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ORIGINAL PAPER

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Another form of 3D Green's function for an elastic solid with exponential inhomogeneity

This paper is dedicated to the memory of Franz Ziegler

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Abstract The 3D Green's function for an elastic solid with exponential inhomogeneity has already been obtained by Martin et al. (Proc R Soc Lond A 458:1931–1947, 2002). But, their Green's function is separated into grading and non-grading parts, and the grading part is in terms of infinite and double finite integrals. In order to obtain a more convenient form for numerical evaluation and for clear insight into physical meanings, the present paper reconsiders the Green's function for an inhomogeneous elastic solid. Applying the Cauchy complex integral theorem to complex-valued Hankel inversion integrals, Green's function is reduced to real-valued finite integrals, whose integrands are characterized by exponential decay, and has no singularity. It is thus more convenient for the numerical evaluation.

1 Introduction

Green's function for an inhomogeneous elastic solid, as a model of a functionally graded material, is important for developing computer codes, such as the boundary element method (BEM) [2]. The simpler the mathematical form of the Green's function the better for computational efficiency. When the Green's function is given in the form of an integral, the simplest form of the integral is also the most useful.

Martin et al. [1] have obtained a 3D Green's function for an exponentially graded elastic solid. Their Green's function is separated into non-graded and graded parts. The non-graded part is just Kelvin's solution and is very simple, but the graded part is very complicated and is given in terms of a double integral and infinite integral. In general terms, their Green's function is very complicated. They also did not show any explicit form of the Green's function, and thus, their Green's function is not user-friendly for coding engineers. The present paper reconsiders the Green's function and presents a simpler form of it. In order to make the present paper more user-friendly, an explicit form of the Green's function, which is given in terms of a finite integral, is presented.

2 Formal solution

Before starting the analysis, it should be noted that a tri-directional/axial inhomogeneity can be reduced to a unidirectional one. For example, let us consider the triaxial exponential inhomogeneity, as $\exp(\kappa_x x' + \kappa_y y' + \kappa_z z')$. Its single variable $\kappa_x x' + \kappa_y y' + \kappa_z z' = const.$ gives a plane in the 3D space. If we rotate the coordinate axes and introduce the new axes (x, y, z), so that the new z-axis is normal to the plane $\kappa_x x' + \kappa_y y' + \kappa_z z' = const.$ and the (x, y)-plane is parallel to it, the triaxial inhomogeneity can be reduced to an unidirectional

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inhomogeneity as $\exp(\kappa z)$ with $\kappa = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$. Thus, when we discuss a solid of infinite extent, it is sufficient to consider Green's function for the solid with a unidirectional inhomogeneity. This has already been discussed for a 2D Green's function by the author [3]. He has obtained an exact closed form of the 2D Green's function as well.

In the present paper, we employ an isotropic elastic solid with Lame's moduli, λ and μ, and their unidirectional inhomogeneity with exponential function of the space variable z,

$$\lambda(z) = \lambda_0 \exp(\kappa z), \quad \mu(z) = \mu_0 \exp(\kappa z) \tag{2.1}$$

where $''\kappa''$ is an inhomogeneity parameter, and the Poisson ratio ν is constant throughout the medium. Employing Cartesian coordinates (x, y, z), we assume that the inhomogeneous solid is of infinite extent and a point body force with the magnitude $P_j,\ j = x, y, z$ is acting at an arbitrary point (x_0, y_0, z_0). Thus, the equilibrium equations are given by

$$\sigma_{ji,j} = -P_i \delta(x - x_0)\delta(y - y_0)\delta(z - z_0); \quad i, j = x, y, z \tag{2.2}$$

where $\delta(.)$ is Dirac's delta function. Hooke's law for the isotropic solid is also employed,

$$\sigma_{ij} = \lambda(z) u_{k,k}\, \delta_{ij} + 2\mu(z)(u_{i,j} + u_{j,i}); \quad i, j, k = x, y, z, \tag{2.3}$$

where δ_{ij} is Kronecker's delta. Lamé's constants are defined by Eq. (2.1).

Substituting Hooke's law into the equilibrium equation, we obtain the displacement equations,

$$\begin{aligned} &\gamma^2 u_{x,xx} + u_{x,yy} + u_{x,zz} + \kappa u_{x,z} + (\gamma^2 - 1) u_{y,xy} + (\gamma^2 - 1) u_{z,xz} + \kappa u_{z,x} \\ &\quad = -\frac{P_x}{\mu_0} \exp(-\kappa z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \end{aligned} \tag{2.4.1}$$

$$\begin{aligned} &(\gamma^2 - 1) u_{x,xy} + u_{y,xx} + \gamma^2 u_{y,yy} + u_{y,zz} + \kappa u_{y,z} + (\gamma^2 - 1) u_{z,yz} + \kappa u_{z,y} \\ &\quad = -\frac{P_y}{\mu_0} \exp(-\kappa z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \end{aligned} \tag{2.4.2}$$

$$\begin{aligned} &(\gamma^2 - 1) u_{x,xz} + \kappa(\gamma^2 - 2) u_{x,x} + (\gamma^2 - 1) u_{y,yz} + \kappa(\gamma^2 - 2) u_{y,y} + u_{z,xx} + u_{z,yy} + \gamma^2 u_{z,zz} + \kappa\gamma^2 u_{z,z} \\ &\quad = -\frac{P_z}{\mu_0} \exp(-\kappa z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \end{aligned} \tag{2.4.3}$$

where the elasticity parameter γ is defined and expressed by the Poisson ratio ν,

$$\gamma^2 = \frac{\lambda + 2\mu}{\mu} = \frac{\lambda_0 + 2\mu_0}{\mu_0} = \frac{2(1 - \nu)}{1 - 2\nu}. \tag{2.5}$$

In order to solve the coupled differential equations (2.4), we introduce new unknowns $U_j(x, y, z)$ defined by

$$u_j(x, y, z) = U_j(x, y, z) \exp(-\kappa z/2); \quad j = x, y, z, \tag{2.6}$$

and then the displacement equations (2.4) are rewritten for the new unknowns,

$$\begin{aligned} &\gamma^2 U_{x,xx} + U_{x,yy} + U_{x,zz} - (\kappa/2)^2 U_x + (\gamma^2 - 1) U_{y,xy} + (\gamma^2 - 1) U_{z,xz} - (\gamma^2 - 3)(\kappa/2) U_{z,x} \\ &\quad = -\frac{P_x}{\mu_0} \exp(-\kappa z/2)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \end{aligned} \tag{2.7.1}$$

$$\begin{aligned} &(\gamma^2 - 1) U_{x,xy} + U_{y,xx} + \gamma^2 U_{y,yy} + U_{y,zz} - (\kappa/2)^2 U_y + (\gamma^2 - 1) U_{z,yz} - (\gamma^2 - 3)(\kappa/2) U_{z,y} \\ &\quad = -\frac{P_y}{\mu_0} \exp(-\kappa z/2)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \end{aligned} \tag{2.7.2}$$

$$\begin{aligned} &(\gamma^2 - 1) U_{x,xz} + (\kappa/2)(\gamma^2 - 3) U_{x,x} + (\gamma^2 - 1) U_{y,yz} + (\kappa/2)(\gamma^2 - 3) U_{y,y} + U_{z,xx} + U_{z,yy} \\ &\quad + \gamma^2 U_{z,zz} - \gamma^2 (\kappa/2)^2 U_z \\ &\quad = -\frac{P_z}{\mu_0} \exp(-\kappa z/2)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0). \end{aligned} \tag{2.7.3}$$

Green's function/dyadic is a particular solution corresponding to the nonhomogeneous term of the body force. In order to obtain the particular solution, we apply a triple Fourier transform with respect to three space variables. Each transform pair is defined by

$$\begin{aligned}
\tilde{f}(\xi) &= \int_{-\infty}^{+\infty} f(x)\exp(+i\xi x)dx \quad \Leftrightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(\xi)\exp(-i\xi x)d\xi, \\
\bar{f}(\eta) &= \int_{-\infty}^{+\infty} f(y)\exp(+i\eta y)dy \quad \Leftrightarrow \quad f(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{f}(\eta)\exp(-i\eta y)d\eta, \\
\hat{f}(\zeta) &= \int_{-\infty}^{+\infty} f(z)\exp(+i\zeta z)dz \quad \Leftrightarrow \quad f(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\zeta)\exp(-i\zeta z)d\zeta.
\end{aligned} \tag{2.8}$$

The convergence condition at infinity,

$$U_i = U_{i,j} = 0; \quad \sqrt{x^2+y^2+z^2} \to \infty, \quad i, j = x, y, z, \tag{2.9}$$

is implicitly incorporated into the transform integral. Then, the simple algebraic equations for the transformed displacement are obtained as

$$\begin{aligned}
&\left\{\gamma^2\xi^2+\eta^2+\zeta^2+(\kappa/2)^2\right\}\hat{\bar{\tilde{U}}}_x + \xi\eta(\gamma^2-1)\hat{\bar{\tilde{U}}}_y - i\xi\left\{+i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)\right\}\hat{\bar{\tilde{U}}}_z \\
&\quad = \frac{P_x}{\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0+i\zeta z_0),
\end{aligned} \tag{2.10.1}$$

$$\begin{aligned}
&\xi\eta(\gamma^2-1)\hat{\bar{\tilde{U}}}_x + \left\{\xi^2+\gamma^2\eta^2+\zeta^2+(\kappa/2)^2\right\}\hat{\bar{\tilde{U}}}_y - i\eta\left\{+i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)\right\}\hat{\bar{\tilde{U}}}_z \\
&\quad = \frac{P_y}{\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0+i\zeta z_0),
\end{aligned} \tag{2.10.2}$$

$$\begin{aligned}
&\left\{-i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)\right\}\left(+i\xi\hat{\bar{\tilde{U}}}_x + i\eta\hat{\bar{\tilde{U}}}_y\right) + \left\{\xi^2+\eta^2+\gamma^2\zeta^2+\gamma^2(\kappa/2)^2\right\}\hat{\bar{\tilde{U}}}_z \\
&\quad = \frac{P_z}{\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0+i\zeta z_0).
\end{aligned} \tag{2.10.3}$$

Solving the above equations for the displacements $\hat{\bar{\tilde{U}}}_j(\xi,\eta,\zeta)$, we apply the Fourier inversion integral with respect to the parameter ζ,

$$\begin{aligned}
\bar{\tilde{U}}_x(\xi,\eta,z) &= \frac{P_x}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0) \\
&\quad \times \int_{-\infty}^{\infty} \frac{\gamma^2(\zeta^2+\alpha_0^2)^2-(\gamma^2-1)\xi^2(\zeta^2+\alpha_0^2)+4(\gamma^2-2)(\kappa/2)^2\eta^2}{(\zeta^2+\alpha_0^2)(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
&\quad + \frac{P_y}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0) \\
&\quad \times (-\xi\eta)\int_{-\infty}^{\infty} \frac{(\gamma^2-1)(\zeta^2+\alpha_0^2)+4(\gamma^2-2)(\kappa/2)^2}{(\zeta^2+\alpha_0^2)(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
&\quad + \frac{P_z}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0) \\
&\quad \times (+i\xi)\int_{-\infty}^{\infty} \frac{+i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)}{(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta,
\end{aligned} \tag{2.11.1}$$

$$\begin{aligned}
\tilde{\tilde{U}}_y(\xi,\eta,z) = {}& \frac{P_x}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times(-\xi\eta)\int_{-\infty}^{\infty}\frac{(\gamma^2-1)(\zeta^2+\alpha_0^2)+4(\gamma^2-2)(\kappa/2)^2}{(\zeta^2+\alpha_0^2)(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
& + \frac{P_y}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times\int_{-\infty}^{\infty}\frac{\gamma^2(\zeta^2+\alpha_0^2)^2-(\gamma^2-1)\eta^2(\zeta^2+\alpha_0^2)+4(\gamma^2-2)(\kappa/2)^2\xi^2}{(\zeta^2+\alpha_0^2)(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
& + \frac{P_z}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times(+i\eta)\int_{-\infty}^{\infty}\frac{+i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)}{(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta,
\end{aligned} \tag{2.11.2}$$

$$\begin{aligned}
\tilde{\tilde{U}}_z(\xi,\eta,z) = {}& \frac{P_x}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times(-i\xi)\int_{-\infty}^{\infty}\frac{-i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)}{(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
& + \frac{P_y}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times(-i\eta)\int_{-\infty}^{\infty}\frac{-i\zeta(\gamma^2-1)+(\gamma^2-3)(\kappa/2)}{(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta \\
& + \frac{P_z}{2\pi\gamma^2\mu_0}\exp(-\kappa z_0/2 + i\xi x_0 + i\eta y_0) \\
& \times\int_{-\infty}^{\infty}\frac{(\zeta^2+\alpha_0^2)+(\gamma^2-1)(\eta^2+\xi^2)}{(\zeta^2+\alpha_1^2)(\zeta^2+\alpha_2^2)}\exp\{-i\zeta(z-z_0)\}d\zeta
\end{aligned} \tag{2.11.3}$$

where

$$\alpha_0 = \sqrt{\xi^2+\eta^2+(\kappa/2)^2}, \tag{2.12}$$

$$\alpha_1 = \sqrt{\left\{\sqrt{\xi^2+\eta^2}+i(\kappa p/2)\right\}^2+(\kappa q/2)^2}, \quad \alpha_2 = \sqrt{\left\{\sqrt{\xi^2+\eta^2}-i(\kappa p/2)\right\}^2+(\kappa q/2)^2}. \tag{2.13}$$

Here, the new material parameters p and q are defined as

$$p^2 = 1 - 2/\gamma^2, \quad q^2 = 1 + p^2 = 2(1 - 1/\gamma^2). \tag{2.14}$$

All integrands in Eq. (2.11) have simple poles at $\zeta = \pm i\alpha_0,\ \pm i\alpha_1,\ \pm i\alpha_2$ and can be evaluated exactly by applying the residue theorem. Otherwise, the integration formulas [4, Vol. 1, p. 8(11) and 65(15)],

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{\zeta^2+\alpha_j^2}\exp\{-i\zeta(z-z_0)\}d\zeta = \frac{\pi}{\alpha_j}\exp(-\alpha_j|z-z_0|), \\
&\int_{-\infty}^{\infty}\frac{+i\zeta}{\zeta^2+\alpha_j^2}\exp\{-i\zeta(z-z_0)\}d\zeta = \begin{cases} +\pi\exp(-\alpha_j|z-z_0|); & z-z_0>0 \\ -\pi\exp(-\alpha_j|z-z_0|); & z-z_0<0 \end{cases},
\end{aligned} \tag{2.15}$$

are applied after partial fractions of the integrand. Then, we have

$$\tilde{\tilde{U}}_j(\xi,\eta,z)=\frac{P_k}{\mu_0}\exp(-\kappa z_0/2+i\xi x_0+i\eta y_0)f_{jk}(\xi,\eta,z-z_0);\quad j,k=x,y,z \tag{2.16}$$

where $f_{jk}(\xi,\eta,z-z_0)$ are given in "Appendix A".

Next, we consider the double Fourier inversion with respect to the parameters ξ and η. Its formal form is given by

$$U_j(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\tilde{\tilde{U}}_j(\xi,\eta,z)\exp(-i\xi x-i\eta y)d\xi d\eta;\quad j=x,y,z, \tag{2.17}$$

and a more explicit form is

$$U_j(x,y,z)=\frac{P_k}{\mu_0}\exp(-\kappa z_0/2)\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{jk}(\xi,\eta,z-z_0)\exp\{-i\xi(x-x_0)-i\eta(y-y_0)\}d\xi d\eta;$$
$$j,k=x,y,z. \tag{2.18}$$

We introduce the polar coordinates (r,θ) in the actual space and (ρ,φ) in the transformed space. They are defined by

$$x-x_0=r\cos\theta,\quad y-y_0=r\sin\theta, \tag{2.19}$$
$$\xi=\rho\sin\varphi,\quad \eta=\rho\cos\varphi. \tag{2.20}$$

Due to these variable transforms, the displacement $U_j(x,y,z)$ is replaced by $U_j(r,\theta,z)$ and is given by the double integral with respect to two variables, ρ and φ,

$$U_j(r,\theta,z)=\frac{P_k}{\mu_0}\exp(-\kappa z_0/2)\frac{1}{(2\pi)^2}\times\int_{\rho=0}^{\infty}\int_{\varphi=0}^{2\pi}f_{jk}(\xi=\rho\sin\varphi,\eta=\rho\cos\varphi,z-z_0)\exp\{-i\rho r\sin(\varphi+\theta)\}\rho d\varphi d\rho. \tag{2.21}$$

The integral with respect to the image angle φ can be evaluated by using the formulas in "Appendix B". The displacement component with cylindrical coordinates is then expressed in terms of the single integral with respect to the image distance ρ as

$$U_i(r,\theta,z)=\frac{P_j}{8\pi\mu_0}\exp(-\kappa z_0/2)g_{ij}(r,\theta,z-z_0);\quad i,j=x,y,z \tag{2.22}$$

where the explicit form of the quasi-dyadic $g_{ij}(r,\theta,z-z_0)$ is given in "Appendix C".

We have now obtained the formal Green's function in terms of an infinite integral, i. e., in the form of Hankel inversion integral, but its integrand is complex-valued. Such integrals are not suitable for numerical evaluation, since their integrands have complex-valued radicals of a fluctuating nature. Thus, we need a simpler form of the integral.

3 Transform to real-valued finite integrals

This Section is the principal part of the present paper. It is concerned with the transformation of the complex-valued infinite integral to a real-valued finite integral. Inspecting the dyadic-like function g_{ij} in "Appendix C", it will be seen that eight integrals have to be discussed. The first two integrals, which include the radical α_0 only,

$$I_0^{(0)}(r,z)=\int_0^{\infty}\frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_0(\rho r)d\rho \tag{3.1}$$

and

$$I_2^{(0)}(r,z)=\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_2(\rho r)d\rho, \tag{3.2}$$

can be evaluated exactly by applying the integration formulas [4, Vol. 2, p. 9 (24) and p. 19 (10)],

$$\int_0^\infty \frac{x}{\sqrt{x^2+\beta^2}}\exp\left(-\alpha\sqrt{x^2+\beta^2}\right)J_0(xy)dx=\frac{1}{\sqrt{\alpha^2+y^2}}\exp\left(-\beta\sqrt{\alpha^2+y^2}\right) \tag{3.3}$$

$$\int_0^\infty \frac{1}{\sqrt{x^2+\beta^2}}\exp\left(-\alpha\sqrt{x^2+\beta^2}\right)J_1(xy)dx=\frac{1}{\beta y}\left\{\exp(-\alpha\beta)-\exp\left(-\beta\sqrt{\alpha^2+y^2}\right)\right\} \tag{3.4}$$

and the recurrence relation for the Bessel function [5, p. 17],

$$J_{n+1}(z)=\frac{2n}{z}J_n(z)-J_{n-1}(z). \tag{3.5}$$

Their results are given by Eqs. (D.1) and (D.2) in "Appendix D".

The other six integrals have to be discussed. They are also in the form of a Hankel transform,

$$I_0^{(+)}(r,z)=\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_0(\rho r)d\rho, \tag{3.6}$$

$$I_2^{(+)}(r,z)=\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_2(\rho r)d\rho, \tag{3.7}$$

$$I_0^{(-)}(r,z)=\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_0(\rho r)d\rho, \tag{3.8}$$

$$I_1^{(-)}(r,z)=\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_1(\rho r)d\rho, \tag{3.9}$$

$$I_2^{(-)}(r,z)=\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_2(\rho r)d\rho, \tag{3.10}$$

$$I_1^{(-)*}(r,z)=\int_0^\infty \rho\left\{\exp(-\alpha_1 z)-\exp(-\alpha_2 z)\right\}J_1(\rho r)d\rho \tag{3.11}$$

where r and z are parameters (not space variables). The real part of both radicals must be positive,

$$\mathrm{Re}(\alpha_j)\geq 0;\ j=1,2. \tag{3.12}$$

This condition guarantees the convergence of the integral. All integrands in these integrals contain two radicals α_1 and α_2. As defined by Eq. (C.11), these radicals have pure imaginary constant $\pm i(\kappa p/2)$ and are complex-valued functions. We have thus no suitable integration formula for these integrals and have to evaluate them numerically. Numerical evaluation is not easy, however, since Bessel and exponential functions in the integrand are of a fluctuating nature. So we shall try to convert the integral to a form more convenient for numerical evaluation.

Now, as a typical example, let us discuss the integral $I_0^{(+)}(r,z)$. In order to obtain a different form of the integral, we consider the complex integral defined by

$$\Phi=\oint_\Gamma \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}H_0^{(1)}(\rho r)d\rho \tag{3.13}$$

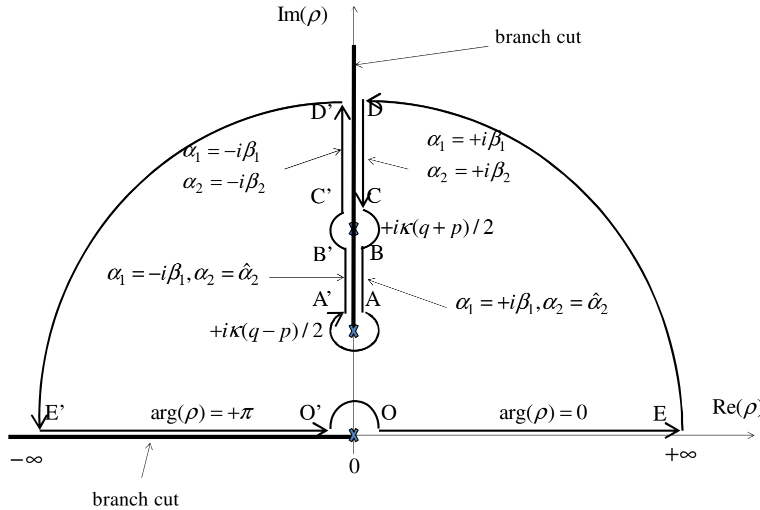

Fig. 1 Integration loop Γ and the argument of the radical

where r and z are positive real constants (not coordinates) and the real parts of two radicals, α_1 and α_2, are positive. It should be noticed that the Bessel function in $I_0^{(+)}(r, z)$ is replaced by the Hankel function of the first kind and of same order. The integration loop Γ will be determined a bit later.

Each radical has two branch points at

$$\alpha_1 : \rho = (\pm iq - ip)(\kappa/2), \quad \alpha_2 : (\pm iq + ip)(\kappa/2), \tag{3.14}$$

and thus, branch cuts are introduced along the imaginary axis in the complex ρ-plane so that the convergence condition, $\mathrm{Re}(\alpha_j) > 0$, can be satisfied. Further, as the Hankel function has no zeros in the upper plane, but has a logarithmic singularity at the origin, one more branch cut along the negative real axis must be introduced. Then, we take the integration loop Γ in the upper half plane so that the Hankel function of the first kind converges at the upper infinity, $\mathrm{Im}(\rho) \to +\infty$. The branch cuts and the integration loop in the upper plane are shown in Fig. 1. It should be understood that the radius ε of small circles around the branch points tends to zero and that the radius R of two large quarter circles goes to infinity. The arguments of the radical and integration variable along each path are listed in Table 1. Then, the integration loop Γ is composed of branch line integrals and the integrals along the circular paths, i.e.

$$\Gamma \equiv \overrightarrow{\mathrm{OE}} + \overset{\cap}{\mathrm{ED}} + \overrightarrow{\mathrm{DC}} + \overset{\cap}{\mathrm{CB}} + \overrightarrow{\mathrm{BA}} + \overset{\cap}{\mathrm{AA'}} + \overrightarrow{\mathrm{A'B'}} + \overset{\cap}{\mathrm{B'C'}} + \overrightarrow{\mathrm{C'D'}} + \overset{\cap}{\mathrm{D'E'}} + \overrightarrow{\mathrm{E'O'}} + \overset{\cap}{\mathrm{O'O}} \tag{3.15}$$

where the upper arrow and half circle denote the linear and circular paths, respectively. There is no singular point within the loop, and thus, the complex integral of Eq. (3.13) vanishes, according to Cauchy's theorem. Written in symbolic form,

$$\Phi = \oint_{\Gamma} = \int_{\overrightarrow{\mathrm{OE}}} + \int_{\overset{\cap}{\mathrm{ED}}} + \int_{\overrightarrow{\mathrm{DC}}} + \int_{\overset{\cap}{\mathrm{CB}}} + \int_{\overrightarrow{\mathrm{BA}}} + \int_{\overset{\cap}{\mathrm{AA'}}} + \int_{\overrightarrow{\mathrm{A'B'}}} + \int_{\overset{\cap}{\mathrm{B'C'}}} + \int_{\overrightarrow{\mathrm{C'D'}}} + \int_{\overset{\cap}{\mathrm{D'E'}}} + \int_{\overrightarrow{\mathrm{E'O'}}} + \int_{\overset{\cap}{\mathrm{O'O}}} = 0. \tag{3.16}$$

Furthermore, the integral along the small circular path around the branch point vanishes in the limit $\varepsilon \to 0$ and that along the large quarter circle also vanishes in the limit $R \to \infty$, since

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\{+i(z - \pi/4)\} \to 0; \quad \mathrm{Im}(z) \to +\infty \text{ in the upper plane.} \tag{3.17}$$

Then, the integral along the real axis is converted to one along the branch line as

$$\int_{\overrightarrow{\mathrm{OE}}} - \int_{\overrightarrow{\mathrm{O'E'}}} = \int_{\overrightarrow{\mathrm{CD}}} - \int_{\overrightarrow{\mathrm{C'D'}}} + \int_{\overrightarrow{\mathrm{AB}}} - \int_{\overrightarrow{\mathrm{A'B'}}}. \tag{3.18}$$

Table 1 Variable and argument of the radical on the integration path

Path	Variable ρ and its differential $d\rho$ *	Radical α_1	Radical α_2	Note
$\overrightarrow{OE}$	$\rho \Rightarrow \rho, d\rho \Rightarrow d\rho,$ $(0 < \rho < \infty)$	α_1	α_2	
$\overset{\cap}{\mathrm{ED}}$	$\rho \Rightarrow Re^{i\varphi},$ $d\rho \Rightarrow iRe^{i\varphi}d\varphi$ $(0 < \varphi < \pi/2)$	$R(\cos\varphi + i\sin\varphi)$	$R(\cos\varphi + i\sin\varphi)$	$R \to \infty$
$\overrightarrow{DC}$	$\rho \Rightarrow \rho e^{\pi i/2} = +i\rho$ $d\rho \Rightarrow id\rho$ $(\infty > \rho > \kappa(q+p)/2)$	$+i\beta_1$	$+i\beta_2$	
$\overset{\cap}{CB}$	$\rho \Rightarrow i\kappa(q+p)/2 + \varepsilon e^{i\varphi}$ $d\rho \Rightarrow i\varepsilon e^{i\varphi}d\varphi$ $(\pi/2 > \varphi > -\pi/2)$	$+i\kappa\sqrt{p(q+p)}$	$\sqrt{\varepsilon\kappa q}e^{i(\varphi/2+\pi/4)}$	$\varepsilon \to 0$
$\overrightarrow{BA}$	$\rho \Rightarrow \rho e^{\pi i/2} = +i\rho$ $d\rho \Rightarrow id\rho$ $(\kappa(q-p)/2 > \rho > \kappa(q+p)/2)$	$+i\beta_1$	$\hat{\alpha}_2$	
$\overset{\cap}{AA'}$	$\rho \Rightarrow i\kappa(q-p)/2 + \varepsilon e^{i\varphi}$ $d\rho \Rightarrow i\varepsilon e^{i\varphi}d\varphi$ $(\pi/2 > \varphi > -3\pi/2)$	$\sqrt{\varepsilon\kappa q}e^{i(\varphi/2+\pi/4)}$	$\kappa\sqrt{p(q+p)}$	$\varepsilon \to 0$
$\overrightarrow{A'B'}$	$\rho \Rightarrow \rho e^{\pi i/2} = +i\rho$ $d\rho \Rightarrow id\rho$ $(\kappa(q-p)/2 < \rho < \kappa(q+p)/2)$	$-i\beta_1$	$\hat{\alpha}_2$	
$\overset{\cap}{B'C'}$	$d\rho \Rightarrow i\kappa(q+p)/2 + \varepsilon e^{i\varphi}$ $d\rho \Rightarrow i\varepsilon e^{i\varphi}d\varphi$ $(-\pi/2 > \varphi > -3\pi/2)$	$-i\kappa\sqrt{p(q+p)}$	$\sqrt{\varepsilon\kappa q}e^{i(\varphi/2+\pi/4)}$	$\varepsilon \to 0$
$\overrightarrow{C'D'}$	$\rho \Rightarrow \rho e^{\pi i/2} = +i\rho$ $d\rho \Rightarrow id\rho$ $(\kappa(q+p)/2 < \rho < \infty)$	$-i\beta_1$	$-i\beta_2$	
$\overset{\cap}{D'E'}$	$\rho \Rightarrow Re^{i\varphi},$ $d\rho \Rightarrow iRe^{i\varphi}d\varphi$ $(\pi/2 < \varphi < \pi)$	$R(\lvert\cos\varphi\rvert - i\sin\varphi)$	$R(\lvert\cos\varphi\rvert - i\sin\varphi)$	$R \to \infty$
$\overrightarrow{E'O'}$	$\rho \Rightarrow \rho e^{\pi i} = -\rho, \quad d\rho \Rightarrow -d\rho$ $(\infty > \rho > 0)$	α_2	α_1	
$\overset{\cap}{O'O}$	$\rho \Rightarrow \varepsilon e^{i\varphi}$ $d\rho \Rightarrow i\varepsilon e^{i\varphi}d\varphi$ $(\pi > \varphi > 0)$	$\kappa/2$	$\kappa/2$	$\varepsilon \to 0$

$\alpha_1 = \sqrt{(\rho + i\kappa p/2)^2 + (\kappa q/2)^2}, \quad \alpha_2 = \sqrt{(\rho - i\kappa p/2)^2 + (\kappa q/2)^2}$
$\beta_1 = \sqrt{(\rho + \kappa p/2)^2 - (\kappa q/2)^2}, \quad \beta_2 = \sqrt{(\rho - \kappa p/2)^2 - (\kappa q/2)^2}$
$\hat{\alpha}_2 = \sqrt{(\kappa q/2)^2 - (\rho - \kappa p/2)^2}$

* Newly defined ρ, ε, and R in this column are positive real

Employing the variable and argument of the radicals listed in Table 1, the integral along the real axis can be expressed as

$$\begin{aligned}&\left(\int\limits_{\overrightarrow{\mathrm{OE}}} - \int\limits_{\overrightarrow{\mathrm{O'E'}}}\right) \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} H_0^{(1)}(\rho r) d\rho \\ &= \int\limits_0^\infty \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} \left\{H_0^{(1)}(\rho r) - H_0^{(1)}(\rho r e^{\pi i})\right\} d\rho.\end{aligned} \tag{3.19}$$

Applying the formula [5, p. 75]

$$H_n^{(1)}(xe^{\pi i}) = e^{-in\pi}\left\{H_n^{(1)}(x) - 2J_n(x)\right\}; \quad n = 0, 1, 2, \ldots \tag{3.20}$$

to the Hankel function, we have two times the requisite integral $I_0^{(+)}(r, z)$,

$$\left(\int_{\overrightarrow{\mathrm{OE}}} - \int_{\overrightarrow{\mathrm{O'E'}}}\right) \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} H_0^{(1)}(\rho r) d\rho$$
$$= 2\int_0^\infty \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} J_0(\rho r) d\rho. \tag{3.21}$$

On the other hand, the branch line integral along $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{A'B'}}$ is given by

$$\begin{aligned}&\left(\int_{\overrightarrow{AB}} - \int_{\overrightarrow{\mathrm{A'B'}}}\right) \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} H_0^{(1)}(\rho r) d\rho \\ &= \int_{(\kappa/2)(q-p)}^{(\kappa/2)(q+p)} i\rho \left\{\frac{1}{\beta_1}\exp(-i\beta_1 z) + \frac{1}{\beta_1}\exp(+i\beta_1 z)\right\} H_0^{(1)}(\rho r e^{\pi i/2}) d\rho\end{aligned} \tag{3.22}$$

where

$$\beta_1 = \sqrt{(\rho + \kappa p/2)^2 - (\kappa q/2)^2}. \tag{3.23}$$

We also apply the formula [5, p. 78]

$$H_n^{(1)}(x e^{\pi i/2}) = -\frac{2i}{\pi} e^{-n\pi i/2} K_n(x); \quad n = 0, 1, 2, \ldots \tag{3.24}$$

where $K_n(x)$ is the modified Bessel function of the second kind. The branch line integral is reduced to

$$\left(\int_{\overrightarrow{AB}} - \int_{\overrightarrow{\mathrm{A'B'}}}\right) \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} H_0^{(1)}(\rho r) d\rho = \frac{4}{\pi}\int_{(\kappa/2)(q-p)}^{(\kappa/2)(q+p)} \frac{\rho}{\beta_1}\cos(\beta_1 z) K_0(\rho r) d\rho. \tag{3.25}$$

Similarly, the other branch line integral along $\overrightarrow{\mathrm{CD}}$ and $\overrightarrow{\mathrm{C'D'}}$ is given by

$$\left(\int_{\overrightarrow{\mathrm{CD}}} - \int_{\overrightarrow{\mathrm{C'D'}}}\right) \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} H_0^{(1)}(\rho r) d\rho$$
$$= \frac{4}{\pi}\int_{(\kappa/2)(q+p)}^{\infty} \rho \left\{\frac{1}{\beta_1}\cos(\beta_1 z) + \frac{1}{\beta_2}\cos(\beta_2 z)\right\} K_0(\rho r) d\rho \tag{3.26}$$

where

$$\beta_2 = \sqrt{(\rho - \kappa p/2)^2 - (\kappa q/2)^2}. \tag{3.27}$$

Substituting Eqs. (3.21), (3.25), and (3.26) into Eq. (3.18), we have the real-valued integral as

$$\begin{aligned}I_0^{(+)}(r, z) &= \int_0^\infty \rho \left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z) + \frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\} J_0(\rho r) d\rho \\ &= \frac{2}{\pi}\int_{(\kappa/2)(q-p)}^{\infty} \frac{\rho}{\beta_1}\cos(\beta_1 z) K_0(\rho r) d\rho + \frac{2}{\pi}\int_{(\kappa/2)(q+p)}^{\infty} \frac{\rho}{\beta_2}\cos(\beta_2 z) K_0(\rho r) d\rho.\end{aligned} \tag{3.28}$$

The integrands in the last two integrals are real-valued, but they are still of an oscillating nature due to the presence of the cosine functions. In order to remove this oscillating nature, we introduce the variable transform defined by

$$u = \beta_1 = \sqrt{(\rho + \kappa p/2)^2 - (\kappa q/2)^2} \tag{3.29}$$

for the first integral and

$$u = \beta_2 = \sqrt{(\rho - \kappa p/2)^2 - (\kappa q/2)^2} \tag{3.30}$$

for the second integral. The last line in Eq. (3.28) is then converted to the infinite integral

$$\begin{aligned} I_0^{(+)}(r,z) = \frac{2}{\pi}\int_0^\infty \left\{1 - \frac{\kappa p/2}{\sqrt{u^2+(\kappa q/2)^2}}\right\} K_0\left(r\left(\sqrt{u^2+(\kappa q/2)^2} - \kappa p/2\right)\right)\cos(uz)du \\ + \frac{2}{\pi}\int_0^\infty \left\{1 + \frac{\kappa p/2}{\sqrt{u^2+(\kappa q/2)^2}}\right\} K_0\left(r\left(\sqrt{u^2+(\kappa q/2)^2} + \kappa p/2\right)\right)\cos(uz)du. \end{aligned} \tag{3.31}$$

Furthermore, we replace the modified Bessel function with its integral form [5, p. 172],

$$K_0(x) = \int_1^\infty \frac{1}{\sqrt{t^2-1}}\exp(-xt)dt, \tag{3.32}$$

and exchange the order of integration. We then have the double integral

$$\begin{aligned} I_0^{(+)}(r,z) =& \frac{2}{\pi}\int_1^\infty \frac{\cosh\{(\kappa p/2)rt\}}{\sqrt{t^2-1}}dt\int_0^\infty \exp\left\{-rt\sqrt{u^2+(\kappa q/2)^2}\right\}\cos(uz)du \\ &- \frac{2}{\pi}\int_1^\infty \frac{\sinh\{(\kappa p/2)rt\}}{\sqrt{t^2-1}}dt\int_0^\infty \frac{\kappa p/2}{\sqrt{u^2+(\kappa q/2)^2}}\exp\left\{-rt\sqrt{u^2+(\kappa q/2)^2}\right\}\cos(uz)du. \end{aligned} \tag{3.33}$$

Now, fortunately, the inner integral with respect to the integration variable u can be evaluated exactly by applying the formulas [5, p. 16 and 17]

$$\int_0^\infty \frac{1}{\sqrt{x^2+\alpha^2}}\exp\left(-\beta\sqrt{x^2+\alpha^2}\right)\cos(xy)dx = K_0\left(\alpha\sqrt{\beta^2+y^2}\right), \tag{3.34}$$

$$\int_0^\infty \exp\left(-\beta\sqrt{x^2+\alpha^2}\right)\cos(xy)dx = \frac{\alpha\beta}{\sqrt{\beta^2+y^2}}K_1\left(\alpha\sqrt{\beta^2+y^2}\right). \tag{3.35}$$

Then, Eq. (3.33) is reduced to the single integral without any oscillating integrand,

$$\begin{aligned} I_0^{(+)}(r,z) = \frac{4}{\pi}\int_1^\infty \frac{1}{\sqrt{t^2-1}}\Bigg[\frac{(\kappa q/2)rt}{\sqrt{(rt)^2+z^2}}\cosh\{(\kappa p/2)rt\}\,K_1\left((\kappa q/2)\sqrt{(rt)^2+z^2}\right) \\ -(\kappa p/2)\sinh\{(\kappa p/2)rt\}\,K_0\left((\kappa q/2)\sqrt{(rt)^2+z^2}\right)\Bigg]dt. \end{aligned} \tag{3.36}$$

Now the integrand is free of oscillating functions and decays exponentially as $t \to \infty$, since $q > p$. However, the integrand has an inverse square root singularity at the lower limit of the integral. So we introduce a variable transformation from t to a new variable φ, defined as

$$t = 1/\sin\varphi. \tag{3.37}$$

The square root singularity at the lower limit is then removed, and the integral is reduced to the finite integral

$$I_0^{(+)}(r,z) = \frac{4}{\pi}\int_0^{\pi/2}\left[\frac{(\kappa q/2)r}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q}{2\sin\varphi}\sqrt{r^2+z^2\sin^2\varphi}\right)\right.$$
$$\left. -(\kappa p/2)\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q}{2\sin\varphi}\sqrt{r^2+z^2\sin^2\varphi}\right)\right]\frac{d\varphi}{\sin\varphi}. \tag{3.38}$$

We have now obtained a convenient form of the integral $I_0^{(+)}(r,z)$. The other five integrals from Eq. (3.7) to (3.11) are similarly converted to the real-valued finite integrals by applying similar mathematical manipulations. The results are given in "Appendix D".

4 Green's function in terms of finite integrals

All integrals have now been transformed to finite integrals, each in a form convenient for the numerical evaluation. Substituting Eqs. (D.1)–(D.8) into the dyadic-like function g_{ij} in "Appendix C" and arranging these equations, we obtain dyadic-like functions in explicit form. They are given in "Appendix E". Finally, recalling Eqs. (2.1) and (2.6), we reach the final form of Green's function as

$$u_i(r,\theta,z) = \frac{P_j}{8\pi\mu(z)}\exp\{(\kappa/2)(z-z_0)\}g_{ij}(r,\theta,z-z_0);\quad i,j=x,y,z. \tag{4.1}$$

It is necessary to examine whether our Green's function can be reduced to Kelvin's solution [6, p. 109] when the inhomogeneity parameter κ vanishes. Unfortunately, it is very hard for the author to derive the approximate form from the finite integral in "Appendix D" or E. Instead, the approximation, i.e. the first term for the small inhomogeneity parameter κ, is derived from the original integral of the Hankel inversion integral. Their approximations are given in "Appendix F". With these, we can confirm that the dyadic-like function g_{ij} in Eq. (2.20) can be reduced to Kelvin's solution. But the author must admit that this is a weak verification.

5 Conclusions

The 3D Green's function for an elastic solid with exponential inhomogeneity has been reconsidered and is obtained in terms of finite integrals. There are no integrands of oscillating nature; thus, the present form of the Green's function is more suitable for numerical evaluation.

As a final comment, the following is pointed out: when considering the deformation of an elastic solid with a simple inhomogeneity, it is better to solve the governing equations directly, rather than separating them into grading and non-grading parts.

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Appendix A

$$\begin{aligned} f_{xx}(\xi,\eta,Z) &= \frac{\eta^2}{2\alpha_0(\xi^2+\eta^2)}\exp(-\alpha_0|Z|) \\ &+ \frac{1}{4}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ &- \frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{i\xi^2}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ &- \frac{\eta^2}{4(\xi^2+\eta^2)}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} f_{xy}(\xi,\eta,Z) =& -\frac{\xi\eta}{2\alpha_0(\xi^2+\eta^2)}\exp(-\alpha_0|Z|) \\ & -\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{+i\xi\eta}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ & +\frac{\xi\eta}{4(\xi^2+\eta^2)}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} f_{xz}(\xi,\eta,Z) =& -\operatorname{sgn}(Z)\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{\xi}{\sqrt{\xi^2+\eta^2}}\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\} \\ & -\frac{\gamma^2-3}{8\gamma^2 p}\frac{\xi}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} f_{yx}(\xi,\eta,Z) =& -\frac{\xi\eta}{2\alpha_0(\xi^2+\eta^2)}\exp(-\alpha_0|Z|) \\ & -\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{+i\xi\eta}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ & +\frac{\xi\eta}{4(\xi^2+\eta^2)}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} f_{yy}(\xi,\eta,Z) =& \frac{\xi^2}{2\alpha_0(\xi^2+\eta^2)}\exp(-\alpha_0|Z|) \\ & +\frac{1}{4}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ & -\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{i\eta^2}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ & -\frac{\xi^2}{4(\xi^2+\eta^2)}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} f_{yz}(\xi,\eta,Z) =& -\operatorname{sgn}(Z)\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\} \\ & -\frac{\gamma^2-3}{8\gamma^2 p}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.6}$$

$$\begin{aligned} f_{zx}(\xi,\eta,Z) =& -\operatorname{sgn}(Z)\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{\xi}{\sqrt{\xi^2+\eta^2}}\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\} \\ & +\frac{\gamma^2-3}{8\gamma^2 p}\frac{\xi}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.7}$$

$$\begin{aligned} f_{zy}(\xi,\eta,Z) =& -\operatorname{sgn}(Z)\frac{\gamma^2-1}{4\kappa p\gamma^2}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\} \\ & +\frac{\gamma^2-3}{8\gamma^2 p}\frac{\eta}{\sqrt{\xi^2+\eta^2}}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.8}$$

$$\begin{aligned} f_{zz}(\xi,\eta,Z) =& \frac{1}{4\gamma^2}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\} \\ & +i\frac{\gamma^2-1}{4\kappa p\gamma^2}\sqrt{\xi^2+\eta^2}\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}, \end{aligned} \tag{A.9}$$

where $Z = z - z_0$ and the sign function is defined by

$$\operatorname{sgn}(Z) = \begin{cases} +1; & Z > 0 \\ -1; & Z < 0. \end{cases} \tag{A.10}$$

Appendix B: Integration formulas

$$\int_0^{2\pi} \exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = 2\pi J_0(\rho r), \tag{B.1}$$

$$\int_0^{2\pi} \sin\varphi\exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = -2\pi i\cos\theta J_1(\rho r), \tag{B.2}$$

$$\int_0^{2\pi} \cos\varphi\exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = -2\pi i\sin\theta J_1(\rho r), \tag{B.3}$$

$$\int_0^{2\pi} \sin\varphi\cos\varphi\exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = -\pi\sin(2\theta)J_2(\rho r), \tag{B.4}$$

$$\int_0^{2\pi} \sin^2\varphi\exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = \pi\left\{J_0(\rho r) - \cos(2\theta)J_2(\rho r)\right\}, \tag{B.5}$$

$$\int_0^{2\pi} \cos^2\varphi\exp\{-i\rho r\sin(\varphi+\theta)\}d\varphi = \pi\left\{J_0(\rho r) + \cos(2\theta)J_2(\rho r)\right\} \tag{B.6}$$

where $J_n(.)$ is the Bessel function of the first kind. The above formulas are derived by applying the Fourier expansion of the mother function,

$$\exp(\pm iz\sin\phi) = \sum_{n=-\infty}^{+\infty} J_n(z)\exp(\pm in\phi). \tag{B.7}$$

Appendix C: Formal form of Green's dyadic-like function

$$\begin{aligned}
g_{xx}(r,\theta,Z) &= \int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_0(\rho r)d\rho \\
&+ \frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|) + \frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho \\
&- i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|) - \frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho \\
&+ \cos(2\theta)\begin{bmatrix}
\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_2(\rho r)d\rho \\
-\frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|) + \frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho \\
+i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|) - \frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho
\end{bmatrix},
\end{aligned} \tag{C.1}$$

$$g_{xy}(r,\theta,Z)=\sin(2\theta)\begin{bmatrix}\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_2(\rho r)d\rho\\ -\frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\\ +i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\end{bmatrix}, \tag{C.2}$$

$$g_{xz}(r,\theta,Z)=\cos\theta\begin{bmatrix}+i\frac{\gamma^2-3}{2\gamma^2 p}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\\ +i\,\mathrm{sgn}(Z)\frac{\gamma^2-1}{\kappa p\gamma^2}\int_0^\infty \rho\left\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\end{bmatrix}, \tag{C.3}$$

$$g_{yx}(r,\theta,Z)=\sin(2\theta)\begin{bmatrix}\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_2(\rho r)d\rho\\ -\frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\\ +i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\end{bmatrix}, \tag{C.4}$$

$$\begin{aligned}g_{yy}(r,\theta,Z)=&\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_0(\rho r)d\rho\\ &+\frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho\\ &-i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho\\ &-\cos(2\theta)\begin{bmatrix}\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0|Z|)J_2(\rho r)d\rho\\ -\frac{1}{2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\\ +i\frac{\gamma^2-1}{2\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_2(\rho r)d\rho\end{bmatrix},\end{aligned} \tag{C.5}$$

$$g_{yz}(r,\theta,Z)=\sin\theta\begin{bmatrix}+i\frac{\gamma^2-3}{2\gamma^2 p}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\\ +i\,\mathrm{sgn}(Z)\frac{\gamma^2-1}{\kappa p\gamma^2}\int_0^\infty \rho\left\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\end{bmatrix}, \tag{C.6}$$

$$g_{zx}(r,\theta,Z)=\cos\theta\begin{bmatrix}-i\frac{\gamma^2-3}{2\gamma^2 p}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\\ +i\,\mathrm{sgn}(Z)\frac{\gamma^2-1}{\kappa p\gamma^2}\int_0^\infty \rho\left\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\end{bmatrix}, \tag{C.7}$$

$$g_{zy}(r,\theta,Z)=\sin\theta\begin{bmatrix}-i\frac{\gamma^2-3}{2\gamma^2 p}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\\ +i\,\mathrm{sgn}(Z)\frac{\gamma^2-1}{\kappa p\gamma^2}\int_0^\infty \rho\left\{\exp(-\alpha_1|Z|)-\exp(-\alpha_2|Z|)\right\}J_1(\rho r)d\rho\end{bmatrix}, \tag{C.8}$$

$$g_{zz}(r,\theta,Z)=\frac{1}{\gamma^2}\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)+\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho,$$

$$+i\frac{\gamma^2-1}{\kappa p\gamma^2}\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1|Z|)-\frac{1}{\alpha_2}\exp(-\alpha_2|Z|)\right\}J_0(\rho r)d\rho \tag{C.9}$$

where

$$\alpha_0=\sqrt{\rho^2+(\kappa/2)^2}, \tag{C.10}$$

$$\alpha_1=\sqrt{\{\rho+i(\kappa p/2)\}^2+(\kappa q/2)^2},\quad \alpha_2=\sqrt{\{\rho-i(\kappa p/2)\}^2+(\kappa q/2)^2}. \tag{C.11}$$

Appendix D: Integration results

$$I_0^{(0)}(r,z)=\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_0(\rho r)d\rho=\frac{1}{\sqrt{r^2+z^2}}\exp\left(-\frac{\kappa}{2}\sqrt{r^2+z^2}\right), \tag{D.1}$$

$$\begin{aligned}I_2^{(0)}(r,z)&=\int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_2(\rho r)d\rho \\ &=\frac{2}{(\kappa/2)r^2}\left\{\exp(-\kappa z/2)-\exp\left(-\frac{\kappa}{2}\sqrt{r^2+z^2}\right)\right\}-\frac{1}{\sqrt{r^2+z^2}}\exp\left(-\frac{\kappa}{2}\sqrt{r^2+z^2}\right),\end{aligned} \tag{D.2}$$

$$\begin{aligned}I_0^{(+)}(r,z)&=\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_0(\rho r)d\rho \\ &=(\kappa q/2)\frac{4}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi} \\ &-(\kappa p/2)\frac{4}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi},\end{aligned} \tag{D.3}$$

$$\begin{aligned}I_2^{(+)}(r,z)&=\int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_2(\rho r)d\rho \\ &=\frac{4}{(\kappa/2)r^2}\exp(-\kappa z/2) \\ &-(\kappa q/2)\frac{4}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi \\ &+(\kappa p/2)\frac{4}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi,\end{aligned} \tag{D.4}$$

$$\begin{aligned}I_0^{(-)}(r,z)&=\int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_0(\rho r)d\rho \\ &=(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{r^2+z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right) \\ &\times\left\{\frac{(\kappa q/2)r^2}{\sin\varphi\sqrt{r^2+z^2\sin^2\varphi}}K_2\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)-K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\right\}d\varphi\end{aligned}$$

$$-2(\kappa p/2)(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$+(\kappa p/2)^2\frac{4i}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}, \tag{D.5}$$

$$I_1^{(-)}(r,z)=\int_0^\infty\rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_1(\rho r)d\rho$$

$$=(\kappa p/2)\frac{4i}{\pi}\int_0^{\pi/2}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi}$$

$$-(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi}, \tag{D.6}$$

$$I_2^{(-)}(r,z)=\int_0^\infty\rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_2 z)\right\}J_2(\rho r)d\rho$$

$$=-(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{r^2+z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)$$

$$\times\left\{\frac{(\kappa q/2)r^2}{\sin\varphi\sqrt{r^2+z^2\sin^2\varphi}}K_2\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)-K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\right\}\frac{1+\cos^2\varphi}{\sin^2\varphi}d\varphi$$

$$+2(\kappa p/2)(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi$$

$$-(\kappa p/2)^2\frac{4i}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi, \tag{D.7}$$

$$I_1^{(*)}(r,z)=\int_0^\infty\rho\left\{\exp(-\alpha_1 z)-\exp(-\alpha_2 z)\right\}J_1(\rho r)d\rho$$

$$=-(\kappa q/2)^2\frac{4i}{\pi}\int_0^{\pi/2}\frac{rz}{r^2+z^2\sin^2\varphi}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_2\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$+(\kappa p/2)(\kappa q/2)\frac{4i}{\pi}\int_0^{\pi/2}\frac{z}{\sqrt{r^2+z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}. \tag{D.8}$$

Appendix E: Dyadic-like functions

$$\begin{aligned} g_{xx}(r,\theta,Z)&=h_{xx}^{(1)}(r,Z)+h_{xx}^{(2)}(r,Z)\cos(2\theta),\\ g_{xy}(r,\theta,Z)&=h_{xy}(r,Z)\sin(2\theta),\\ g_{xz}(r,\theta,Z)&=h_{xz}(r,Z)\cos\theta, \end{aligned} \tag{E.1}$$

$$\begin{aligned} g_{yx}(r,\theta,Z)&=h_{yx}(r,Z)\sin(2\theta),\\ g_{yy}(r,\theta,Z)&=h_{yy}^{(1)}(r,Z)-h_{yy}^{(2)}(r,Z)\cos(2\theta),\\ g_{yz}(r,\theta,Z)&=h_{yz}(r,Z)\sin\theta, \end{aligned} \tag{E.2}$$

$$
\begin{aligned}
g_{zx}(r,\theta,Z) &= h_{zx}(r,Z)\cos\theta,\\
g_{zy}(r,\theta,Z) &= h_{zy}(r,Z)\sin\theta,\\
g_{zz}(r,\theta,Z) &= h_{zz}(r,Z)
\end{aligned}
\tag{E.3}
$$

where

$$
\begin{aligned}
h_{xx}^{(1)}(r,Z) &= h_{yy}^{(1)}(r,Z)\\
&= \frac{1}{\sqrt{r^2+Z^2}}\exp\left\{-(\kappa/2)\sqrt{r^2+Z^2}\right\}\\
&\quad+\frac{(\kappa q/2)}{\gamma^2}\frac{2}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+Z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}\\
&\quad-\frac{\gamma^2+1}{2\gamma^2}(\kappa p/2)\frac{2}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}\\
&\quad+\frac{\gamma^2-1}{2\gamma^2}\frac{q}{p}\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{r^2+Z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)\\
&\quad\times\left\{\frac{(\kappa q/2)r^2}{\sin\varphi\sqrt{r^2+Z^2\sin^2\varphi}}K_2\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)-K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\right\}d\varphi,
\end{aligned}
\tag{E.4}
$$

$$
\begin{aligned}
h_{xx}^{(2)}(r,Z) &= h_{yy}^{(2)}(r,Z) = h_{xy}(r,Z) = h_{yx}(r,Z)\\
&= -\left\{\frac{2}{(\kappa/2)r^2}+\frac{1}{\sqrt{r^2+Z^2}}\right\}\exp\left\{-(\kappa/2)\sqrt{r^2+Z^2}\right\}\\
&\quad+\frac{(\kappa q/2)}{\gamma^2}\frac{2}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+Z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi\\
&\quad-\frac{\gamma^2+1}{2\gamma^2}(\kappa p/2)\frac{2}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{1+\cos^2\varphi}{\sin^3\varphi}d\varphi\\
&\quad+\frac{\gamma^2-1}{2\gamma^2}\frac{q}{p}\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{r^2+Z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)\\
&\quad\times\left\{\frac{(\kappa q/2)r^2}{\sin\varphi\sqrt{r^2+Z^2\sin^2\varphi}}K_2\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)-K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\right\}\frac{1+\cos^2\varphi}{\sin^2\varphi}d\varphi,
\end{aligned}
\tag{E.5}
$$

$$
\begin{aligned}
h_{xz}(r,Z) &= h_{yz}(r,Z)\\
&= \frac{\gamma^2-1}{\gamma^2}\frac{q}{p}(\kappa q/2)\frac{2}{\pi}\int_0^{\pi/2}\frac{rZ}{r^2+Z^2\sin^2\varphi}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_2\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}\\
&\quad-\frac{\gamma^2-1}{\gamma^2}(\kappa q/2)\frac{2}{\pi}\int_0^{\pi/2}\frac{Z}{\sqrt{r^2+Z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}\\
&\quad+\frac{\gamma^2-3}{\gamma^2}\frac{(\kappa q/2)}{p}\frac{2}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+Z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi}\\
&\quad-\frac{\gamma^2-3}{\gamma^2}(\kappa/2)\frac{2}{\pi}\int_0^{\pi/2}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi},
\end{aligned}
\tag{E.6}
$$

$$
h_{zx}(r,Z) = h_{zy}(r,Z)
$$

$$= \frac{\gamma^2-1}{\gamma^2}\frac{q}{p}(\kappa q/2)\frac{2}{\pi}\int_0^{\pi/2}\frac{rZ}{r^2+Z^2\sin^2\varphi}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_2\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$-\frac{\gamma^2-1}{\gamma^2}(\kappa q/2)\frac{2}{\pi}\int_0^{\pi/2}\frac{Z}{\sqrt{r^2+Z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$-\frac{\gamma^2-3}{\gamma^2}\frac{(\kappa q/2)}{p}\frac{2}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+Z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi} \tag{E.7}$$

$$+\frac{\gamma^2-3}{\gamma^2}(\kappa/2)\frac{2}{\pi}\int_0^{\pi/2}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin^2\varphi},$$

$$h_{zz}(r,Z) = -\frac{\gamma^2+1}{\gamma^2}(\kappa p/2)\frac{2}{\pi}\int_0^{\pi/2}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_0\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$+(\kappa q)\frac{2}{\pi}\int_0^{\pi/2}\frac{r}{\sqrt{r^2+Z^2\sin^2\varphi}}\cosh\left(\frac{\kappa pr}{2\sin\varphi}\right)K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\frac{d\varphi}{\sin\varphi}$$

$$-\frac{\gamma^2-1}{\gamma^2}\frac{q}{p}\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{r^2+Z^2\sin^2\varphi}}\sinh\left(\frac{\kappa pr}{2\sin\varphi}\right) \tag{E.8}$$

$$\times\left\{\frac{(\kappa q/2)r^2}{\sin\varphi\sqrt{r^2+Z^2\sin^2\varphi}}K_2\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)-K_1\left(\frac{\kappa q\sqrt{r^2+Z^2\sin^2\varphi}}{2\sin\varphi}\right)\right\}d\varphi.$$

Appendix F: Approximation of the integral as $\kappa \to 0$

$$I_0^{(0)}(r,z) = \int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_0(\rho r)d\rho \approx \frac{1}{\sqrt{r^2+z^2}} + O(\kappa), \tag{F.1}$$

$$I_2^{(0)}(r,z) = \int_0^\infty \frac{\rho}{\alpha_0}\exp(-\alpha_0 z)J_2(\rho r)d\rho \approx \frac{1}{r^2}\left(\sqrt{r^2+z^2}-2z+\frac{z^2}{\sqrt{r^2+z^2}}\right) + O(\kappa), \tag{F.2}$$

$$I_0^{(+)}(r,z) = \int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_1 z)\right\}J_0(\rho r)d\rho \approx \frac{2}{\sqrt{r^2+z^2}} + O(\kappa), \tag{F.3}$$

$$I_2^{(+)}(r,z) = \int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)+\frac{1}{\alpha_2}\exp(-\alpha_1 z)\right\}J_2(\rho r)d\rho \approx \frac{2}{r^2}\left(\sqrt{r^2+z^2}-2z+\frac{z^2}{\sqrt{r^2+z^2}}\right) + O(\kappa), \tag{F.4}$$

$$I_0^{(-)}(r,z) = \int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_1 z)\right\}J_0(\rho r)d\rho \approx (-2i)(\kappa p/2)\frac{1}{\sqrt{r^2+z^2}}\left(1+\frac{z^2}{r^2+z^2}\right) + O(\kappa^2), \tag{F.5}$$

$$I_1^{(-)}(r,z) = \int_0^\infty \rho\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_1 z)\right\}J_1(\rho r)d\rho \approx (-2i)(\kappa p/2)\frac{r}{\sqrt{r^2+z^2}} + O(\kappa^2), \tag{F.6}$$

$$I_2^{(-)}(r,z) = \int_0^\infty \rho^2\left\{\frac{1}{\alpha_1}\exp(-\alpha_1 z)-\frac{1}{\alpha_2}\exp(-\alpha_1 z)\right\}J_2(\rho r)d\rho \approx (-2i)(\kappa p/2)\frac{r^2}{(r^2+z^2)^{3/2}} + O(\kappa^2), \tag{F.7}$$

$$I_1^{(*)}(r,z) = \int_0^\infty \rho\left\{\exp(-\alpha_1 z)-\exp(-\alpha_1 z)\right\}J_1(\rho r)d\rho \approx (-2i)(\kappa p/2)\frac{rz}{(r^2+z^2)^{3/2}} + O(\kappa^2) \tag{F.8}$$

where the following integration formulas [3, Vol. 2, p. 9 and 19] are applied:

$$\int_0^\infty \exp(-\rho z) J_0(\rho r) d\rho = \frac{1}{\sqrt{r^2+z^2}}, \tag{F.9}$$

$$\int_0^\infty \exp(-\rho z) J_1(\rho r) d\rho = \frac{1}{r}\left(1 - \frac{z}{\sqrt{r^2+z^2}}\right), \tag{F.10}$$

$$\int_0^\infty \exp(-\rho z) J_2(\rho r) d\rho = \frac{1}{r^2}\left(\sqrt{r^2+z^2} - 2z + \frac{z^2}{\sqrt{r^2+z^2}}\right), \tag{F.11}$$

$$\int_0^\infty \rho \exp(-\rho z) J_0(\rho r) d\rho = \frac{z}{(r^2+z^2)^{3/2}}, \tag{F.12}$$

$$\int_0^\infty \rho \exp(-\rho z) J_1(\rho r) d\rho = \frac{r}{(r^2+z^2)^{3/2}}, \tag{F.13}$$

$$\int_0^\infty \rho \exp(-\rho z) J_2(\rho r) d\rho = \frac{1}{r^2}\left\{2 - \frac{2z}{\sqrt{r^2+z^2}} + \frac{r^2 z}{(r^2+z^2)^{3/2}}\right\}, \tag{F.14}$$

$$\int_0^\infty \frac{1}{\rho} \exp(-\rho z) J_1(\rho r) d\rho = \frac{1}{r}\left(\sqrt{r^2+z^2} - z\right). \tag{F.15}$$

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