\frac{\theta}{3}\right)}{3a} \end{cases} \end{cases} \tag{A.1}$$

where $Y_{1,2}=Bb+3a\left(\frac{-C\pm\sqrt{C^2-4BF}}{2}\right)$, $i^2=-1$, $\theta=\arccos\frac{2Bb-3aC}{2\sqrt{B^3}}$.

Appendix B

The fundamental solutions of Eq. (13) are expressed as

Case1: for $B=C=0$:

$$\begin{cases} X_1=0, & t_1=1, t_2=\xi, t_3=\xi^2, t_4=\xi^3, t_5=\xi^4, t_6=\xi^5, \\ X_1>0, & t_1=e^{(\xi\sqrt{X_1})}, t_2=\xi e^{(\xi\sqrt{X_1})}, t_3=\xi^2 e^{(\xi\sqrt{X_1})}, t_4=e^{(-\xi\sqrt{X_1})}, \\ & t_5=\xi e^{(-\xi\sqrt{X_1})}, t_6=\xi^2 e^{(-\xi\sqrt{X_1})}, \\ X_1<0, & t_1=\cos(\xi\sqrt{-X_1}), t_2=\xi\cos(\xi\sqrt{-X_1}), t_3=\xi^2\cos(\xi\sqrt{-X_1}), \\ & t_4=\sin(\xi\sqrt{-X_1}), t_5=\xi\sin(\xi\sqrt{-X_1}), t_6=\xi^2\sin(\xi\sqrt{-X_1}), \end{cases} \tag{B.1}$$

Case 2: for $B\neq 0$ or $C\neq 0$:

$$\Delta>0 \begin{cases} X_1=0, & t_1=1, t_2=\xi, \\ X_1>0, & t_1=e^{(\xi\sqrt{X_1})}, t_2=e^{(-\xi\sqrt{X_1})}, \\ X_1<0, & t_1=\cos(\xi\sqrt{-X_1}), t_2=\sin(\xi\sqrt{-X_1}), \\ & t_3=e^{[\mathrm{Re}(\sqrt{X_2})\xi]}\cos[|\mathrm{Im}(\sqrt{X_2})|\xi], t_4=e^{[\mathrm{Re}(\sqrt{X_2})\xi]}\sin[|\mathrm{Im}(\sqrt{X_2})|\xi], \\ & t_5=e^{[-\mathrm{Re}(\sqrt{X_2})\xi]}\cos[|\mathrm{Im}(\sqrt{X_2})|\xi], t_6=e^{[-\mathrm{Re}(\sqrt{X_2})\xi]}\sin[|\mathrm{Im}(\sqrt{X_2})|\xi], \end{cases} \tag{B.2}$$

$$\Delta=0 \begin{cases} X_1=0, & t_1=1, t_2=\xi, \\ X_1>0, & t_1=e^{(\xi\sqrt{X_1})}, t_2=e^{(-\xi\sqrt{X_1})}, \\ X_1<0, & t_1=\cos(\xi\sqrt{-X_1}), t_2=\sin(\xi\sqrt{-X_1}), \\ X_2>0, & t_3=e^{(\xi\sqrt{X_2})}, t_4=e^{(-\xi\sqrt{X_2})}, t_5=\xi e^{(\xi\sqrt{X_2})}, t_6=\xi e^{(-\xi\sqrt{X_2})}, \\ X_2<0, & t_3=\cos(\xi\sqrt{-X_2}), t_4=\sin(\xi\sqrt{-X_2}), \\ & t_5=\xi\cos(\xi\sqrt{-X_2}),\; t_6=\xi\sin(\xi\sqrt{-X_2}), \end{cases} \tag{B.3}$$

$$\Delta<0 \begin{cases} X_1=0, & t_1=1, t_2=\xi, \\ X_1>0, & t_1=e^{(\xi\sqrt{X_1})}, t_2=e^{(-\xi\sqrt{X_1})}, \\ X_1<0, & t_1=\cos(\xi\sqrt{-X_1}), t_2=\sin(\xi\sqrt{-X_1}), \\ X_2=0, & t_3=1, t_4=\xi, \\ X_2>0, & t_3=e^{(\xi\sqrt{X_2})}, t_4=e^{(-\xi\sqrt{X_2})}, \\ X_2<0, & t_3=\cos(-\xi\sqrt{X_2}), t_4=\sin(\xi\sqrt{-X_2}), \\ X_3=0, & t_5=1, t_6=\xi, \\ X_3>0, & t_5=e^{(\xi\sqrt{X_3})}, t_6=e^{(-\xi\sqrt{X_3})}, \\ X_3<0, & t_5=\cos(-\xi\sqrt{X_3}), t_6=\sin(\xi\sqrt{-X_3}). \end{cases} \tag{B.4}$$

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