Acta Mech 229, 659–675 (2018)
https://doi.org/10.1007/s00707-017-1991-4

ORIGINAL PAPER

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Graph transformations for efficient structural analysis

This paper is dedicated to the memory of Franz Ziegler

Received: 28 February 2017 / Revised: 5 June 2017 / Published online: 15 November 2017

Abstract In this paper, some graph theoretical (topological) transformations are presented for simplifying certain problems involved in structural analysis. For each case, the main problem is stated and the proposed topological transformation is established. Once the required topological analysis is completed, a back transformation results in the solution for the main problem. The transformations studied here employ (i) models drawn on a lower dimensional space, (ii) models embedded on higher dimensional spaces and (iii) interchange models which have simpler connectivity properties than the corresponding original structural models. All these transformations are illustrated utilizing simple examples.

1 Introduction

An analysis of systems and in particular structures can be decomposed into three phases:

(i) Approximation, followed by choosing an appropriate model;
(ii) Specifying topological properties followed by a topological analysis;
(iii) Assigning algebraic variables, followed by an algebraic analysis.

Such a decomposition results in a considerable simplification in the analysis and leads to a clear understanding of the structural behavior.

For an optimal analysis of a structure, three conditions should be fulfilled. The structural (stiffness or flexibility) matrices should be sparse, properly structured (e.g., banded) and well conditioned. The latter property is not purely topological and is treated elsewhere (Kaveh [1]).

Pattern equivalence of structural matrices and graph matrices simplifies structural problems and allows advances in the field of graph theory to be transferred to structural mechanics. As an example, for rigid-jointed frames the sparsity of flexibility matrices can be provided by the construction of sparse cycle adjacency matrices. Similarly, using sparse cut-set bases, the formation of sparse stiffness matrices becomes feasible. Proper structuring of the flexibility and stiffness matrices of a structure can also be achieved by structuring the pattern of the cycle and cut-set adjacency matrices of its graph model, respectively.

This paper is devoted to the study of some structural problems in which topological graph theory plays an important role. Topological graph theory is primarily concerned with representing graphs on surfaces. An

This paper is dedicated to late Professor Dr. F. Ziegler who had a great influence on my academic life. He encouraged me to write my first two international books, and with his excellent question, he attracted my attention to the conditioning of structural matrices, resulting in completion of a topic entitled: *optimal analysis of structures*. He will stay in my heart and my mind as a highly influential academician.

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embedding or a drawing of a graph can be considered as identification of the graph with a subset of a surface in an appropriate fashion. For some problems, it is beneficial to define a new graph with simpler connectivity properties than the original model.

The article is structured as follows. Section 2 presents some of the basic definitions from the theory of graphs. The optimal analysis of structures is defined, and methods are presented for such an analysis in Sect. 3. In Sect. 4, transformation of the structural models to spaces of lower dimensions is discussed. Transformations of the models to higher dimensions are studied in Sect. 5. Transformations to identical spaces are presented in Sect. 6. Some concluding remarks are provided in Sect. 7.

2 Basic definitions from graph theory

In order to describe the concepts and methods of this paper in a self-contained manner, a number of definitions are presented in this section, which generally follows those of Refs. [1,2]:

A *graph S* consists of a set of elements called *nodes* (vertices) and a set of elements called *members* (edges), together with a relation of incidence, which associates two distinct nodes with each member. A graph is called *connected* if every pair of its nodes is joined together by a path. A *subgraph* of S is a graph having all its nodes and members in S. Two nodes of S are called *adjacent* if these nodes are the end nodes of a member. A member is *incident* to a node if the node is an end node of the member.

A *path* is a finite sequence of alternately distinct nodes and members of the graph. A path becomes a *cycle* if the first node and the last node of the path coincide. A *cut-set* is a set of members of S such that the removal of these members from S results in a disconnected graph.

A maximal set of independent cycles (cut-sets) is known as a *cycle (cut-set) basis* of S. The cardinality of a cycle basis is the same as the first Betti number $b_1(S) = M(S) - N(S) + b_0(S)$ of S, where $M(S)$, $N(S)$ and $b_0(S)$ are the number of members, nodes, and components of S, respectively. A *cycle adjacency matrix* is a $b_1(S) \times b_1(S)$ matrix consisting of 0 and 1 entries. An entry is 1 if the corresponding cycles have at least a member in common and 0 otherwise. A *cut-set adjacency matrix* has $N(S) + b_0(S)$ columns and rows and is defined analogously.

A graph is called *planar* if it can be embedded in the plane with no members crossing each other. A bipartite graph consists of two sets of nodes A and B such that only the nodes of A are joined to the nodes of B by members of the graph. A graph is called *clique graph* if all of its nodes are connected to each other.

Further definitions, concepts, and theorems on graph theory can be found in basic books by Harary [3], Berge [4], West [5], and those of graph products can be found in Hammack et al. [6].

3 Optimal structural analysis

In the analysis of a structure, the main aim is to find the response of the structure under external actions. The external actions may consist of loading, change in temperature, and support settlements. Tools for the analysis of a structure consist of the equilibrium of forces, compatibility of displacements, and force-displacement relationships. A comprehensive treatise on the mechanics of solids and fluids with a significant application to structural mechanics is due to Ziegler [7]. In this book, the author offers a unified presentation of the concepts and most of the practicable principles common to all branches of solid and fluid mechanics.

Two main dual methods of analysis can briefly be defined as (Henderson [8]):

Force Method: In this method, we assume the equilibrium to hold, and we proceed to satisfy the compatibility.

Displacement Method: In this method, we assume the compatibility to hold, and we proceed to satisfy the equilibrium.

The duality of these methods is shown by Langefors [9], Argyris and Kelsey [10], Samuelson [11] and Maunder [12].

An analysis is called optimal if the structural (flexibility or stiffness) matrices are (Kaveh [2]):

(i) Sparse,
(ii) Well structured,
(iii) Well conditioned.

These three properties are briefly discussed, and available methods are presented in the following:

Sparsity of structural matrices is a pure graph theoretical (topological) property. In order to have sparse structural matrices, a special statical basis (for the force method) and a special kinematical basis (for the displacement approach) should be selected. Corresponding to such statical kinematical bases, there exist graph theoretical counterparts known as the cycle basis (generalized cycle basis for non-frame structures) and the cut-set basis, respectively. Methods for the formation of such bases can be found in the works of Kaveh [13–15].

The problem of well-structured matrices is also a pure graph theoretical (topological) property. Sparse structural (flexibility and stiffness) matrices can be given special patterns depending on the solution scheme being used. There are many canonical patterns, and the most common of these is a banded form. These patterns can be obtained by different ordering schemes. An efficient method can be found in the books on sparse matrix technology [16–18].

There are other important patterns arising from symmetry and regularity of the structural matrices [19]. A structure is considered as symmetric or regular if the corresponding models are in the form of the product of two or more graphs. Such matrices can easily be solved using graph theory [20,21] and group theory methods [22–25].

The problem of well-conditioned matrices involves both graph theoretical (topological) and mechanical properties. For this purpose, special weighted cycle bases and cut-set bases should be found. Such bases require the formation of cycle bases and cut-set bases leading to small off-diagonal entries for the flexibility and stiffness matrices (Kaveh [26,27]).

In the subsequent sections, transformations are defined to achieve the above goals. Here, the sparsity and well structuring will be mainly considered, and the well conditioning can easily be added by considering weighted graphs.

4 Transformation to a lower dimensional space

4.1 Degree of static indeterminacy of space structures

The first step in the analysis of a structure by means of the force method consists of determining the degree of static indeterminacy (DSI) of the structure. For space structures, an efficient approach is developed by drawing the structural model on the plane, using the two simple theorems presented in this section [28].

Definitions A *drawing* S^p of S is a mapping of S into a surface. The nodes of S go into distinct nodes of S^p. A member and incident nodes map into a homeomorphic image of the closed interval [0,1] with the relevant nodes. A *good drawing* is one in which no two members are incident with a common point, and no two members have more than one point in common. A common point of two members is a *crossing*. An *optimal drawing* on a given surface is the one which exhibits the least possible crossings. The number of crossing points of S after drawing on a plane or a sphere, S^p, is denoted by $\nu(S^p)$. For cases when the drawing is optimal, $\nu(S^p)$ becomes the *crossing number* of the graph S.

Theorem 1 (Euler [29]) *Let S be a connected planar graph. Then:*

$$R(S) - M(S) + N(S) = 2. \tag{1}$$

The following theorems can easily be proved using Euler's theorem and considering the additional members and node which are created by each crossing node (Kaveh [1,13]*).*

Theorem 2 (Kaveh [30]) *For a space frame S, the degree of static indeterminacy is given by*

$$\gamma(S) = 6b_1\left(S^p\right) = 6\left[R_i\left(S^p\right) - \nu\left(S^p\right)\right] \tag{2}$$

where $R_i(S^p)$ is the number of internal regions of S^p, i.e.,

$$R_i(S^p) = R\left(S^p\right) - 1. \tag{3}$$

Example For a space frame depicted in Fig. 1a, a drawing may be considered as shown in Fig. 1b. Using Eq. (1) results in

$$\gamma(S) = 6\left[31 - 10\right] = 126.$$

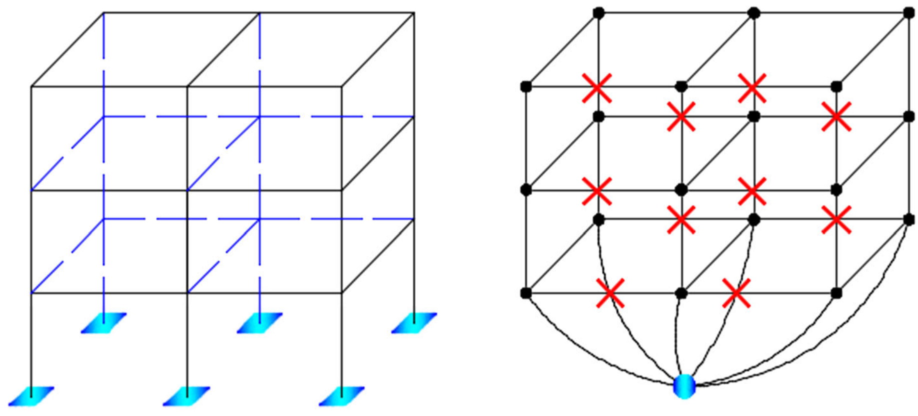


Fig. 1 A space frame with an arbitrary drawing. **a** A space frame S. **b** A drawing S^p of S

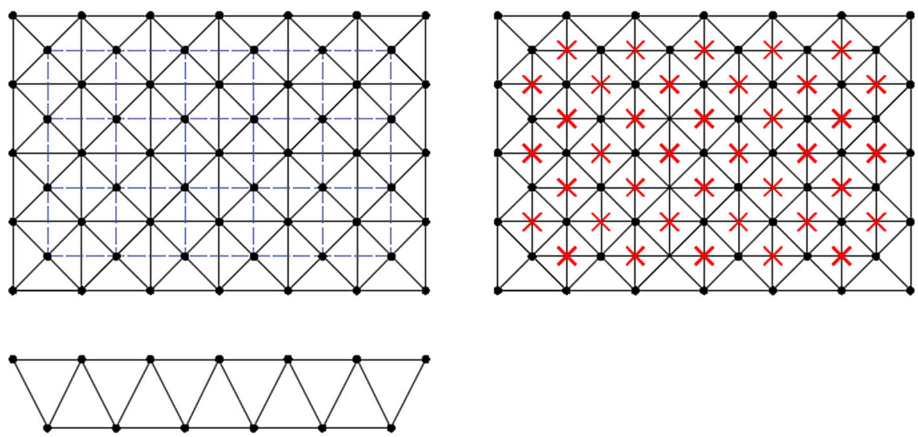


Fig. 2 A space truss S and an arbitrary drawing S^p of S. **a** A double layer grid S. **b** An arbitrary drawing S^p of S

Theorem 2 (Kaveh [27]) *For a space truss the degree of static indeterminacy is given by*

$$\gamma(S) = \nu\left(S^p\right) - M_c\left(S^p\right) \tag{4}$$

where $M_c\left(S^p\right)$ is the number of members required for full triangulation of S^p.

Example A space truss S supported in a statically determinate fashion together with a drawing S^p of S is shown in Fig. 2. Employing Eq. (3) leads to

$$\gamma(S) = 38 - 17 = 21.$$

Naturally, it is advantageous to use optimal drawings in order to reduce the number of counting for calculating the DSI of the space structures.

Apart from structural application, the above two theorems provide means for establishing upper and lower bounds for the crossing number of a general graph. It should be added that there is no definite formula for finding the crossing number of a general graph.

5 Transformation to a higher dimensional space

5.1 Cycle basis selection; disk embedding

The force method of frame analysis requires the formation of suitable statical bases corresponding to sparse flexibility matrices. Due to the pattern equivalence of the flexibility matrix of a frame and the cycle adjacency

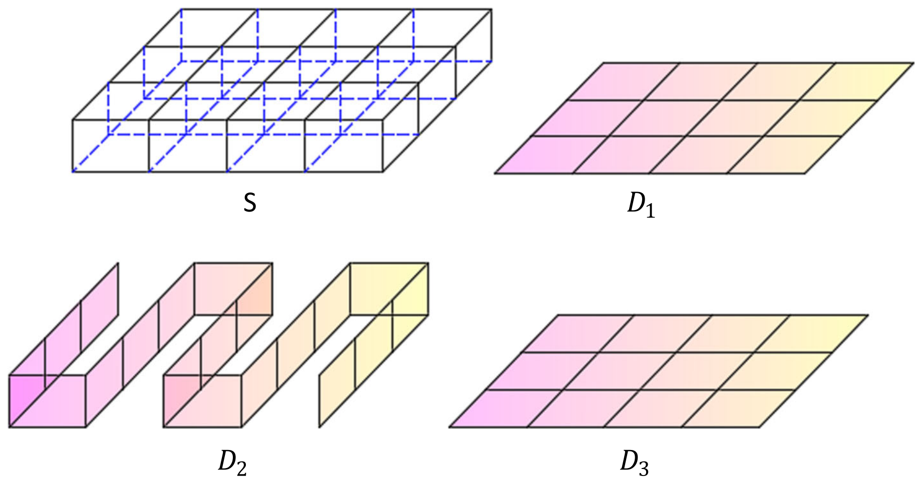


Fig. 3 A space graph S and the three identified disks

matrix of its graph model, the problem can be transformed to the formation of a maximal set of independent cycles, known as a *cycle basis* (Kaveh [13–15]). In order to have a sparse flexibility matrix for a frame structure, the elements which have the least overlaps should be selected (*optimal cycle basis*). The formation of an optimal cycle basis is not simple; however, such a basis is quite often among cycle bases of the least length (*minimal cycle bases*). There are various methods for the selection of subminimal cycle bases, some of which will be described in the next three sections.

The process of embedding a graph S on a union of disks can be summarized as follows:

Step 1. Identify a planar subgraph and embed it on the first disk whose dissection is isomorphic to the selected subgraph.
Step 2. Select the second subgraph such that the corresponding dissection has a 2-cell with a free 1-face, and its intersection with the previous dissection is a connected subspace of the frontier of the first disk.
Step k. Repeat the process of the second step, identifying the ith planar subgraph, whose dissection has a 2-cell with a free 1-face and the intersection of its dissection with the previously selected dissections is a connected subspace of the frontier of the ith disk. Repeat step k until all members (1-cells) of S are embedded, and the regional cycles forming a cycle basis are obtained.

Example A space frame structure S is considered and embedded on the union of three disks D_1, D_2, and D_3 as depicted in Fig. 3.

In order to reduce the overlaps of the selected cycles, it is ideal to embed S on a minimum number of disks. This number is known as the *thickness* of the graph. However, the restrictions imposed, and the lack of efficiency of the available methods for embedding reduces the chance of the minimality of such an embedding (Henderson [30]).

5.2 Cycle basis selection; a manifold embedding

The planar embedding of a graph results in a set of independent regional cycles forming a *mesh basis*. This is the simplest subminimal cycle basis of a planar graph. However, for non-planar graphs other embeddings should be employed. A manifold embedding is an example of this kind. A cycle basis can be obtained by embedding S on an admissible manifold (Henderson and Maunder [31]). An orientable manifold may be viewed as a sphere with h handles. In order to guarantee the independence of the corresponding regional cycles, 2-h fillings and one perforation of order 2 should be made, resulting in an admissible manifold embedding (Henderson and Maunder [31]).

Example A hollow box S is considered as shown in Fig. 4a, which is embedded on a sphere with two handles. Therefore, four fillings (shaded) and one perforation have been considered, Fig. 4b. The selected cycle basis consists of 79 four-sided cycles and two eight-sided cycles.

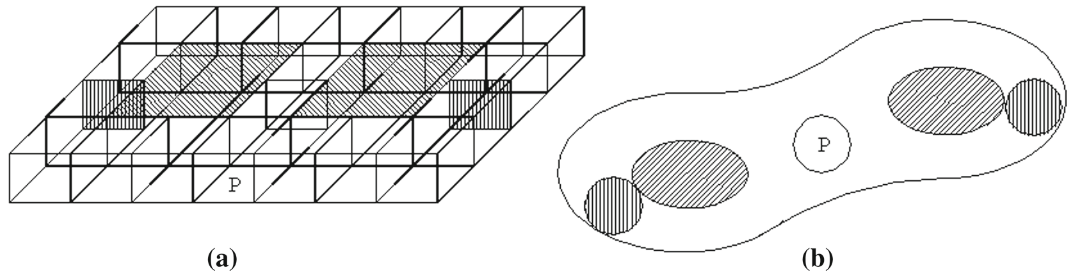


Fig. 4 An admissible manifold embedding of S. **a** A space structure S. **b** A manifold embedding

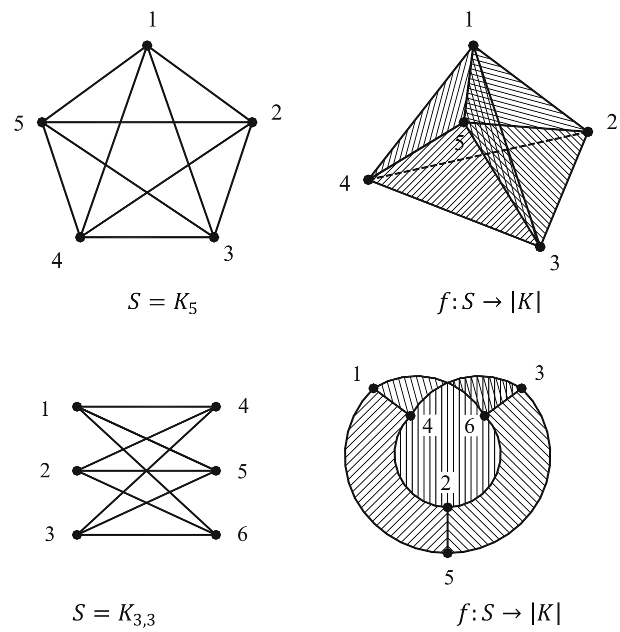


Fig. 5 Two non-planar graphs K_5 and $K_{3,3}$ are embedded into polyhedrons

In manifold embedding, the quality of the selected cycle basis depends on the number of handles being used. It is ideal to embed S on a sphere with minimum number of handles. This number is known as the *genus* of the graph. Again there is no efficient method for such an embedding (Kaveh [15]).

5.3 Cycle basis selection; collapsible embedding

A graph can be viewed as the 1-skeleton of a 3-complex. An n-cell is called *collapsible* if it can be shrunk into the remainder of its $n-1$ cells through a free $n-1$ cell. If a 3-complex can be collapsed into a point, then it is called *collapsible*. It can be proven that a collapsible 3-complex can be used for the formation of a cycle basis of its 1-skeleton. This can be achieved by collapsing all the 3-cells through free 2-cells or 2-cells being freed in subsequent steps (Maunder [21] and Henderson and Maunder [31]).

Example 1 The two non-planar graphs K_5 and $K_{3,3}$ are embedded into polyhedrons whose dissections are collapsible. In both cases, we have initially free 1-faces, and therefore the embeddings are trivial, Fig. 5.

Example 2 Consider a space structure as shown in Fig. 6a. This graph can be viewed as the 1-skeleton of a 3-complex as depicted in Fig. 6b. After collapsing all the 3-cells through the shaded 2-cells, the bounding cycles of the remaining 2-complex ($16 + 10 \times 24 = 256$ cycles) form a cycle basis of S.

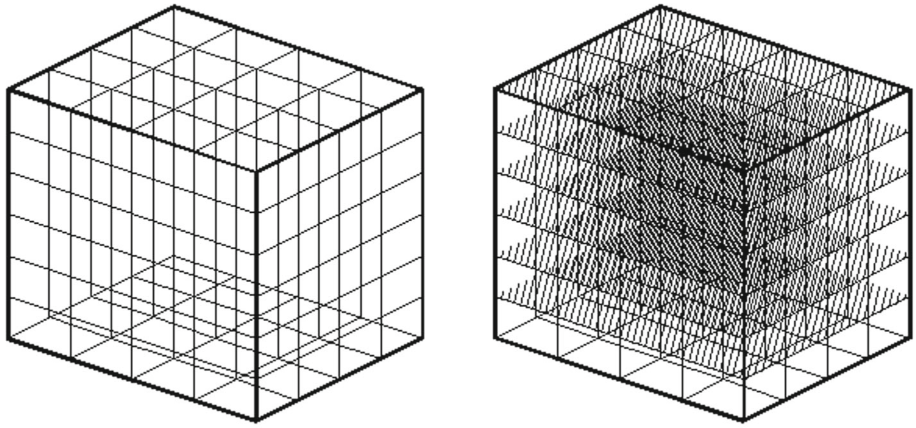


Fig. 6 A space structure and its collapsible embedding. **a** A space structure S. **b** S embedded on a 3-complex

5.4 Cycle basis selection; single-cycle disk embedding

This algorithm selects one cycle at a time, equivalent to embedding a single-cycle on a disk. The interface of a newly added disk with the previously generated disks should either be empty (disjoint) or a single connected path. In such an expansion process, the disk and the corresponding boundary (cycle) are called admissible.

Formation of a minimal disk (cycle) on a member (generator) m_j:

Generate a shortest route tree starting from an end node n_s of the generator m_j until the other end (not through the generator) n_t is reached. Backtracking from n_t to n_s, a shortest path will be obtained that together with m_j forms a minimal cycle on m_j.

Step 1: Generate as many admissible cycles (disks) of length 3 as possible. Denote the union of the selected cycles (disks) by C^n.

Step 2: Select an admissible cycle (disk) of length 4 on an unused member. Once such a cycle (disk) C_{n+1} is found, check the other unused members for possible admissible cycles (disks) of length 3. Again select an admissible cycle (disk) of length 4 followed by the formation of possible 3-sided cycles (disks). This process is repeated until no admissible cycles (disk) of length 3 and 4 can be formed. Denote the generated cycles (disks) by C^m.

Step 3: Select an admissible cycle (disk) of length 5 on an unused member. Then, check the unused members for the formation of 3-sided admissible cycles (disks). Repeat Step 2 until no cycle (disk) of length 3 or 4 can be generated. Repeat Step 3 until no cycle (disk) of length 3, 4 or 5 can be found.

Step 4: Repeat similar steps to Step 3, considering higher-length cycles (disks), until $b_1(S)$ admissible cycles (disks) forming a subminimal cycle basis are generated.

It should be mentioned that in this algorithm there is no need to embed on disks, and graph theoretical formulation is sufficient. However, the single-cycle disk embedding is utilized to relate the latter algorithm to the previously developed approaches.

Other algorithms for the formation of subminimal and suboptimal cycle bases can be found in Kaveh [13–15].

5.5 Force method of frame analysis

The transformations of Sects. 4 and 5 can be utilized in optimal analysis of frame structures as shown in Fig. 7. This chart shows how three graph invariants consisting of crossing number, genus, and thickness are involved in the force method of frame analysis. However, it should be mentioned that in structural engineering one is not necessarily dependent on these numbers, and any subminimal values of these invariants will be sufficient. In Fig. 7, $\boldsymbol{r}$ is the stress resultant of the frame structure under the applied load $\boldsymbol{P}$, and $\boldsymbol{X}$ is the column of redundant forces. $\boldsymbol{B}_0\boldsymbol{P}$ is the particular solution, and $\boldsymbol{B}_1\boldsymbol{X}$ is known as the complementary solution. A sparse $\boldsymbol{B}_0$ matrix can easily be formed using a shortest route tree (SRT), and the columns of $\boldsymbol{B}_1$ can be constructed

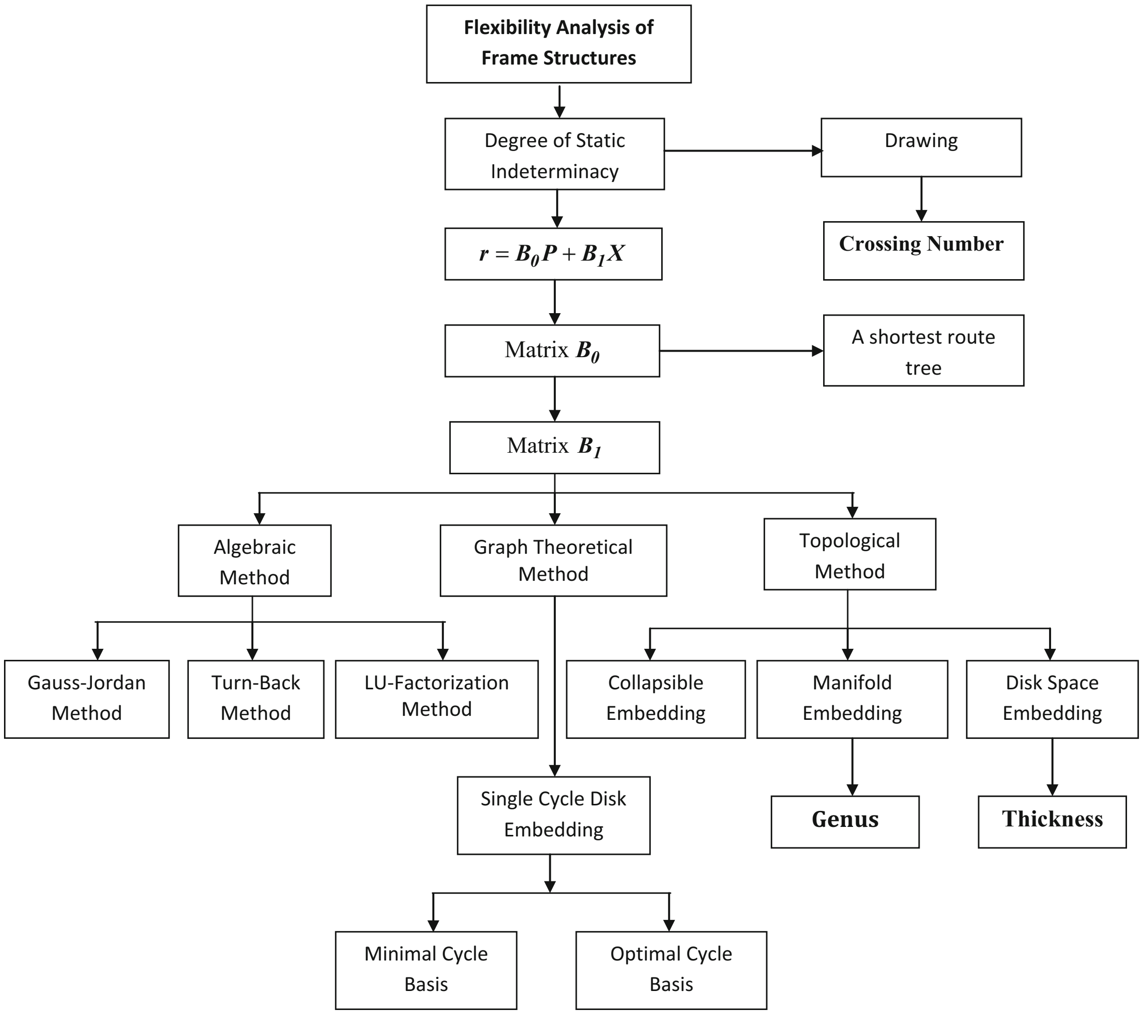


Fig. 7 Flowchart of the flexibility analysis for frame structures

on cycles of the selected cycle basis. Here $\boldsymbol{r} = \boldsymbol{B}_0\boldsymbol{P} + \boldsymbol{B}_1\boldsymbol{X}$ is the equilibrium equation, and the redundant vector $\boldsymbol{X}$ can be found using the compatibility conditions.

Though the above flowchart is prepared for frame structures, a similar formulation is made for finite element models, and the interested reader may refer to Kaveh [32].

From this flowchart, it can be observed that the three important graph invariants, namely crossing number, thickness and genus, find application in the force method of structural analysis.

Naturally, the future progress in these graph theoretical problems can directly be applied to the force method, enhancing the formation methods of more efficient cycle bases leads to better statical bases corresponding to more sparse flexibility matrices.

Russopoulos was the first to generalize the analysis of skeletal structures and presented a theory of elastic complexes, applicable to finite element models [33]. A comprehensive treatment of the theory and practice of equilibrium finite element analysis can be found in the work of Kaveh [32] and Moitinho de Almeida and Maunder [34].

5.6 Generalized cycle basis; interchange graph

For a general skeletal structure, a statical basis can be formed on a maximal set of subgraphs defined as a *generalized cycle basis* (GCB) of S (Kaveh [35]). Such a basis has been defined as the consequence of generalizing the first Betti number $b_1(S) = M(S) - N(S) + b_0(S)$ to $\gamma_1(S) = aM(S) + bN(S) + c\gamma_0(S)$. The

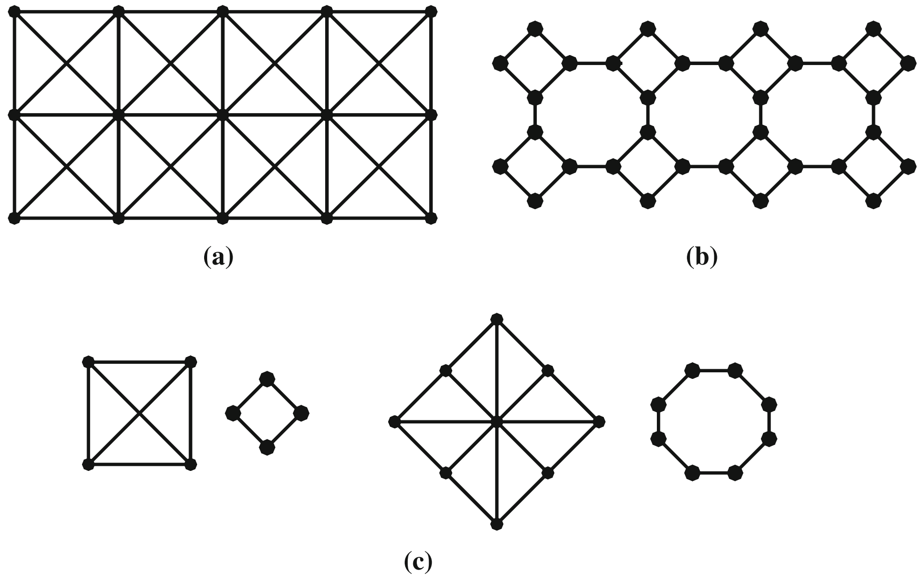


Fig. 8 A planar truss and typical elements of the selected GCB. **a** A planar truss. **b** The interchange graph. **c** Typical elements of the GCB

formation of a generalized cycle basis can be time-consuming; however, for planar trusses the problem can be simplified by using a special graph, known as the *interchange graph*. An interchange graph $I(S)$ of S is a graph whose vertices are in a one-to-one correspondence with the triangular regions of S (when S is embedded in the plane), and two nodes are connected by an edge if the corresponding triangles have a common member.

In order to form a generalized cycle basis of S, one can generate a cycle basis of $I(S)$, and with a back transformation, the elements of the generalized cycle basis can be obtained (Kaveh [36]).

Example A planar truss as shown in Fig. 8a is considered. The interchange graph of S is formed as depicted in bold lines in Fig. 8b. A cycle basis of $I(S)$ consists of 11 regional cycles leading to 18 subgraphs forming a GCB of S. On each subgraph one self-equilibrating stress system (S.E.S) can be constructed, corresponding to a suitable statical basis. Typical elements of the selected GCB are shown in Fig. 8c.

The regions of S after being embedded on the plane do not need to be all triangulated. For such models, however, different types of cycles for $I(S)$ can be employed (Kaveh [35]).

6 Transformations in an identical dimensional space

6.1 Graph models of finite element meshes

In order to transform the nodal numbering of a finite element mesh (model) into the graph nodal ordering, six transformations are presented in this section (Kaveh and Roosta [37]). Such transformations can also be utilized for the formation of localized self-equilibrating stress systems in the force method of analysis leading to sparse flexibility matrices $\boldsymbol{G} = \boldsymbol{B}_1^t \boldsymbol{F}_m \boldsymbol{B}_1$. Examples can be found in Kaveh [32].

6.1.1 Element clique graph method

Definition The *element clique graph S* of an FEM, denoted by ECG, is a graph whose nodes are the same as those of the finite element model (FEM) and two nodes n_i and n_j of S are connected by an edge (member) if n_i and n_j belong to the same element in the FEM.

This model is exceptionally easy because it provides a one-stage process for nodal numbering of FEMs with elements having mid-side nodes (higher-order elements). This definition is directly applicable to both 2D and 3D finite element models. The ECG of the FEM shown Fig. 9a is illustrated in Fig. 9b.

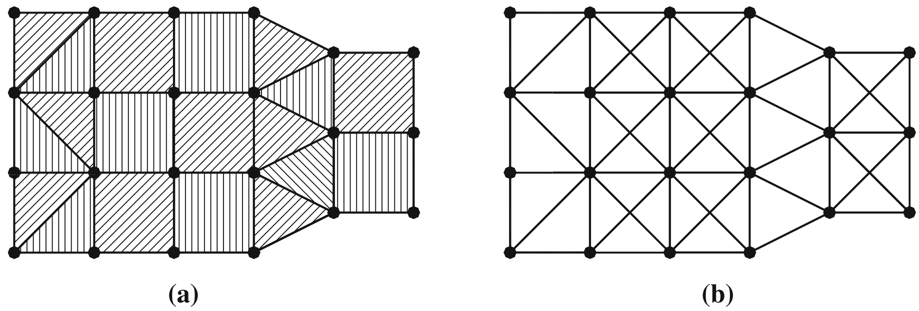


Fig. 9 An FEM and the corresponding element click graph. **a** A finite element model. **b** Element clique graph of the FEM

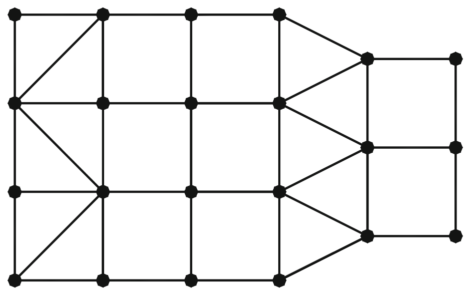


Fig. 10 Skeleton graph of the FEM

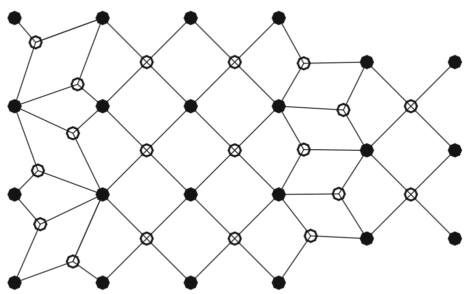


Fig. 11 Element star graph of the FEM

6.1.2 Skeleton graph method

Definition The 1-skeleton graph (*skeleton graph) S* of an FEM, denoted by SKG, is a graph whose nodes are the same as those of the FEM, and its edges are the members of the FEM.

This model is very simple, and it provides a one-stage process for nodal numbering of the FEMs; however, it transfers only partial connectivity properties of the FEMs to that of the graph. Naturally, the numbering obtained using this graph cannot be considered efficient in particular for models with higher-order elements. The SKG of the FEM shown Fig. 9a is illustrated in Fig. 10.

6.1.3 Element star graph method

Definition The *element star graph S* of an FEM, denoted by ESG, has two sets of nodes; namely the main set containing the same nodes as those of the FEM and the virtual set consisting of the virtual nodes in a one-to-one correspondence with the elements of the FEM. The edge set of *S* is constructed by connecting the virtual node of each element i to all nodes of the element i. The ESG of the FEM shown Fig. 9a is illustrated in Fig. 11.

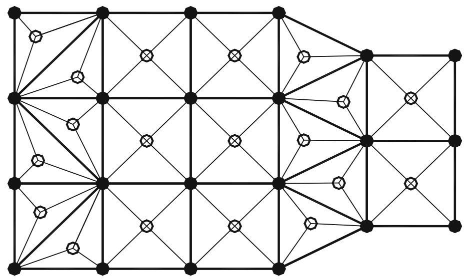


Fig. 12 Element wheel graph of the FEM

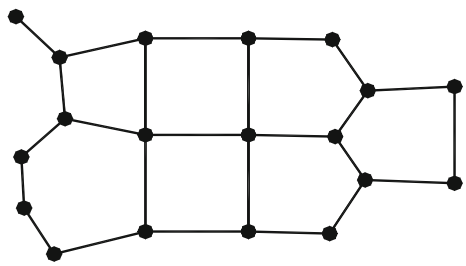


Fig. 13 Natural associate graph of the FEM

6.1.4 Element wheel graph method

Definition The *element wheel graph S* of an FEM, denoted by EWG, is the union of the element star graph and the skeleton graph of the FEM. The element wheel graph of the FEM shown in Fig. 9a is illustrated in Fig. 12. The virtual nodes are shown by bigger dots.

This model is in effect the combination of a skeletal graph and the star graph. Naturally, it provides a better connectivity property for an FEM.

6.1.5 Natural associate graph method

Definition The *natural associate graph* S of an FEM, denoted by NAG, has its nodes in a one-to-one correspondence with elements of the FEM, and two nodes of S are connected by an edge if the corresponding elements have a common boundary. The natural associate graph of the FEM shown in Fig. 9a is illustrated in Fig. 13.

This model is very simple, and it provides a powerful two-stage process for nodal numbering of the FEMs. In the first stage, the nodes of the associate graph are ordered, and in the second stage the nodes within each element are numbered in the same sequence obtained in the first stage. In the latter stage, priority should be defined for ordering the nodes within each element.

6.1.6 Incidence graph method

Definition The *incidence graph S* of an FEM, denoted by ING, has its nodes in a one-to-one correspondence with the elements of the FEM, and two nodes are connected with an edge if the corresponding elements have a common node.

This graph is simple and provides a better connectivity than the associate graph in the expense of additional edges. The ING of the FEM shown Fig. 9a is illustrated in Fig. 14.

6.2 Rigidity of grid-shaped planar trusses; a bipartite graph

The study of the rigidity of planar trusses is due to Laman [38], who found the necessary and sufficient conditions for the rigidity of this type of structures. Lovasz and Yemini [39] and Sugihara [40] developed

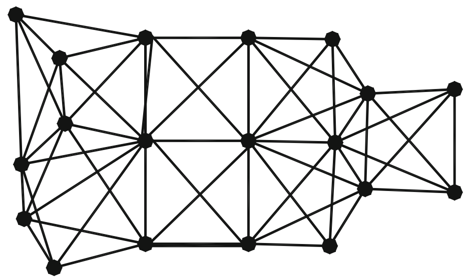


Fig. 14 Incidence graph of the FEM

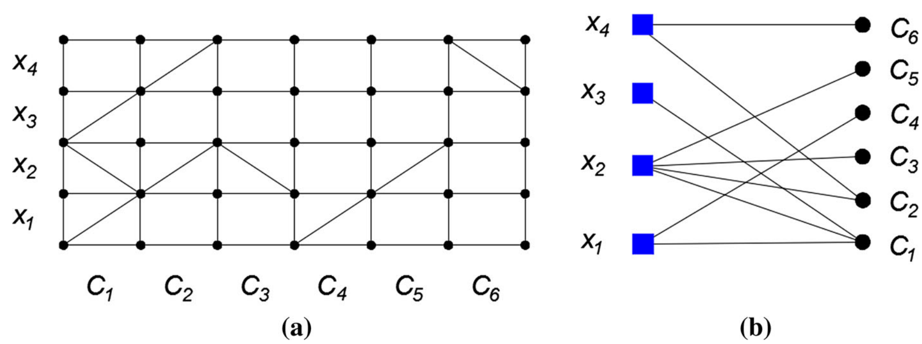


Fig. 15 A grid-shaped planar truss and its bipartite graph. **a** A planar truss S. **b** The bipartite graph $B(S)$ of S

algorithms for controlling the rigidity. There is a special type of planar trusses for which the rigidity can be checked more efficiently. In this approach, a bipartite graph is defined for the truss, and the connectedness of this graph indicates the rigidity of the main truss (Bolker and Crapo [41]).

Consider a grid-shaped planar truss with m rows and n columns with some diagonal bracings. Associate one vertex with each row and one vertex with each column. Connect a row vertex to a column vertex if the corresponding panel has a diagonal member (edge), resulting in a bipartite graph $B(S)$. It can easily be proved that S is rigid if $B(S)$ is connected. Furthermore, at least $m + n - 1$ diagonal members are needed for minimal rigidity of S.

Example A grid-shaped planar truss is shown in Fig. 15a. The bipartite graph $B(S)$ of S is illustrated in Fig. 15b. Since $B(S)$ is connected, S is rigid. It can be seen that the removal of any diagonal bracing member of S will result in a disconnected $B(S)$, destroying the rigidity of S.

This method is generalized and applied to the formation of generalized cycle bases of this type of planar trusses (Kaveh [35]).

6.3 Generalized ordering

Efficient graph theoretical methods were developed for structuring square sparse matrices [42]. In some cases, it is necessary to structure the rectangular matrices. An example of this is the algebraic force method of structural analysis where it is necessary to make the equilibrium matrices banded. This makes the efficient application of the turn-back method feasible (Kaveh [43]).

6.3.1 Element ordering for frontwidth reduction; a line graph

In frontal solution, the elements of the model should be ordered in place of its nodes. This can easily be achieved by defining a line graph of the model. A *line graph* $L(S)$ of S has its vertices in a one-to-one correspondence with the elements of the model, and two vertices are connected by an edge if the corresponding elements are incident (if the model is a skeletal structure) or have a common boundary (if the model is an FE model). Nodal ordering of the line graph leads to the element ordering of the original model, corresponding to a reasonably narrow frontwidth.

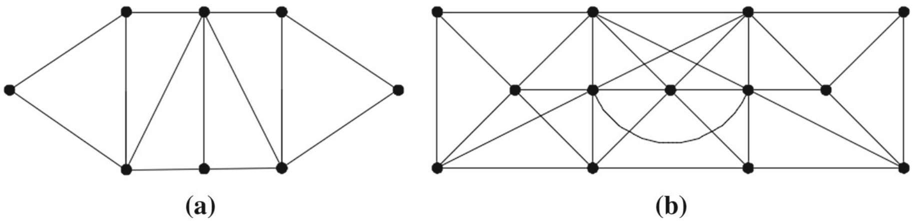


Fig. 16 A planar structural model and its line graph. **a** A planar model S. **b** The line graph of S

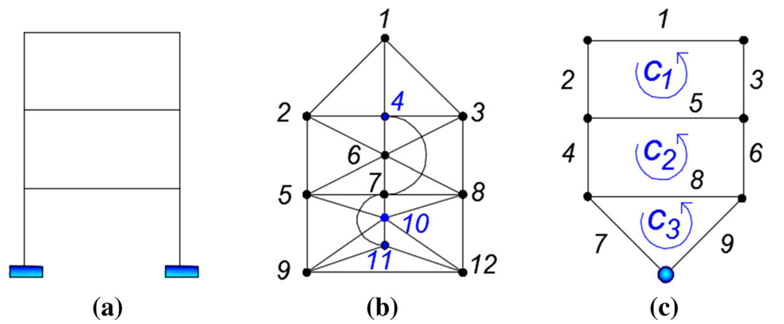


Fig. 17 Member and cycle ordering of a frame structure. **a** Frame structure S. **b** The corresponding K-total graph. **c** Ordered members and cycles

Example A planar graph S is considered as the model of a skeletal structure, Fig. 16a. The corresponding line graph is constructed in Fig. 16b. The nodes of $L(S)$ are ordered. This gives a favorable ordering for the elements of the original model, although the corresponding frontwidth may differ from minimum by a small amount.

6.3.2 Element and nodal ordering; K-total graph

In structural analysis, it may be desirable to reduce the bandwidth of some sparse rectangular matrices. As an example, the bandwidth of the coefficient matrix of the equilibrium equations can be reduced in the algebraic force method when the turn-back technique is utilized. Similarly the bandwidth of the cycle-member incidence matrix may be reduced for a compact storage (Kaveh [43]). For reducing the bandwidth of such matrices, simultaneous ordering of the elements of cycle (cut-set) basis and members will be required. The K-*total graph* of a graph is defined for this purpose as follows:

Associate one vertex with each member and with each element of the selected cycle (cut-set) basis of S. Connect two vertices with an edge if (a) the corresponding members are incident, (b) the corresponding cycles (cut-sets) are adjacent, and (c) the corresponding member and cycle (cut-set) are incident. A nodal numbering of this graph leads to the simultaneous ordering of the members and the elements of the selected basis of S.

Example Consider a planar frame as shown in Fig. 17a. For the selected cycle basis of the graph model, the corresponding K-total graph and its nodal ordering are shown in Fig. 17b. The final ordering of the members and elements (a cycle basis in this case) are depicted in Fig. 17c.

6.4 Coarsening of large-scale graph models using matching

For multi-member (large-scale) graphs, application of a method might be infeasible or highly time-consuming. In such a case, the size of the graph can be reduced in a sequential coarsening process using graph matching. The intended method can be used to the coarsened graph, and the results can be obtained. These results can then be uncoarsened to obtain the final result for the initial large-scale model.

A matching of a graph is a set of edges, no two of which are incident on the same vertex. The graph S can be transformed into a sequence of graphs $S_1; S_2; \ldots; S_m$ with smaller number of nodes such that $|n_0| > |n_1| > |n_2| > \cdots > |n_m|$. Graph coarsening can be achieved in different ways. When the nodes $u, v \in n_i$ are contracted to form vertex $w \in n_{i+1}$, the weight of vertex w is set to the sum of the weights of

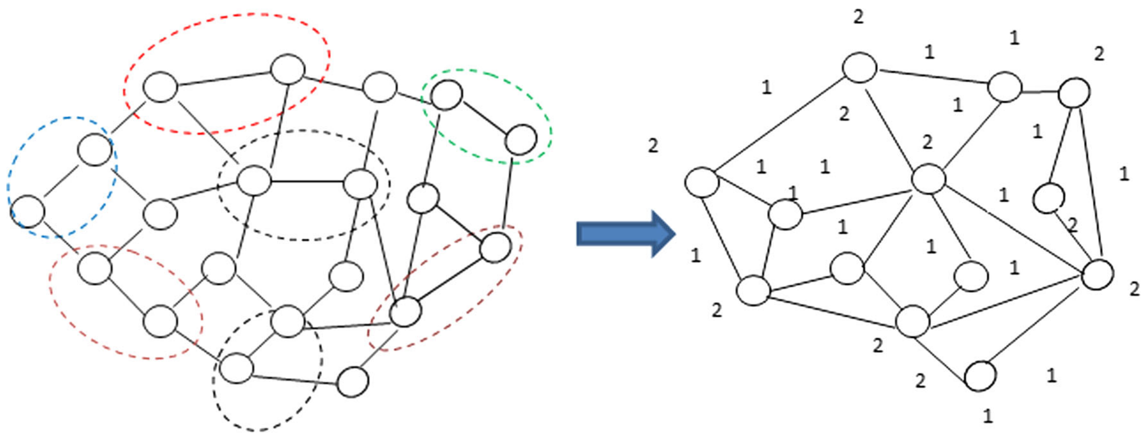


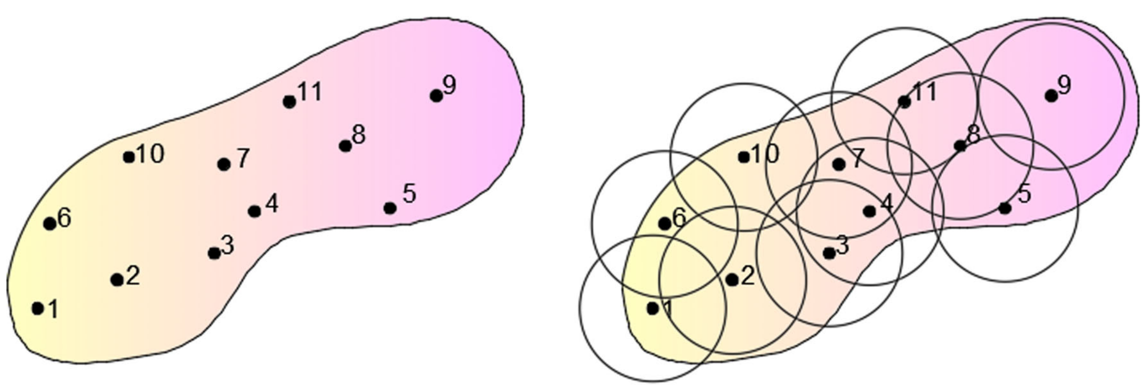
Fig. 18 A Coarsening process by the heavy edge matching

vertices u and v, while the weights of the edges incident on w are set equal to the sum of the weights of the edges incident on u and v minus the weight of the edge $(u; v)$. If there is an edge that is interceding both u and v, then the weight of this edge is set equal to the sum of the weights of these edges. Nodes that are not incident on any edge of the matching are simply copied over to S_{i+1}. For this reason, maximal matchings are used to obtain successively coarser graphs. A matching is maximal if any edge in the graph that is not in the matching has at least one of its endpoints matched. Since maximal matchings are used to coarsen the graph, the number of vertices in S_{i+1} cannot be less than half of the number of vertices in S_i. When vertices $u; v \in n_i$ are contracted to form the vertex $w \in n_{i+1}$, the weight of vertex w is set to the sum of the weights of these vertices.

There are different methods for selecting the maximal matching. An example of heavy edge matching (HEM) is shown in Fig. 18. Finding a maximal matching that contains edges with large weights is the idea behind the HEM. An HEM is computed using a randomized algorithm similar to that for computing a random matching (Birn et al. [44]). The vertices are again visited in random order. However, instead of randomly matching a vertex u with one of its adjacent unmatched vertices, u is matched with the vertex v such that the weight of the edge $(u; v)$ is maximum over all valid incident edges (heavier edge). This process can be repeated until the model becomes manageable for the identical purpose. For two different applications, one can refer to Kaveh and Rahimi Bondarabady [45] and Kaveh and Ghobadi [46].

6.5 Graph models for meshless discretization

In this section, four graphs are defined for representing the connectivity of a meshless model, Ref. [47]. Consider a domain as shown in Fig. 19a with the meshless model consisting of eleven nodes with the domains of influence shown in Fig. 19b. For the sake of clarity, the associated graphs for this model are presented. These graphs are defined as follows:



(a) **(b)**

Fig. 19 A domain and its meshless model. **a** A simple domain. **b** The meshless model consisting of eleven nodes with the domains of influence

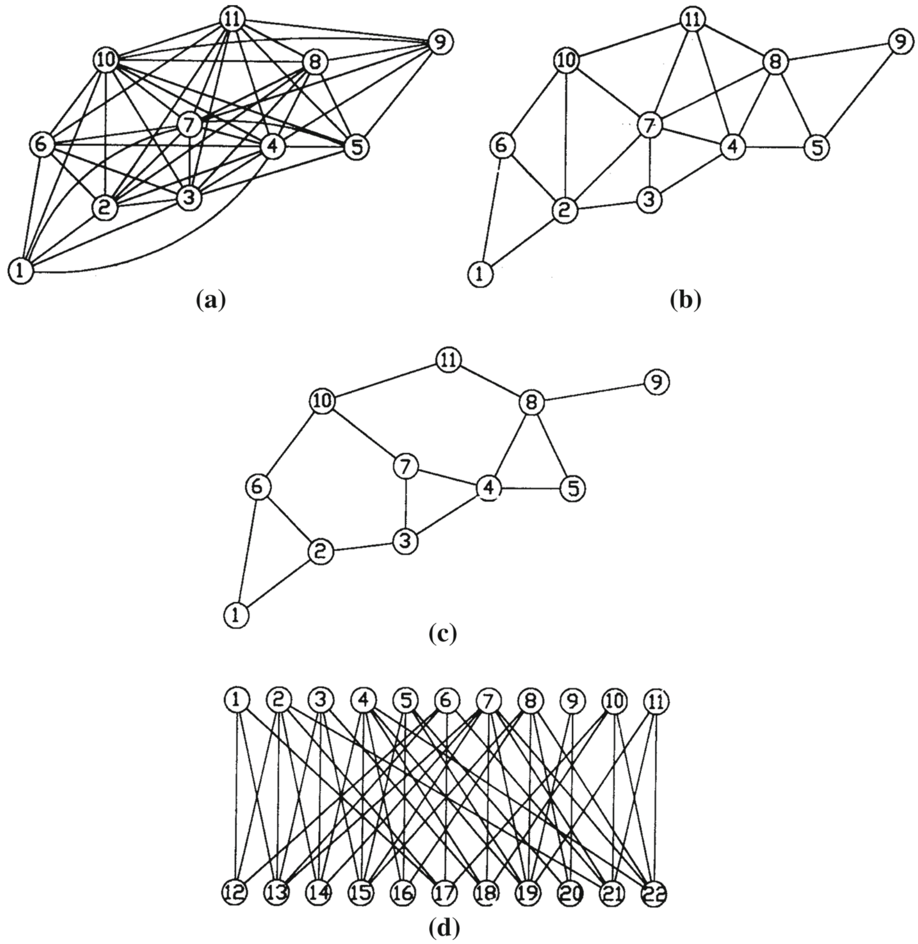


Fig. 20 Four different graphs associated with the meshless model. **a** The SCAG of a meshless model. **b** The PCAG of a meshless model. **c** The WCAG of a meshless model. **d** Associate bipartite graph of the model

The *Strongly Connected Associate graph* (SCAG) of a meshless model is a graph whose nodes are the same as those of the meshless model and two nodes n_i and n_j of the SCAG are connected with an edge if and only if $\Omega_i \cap \Omega_j \neq 0$ in the meshless model. Figure 20a shows the strongly connected associate graph of the model.

The *Partially Connected Associate Graph* (PCAG) of a meshless model is a graph whose nodes are the same as those of the meshless model and two nodes n_i and n_j of the PCAG are connected with an edge if and only if $I \in \Omega_i$ or $J \in \Omega_j$ in the meshless model. Figure 20b shows the partially connected associate graph of the model.

The *Weakly Connected Associate Graph* (WCAG) of a meshless model is a graph whose nodes are the same as those of the meshless model and two nodes n_i and n_j of the PCAG are connected with an edge if and only if $I \in \Omega_i$ and $J \in \Omega_j$ in the meshless model. Figure 20c shows the weakly connected associate graph of the model.

The *Associate Bipartite Graph* (ABG) has two sets A and B corresponding to nodes and influence domains, respectively. A node n_i of A is connected to $n_j \in B$ by an edge if and only if $I \in \Omega_j$. Figure 20d shows the associate bipartite graph of the model.

It can be proven that the strongly connected associate graph is always a connected graph. For the model with 11 nodes shown in Fig. 19a, b, the number of edges for SCAG, PCAG, WCAG, and ABG are 46, 31, 15, and 47, respectively, as depicted in Fig. 20a–d.

7 Concluding remarks

A collection of graph transformations is presented for the study of different topological properties of structures. These transformations provide useful tools for an optimal analysis of the structures. The presented methods reduce the computational time and storage requirements for the analysis of structures and finite element models. These transformations provide additional information and also make the swift analysis of large structures feasible.

The developed methods form a solid basis for the design of various efficient methods for the different purposes. As an example, a suitable cycle basis of a graph for the flexibility analysis can be generated using the author's expansion process, which can be considered as a single-cycle disk embedding with simple conditions for interfaces.

Weighted graphs and their transformations provide efficient means for selecting special static and kinematical bases leading to well-conditioned structural matrices. In the process, an attempt is made to select bases leading to small off-diagonal entries for the structural matrices. The developed methods are also applied to finite element models (Kaveh [32]). Graph theoretical methods are also applied to form-finding and design of tensegrity structures by Koohestani [48,49] and Koohestani and Guest [50].

Graph theory is also applied to the optimal design of structures using mathematical programming. Examples of this can be found in the application of graphs in plastic analysis and design of frame structures [1]. Some other transformations can facilitate the future new developments in the theory of graphs to be transferred to solving other problems in structural mechanics.

Though other tools like meta-heuristic optimization algorithms are applied to similar problems discussed in this paper, the efficiency of the combinatorial optimization methods is much higher than that of the meta-heuristic approaches [51].

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