Acta Mech 229, 1773–1781 (2018)
https://doi.org/10.1007/s00707-017-2076-0

ORIGINAL PAPER

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Symmetry and conserved quantities for non-material volumes

Received: 6 June 2017 / Revised: 11 October 2017 / Published online: 23 December 2017

Abstract This paper investigates the Lie symmetry and conserved quantities of non-material volumes. The Lie symmetrical determining equations of the system are presented by introducing the invariance of equations of motion for the system under general infinitesimal transformation of Lie groups. The structure equations and the form of conserved quantities are calculated. And three kinds of conserved quantities, i.e., Noether, Lutzky and Mei conserved quantities of the systems, are derived. In addition, the Hojman conserved quantity of the systems is proposed under the special infinitesimal transformations. An example is given to illustrate the application of the method and result, and four kinds of conserved quantities are obtained under the Lie symmetrical transformations.

1 Introduction

Non-material volumes have flourished in recent years, with important theoretical significance and engineering background. They are greatly used in rocketry, mechanical engineering, civil engineering and fluid–structure interaction [1,2]. Recently, there have been many researchers who have addressed the fundamental principles of the non-material volumes. Irschik and Holl [3] derived the Lagrange's equation of a non-material volume which instantaneously coincides with some part of a continuous and possibly deformable body. Casetta and Pesce [4] established the generalized Hamilton's principle for a non-material volume by introducing Reynolds' transport theorem. They [5] also investigated the inverse problem of Lagrangian mechanics connected to Meshchersky's equation and stated a variational formulation and a Hamiltonian formulation. Casetta [6] reported the inverse problem of Lagrangian mechanics for a non-material volume via the method of Darboux and proposed Hamiltonian formalism and a conservation law. Irschik and Holl [7] proposed a formulation of Lagrange's equations for non-material volumes and calculated local forms and global form of Lagrange's equations in the framework of the Lagrange description of Continuum Mechanics. Casetta et al. [8] developed the generalization of Noether's theorem for a non-material volume and proposed a Noether conserved quantity and the corresponding Killing equations. However, to the authors' knowledge, the Lie symmetry and conserved quantities of the non-material volumes have not been investigated yet.

Symmetries and conserved quantities of dynamical systems have profound theoretical significance and important practical background. There are mainly three kinds of symmetries, i.e., Noether symmetry [9–14], Lie symmetry [15–17] and form invariance [18,19]. In 1918, Noether [9] initiated the Noether symmetry which is the invariance of Hamiltonian actions under infinitesimal transformations group. Djukić and Vujanović [10]

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proposed the Noether's theory in a classical non-conservative system. Vujanovic [11] presented a conservation law of dynamical systems by introducing the variational principles of Jourdain and Gauss. Li and Li [12] derived generalized Noether theorem of non-holonomic system. Luo [13] extended Noether theorem for variable mass high-order non-holonomic mechanical system under a non-inertial reference system. Zhou and Zhang [14] employed Noether's theorems of fractional Birkhoffian mechanics. In 1979, Lutzky [15] presented a new symmetry which is the invariance of equations of motion under Lie group's infinitesimal transformations. Sen and Tabor [16] considered Lie symmetries of the Lorenz model. Mei [17] employed Lie symmetries of constrained systems. In 2000, Mei [18] first reported a new symmetry, i.e., form invariance. Xia et al. [19] promoted the form invariance to non-holonomic mechanical systems. Analogously, there are chiefly the Noether conserved quantity [20–22], Lutzky conserved quantity [23–25], Hojman conserved quantity [26–28], and Mei conserved quantity [29–32]. Fu and Chen [20] reported the Noether conserved quantity of mechanico-electrical system. Xia et al. [21,22] investigated the Noether conserved quantity of a difference equation. Lutzky [23,24] pioneered a new conserved quantity of Lagrangian systems. Fu and Chen [25] extended the Lutzky conserved quantity for a non-conservative dynamical system. Hojman [26] shown a new conserved quantity by using a symmetry transformation vector instead of the Lagrangian or Hamiltonian. Chen et al. [27] considered Hojman conserved quantities of first-order Lagrange systems. Jiang and Luo [28] derived Hojman conserved quantities of generalized Hamiltonian systems. Xia and Chen [29,30] investigated Mei conserved quantities of discrete systems. Zhang et al. [31] described Mei conserved quantities of generalized Hamilton systems with additional terms. Wang and Xue [32] considered Mei conserved quantities of thin elastic rod. So far, to the authors' best knowledge, there is no symmetry analysis on non-material volumes. To address the lack of research in this aspect, the present work develops the Lie symmetry technique to determine the conserved quantities of non-material volumes.

The paper is organized as follows. Section 2 reviews the differential equations of non-material volumes. Section 3 considers the Lie symmetrical definition of non-material volumes and obtains the determining equations of Lie symmetry of non-material volumes. Section 4 presents the Noether conserved quantity of non-material volumes under Lie symmetrical transformations. Section 5 treats the Hojman conserved quantity of the non-material volumes under particular Lie symmetrical transformations. Section 6 proposes the Lutzky conserved quantity of the non-material volumes under general Lie symmetrical transformations. Section 7 reports the Mei conserved quantity of non-material volumes and proposes a necessary and sufficient condition of the Lie symmetry which is a form invariance. Section 8 gives an example to illustrate the application of the method and obtains four kinds of conserved quantities of the system under the Lie symmetrical transformations. Section 9 contains the concluding remarks.

2 The differential equations of the non-material volumes

The Lagrange's equation of non-material volumes is pioneered by Irschik and Holl [2, p. 243, Eq. (5.6)], and can be given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k}-\frac{\partial T_u}{\partial q_k}-\int\limits_{\partial V_u}\frac{1}{2}\rho\upsilon^2\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}-\frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot\boldsymbol{n}d\partial V_u+\int\limits_{\partial V_u}\rho\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}\left(\boldsymbol{v}-\boldsymbol{u}\right)\cdot\boldsymbol{n}d\partial V_u=Q_k, \tag{1}$$

where $T_u = T_u\left(\dot{q}_k, q_k, t\right)$ is the total kinetic energy of the material particles come with non-material volume V_u , q_k represents generalized coordinates, $\boldsymbol{v}$ is the velocity of the material particles, $\boldsymbol{u}$ is the velocity of the fictitious particles, ρ is the volumetric mass density, Q_k is generalized force applied to the material body, $\boldsymbol{n}$ is the outer normal unit vector at the surface of V_u and ∂V_u denotes the bounding surface of V_u.

Suppose that the system is non-singular, i.e.,

$$\Lambda=\det\left(\frac{\partial^2 T_u}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0. \tag{2}$$

Then, the differential equations of the non-material volumes have the form

$$\ddot{q}_s=\alpha_s(t,q,\ldots q),\quad (s=1,2,\ldots,n), \tag{3}$$

where

$$\alpha_s(t, q, \ldots q) = \frac{\Lambda_{ks}}{\Lambda}\left(\frac{\partial T_u}{\partial q_k} + Q_k + Z_k - \frac{\partial^2 T_u}{\partial \dot{q}_k \partial q_r}\dot{q}_r - \frac{\partial^2 T_u}{\partial \dot{q}_k \partial t}\right),$$

$$Z_k = \int_{\partial V_u} \frac{1}{2}\rho \upsilon^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot \boldsymbol{n} d\partial V_u - \int_{\partial V_u} \rho \boldsymbol{v} \frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot \boldsymbol{n} d\partial V_u.$$

3 Lie symmetry of the non-material volumes

Choose the infinitesimal transformations t and q_i of Lie group as the following

$$t^* = t + \Delta t,\ \ q_i^*(t^*) = q_i(t) + \Delta q_i, (i = 1, 2, \ldots, n), \tag{4}$$

and their expanding forms are

$$t^* = t + \varepsilon\tau(t, q_j), q_i^*(t^*) = q_i(t) + \varepsilon\xi_i(t, q_j), \tag{5}$$

where ε is an infinitesimal parameter and τ and ξ_i represent the infinitesimal generators. Take the infinitesimal generator vector

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} \tag{6}$$

and its first extension vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s} \tag{7}$$

and the second extension vector

$$X^{(2)} = X^{(1)} + \left(\ddot{\xi}_s - \dot{q}_s\ddot{\tau} - 2\dot{\tau}\alpha_s\right)\frac{\partial}{\partial \ddot{q}_s}. \tag{8}$$

By using the definition of Lie symmetries [15], the invariance of non-material volumes Eq. (3) under infinitesimal transformation (4) leads to

$$X^{(2)}(\ddot{q}_s) = X^{(1)}(\alpha_s). \tag{9}$$

Substituting Eq. (8) into Eq. (9) yields

$$\ddot{\xi}_s - \dot{q}_s\ddot{\tau} - 2\dot{\tau}\alpha_s = \tau\frac{\partial \alpha_s}{\partial t} + \xi_k\frac{\partial \alpha_s}{\partial q_k} + (\dot{\xi}_k - \dot{q}_k\dot{\tau})\frac{\partial \alpha_s}{\partial \dot{q}_k}. \tag{10}$$

Equation (10) is called the determining equations of Lie symmetry of non-material volumes (3).

If the generators τ and ξ_i of infinitesimal transformations (4) satisfy the determining equation (10), then the corresponding transformations are called the Lie symmetrical transformations of non-material system (3).

4 Lie symmetry and Noether conserved quantity of the non-material volumes

For the non-material volumes, the Lie symmetries can indirectly lead to Noether conserved quantities. The following theorem gives the condition of the existence of conserved quantities indirectly induced by Lie symmetries.

Theorem 1 *For non-material volumes (3), if the generators of infinitesimal transformations (4) satisfy the determining equation (10), and there exists a gauge function $G_{\mathrm{N}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following condition*

$$T_u\dot{\tau} + X^{(1)}(T_u) + (Q_k + Z_k)(\xi_s - \dot{q}_s\tau) + \dot{G}_{\mathrm{N}} = 0, \tag{11}$$

then system (3) furnishes the following Noether conserved quantities:

$$I_{\mathrm{N}} = T_u\tau + \frac{\partial T_u}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\tau) + G_{\mathrm{N}} = \mathrm{const}. \tag{12}$$

Proof Differentiating Eq. (12) with respect to t, and by virtue of Eq. (11), we have

$$
\begin{aligned}
\frac{\mathrm{d}I_{\mathrm{N}}}{\mathrm{d}t} &= \dot{T}_u\tau + T_u\dot{\tau} + \frac{\partial T_u}{\partial \dot{q}_s}\left(\dot{\xi}_s - \ddot{q}_s\tau - \dot{q}_s\dot{\tau}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_s}\left(\xi_s - \dot{q}_s\tau\right) - T_u\dot{\tau} - X^{(1)}\left(T_u\right) \\
&\quad - \left(Q_k + Z_k\right)\left(\xi_s - \dot{q}_s\tau\right) \\
&= \left(\frac{\partial T_u}{\partial t} + \frac{\partial T_u}{\partial q_s}\dot{q}_s + \frac{\partial T_u}{\partial \dot{q}_s}\ddot{q}_s\right)\tau - \frac{\partial T_u}{\partial \dot{q}_s}\ddot{q}_s\tau + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_s}\left(\xi_s - \dot{q}_s\tau\right) - \frac{\partial T_u}{\partial t}\tau \\
&\quad - \frac{\partial T_u}{\partial q_s}\xi_s - \left(Q_k + Z_k\right)\left(\xi_s - \dot{q}_s\tau\right) \\
&= \left(\xi_s - \dot{q}_s\tau\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k} - \frac{\partial T_u}{\partial q_k} - Q_k - Z_k\right) \\
&= 0.
\end{aligned} \tag{13}
$$

Consequently, I_{N} is a conserved quantity.

5 Lie symmetry and Hojman conserved quantity of the non-material volumes

Lie symmetries can directly lead to Hojman conserved quantities under the particular infinitesimal transformations in which time is not varied as

$$t^* = t, \quad q_i^*(t^*) = q_i(t) + \Delta q_i, \quad (i = 1, 2, \ldots, n). \tag{14}$$

And the corresponding determining equations of non-material volumes (3) was reduced to the following form:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k, \tag{15}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}.$$

Theorem 2 *For non-material volumes* (3), *if the generators of particular infinitesimal transformations* (14) *satisfy the reduced Lie symmetrical determining* Eq. (15), *and there exists a function* $\mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying the condition*

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{dt}}\ln \mu = 0, \tag{16}$$

then system (3) *gives the following Hojman conserved quantities,*

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\dot{\xi}_s\right) = \mathrm{const}. \tag{17}$$

Proof Differentiating Eq. (17) with respect to time yields

$$\frac{\bar{\mathrm{d}}I_{\mathrm{H}}}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} + \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}\dot{\xi}_s}{\mathrm{d}t}\right]. \tag{18}$$

We introduce the following relations:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\right) = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial \dot{q}_s}\left(\frac{\partial}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \tag{19}$$

and

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}. \tag{20}$$

Taking the partial derivative of Eq. (15) with respect to $\dot{q}_s$ yields

$$\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right). \tag{21}$$

Substituting Eqs. (19), (20), (21) and (16) into Eq. (18), one has

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_H}{\mathrm{d}t} &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}\right) + \frac{\partial^2 \alpha_s}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{\partial^2 \alpha_s}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \\ &= \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_s}{\partial q_k}\xi_k - \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right] \\ &= 0. \end{aligned} \tag{22}$$

6 Lie symmetry and Lutzky conserved quantity of the non-material volumes

Lie symmetries can also directly lead to Lutzky conserved quantities under generally infinitesimal transformations (4).

Theorem 3 *For non-material volumes (3), if the generators of generally infinitesimal transformations (4) satisfy the Lie symmetrical determining equation (10), and there exists a function* $G_{\mathrm{L}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ *satisfying the following condition*

$$\frac{\partial}{\partial \dot{q}_s}\left[\frac{\Lambda_{sk}}{\Lambda}(Q_k + Z_k)\right] = \frac{\bar{\mathrm{d}}G_{\mathrm{L}}}{\mathrm{d}t}, \tag{23}$$

then system (3) provides the following Lutzky conserved quantities:

$$I_{\mathrm{L}} = 2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \tau}{\partial q_s}\right) - n\frac{\bar{d}\tau}{dt} + X^{(1)}(\ln \Lambda) - X^{(1)}(\ln G_{\mathrm{L}}) = \text{const.}. \tag{24}$$

Proof One can easily demonstrate a relation

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} - \frac{\partial}{\partial \dot{q}_s}\left[\frac{\Lambda_{sk}}{\Lambda}(Q_k + Z_k)\right] + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln \Lambda) = 0. \tag{25}$$

In addition, if τ and ξ_s satisfy Eq. (4), for any function $\Phi(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, Fu and Chen [25, p. 256, Eq. (11)] presented the relationship that

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X^{(1)}(\Phi) = X^{(1)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Phi\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\Phi. \tag{26}$$

From Eq. (15), one can obtain

$$\begin{aligned}&\frac{\partial}{\partial \dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau - X^{(1)}(\alpha_s)\right] \\ &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \tau}{\partial q_s}\right) - n\frac{\bar{\mathrm{d}}\tau}{\mathrm{d}t}\right] - X^{(1)}\left(\frac{\partial \alpha_s}{\partial \dot{q}_s}\right) - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau\frac{\partial \alpha_s}{\partial \dot{q}_s}. \end{aligned} \tag{27}$$

According to Eqs. (25) and (26), using Eq. (23) yields

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_L &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \tau}{\partial q_s}\right) - n\frac{\bar{\mathrm{d}}\tau}{\mathrm{d}t} + X^{(1)}(\ln \Lambda) - X^{(1)}(\ln G_L)\right] \\ &= \frac{\partial}{\partial \dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau - X^{(1)}(\alpha_s)\right] \\ &= 0. \end{aligned} \tag{28}$$

7 Lie symmetry and Mei conserved quantity of the non-material volumes

If the Lie symmetry of the non-material volumes is a form invariance, then the Lie symmetry can lead to a Mei conserved quantity. The necessary and sufficient condition under which the Lie symmetry of the non-material volumes is a form invariance is that the generating functions satisfy the criterion equations of the form invariance

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s} - \frac{\partial X^{(1)}(T_u)}{\partial q_s} = X^{(1)}(Q_k + Z_k). \tag{29}$$

Theorem 4 *For non-material volumes* (3)*, if the generators τ and ξ_i of particular infinitesimal transformations* (14) *satisfy the Lie symmetrical determining equation* (15)*, the criterion equation* (29) *of the form invariance, and there exists a function $G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following condition*

$$X^{(1)}\left\{X^{(1)}(T_u)\right\} + X^{(1)}(T_u)\frac{\bar{d}\tau}{dt} + X^{(1)}(Q_k + Z_k)(\xi_s - \dot{q}_s\tau) + \frac{\bar{\mathrm{d}}G_M}{\mathrm{d}t} = 0, \tag{30}$$

then system (3) *produces the following Mei conserved quantity:*

$$I_{\mathrm{M}} = X^{(1)}(T_u)\,\tau + \frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\tau) + G_{\mathrm{M}} = \mathrm{const.} \tag{31}$$

Proof Differentiating Eq. (31) with respect to time, using Eq. (15) and Eqs. (29–30) yields

$$\begin{aligned}
\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{M}} &= \left[\frac{\partial X^{(1)}(T_u)}{\partial t} + \dot{q}_s\frac{\partial X^{(1)}(T_u)}{\partial q_s} + \alpha_s\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\right]\tau + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\right)(\xi_s - \dot{q}_s\tau) \\
&\quad + X^{(1)}(T_u)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau + \frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \alpha_s\tau - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau\right) - X^{(1)}(T_u)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\tau \\
&\quad - X^{(1)}\left\{X^{(1)}(T_u)\right\} - X^{(1)}(Q_k + Z_k)(\xi_s - \dot{q}_s\tau) \\
&= \left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_s} - \frac{\partial X^{(1)}(T_u)}{\partial q_s} - X^{(1)}(Q_k + Z_k)\right\}(\xi_s - \dot{q}_s\tau) \\
&= 0.
\end{aligned} \tag{32}$$

8 Example

To illustrate the applicability of the Lie symmetry and conserved quantity of Sects. 4–7, an ideal two-dimensional problem of a rotating drum which has been derived by Casetta et al. [8, p. 705, Eq. (67)] is considered, and the governing equation of motion for the system can be described as

$$\frac{1}{2}\rho l\pi\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4\ddot{\phi} = \Pi, \tag{33}$$

where $\Pi = \Pi(\phi)$ is the torque acting upon the control volume, the corresponding kinetic energy is

$$T_u = \frac{1}{4}\rho l\pi\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4\dot{\phi}^2, \tag{34}$$

the flux of kinetic energy is

$$\int\limits_{\partial V_u}\frac{1}{2}\rho\upsilon^2\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot\boldsymbol{n}d\partial V_u = \frac{1}{2}\rho\varepsilon l\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3\dot{\phi}^2, \tag{35}$$

and the flux of linear momentum of the systems is

$$\int\limits_{\partial V_u}\rho\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u})\cdot\boldsymbol{n}d\partial V_u = \rho\varepsilon l\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3\dot{\phi}^2. \tag{36}$$

It is noted that ε represents the thickness of a thin strip, l its width, ρ describes mass density, $\phi = \phi(t)$ is the rotation angle, R_0 is the original radius, V_u describes the control volume, ∂V_u is the control surface, v is the velocity of the material particles instantaneously included in V_u, and $\partial V_u = \varepsilon l$.

According to the definition of Lie symmetries, the Lie symmetrical determining equation (10) gives

$$\ddot{\xi}_1 - \dot{\phi}\ddot{\tau} - 2\dot{\tau}\frac{2\Pi}{\rho l\pi}\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-4} = \xi_1\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-5}\left[\frac{2}{\rho l\pi}\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)\frac{\partial\Pi}{\partial\phi} + \frac{4\varepsilon\Pi}{\rho l\pi^2}\right]. \tag{37}$$

Due to $\phi = \phi(t)$ being an implicit function, it is difficult to find the exact solution of Eq. (37). We can find an approximate series solution of Eq. (37) as

$$\tau = 1, \quad \xi_1 = \sum_{i,j=0}^{n} a_{ij}\phi^i t^j. \tag{38}$$

Note that Eq. (68) of Casetta et al. [8] can be simplified as

$$\begin{aligned}\dot{\phi}^2 &= C + \frac{4}{\rho l\pi}\int\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-4}\Pi(\phi)\,\dot{\phi}\mathrm{d}t \\ &= C + \frac{4}{\rho l\pi}\int\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-4}\Pi(\phi)\,\mathrm{d}\phi \\ &= \Psi(\phi)\,.\end{aligned} \tag{39}$$

Substituting Eqs. (33), (38) and (39) into Eq. (37) and equating coefficients of t and ϕ in the resulting equations obtain the coefficients a_{ij}.

Theorem 1 gives the Noether conserved quantity as follows:

$$I_{\mathrm{N}} = \frac{1}{4}\rho l\pi\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^4\dot{\phi}^2 + G_{\mathrm{N}} = \text{const.}, \tag{40}$$

where

$$G_{\mathrm{N}} = \int\left[\frac{1}{2}\varepsilon\rho l\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3\dot{\phi}^2\xi_1 + \left(\Pi + \int\frac{1}{2}\varepsilon\rho l\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^3\dot{\phi}^2\right)\left(\xi_1 - \dot{\phi}\right)\right]\mathrm{d}t.$$

Theorem 2 presents the Hojman conserved quantity as

$$I_{\mathrm{H}} = \frac{\xi_1}{\mu}\frac{\partial\mu}{\partial\phi} + \frac{\partial\xi_1}{\partial\phi} + \frac{\dot{\xi}_1}{\mu}\frac{\partial\mu}{\partial\dot{\phi}} + \frac{\partial\dot{\xi}_1}{\partial\dot{\phi}} = \text{const.}, \tag{41}$$

where

$$\mu = \exp\left[\frac{1}{4}\rho l\pi\dot{\phi}^2 - \int\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-4}\Pi\mathrm{d}\phi\right].$$

Theorem 3 leads to the Lutzky conserved quantity as follows:

$$I_{\mathrm{L}} = 2\frac{\partial\xi_1}{\partial\phi} + X^{(1)}(\ln\Lambda) - X^{(1)}(\ln G_L) = \text{const.}, \tag{42}$$

where

$$G_{\mathrm{L}} = \frac{1}{4}\rho l\pi\dot{\phi}^2 - \int\left(R_0 - \frac{\varepsilon\phi}{2\pi}\right)^{-4}\Pi\mathrm{d}\phi.$$

Theorem 4 produces the Mei conserved quantity:

$$I_{\mathrm{M}} = X^{(1)}(T_u) + \frac{\partial X^{(1)}(T_u)}{\partial\dot{\phi}}\left(\xi_1 - \dot{\phi}\right) + G_{\mathrm{M}} = \text{const.}, \tag{43}$$

where

$$G_{\mathrm{M}} = \int \left\{ X^{(1)} \left[X^{(1)} (T_u) \right] + X^{(1)} \left(\Pi - \frac{1}{2} \varepsilon \rho l \left(R_0 - \frac{\varepsilon \phi}{2\pi} \right)^3 \right) (\xi_1 - \dot{\phi}) \right\} \mathrm{d}t.$$

Let

$$\xi_1 = 0. \tag{44}$$

Since α does not explicitly depend on time and $\dot{\phi}$, the determining equation (10) is reformatted in the following form

$$-\dot{\phi}\ddot{\tau} - \Xi \dot{\tau} = 0. \tag{45}$$

So one particular solution can be written as

$$\tau = \int e^{-\int \frac{\Xi}{\dot{\phi}} \mathrm{d}t} \mathrm{d}t, \tag{46}$$

where

$$\Xi = \frac{4\Pi}{\rho l \pi} \left(R_0 - \frac{\varepsilon \phi}{2\pi} \right)^{-4}.$$

Then, corresponding conserved quantities (40), (42) and (43) will also be given as the following: Noether conserved quantity

$$I_N = T_u \tau - \frac{\partial T_u}{\partial \dot{\phi}} \dot{\phi} \tau + G_N = \text{const.}, \tag{47}$$

where

$$G_N = \int \left[\Pi \dot{\phi} \tau - T_u \dot{\tau} - X^{(1)} (T_u) \right] \mathrm{d}t,$$

Lutzky conserved quantity

$$I_L = -2\dot{\phi} \frac{\partial \tau}{\partial \phi} - n \frac{\bar{\mathrm{d}}\tau}{\mathrm{d}t} + X^{(1)} (\ln \Delta) - X^{(1)} (\ln G_L) = \text{const.}, \tag{48}$$

where

$$G_L = \int \frac{\partial}{\partial \dot{\phi}} \left(\frac{\Delta_{sk}}{\Delta} \Pi \right) \mathrm{d}t,$$

and Mei conserved quantity

$$I_M = X^{(1)} (T_u) \tau - \frac{\partial X^{(1)} (T_u)}{\partial \dot{\phi}} \dot{\phi} \tau + G_M = \text{const.}, \tag{49}$$

where

$$G_M = -\int \left\{ X^{(1)} \left[X^{(1)} (T_u) \right] + X^{(1)} (T_u) \frac{\bar{\mathrm{d}}\tau}{\mathrm{d}t} - X^{(1)} (\Pi) \dot{\phi} \tau \right\} \mathrm{d}t.$$

9 Conclusions

This paper focuses on the Lie symmetry and conserved quantity of non-material volumes under Lie symmetrical transformations. The determining equations of Lie symmetry of non-material volumes are derived by employing the definition of Lie symmetry. Noether, Hojman, Lutzky and Mei conserved quantities of the non-material volumes are calculated under Lie symmetrical transformations. The proof of the theorem on four kinds of conserved quantities is given. The example of a rotating drum uncoiling a strip is employed to illustrate the applicability of the Lie symmetry of non-material volumes, and four kinds of new conserved quantities are proposed.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Nos. 11702119, 51779109 and 11502071), the Natural Science Foundation of Jiangsu Province (Nos. BK20170565 and BK20171306) and the Innovation Foundation of Jiangsu University of Science and Technology (1012931609 and 1014801501-6).

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