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ORIGINAL PAPER

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A weighted dual criterion of the equivalent linearization method for nonlinear systems subjected to random excitation

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Abstract This paper introduces and discusses a weighted form of the dual mean-square error criterion by considering weighted contributions of forward and return replacements adjusted by a specific non-dimensional weight coefficient. The simplicity and accuracy of the proposed weighted dual equivalent linearization technique are checked on several random vibration systems in comparison with other criteria. It is shown that the corresponding accuracy can be significantly improved for a large range of nonlinearity of investigated vibration systems.

1 Introduction

Most phenomena in our world are essentially nonlinear and described by nonlinear equations. We might simplify nonlinear phenomena as linear ones to make them easier to understand; however, for further investigation nonlinear phenomena should be treated as nonlinear problems. Thus, the study of nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and in other disciplines. In particular, it appears that the analysis of vibration based on nonlinear mathematical models requires appropriate methods. In the theory of random vibration, the stochastic equivalent linearization method that replaces a nonlinear system by an equivalent linear one is a popular method since it preserves some essential properties of the original nonlinear system. The method has been described in numerous review articles [1,2], and was summarized in the monographs by Roberts and Spanos [3] and Socha [4]. About the efficiency and versatility of the stochastic linearization method, Elishakoff and Crandall have written: It allows obtaining estimates of the response of the system when the exact solution is unavailable; in contrast to the perturbation technique, its realization does not demand smallness of the parameter; on the other hand, unlike the Monte Carlo simulation, it does not involve extensive computational cost. Although its accuracy may be not very high, this is remedied by the fact that the stochastic excitation itself need not be known quite precisely [2]. Canor et al. [5] also have written: Owing to its accessible implementation and rapidity, the equivalent linearization has become a common probabilistic approach for the analysis of large-dimension nonlinear structures.

The equivalent linearization method has been used in many research papers. An equivalent linearization-based analytical approach is developed in [6,7] for the analysis of nonlinear energy harvesters under random

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excitations. The nonlinear flutter system of an airfoil is investigated in [8,9] using the equivalent linearization method. An approach based on the equivalent linearization method is proposed in [10] to determine the approximate analytical solution for the H_∞ optimization of the three-element dynamic vibration absorber attached to the damped primary structure. Jalali proposed in [11] a method for equivalent linearization of nonlinear restoring forces where numerical investigations revealed that the proposed method is efficient in the analysis of weakly nonlinear hysteretic systems. Silva-Gonzlez et al. [12] have used the stochastic equivalent linearization method for investigating inelastic nonlinear structural systems subjected to seismic ground motions. Su et al. [13] have developed an efficient procedure based on the equivalent linearization method for nonlinear structures subjected to nonstationary random excitations. The technique of equivalent linearization is used in [14] to investigate the thermal radiation of small satellites with a single-node model. A new equivalent linearization method using a Gaussian mixture distribution model has been developed in [15] for nonlinear random vibration analysis.

The essential of the equivalent linearization method is how to find the linearization coefficients for a given nonlinear system. In the literature, several criteria of equivalent linearization have been suggested to define the linearization coefficients where the original version is the conventional criterion [16] that minimizes the mean-square of equation error. Despite the aforementioned advantages, the main disadvantage of this criterion is that its accuracy decreases as the nonlinearity is increasing, in many cases it results in unacceptable errors. Hence, other different equivalent linearization criteria have been developed in order to improve the accuracy of an approximate solution, see for example [1,4,17,18]. Recently, a dual criterion was proposed in [19,20] where the linearization coefficients are determined from the dual replacement involving forward and return replacements. Application of the dual criterion to three nonlinear systems, namely Duffing, Van der Pol and Lutes–Sarkani oscillators, has shown an improved accuracy of the approximate solutions for cases where the nonlinearity is of intermediate level [20,21]. A possible reason may be the fact that the forward and return replacements would have different roles in adjusting the replacement error rather than being the same. In [22,23], a weighted dual criterion is investigated by considering the weight parameter as a piecewise linear function of the squared correlation coefficient which is defined by the interpolation method of least squares from available exact solutions of several nonlinear restoring oscillators. The main restriction of the piecewise linear weight parameter is that it represents only a limited class of nonlinear oscillators. In this paper, we therefore develop a more sophisticate form of the weight parameter. The simplicity and accuracy of the proposed weighted dual error criterion are then checked on several random vibration systems with nonlinear restoring or nonlinear damping. It is obtained that the corresponding accuracy can be significantly improved for a large range of nonlinearity of investigated vibration systems.

2 Weighted dual criterion

We consider firstly the problem of equivalent replacement: Given random functions A and B with zero mean and A is replaced by kB where k is an equivalent replacement coefficient to be found from an equivalent replacement criterion. When the function B is considered as linear one, the equivalent replacement is known as equivalent linearization. Let us denote:

$$r^2 = \frac{\langle AB\rangle^2}{\langle A^2\rangle\langle B^2\rangle} \tag{1}$$

where $\langle\cdot\rangle$ is the expectation operator. One notes that the parameter r^2 is precisely the squared correlation coefficient that is used as a measure of the linear dependence between two functions A, B [24]. Following the Cauchy–Schwarz inequality, one has

$$r^2 \leq 1, \tag{2}$$

and $r^2 = 1$ when A is proportional to B, $A = \alpha B, \alpha = \text{const}$. It has been observed that the conventional equivalent replacement has a one-way sense. Indeed, the conventional replacement of A by kB can be considered as the forward replacement from the original function A to its counterpart B and illustrated schematically as follows:

$$A \to kB. \tag{3}$$

For the conventional mean-square error criterion [16], the equivalent replacement coefficient k is found from the condition:

$$e_{\mathrm{f}} \equiv \left\langle (A - kB)^2 \right\rangle \to \min_{k} \tag{4}$$

where the index "$_{\mathrm{f}}$" denotes the forward replacement. The criterion (4) gives the forward equivalent replacement coefficient:

$$k_{\mathrm{f}} = \frac{\langle AB \rangle}{\left\langle B^2 \right\rangle}. \tag{5}$$

Supposing now $k_{\mathrm{f}} B$ is found one gets back to A using the return replacement

$$k_{\mathrm{f}} B \to \lambda A \tag{6}$$

where the return coefficient λ is found from the corresponding mean-square replacement error criterion:

$$e_{\mathrm{r}} \equiv \left\langle (kB - \lambda A)^2 \right\rangle \to \min_{\lambda} \tag{7}$$

where $k = k_{\mathrm{f}}$. Using (7), (5), and (1), one obtains

$$\lambda_{\mathrm{r}} = k_{\mathrm{f}} \frac{\langle AB \rangle}{\left\langle A^2 \right\rangle} = \frac{\langle AB \rangle^2}{\left\langle A^2 \right\rangle \left\langle B^2 \right\rangle} = r^2. \tag{8}$$

The index "$_{\mathrm{r}}$" denotes the return replacement. The minimal mean-square errors of forward and return replacements can be calculated using (1), (4), (5), (7), (8), respectively,

$$e_{\mathrm{f\,min}} = \left\langle (A - k_{\mathrm{f}} B)^2 \right\rangle = \left\langle A^2 \right\rangle - 2k_{\mathrm{f}} \langle AB \rangle + k_{\mathrm{f}}^2 \left\langle B^2 \right\rangle = (1 - r^2) \left\langle A^2 \right\rangle , \tag{9}$$

$$e_{\mathrm{r\,min}} = \left\langle (k_{\mathrm{f}} B - \lambda_r A)^2 \right\rangle = k_{\mathrm{f}}^2 \left\langle B^2 \right\rangle - 2k_{\mathrm{f}} \lambda_{\mathrm{r}} \langle AB \rangle + \lambda_{\mathrm{r}}^2 \left\langle A^2 \right\rangle = \mathrm{r}^2 (1 - \mathrm{r}^2) \left\langle A^2 \right\rangle . \tag{10}$$

It is observed from (9) and (10) that the squared correlation coefficient r^2 can play a key role in the equivalent replacement problem since it appears naturally in minimal square errors of forward and return replacements.

In order to improve the accuracy of the equivalent replacement coefficient, the dual approach was recently proposed in [19] and has been developed in [20–23,25]. One of the significant advantages of the dual approach is its consideration of two different aspects of a problem in question allowing the investigation to be more appropriate. Application of the dual approach to the problem of replacement means this problem should combine two forward and return replacements (3) and (6) in a weighted dual replacement:

$$A \to k_{\mathrm{w}} B \to \lambda_{\mathrm{w}} A \tag{11}$$

characterized by a weighted dual mean-square error criterion [22,23]:

$$e_{\mathrm{w}} \equiv (1 - p) e_{\mathrm{f}} + p e_{\mathrm{r}} = (1 - p) \left\langle (A - k_{\mathrm{w}} B)^2 \right\rangle + p \left\langle (k_{\mathrm{w}} B - \lambda_{\mathrm{w}} A)^2 \right\rangle \to \min_{k_{\mathrm{w}}, \lambda_{\mathrm{w}}} \tag{12}$$

where the index "$_{\mathrm{w}}$" denotes the weighted dual replacement, p is a non-dimensional weight parameter which plays a role in adjusting the contributions of forward and return replacements in order to obtain the best replacement of A by kB. When $p = 0$ the criterion (12) leads to the conventional mean-square error criterion [16]; and when $p = 1/2$ it leads to the dual mean-square error criterion investigated in [20]. An expression of the weight parameter p is proposed in [22,23] as a piecewise linear function of r^2. In this paper, we consider a more sophisticated form of the weight parameter p satisfying the condition:

$$0 \le p \le 1. \tag{13}$$

Given a weight parameter p, the criterion (12) yields minimum necessary and sufficient conditions,

$$\frac{\partial e_{\mathrm{w}}}{\partial k_{\mathrm{w}}} = 0, \quad \frac{\partial e_{\mathrm{w}}}{\partial \lambda_{\mathrm{w}}} = 0,$$

which gives k_{w} and λ_{w}, respectively,

$$k_w = (1 - p + p\lambda_{\mathrm{w}}) \frac{\langle AB \rangle}{\langle B^2 \rangle}, \ \lambda_{\mathrm{w}} = k_{\mathrm{w}} \frac{\langle AB \rangle}{\langle A^2 \rangle}. \tag{14}$$

Equation (14) shows that the return replacement can contribute to the equivalent replacement coefficient k_{w} through the return coefficient λ_{w} that plays a complementary role. Solving (14), one obtains:

$$k_{\mathrm{w}} = \frac{1-p}{1-pr^2} \frac{\langle AB \rangle}{\langle B^2 \rangle}, \ \lambda_{\mathrm{w}} = \frac{1-p}{1-pr^2} r^2. \tag{15}$$

Thus, it is shown that if the weight parameter p is known, using the weighted dual mean-square error criterion (12), the original function A can be replaced by $k_{\mathrm{w}} B$ where the equivalent replacement coefficient k_{w} can be found from (15). The squared correlation coefficient r^2 also appears naturally in both k_{w} and λ_{w}. Thus, the main problem is now reduced to how the weight parameter p can be chosen. This problem is complicated and needs an intensive study. Here we use an empirical research approach based on actual observations and experiences that help to suggest a formula for the weight parameter. The validity of the formula will be tested by different examples. For this purpose, firstly it is observed that the return replacement e_{r} is introduced in (12) due to that the minimal mean-square forward error determined by (9) differs from zero. Hence, one assumes that the weight parameter p can be a function of $e_{\mathrm{f\,min}}$, and a linear relation is of the first choice:

$$p = \eta e_{\mathrm{f\,min}} \tag{16}$$

where η is a dimensional coefficient which makes p non-dimensional satisfying (13). Substituting (9) into (16) yields

$$p = \eta(1 - r^2) \langle A^2 \rangle. \tag{17}$$

The further discussion has large dispersion. Here a possible inference is used as follows. The original function A is given, and its counterpart $k_{\mathrm{f}} B$ has been extracted from A. Both of them play certainly key roles in the equivalent replacement problem. Thus, the dimensional coefficient η can be taken as an effect sum of these two things as Eq. (18), noting (1) and (5),

$$\eta = \frac{1}{\langle A^2 \rangle + \langle (k_{\mathrm{f}} B)^2 \rangle} = \frac{1}{\langle A^2 \rangle + \left(\frac{\langle AB \rangle}{\langle B^2 \rangle} \right)^2 \langle B^2 \rangle} = \frac{1}{(1 + r^2) \langle A^2 \rangle}. \tag{18}$$

Finally, substituting (18) into (17) yields the following expression for the non-dimensional weight parameter p:

$$p = \frac{1 - r^2}{1 + r^2} \tag{19}$$

which satisfies (13). It is obtained in (19) again that the non-dimensional weight parameter p depends on the parameter r^2 indicating the difference level between A and B. Substituting (19) into (15) yields the weighted dual equivalent replacement coefficient:

$$k_{\mathrm{w}} = \frac{2r^2}{1 + r^4} \frac{\langle AB \rangle}{\langle B^2 \rangle}. \tag{20}$$

Thus, using the weighted dual mean-square error criterion (12) with the weight parameter (19) the original random function A can be replaced by the equivalent random function $k_{\mathrm{w}} B$, where the weight dual equivalent replacement coefficient k_{w} is found from (20) and r^2 takes form (1). In the next Section, the accuracy of the proposed weighted dual mean-square error criterion will be investigated for several nonlinear systems.

3 Applications

3.1 Power-law oscillators

Consider the nonlinear system consisting of a linear damper with damping coefficient $2h$, a linear restoring force $\omega_0^2 x$, and a nonlinear restoring force $g(x)$,

$$\ddot{x} + 2h\dot{x} + \omega_0^2 x + g(x) = \sigma\dot{\xi}(t) \tag{21}$$

where h, ω_0, and σ are real positive constants, $g(x)$ is an odd function, and $\dot{\xi}(t)$ is Gaussian white noise excitation with zero mean and unit intensity. Indeed, Eq. (21) may represent a class of power-law oscillators. For instance, when the restoring force $g(x)$ in (21) has the form (22):

$$g(x) = \gamma x^3 \tag{22}$$

with γ a real positive constant, the equation is said to be a Duffing oscillator [3]. When the restoring force in (21) has the form (23),

$$g(x) = \gamma x^{1/3}, \tag{23}$$

then the equation is said to be the cubic-root oscillator [1,3]. For any odd restoring force $g(x)$, the probability density function of the nonlinear oscillator (21) takes the form (24):

$$W(x) = \frac{\exp\left\{-\frac{4h}{\sigma^2}G(x)\right\}}{\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{4h}{\sigma^2}G(x)\right\}\mathrm{d}x} \tag{24}$$

where $G(x)$ is the potential energy function for the restoring force, $G(x) = \int_0^x \left[\omega_0^2 u + g(u)\right]\mathrm{d}u$ [3]. Due to the zero mean, the exact mean-square displacement response coincides with the variance one,

$$\left\langle x^2\right\rangle_{\mathrm{e}} = \sigma_{\mathrm{x,e}}^2 = \frac{\int\limits_{-\infty}^{\infty} x^2\exp\left\{-\frac{4h}{\sigma^2}G(x)\right\}\mathrm{d}x}{\int\limits_{-\infty}^{\infty}\exp\left\{-\frac{4h}{\sigma^2}G(x)\right\}\mathrm{d}x}. \tag{25}$$

Applying the equivalent linearization method, the desired mean-square response of the nonlinear system (21) with two forms of the restoring force $g(x)$ that are described in (22), (23) can be estimated by the mean-square response of an equivalent linearization system:

$$\ddot{x} + 2h\dot{x} + \left(\omega_0^2 + k\right)x = \sigma\dot{\xi}(t) \tag{26}$$

where the nonlinear component $g(x)$ is replaced by a linear one kx, k being the equivalent linearization coefficient. The mean-square response defined from (26) is [3]:

$$\left\langle x^2\right\rangle = \sigma_{\mathrm{x}}^2 = \frac{\sigma^2}{4h\left(\omega_0^2 + k\right)}. \tag{27}$$

Applying the weighted dual criterion with $A = g(x)$, $B = x$, first calculate the moments $\left\langle A^2\right\rangle = \left\langle g^2\right\rangle$, $\langle AB\rangle = \langle xg\rangle$ and the squared correlation coefficient r^2, and then the weight parameter (19) and the weighted dual equivalent linearization coefficient (20) are defined next.

For a Duffing oscillator with $A = \gamma x^3$, $B = x$, one gets

$$\begin{aligned}\left\langle A^2\right\rangle &= \gamma^2\left\langle x^6\right\rangle = 15\gamma^2\left\langle x^2\right\rangle^3,\\ \langle AB\rangle &= \gamma\left\langle x^4\right\rangle = 3\gamma\left\langle x^2\right\rangle^2,\\ \left\langle B^2\right\rangle &= \left\langle x^2\right\rangle,\end{aligned}$$

Table 1 The errors of the approximate mean-square responses of a Duffing oscillator for $h = 0.5$; $\omega_0 = 1$; $\sigma = \sqrt{2}$ and various values of γ

γ	$\langle x^2\rangle_{\rm e}$	$\langle x^2\rangle_{\rm c}$	err (%)	$\langle x^2\rangle_{\rm regI}$	err (%)	$\langle x^2\rangle_{\rm regII}$	err (%)	$\langle x^2\rangle_{\rm d}$	err (%)	$\langle x^2\rangle_{\rm w}$	err (%)
0.01	0.9721	0.9717	0.05	0.9777	0.57	0.9810	0.91	0.9794	0.75	0.9748	0.28
0.05	0.889	0.883	0.64	0.905	1.78	0.917	3.18	0.911	2.52	0.894	0.62
0.1	0.818	0.805	1.49	0.837	2.34	0.856	4.64	0.846	3.54	0.821	0.47
0.5	0.579	0.549	5.29	0.592	2.15	0.620	7.09	0.606	4.67	0.570	1.59
1.0	0.468	0.434	7.19	0.475	1.42	0.502	7.32	0.489	4.41	0.454	2.95
5.0	0.254	0.227	10.74	0.253	0.51	0.272	6.81	0.262	3.17	0.240	5.76
10.0	0.189	0.167	11.77	0.187	1.17	0.201	6.48	0.194	2.67	0.176	6.62

$$r^2 = 3/5. \tag{28}$$

Using the results in (28) to calculate the weight parameter (19) and the weighted dual equivalent linearization coefficient (20) gives

$$p = 1/4, \tag{29}$$

$$k_{\rm w} = \frac{45}{17}\gamma\langle x^2\rangle. \tag{30}$$

Substituting the defined equivalent linearization coefficient from (30) to (27), yields

$$\frac{45}{17}\gamma\langle x^2\rangle_{\rm w}^2 + \omega_0^2\langle x^2\rangle_{\rm w} - \frac{\sigma^2}{4h} = 0. \tag{31}$$

Solving (31), one obtains the approximate mean-square response,

$$\langle x^2\rangle_{\rm w} = \frac{17}{90\gamma}\left[-\omega_0^2 + \sqrt{\omega_0^4 + \frac{45}{17}\gamma\frac{\sigma^2}{h}}\right]. \tag{32}$$

The approximate mean-square response (32) is compared in Table 1 with the exact solution (25), also with the ones obtained by other equivalent linearizations, i.e., conventional, one-step regulation, two-step regulation, and dual criteria, with the corresponding indexes "$_{\rm c,\ regI,\ regII,\ d}$", respectively. Those solutions are available in [20]:

$$\langle x^2\rangle_{\rm c} = \frac{1}{6\gamma}\left(-\omega_{\rm o}^2 + \sqrt{\omega_o^4 + 3\frac{\gamma\sigma^2}{h}}\right), \tag{33}$$

$$\langle x^2\rangle_{\rm regI} = \frac{3}{14\gamma}\left(-\omega_{\rm o}^2 + \sqrt{\omega_o^4 + \frac{7}{3}\frac{\gamma\sigma^2}{h}}\right), \tag{34}$$

$$\langle x^2\rangle_{\rm regII} = \frac{39}{154\gamma}\left(-\omega_{\rm o}^2 + \sqrt{\omega_o^4 + \frac{77}{39}\frac{\gamma\sigma^2}{h}}\right), \tag{35}$$

$$\langle x^2\rangle_{\rm d} = \frac{7}{30\gamma}\left(-\omega_o^2 + \sqrt{\omega_{\rm o}^4 + \frac{15}{7}\frac{\gamma\sigma^2}{h}}\right). \tag{36}$$

For the cubic-root oscillator, the calculations are similar to those for the Duffing oscillator. With $A = \gamma x^{1/3}$, $B = x$, the calculated moments and the squared correlation coefficient are, respectively,

$$\langle A^2\rangle = \gamma^2\langle x^{2/3}\rangle = \frac{2^{5/6}}{\sqrt{2\pi}}\Gamma\left(\frac{5}{6}\right)\gamma^2\langle x^2\rangle^{1/3},$$

$$\begin{aligned}\langle AB\rangle &= \gamma\left\langle x^{4/3}\right\rangle = \frac{2^{1/6}}{3\sqrt{2\pi}}\Gamma\left(\frac{1}{6}\right)\gamma\left\langle x^2\right\rangle^{2/3},\\ \left\langle B^2\right\rangle &= \left\langle x^2\right\rangle,\\ r^2 &= \frac{1}{18\sqrt{\pi}}\frac{\left[\Gamma\left(\frac{1}{6}\right)\right]^2}{\Gamma\left(\frac{5}{6}\right)}.\end{aligned} \tag{37}$$

In the above equations, we use the integration formula [3]:

$$\int_0^\infty u^n \exp\left(-au^b\right)\mathrm{d}u = \frac{1}{b}a^{-\frac{n+1}{b}}\Gamma\left(\frac{n+1}{b}\right) \tag{38}$$

where the Gamma function $\Gamma(s)$ is given by:

$$\Gamma(s) = \int_0^\infty u^{s-1}\exp(-u)\,\mathrm{d}u. \tag{39}$$

Using the results in (37) to calculate the weight parameter (19) and the weighted dual equivalent linearization coefficient (20), one obtains:

$$p = \frac{18\sqrt{\pi}\Gamma\left(\frac{5}{6}\right) - \left[\Gamma\left(\frac{1}{6}\right)\right]^2}{18\sqrt{\pi}\Gamma\left(\frac{5}{6}\right) + \left[\Gamma\left(\frac{1}{6}\right)\right]^2}, \tag{40}$$

$$k_{\mathrm{w}} = R\frac{2^{1/6}}{3\sqrt{2\pi}}\Gamma\left(\frac{1}{6}\right)\gamma\frac{1}{\left\langle x^2\right\rangle^{1/3}} \tag{41}$$

where it is denoted

$$R = \frac{2r^2}{1+r^4} = \left[\frac{1}{9\sqrt{\pi}}\frac{\left[\Gamma\left(\frac{1}{6}\right)\right]^2}{\Gamma\left(\frac{5}{6}\right)}\right]\left[1+\left[\frac{1}{18\sqrt{\pi}}\frac{\left[\Gamma\left(\frac{1}{6}\right)\right]^2}{\Gamma\left(\frac{5}{6}\right)}\right]^2\right]^{-1}. \tag{42}$$

Substituting (41) into (27) yields Eq. (43),

$$\omega_0^2\left\langle x^2\right\rangle + R\frac{2^{1/6}}{3\sqrt{2\pi}}\Gamma\left(\frac{1}{6}\right)\gamma\left\langle x^2\right\rangle^{2/3} - \frac{\sigma^2}{4h} = 0. \tag{43}$$

Solving Eq. (43), with R given by (42), one obtains the approximate mean-square response $\left\langle x^2\right\rangle_w$. The approximate mean-square responses determined by conventional and dual criteria are also calculated from Eq. (43) where $R = 1$ and $R = 1/\left(2-r^2\right)$, respectively. Note that the regulation linearization procedures for nonlinear functions with positive non-integer power are not available.

Tables 1 and 2 display the resulting comparison of the mean-square responses for $h = 0.5$; $\omega_0 = 1$; $\sigma = \sqrt{2}$, and various values of γ. Among the largest error of approximations, in the case of a Duffing oscillator (Table 1), the one-step regulation yields the smallest error: 2.34%, the dual criterion gives: 4.67%, and the present weighted dual one gives a little higher one: 6.62%, while the conventional linearization provides the worst 11.77%; in the case of the cubic-root oscillator (Table 2), the weighted dual criterion yields the smallest error: 4.92% while the one by the dual criterion becomes the worst: 12.44% in comparison with: 6.43% by the conventional criterion.

Table 2 The errors of the approximate mean-square responses of a cubic-root oscillator for $h = 0.5; \omega_0 = 1; \sigma = \sqrt{2}$ and various values of γ

γ	$\langle x^2\rangle_{\mathrm{e}}$	$\langle x^2\rangle_{\mathrm{c}}$	err (%)	$\langle x^2\rangle_{\mathrm{d}}$	err (%)	$\langle x^2\rangle_{\mathrm{w}}$	err (%)
0.01	0.99174	0.99174	0.00	0.9927	0.10	0.9918	0.01
0.05	0.95964	0.95958	0.01	0.964	0.50	0.960	0.04
0.1	0.9215	0.9213	0.02	0.9305	0.97	0.9222	0.07
0.5	0.6821	0.6791	0.45	0.7099	4.08	0.6818	0.04
1.0	0.493	0.486	1.29	0.525	6.65	0.490	0.60
5.0	0.106	0.101	5.31	0.119	11.73	0.102	3.93
10.0	0.042	0.039	6.43	0.047	12.44	0.040	4.92

3.2 Lutes–Sarkani dynamical system

Consider another power-law dynamical system that is governed by the first-order stochastic differential equation

$$\dot{x} + \gamma |x|^a \operatorname{sgn}(x) = f(t) \tag{44}$$

where a is a real positive number, $f(t)$ is a zero mean, stationary Gaussian white noise with spectral density $S_{\mathrm{o}} = \text{const}$. System (44) is called the Lutes–Sarkani system [18,27], which has the probability density function [18]:

$$W(x) = C \exp\left\{-\frac{\gamma |x|^{a+1}}{(a+1)\pi S_0}\right\} \tag{45}$$

where C is the normalization coefficient. Due to zero mean, the mean-square response coincides with the variance one which is formulated by [18]

$$\langle x^2\rangle_{\mathrm{e}} = \sigma^2_{\mathrm{x,e}} = \left(\frac{\pi S_{\mathrm{o}}}{\gamma}\right)^{\frac{2}{a+1}} h_{\mathrm{e}}(a) \tag{46}$$

where

$$h_{\mathrm{e}}(a) = (a+1)^{\frac{2}{a+1}} \Gamma\left(\frac{3}{a+1}\right)\left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{47}$$

The equivalent linearization equation of (44) is of the form

$$\dot{x} + kx = f(t) \tag{48}$$

where k is the linearization coefficient. The variance response of (48) is [3]:

$$\sigma^2_{\mathrm{x}} = \frac{\pi S_{\mathrm{o}}}{k}. \tag{49}$$

Using the weighted dual criterion with $A = \gamma |x|^{\mathrm{a}} \operatorname{sgn}(x)$, $B = |x| \operatorname{sgn}(x) = x$, the calculated moments and squared correlation coefficient are, respectively,

$$\begin{aligned}
\langle A^2\rangle &= \gamma^2 \langle |x|^{2\mathrm{a}}\rangle = \frac{1}{\sqrt{2\pi}} \gamma^2 2^{\left(a+\frac{1}{2}\right)} \Gamma\left(a+\frac{1}{2}\right) \sigma^{2\mathrm{a}}_{\mathrm{x}}, \\
\langle AB\rangle &= \gamma \langle |x|^{\mathrm{a}+1}\rangle = \frac{1}{\sqrt{2\pi}} \gamma 2^{\mathrm{a}/2} a \Gamma\left(\frac{\mathrm{a}}{2}\right) \sigma^{a+1}_{\mathrm{x}}, \\
\langle B^2\rangle &= \langle x^2\rangle = \sigma^2_{\mathrm{x}}, \\
r^2 &= \frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}
\end{aligned} \tag{50}$$

where it is used the Gamma function $\Gamma(s)$ (39) to calculate the integral $\langle |x|^a \rangle = \sigma_x^a \int_{-\infty}^{\infty} \frac{|u|^a}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du$ [18,27]. Next, using (50) to define the weight parameter (19) and the weighted dual equivalent linearization coefficient (20) gives:

$$p = \left[1 - \frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right]\left[1 + \frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right]^{-1}, \tag{51}$$

$$k_w = \frac{2r^2}{1+r^4}\frac{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}{\sqrt{2\pi}}\gamma\sigma_x^{a-1} = \frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\left[1 + \left(\frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right)^2\right]^{-1}\frac{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}{\sqrt{2\pi}}\gamma\sigma_x^{a-1}. \tag{52}$$

Substituting (52) into (49) yields the approximate variance response

$$\sigma_{x,w}^2 = \left(\frac{\pi S_o}{\gamma}\right)^{\frac{2}{a+1}} h_w(a) \tag{53}$$

where it is denoted

$$\begin{aligned} h_w(a) &= \left[\frac{1}{\frac{2r^2}{1+r^4}\frac{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}{\sqrt{2\pi}}}\right]^{\frac{2}{a+1}} \\ &= \left[1 + \left(\frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right)^2\right]^{\frac{2}{a+1}}\left[\frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right]^{-\frac{2}{a+1}}\left[\frac{\sqrt{2\pi}}{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}\right]^{\frac{2}{a+1}}. \end{aligned} \tag{54}$$

The approximate variance σ_x^2 determined by other equivalent linearizations, i.e., conventional, one-step regulation, two-step-regulation, and dual criteria are available in [20,27], and they have the form (46) with the corresponding functions $h(a)$, respectively,

$$h_c(a) = \left[\frac{\sqrt{2\pi}}{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}\right]^{\frac{2}{a+1}}, \tag{55}$$

$$h_{regI}(a) = \left[\sqrt{\pi}2^{-\frac{a+1}{2}}\frac{\Gamma\left(\frac{4a-1}{2}\right)\Gamma\left(\frac{2a+1}{2}\right)}{\left[\Gamma\left(\frac{3a}{2}\right)\right]^2\Gamma\left(\frac{a+2}{2}\right)}\right]^{\frac{2}{a+1}}, \tag{56}$$

$$h_{regII}(a) = \left[\sqrt{\pi}2^{-\frac{a+1}{2}}\frac{\Gamma\left(\frac{6a-3}{2}\right)\Gamma\left(\frac{2a+1}{2}\right)\left[\Gamma\left(\frac{4a-1}{2}\right)\right]^2}{\left[\Gamma\left(\frac{5a-2}{2}\right)\right]^2\left[\Gamma\left(\frac{3a}{2}\right)\right]^2\Gamma\left(\frac{a+2}{2}\right)}\right]^{\frac{2}{a+1}}, \tag{57}$$

$$h_d(a) = \left[2 - \frac{a^2\left[\Gamma\left(\frac{a}{2}\right)\right]^2}{2\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)}\right]^{\frac{2}{a+1}}\left[\frac{\sqrt{2\pi}}{2^{a/2}a\Gamma\left(\frac{a}{2}\right)}\right]^{\frac{2}{a+1}}. \tag{58}$$

For the Lutes–Sarkani system with various values of a, the percentage errors of the mentioned approximate variances through the values of $h(a)$ in comparison with the exact one (47) are given in Table 3 where for information the corresponding squared correlation coefficient r^2 is also given. It is noted that the application of the regulation criteria introduced in [26,27] is only available for the case of positive integer exponents a. As seen in Table 3, the weighted dual criterion yields the best approximation with 9.7% of the largest error in comparison with 41%, 25.8%, 14.6%, and 30.5% provided by the conventional, one-step regulation, two-step regulation, and dual criteria, respectively. Table 3 also shows that the ranges of the value of r^2 for which the errors are less than 10% are: from 0.82 to 1 by conventional linearization; from 0.39 to 1 corresponding to the exponent $a > 1$ by one-step regulation and dual criteria; from 0.14 to 1 corresponding to $a > 1$ by two-step regulation criterion; and from 0.08 to 1 by the proposed criterion, respectively.

Table 3 The errors of approximate responses $h(a)$ of the Lutes–Sarkani system for various a (N/A: not available)

a	$h_e\,(a)$	$h_c\,(a)$	err (%)	$h_{regI}\,(a)$	err (%)	$h_{regII}\,(a)$	err (%)	$h_d\,(a)$	$err\,(\%)$	$h_w\,(a)$	$err\,(\%)$	r^2
1/7	1.689	1.453	13.98					2.150	27.28	1.561	7.55	0.75
1/6	1.648	1.435	12.96					2.060	24.95	1.525	7.51	0.77
1/5	1.596	1.410	11.62					1.947	21.98	1.479	7.32	0.79
1/4	1.525	1.375	9.80	N/A		N/A		1.800	18.08	1.421	6.81	0.82
1/3	1.423	1.320	7.24					1.606	12.85	1.343	5.66	0.86
1/2	1.268	1.223	3.58					1.343	5.91	1.227	3.21	0.93
1.0	1.000	1.000	0.00	1.000	0.00	1.000	0.00	1.000	0.00	1.000	0.00	1.00
2.0	0.776	0.732	5.69	0.782	0.77	0.820	5.67	0.804	3.59	0.739	4.84	0.85
3.0	0.676	0.577	14.59	0.655	3.15	0.712	5.28	0.683	1.06	0.615	9.07	0.60
4.0	0.618	0.476	22.85	0.562	8.99	0.625	1.22	0.577	6.61	0.558	9.69	0.39
5.0	0.579	0.405	29.92	0.492	15.02	0.555	4.01	0.490	15.36	0.529	8.60	0.24
6.0	0.551	0.353	35.90	0.437	20.68	0.499	9.40	0.421	23.48	0.509	7.47	0.14
7.0	0.529	0.312	40.96	0.393	25.82	0.452	14.55	0.368	30.52	0.492	6.96	0.08

3.3 Van der Pol oscillator and cubic damping oscillator

Consider two nonlinear damped oscillators in which the damping forces are presented by cubic monomials. The first one is the Van der Pol oscillator governed by

$$\ddot{x} + \left(-\alpha + \gamma x^2\right)\dot{x} + \omega_o^2 x = \sigma\dot{\xi}\,(t) \tag{59}$$

where α, γ, ω_o, and σ are real positive constants, and $\dot{\xi}\,(t)$ is Gaussian white noise excitation with zero mean and unit intensity. Using the weighted dual error criterion with $A = \gamma x^2\dot{x}$, $B = \dot{x}$, the calculated moments and squared correlation coefficient are, respectively,

$$\begin{aligned}
\left\langle A^2\right\rangle &= \gamma^2\left\langle x^4\dot{x}^2\right\rangle = 3\gamma^2\omega_0^2\left\langle x^2\right\rangle^3,\\
\langle AB\rangle &= \gamma\left\langle x^2\dot{x}^2\right\rangle = \gamma\omega_0^2\left\langle x^2\right\rangle^2,\\
\left\langle B^2\right\rangle &= \left\langle\dot{x}^2\right\rangle = \omega_0^2\left\langle x^2\right\rangle,\\
r^2 &= 1/3.
\end{aligned} \tag{60}$$

Substituting (60) into (19) yields

$$p = 1/2. \tag{61}$$

The result (61) shows that the weighted dual linearization gives the same value of the weight parameter p as the dual linearization does for the Van der Pol oscillator. A comparison of the approximate solutions obtained by Monte Carlo simulation and by conventional, dual, and regulation criteria is available in [20]. The result shows that the largest error of the dual criterion is 5.5% and of the two-step regulation linearization is 7.3%, whereas the smallest error of the conventional linearization method is 23.3% and of the one-step regulation linearization is 5.5%, respectively.

The second cubic damped oscillator is

$$\ddot{x} + \alpha\dot{x} + \gamma\dot{x}^3 + x = f(t) \tag{62}$$

where α, γ are real positive constants, $f(t)$ is a zero mean, stationary Gaussian white noise with spectral density S_o = const. The probability density function defined by the equivalent nonlinearization method (ENLE) [3,17] is considered as being the exact one,

$$W(x) = C\int_{-\infty}^{\infty} e^{-\frac{1}{\pi S_0}\left[\alpha\left(x^2+\dot{x}^2\right)+\frac{3}{4}\gamma\left(x^2+\dot{x}^2\right)^2\right]}\mathrm{d}\dot{x}, \tag{63}$$

where C is the normalization coefficient. Then the exact mean-square displacement is given by Eq. (64):

$$\left\langle x^2\right\rangle_{ENLE} = \int_{-\infty}^{\infty} x^2 W(x)\,\mathrm{d}x. \tag{64}$$

The equivalent linearization equation to (62) is of the form

$$\ddot{x} + (\alpha + b)\,\dot{x} + x = f(t) \tag{65}$$

where b is the linearization coefficient, and the corresponding mean-square displacement is given by [3]:

$$\langle x^2 \rangle = \frac{\pi S_0}{\alpha + b}. \tag{66}$$

Using the weighted dual criterion with $A = \gamma \dot{x}^3$, $B = \dot{x}$, the calculated moments and squared correlation coefficient are, respectively,

$$\begin{aligned} \langle A^2 \rangle &= \gamma^2 \langle \dot{x}^6 \rangle = 15\gamma^2 \langle \dot{x}^2 \rangle^3, \\ \langle AB \rangle &= \gamma \langle \dot{x}^4 \rangle = 3\gamma \langle \dot{x}^2 \rangle^2, \\ \langle B^2 \rangle &= \langle \dot{x}^2 \rangle = \langle x^2 \rangle, \\ r^2 &= 3/5. \end{aligned} \tag{67}$$

Using (67) one defines the weight parameter (19) and the weighted dual equivalent linearization coefficient (20):

$$p = 1/4, \tag{68}$$

$$b_{\mathrm{w}} = \frac{15}{17} 3\gamma \langle \dot{x}^2 \rangle. \tag{69}$$

Substituting (69) into (66) yields

$$\frac{45}{17}\gamma \langle x^2 \rangle_{\mathrm{w}}^2 + \alpha \langle x^2 \rangle_{\mathrm{w}} - \pi S_0 = 0. \tag{70}$$

Solving (70) one obtains the approximate mean-square displacement:

$$\langle x^2 \rangle_{\mathrm{w}} = \frac{17}{90\gamma} \left[-\alpha + \sqrt{\alpha^2 + \frac{180}{17}\gamma \pi S_0} \right]. \tag{71}$$

The approximate mean-square response $\langle x^2 \rangle_{\mathrm{w}}$ (71) is compared with the exact solution (64), also with the ones obtained by other equivalent linearizations, i.e., conventional and dual criteria. These approximate mean-square displacements are, respectively,

$$\langle x^2 \rangle_{\mathrm{c}} = \frac{1}{6\gamma} \left[-\alpha + \sqrt{\alpha^2 + 12\gamma \pi S_0} \right], \tag{72}$$

$$\langle x^2 \rangle_{\mathrm{d}} = \frac{7}{30\gamma} \left[-\alpha + \sqrt{\alpha^2 + \frac{60}{7}\gamma \pi S_0} \right]. \tag{73}$$

Table 4 displays the resulting comparison of the mean-square solutions for $\pi S_0 = \alpha$; $\gamma = \beta\alpha$, and various values of β. Among the largest error of approximations, the proposed criterion yields the smallest one 3.87%, the dual criterion gives about 6.10%, while the conventional criterion gives 9.17%.

Table 4 The errors of the approximate mean-square displacements of a cubic damping oscillator for $\pi S_0 = \alpha$; $\gamma = \beta\alpha$ and various values of β

β	$\langle x^2\rangle_e$	$\langle x^2\rangle_c$	err (%)	$\langle x^2\rangle_d$	err (%)	$\langle x^2\rangle_w$	err (%)
1	0.460	0.434	5.66	0.489	6.14	0.454	1.34
2	0.358	0.333	6.99	0.380	6.10	0.350	2.26
3	0.306	0.282	7.66	0.324	6.02	0.297	2.73
4	0.272	0.250	8.09	0.288	5.95	0.264	3.05
5	0.248	0.227	8.43	0.262	5.84	0.240	3.32
6	0.229	0.210	8.65	0.243	5.80	0.221	3.48
7	0.215	0.196	8.85	0.227	5.73	0.207	3.64
8	0.203	0.184	8.96	0.214	5.73	0.195	3.72
9	0.192	0.175	9.09	0.203	5.70	0.185	3.82
10	0.184	0.167	9.17	0.194	5.69	0.176	3.87

3.4 Oscillator with nonlinear damping by displacement and velocity

Consider the nonlinear stochastic oscillator governed by Eq. (74),

$$\ddot{x} + \zeta\left(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2\right)^a \dot{x} + \omega_0^2 x = f(t), \tag{74}$$

where ζ is the damping constant, ω_0 is the undamped natural frequency, a is a positive constant, and $f(t)$ is a Gaussian white noise process with spectral density $S_o = \text{const}$. The exact mean-square response of the oscillator (74) is (see [3]):

$$\langle x^2\rangle_e = \frac{1}{\omega_0^2}\left(\frac{\pi S_0}{\zeta}\right)^{\frac{1}{a+1}} h_e(a) \tag{75}$$

where it is denoted:

$$h_e(a) = (a+1)^{1/(a+1)}\Gamma\left(\frac{2}{a+1}\right)\left[\Gamma\left(\frac{1}{a+1}\right)\right]^{-1}. \tag{76}$$

The equivalent linearization equation to (74) is of the form:

$$\ddot{x} + b\dot{x} + \omega_0^2 x = f(t) \tag{77}$$

where b is the linearization coefficient, and the mean-square response is given by [3]:

$$\langle x^2\rangle = \frac{\pi S_0}{b\omega_0^2}. \tag{78}$$

Using the weighted dual error criterion with $A = \zeta\left(\dot{x}^2/2 + \omega_0^2 x^2/2\right)^a \dot{x}$; $B = \dot{x}$, the calculated moments and squared correlation coefficient are, respectively,

$$\begin{aligned}
\langle A^2\rangle &= \zeta^2\left\langle\left(\dot{x}^2/2 + \omega_0^2 x^2/2\right)^{2a}\dot{x}^2\right\rangle = \zeta^2\Gamma(2a+2)\langle\dot{x}^2\rangle^{2a+1},\\
\langle AB\rangle &= \zeta\left\langle\left(\dot{x}^2/2 + \omega_0^2 x^2/2\right)^{a}\dot{x}^2\right\rangle = \zeta\Gamma(a+2)\langle\dot{x}^2\rangle^{a+1},\\
\langle B^2\rangle &= \langle\dot{x}^2\rangle = \omega_0^2\langle x^2\rangle,\\
r^2 &= [\Gamma(a+2)]^2[\Gamma(2a+2)]^{-1}.
\end{aligned} \tag{79}$$

Using (79) one defines the weight parameter (19) and the weighted dual equivalent linearization coefficient (20):

$$p = \frac{\Gamma(2a+2) - [\Gamma(a+2)]^2}{\Gamma(2a+2) - [\Gamma(a+2)]^2}, \tag{80}$$

Table 5 The errors of the approximate responses $h(a)$ of the considered oscillator with various values of a

a	$h_{\mathrm{e}}(a)$	$h_{\mathrm{c}}(a)$	err (%)	$h_{\mathrm{d}}(a)$	err (%)	$h_{\mathrm{w}}(a)$	err (%)	r^2
0.0	1.000	1.000	0.00	1.000	0.00	1.00	0.00	1.000
0.5	0.864	0.827	4.28	0.890	3.01	0.83	3.79	0.884
1.0	0.798	0.707	11.38	0.816	2.33	0.74	7.76	0.667
1.5	0.757	0.619	18.31	0.735	2.92	0.69	8.81	0.460
2.0	0.729	0.550	24.51	0.657	9.91	0.67	7.89	0.300
2.5	0.708	0.496	29.93	0.588	16.96	0.66	6.40	0.188
3.0	0.691	0.452	34.65	0.529	23.42	0.66	5.18	0.114
3.5	0.678	0.415	38.78	0.480	29.14	0.65	4.52	0.068
4.0	0.667	0.384	42.42	0.439	34.12	0.64	4.40	0.040
4.5	0.657	0.357	45.64	0.404	38.47	0.63	4.72	0.023
5.0	0.649	0.334	48.51	0.375	42.27	0.61	5.39	0.013

$$b_w = \frac{2[\Gamma(a+2)]^2[\Gamma(2a+2)]^{-1}}{1+[\Gamma(a+2)]^4[\Gamma(2a+2)]^{-2}}\zeta\,\Gamma(a+2)\left\langle \dot{x}^2\right\rangle^a. \tag{81}$$

Substituting (81) into (78) one obtains the approximate mean-square response as:

$$\left\langle x^2\right\rangle_{\mathrm{w}} = \frac{1}{\omega_0^2}\left(\frac{\pi S_0}{\zeta}\right)^{\frac{1}{a+1}} h_{\mathrm{w}}(a) \tag{82}$$

where

$$h_{\mathrm{w}}(a) = \left[\frac{1+[\Gamma(a+2)]^4[\Gamma(2a+2)]^{-2}}{2[\Gamma(a+2)]^2[\Gamma(2a+2)]^{-1}\Gamma(a+2)}\right]^{\frac{1}{a+1}}. \tag{83}$$

The approximate mean-square responses $\left\langle x^2\right\rangle$ determined by conventional and dual criteria also take the form (75) where $h_{\mathrm{e}}(a)$ is replaced, respectively, by [3]

$$h_{\mathrm{c}}(a) = \left[\frac{1}{\Gamma(a+2)}\right]^{\frac{1}{a+1}} \tag{84}$$

and

$$h_{\mathrm{d}}(a) = \left[\frac{2-[\Gamma(a+2)]^2[\Gamma(2a+2)]^{-1}}{\Gamma(a+2)}\right]^{\frac{1}{a+1}}. \tag{85}$$

The percentage errors of the considered approximate mean-square responses through the values of $h(a)$ in comparison with the exact one (76) are given in Table 5. It is seen that the weighted dual criterion gives the largest error as 8.81% in comparison with 42.27 and 48.51% provided by the dual and conventional criteria, respectively.

4 Conclusions

The study of nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and in other disciplines. In the theory of random vibration, the stochastic equivalent linearization method that replaces a nonlinear system by an equivalent linear one is a popular method since it preserves some essential properties of the original nonlinear system. The essential of the equivalent linearization method is how to find the linearization coefficients for a given nonlinear system. The accuracy of the linearization coefficients can be improved by using the dual approach that combines two forward and return replacements in a weighted dual replacement. In this paper a weighted dual mean-square error criterion is introduced and discussed by considering weighted contributions of forward and return replacements. Introducing the weight parameter p makes the weighted dual mean-square error criterion more flexible than the conventional and dual mean-square error criteria. An empirical research approach is used to choose a specific form of p. The application to

several nonlinear random systems has shown the improved accuracy of the proposed weighted dual equivalent linearization technique for a quite large range of nonlinearity. It appears that the weighted dual approach has a potential to become an effective technique and it ought to be explored for further studies.

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