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ORIGINAL PAPER

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Noether's theorems for dynamical systems of two kinds of non-standard Hamiltonians

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Abstract This paper focuses on studying Noether's theorems for dynamical systems with two kinds of non-standard Hamiltonians, respectively, namely exponential Hamiltonian and power-law Hamiltonian. Firstly, the differential equations of motion for dynamical systems with exponential Hamiltonian and power-law Hamiltonian are established. Secondly, according to the invariance of the action under the infinitesimal transformations, the definitions and criteria of Noether symmetric transformations and Noether quasi-symmetric transformations are given. Then, Noether's theorems for dynamical systems with exponential Hamiltonian and power-law Hamiltonian are obtained, respectively. Finally, two examples are given to illustrate the applications of the results.

1 Introduction

The method of symmetry or invariance is an important branch of analytical dynamics. In 1918, Emmy Noether [1] proved a general theorem of the calculus of variations that reveals the interrelation between a variational symmetry and a conserved quantity. The symmetry is described by infinitesimal transformations of the system, which results in the same object after the transformation is carried out. Noether's theorem explains all conservation laws of classical mechanics; for example, the conservation of energy comes from the invariance of the system under time translations, the conservation of momentum comes from the invariance of the system under spatial translations, and the conservation of the moment of momentum comes from the invariance of the system under spatial rotations. Nowadays, the celebrated Noether's theorem is a well-known tool in constrained mechanical systems, such as holonomic systems [2–5], non-holonomic systems [2,6–9], Birkhoffian systems [10–13], dynamical systems with time delay [14–17], fractional calculus of variations [18–22], and variational problems of Herglotz type [23,24]. However, non-standard Lagrangians and non-standard Hamiltonians may make the description easier in some cases, for example, when dealing with nonlinear dynamics.

Hence, there has been a successful formulation for nonlinear dynamical systems, known as non-standard Lagrangians that are characterized by a deformed Lagrangian or deformed kinetic and potential energy terms. It, entitled "non-natural Lagrangian" by Arnold in 1978 [25], has not motivated a large number of studies until Alekseev [26] applied the non-standard Lagrangians to the Yang–Mills quantum field theory where they are used to describe large-distance interactions in the region of applicability of classical theory. In the progress of years, it is observed that non-standard Lagrangians play an important role in some dynamical problems such

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as nonlinear dynamics [27,28], dissipative systems [29–32], cosmology [33], and quantum field theory [34]. Non-standard Lagrangians have much more terms, compared with the standard ones, which can be considered as a special case of non-standard Lagrangians. In the recent past, it is worth pointing out that some scholars have studied their properties and applications. Zhang and Zhou [35,36] studied Noether's theorem and its inverse for dynamical system with non-standard Lagrangians, and extended the celebrated Routh method of reduction to non-standard Lagrangians. Musielak [31,32] studied the method of obtaining the non-standard Lagrangians for dissipative systems and its existence conditions. El-Nabulsi [27,29,33,34,37] studied the action and differential equation of motion with non-standard Lagrangians in nonlinear dynamical systems, and applied them to Friedmann–Robertson–Walker space-time, and discussed their implications in classical and quantum theory by aid of the modified Hamilton–Jacobi equation and Schrödinger equation, and generalized dynamical systems with higher order derivatives. Carinena and Nunez [38] studied the relationship of equations of motion of a deformed Lagrangian.

Non-standard Hamiltonians may take, depending on the problem, exponential form $\exp(p_k\dot{q}_k - H)$, logarithm form $\log_a(p_k\dot{q}_k - H)$ [27], power-law form $(p_k\dot{q}_k - H)^{1+\gamma}$, etc.; H is the standard Hamiltonian, q_k is the generalized coordinate with $\dot{q}_k = \frac{\mathrm{d}q_k}{\mathrm{d}t}$, and p_k is the generalized momentum corresponding to the generalized coordinate q_k. In 2017, Liu and his coworkers [39] studied exponential Hamiltonians. By using the standard variational method, some dynamical equations with different exponential Hamiltonians and some new dynamical properties that nonlinear dynamics hold were obtained. However, scholars have not studied the relationship between Noether's theorem and non-standard Hamiltonians yet. In this paper, we will get Noether's theorem for dynamical systems with two kinds of non-standard Hamiltonians, namely exponential Hamiltonian and power-law Hamiltonian.

The organization of this paper is demonstrated as follows. In Sects. 2 and 3, the actions based on two kinds of non-standard Hamiltonians are introduced, and differential equations of motion are derived, respectively. According to the invariance of actions for non-standard Hamiltonians under the infinitesimal transformations, Noether's theorems are obtained. Two examples are given to illustrate the applications of the results.

2 Noether's theorem for dynamical system with exponential Hamiltonian

2.1 Differential equations of motion

Suppose that the configuration of a dynamical system is determined by n generalized coordinates $q_k\ (k = 1, 2, \ldots, n)$, the action with exponential Hamiltonian is [27,39]

$$S = \int_a^b \exp(p_k\dot{q}_k - H)\,\mathrm{d}t \tag{1}$$

where $H\colon R \times R^n \times R^n$ is of class C^2 and $H = H(t, q_k, p_k) = p_k\dot{q}_k - L(t, q_k, \dot{q}_k)$ is the standard Hamiltonian, $\dot{q}_k$ is the generalized velocity with $\dot{q}_k = \frac{\mathrm{d}q_k}{\mathrm{d}t}$, and p_k is the generalized momentum corresponding to the generalized coordinate q_k. Here, we comply with Einstein's summation convention.

The variational principle with exponential Hamiltonian is

$$\delta S = 0 \tag{2}$$

which satisfies the commutation relations

$$\mathrm{d}\delta q_k = \delta \mathrm{d}q_k \quad (k = 1, 2, \ldots, n) \tag{3}$$

and the given terminal conditions

$$\delta q_k|_{t=a} = \delta q_k|_{t=b} = 0 \quad (k = 1, 2, \ldots, n). \tag{4}$$

By Eqs. (1)–(4), it is easy to get

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0, -\dot{p}_k - p_k\dot{p}_i\frac{\partial H}{\partial p_i} - p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + p_k\frac{\mathrm{d}H}{\mathrm{d}t} - \frac{\partial H}{\partial q_k} = 0, \quad (k = 1, 2, \ldots, n). \tag{5}$$

Making use of the total differential of $H(t, q_k, p_k)$,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k, \tag{6}$$

Eq. (5) can be expressed as

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0, -\dot{p}_k - p_k p_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + p_k \frac{\partial H}{\partial t} + p_k \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_k} = 0, \quad (k = 1, 2, \ldots, n). \tag{7}$$

Equations (5) or (7) are the differential equations of motion for dynamical systems with exponential Hamiltonian.

2.2 Noether symmetry

Let us introduce the infinitesimal transformations with respect to time t, generalized coordinates q_k, and generalized momentum p_k, i.e.,

$$\bar{t} = t + \Delta t, \bar{q}_k(\bar{t}) = q_k(t) + \Delta q_k, \bar{p}_k(\bar{t}) = p_k(t) + \Delta p_k, \quad (k = 1, 2, \ldots, n) \tag{8}$$

and their expansion formulae

$$\begin{aligned} &\bar{t} = t + \varepsilon_\alpha \xi_0^\alpha(t, q_s, p_s), \quad \bar{q}_k(\bar{t}) = q_k(t) + \varepsilon_\alpha \xi_k^\alpha(t, q_s, p_s), \\ &\bar{p}_k(\bar{t}) = p_k(t) + \varepsilon_\alpha \eta_k^\alpha(t, q_s, p_s), (\alpha = 1, 2, \ldots, r;\ s, k = 1, 2, \ldots, n) \end{aligned} \tag{9}$$

where ε_α $(\alpha = 1, 2, \ldots, r)$ is the infinitesimal parameter, and $\xi_0^\alpha, \xi_k^\alpha, \eta_k^\alpha$ are the generators of the infinitesimal transformations. Under the infinitesimal transformations (8), the action (1) is transformed to

$$\begin{aligned} \Delta S &= \delta S + \dot{S}\Delta t \\ &= \int_a^b \left[\exp(p_s\dot{q}_s - H)\left(p_k\delta\dot{q}_k + \dot{q}_k\delta p_k - \frac{\partial H}{\partial q_k}\delta q_k - \frac{\partial H}{\partial p_k}\delta p_k\right)\right]\mathrm{d}t + \exp(p_s\dot{q}_s - H)\Delta t \\ &= \int_a^b \left\{\exp(p_s\dot{q}_s - H)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\delta p_k + \exp(p_s\dot{q}_s - H)\left[-\frac{\partial H}{\partial q_k} + p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k\frac{\partial H}{\partial t}\right.\right. \\ &\quad \left.\left. -p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\delta q_k + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(p_s\dot{q}_s - H)(p_k\delta q_k + \Delta t)\right]\right\}\mathrm{d}t \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Delta S &= \int_a^b \left[\Delta\exp(p_s\dot{q}_s - H) + \exp(p_s\dot{q}_s - H)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t \\ &= \int_a^b \exp(p_s\dot{q}_s - H)\left[p_k\Delta\dot{q}_k + \dot{q}_k\Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k - \frac{\partial H}{\partial p_k}\Delta p_k + \frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t. \end{aligned} \tag{11}$$

Considering

$$\Delta t = \varepsilon_\alpha\xi_0^\alpha, \delta q_k = \Delta q_k - \dot{q}_k\Delta t = \varepsilon_\alpha\left(\xi_k^\alpha - \dot{q}_k\xi_0^\alpha\right), \delta p_k = \Delta p_k - \dot{p}_k\Delta t = \varepsilon_\alpha\left(\eta_k^\alpha - \dot{p}_k\xi_0^\alpha\right) \tag{12}$$

and from Eq. (10), we obtain

$$\begin{aligned} \Delta S = \int_a^b \varepsilon_\alpha &\left\{\exp(p_s\dot{q}_s - H)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + \exp(p_s\dot{q}_s - H)\left[-\frac{\partial H}{\partial q_k} + p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k\frac{\partial H}{\partial t}\right.\right. \\ &\left.\left. -p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(p_s\dot{q}_s - H)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right)\right]\right\}\mathrm{d}t \end{aligned} \tag{13}$$

where

$$\bar{\xi}_k^\alpha = \xi_k^\alpha - \dot{q}_k\xi_0^\alpha, \bar{\eta}_k^\alpha = \eta_k^\alpha - \dot{p}_k\xi_0^\alpha. \tag{14}$$

Equations (11) and (13) are the basic formulae for the variation in the action (1).

Now, we give the definitions and criteria of Noether symmetry for dynamical systems with exponential Hamiltonian.

Definition 1 For a dynamical system with exponential Hamiltonian (5), the transformations (8) are called the Noether symmetric transformations if and only if

$$\Delta S = 0 \tag{15}$$

for each of the infinitesimal transformations.

Criterion 1 For the infinitesimal transformations (8), if the condition

$$\exp\left(p_s \dot{q}_s - H\right)\left[p_k \Delta \dot{q}_k + \dot{q}_k \Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k - \frac{\partial H}{\partial p_k}\Delta p_k + \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta t\right)\right] = 0 \tag{16}$$

is satisfied, then the transformations (8) are the Noether symmetric transformations for dynamical systems with exponential Hamiltonian.

Criterion 2 For the infinitesimal transformations (9), if the condition

$$\begin{aligned} &\exp\left(p_s \dot{q}_s - H\right)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^{\alpha} + \exp\left(p_s \dot{q}_s - H\right)\left[-\frac{\partial H}{\partial q_k} + p_k \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k \frac{\partial H}{\partial t}\right. \\ &\left. - p_k p_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\bar{\xi}_k^{\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s \dot{q}_s - H\right)\left(p_k \bar{\xi}_k^{\alpha} + \xi_0^{\alpha}\right)\right] = 0, \quad (\alpha = 1, 2, \ldots, r) \end{aligned} \tag{17}$$

is satisfied, then the transformations (9) are the Noether symmetric transformations for dynamical systems with exponential Hamiltonian.

Considering

$$\Delta \dot{q}_k = \varepsilon_{\alpha}\left(\dot{\xi}_k^{\alpha} - \dot{q}_k \dot{\xi}_0^{\alpha}\right) \tag{18}$$

and the former of Eq. (7), condition (16) can be expressed as

$$\exp\left(p_s \dot{q}_s - H\right)\left[p_k \dot{\xi}_k^{\alpha} - \frac{\partial H}{\partial t}\xi_0^{\alpha} - \frac{\partial H}{\partial q_k}\xi_k^{\alpha} + \left(1 - p_k \frac{\partial H}{\partial p_k}\right)\dot{\xi}_0^{\alpha}\right] = 0. \tag{19}$$

Equation (19) is also the criterion of the Noether symmetric transformations for dynamical systems with exponential Hamiltonian. When $\alpha = 1$, Eq. (19) is transformed to

$$\exp\left(p_s \dot{q}_s - H\right)\left[p_k \dot{\xi}_k - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_k}\xi_k + \left(1 - p_k \frac{\partial H}{\partial p_k}\right)\dot{\xi}_0\right] = 0. \tag{20}$$

Equation (20) is called the Noether identity for dynamical systems with exponential Hamiltonian.

Using Criteria 1 and 2, one can find the Noether symmetry for a dynamical system with power-law Hamiltonian

Definition 2 For a dynamical system with exponential Hamiltonian (5), the transformations (8) are called the Noether quasi-symmetric transformations if and only if

$$\Delta S = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G\right)\mathrm{d}t \tag{21}$$

where $G = G\left(t, q_k, p_k\right)$, for each of the infinitesimal transformations.

Criterion 3 For the infinitesimal transformations (8), if the condition

$$\exp\left(p_s \dot{q}_s - H\right)\left[p_k \Delta \dot{q}_k + \dot{q}_k \Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k - \frac{\partial H}{\partial p_k}\Delta p_k + \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta t\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G\right) \tag{22}$$

is satisfied, then the transformations (8) are the Noether quasi-symmetric transformations for dynamical systems with exponential Hamiltonian.

Criterion 4 For the infinitesimal transformations (9), if the condition

$$\exp\left(p_s\dot{q}_s - H\right)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + \exp\left(p_s\dot{q}_s - H\right)\left[-\frac{\partial H}{\partial q_k} + p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k\frac{\partial H}{\partial t}\right.$$
$$\left. - p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}t}G^\alpha, \quad (\alpha = 1, 2, \ldots, r) \quad (23)$$

is satisfied, where $\Delta G = \varepsilon_\alpha G^\alpha$, then the transformations (9) are the Noether quasi-symmetric transformations for dynamical systems with exponential Hamiltonian.

Considering Eqs. (12) and (18), and the former of Eq. (7), condition (22) can be expressed as

$$\exp\left(p_s\dot{q}_s - H\right)\left[p_k\dot{\xi}_k^\alpha - \frac{\partial H}{\partial t}\xi_0^\alpha - \frac{\partial H}{\partial q_k}\xi_k^\alpha + \left(1 - p_k\frac{\partial H}{\partial p_k}\right)\dot{\xi}_0^\alpha\right] = -\dot{G}^\alpha \quad (24)$$

where $G^\alpha = G^\alpha\left(t, q_k, p_k\right)$. Equation (24) is also the criterion of the Noether quasi-symmetric transformations for dynamical systems with exponential Hamiltonian. When $\alpha = 1$, Eq. (24) gives the Noether identity for dynamical systems with exponential Hamiltonian,

$$\exp\left(p_s\dot{q}_s - H\right)\left[p_k\dot{\xi}_k - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_k}\xi_k + \left(1 - p_k\frac{\partial H}{\partial p_k}\right)\dot{\xi}_0\right] = -\dot{G}. \quad (25)$$

Using Criteria 3 and 4, one can find the Noether quasi-symmetry for dynamical systems with exponential Hamiltonian.

2.3 Noether's theorem

Under the infinitesimal transformations (9), from Eqs. (13) and (15), we have

$$\exp\left(p_s\dot{q}_s - H\right)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + \exp\left(p_s\dot{q}_s - H\right)\left[-\frac{\partial H}{\partial q_k} + p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k\frac{\partial H}{\partial t}\right.$$
$$\left. - p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right)\right] = 0, \quad (\alpha = 1, 2, \ldots, r). \quad (26)$$

Substituting Eq. (7) into Eq. (26), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right)\right] = 0. \quad (27)$$

Integrating Eq. (27), we get the Noether conserved quantity

$$I^\alpha = \exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right) = \text{const.}, \quad (\alpha = 1, 2, \ldots, r). \quad (28)$$

Therefore, we have

Theorem 1 *For the dynamical system with exponential Hamiltonian (5), if the infinitesimal transformations (9) are the Noether symmetric transformations in the sense of Definition 1, then the system admits r linearly independent Noether conserved quantities (28).*

Under the infinitesimal transformations (9), from Eqs. (13) and (21), we have

$$\exp\left(p_s\dot{q}_s - H\right)\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + \exp\left(p_s\dot{q}_s - H\right)\left[-\frac{\partial H}{\partial q_k} + p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + p_k\frac{\partial H}{\partial t}\right.$$
$$\left. - p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) - \dot{p}_k\right]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right) + G^\alpha\right] = 0, \quad (\alpha = 1, 2, \ldots, r). \quad (29)$$

Substituting Eq. (7) into Eq. (29), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(p_s\dot{q}_s - H\right)\left(p_k\bar{\xi}_k^\alpha + \xi_0^\alpha\right) + G^\alpha\right] = 0. \quad (30)$$

Integrating Eq. (30), we get the Noether conserved quantity

$$I^{\alpha} = \exp\left(p_s \dot{q}_s - H\right)\left(p_k \bar{\xi}_k^{\alpha} + \xi_0^{\alpha}\right) + G^{\alpha} = \text{const.}, \quad (\alpha = 1, 2, \ldots, r). \tag{31}$$

Therefore, we have

Theorem 2 *For the dynamical system with exponential Hamiltonian (5), if the infinitesimal transformations (9) are the Noether quasi-symmetric transformations in the sense of Definition 2, then the system admits r linearly independent Noether conserved quantities (31).*

Theorems 1 and 2 are called Noether's theorem for dynamical systems with exponential Hamiltonian (5). The theorem shows that one can obtain a conserved quantity of the system if one can find a Noether symmetry transformation or a Noether quasi-symmetry transformation.

If the Hamiltonian H of the dynamical system is not dependent on time t, i.e., $H = H(q_k, p_k)$, and letting

$$\xi_0 = 1, \xi_k = 0, \quad (k = 1, 2, \ldots, n), \tag{32}$$

by Theorem 1, we obtain

$$I = \exp\left(p_s \dot{q}_s - H\right)\left(1 - p_k \frac{\partial H}{\partial p_k}\right) = \text{const.} \tag{33}$$

Equation (33) is called the generalized energy integral for the dynamical system with exponential Hamiltonian (5).

2.4 Example

Consider the dynamical system with exponential Hamiltonian whose action is defined by

$$S = \int_a^b \exp\left(p\dot{q} - H\right)\mathrm{d}t \tag{34}$$

where $H = tqp$ [39] is the standard Hamiltonian. By Eq. (7), we have

$$\dot{q} - qt = 0, \dot{p} + pt = 0. \tag{35}$$

Let us study the Noether symmetry and conserved quantity. The generalized Noether identity (25) gives

$$p\dot{\xi} - qp\xi_0 - tp\xi + (1 - tqp)\,\dot{\xi}_0 = -\dot{G}. \tag{36}$$

Equation (36) has the following solutions:

$$\xi_0^1 = 0, \xi^1 = q, G^1 = 0, \tag{37}$$

$$\xi_0^2 = \frac{1}{t}, \xi^2 = q, G^2 = -\frac{1}{t}. \tag{38}$$

The generator (37) corresponds to the Noether symmetry for the dynamical system, and the generator (38) corresponds to the Noether quasi-symmetry for the dynamical system. By Noether's theorem we have obtained that the system has the following conserved quantities:

$$I^1 = qp, \tag{39}$$

$$I^2 = 0. \tag{40}$$

It is obvious that the conserved quantity (40) is trivial.

3 Noether's theorem for a dynamical system with power-law Hamiltonian

3.1 Differential equations of motion

The action with power-law Hamiltonian is [27]

$$A=\int_a^b\left(p_k\dot{q}_k-H\right)^{1+\gamma}\,\mathrm{d}t. \tag{41}$$

The variational principle with power-law Hamiltonian is

$$\delta A=0. \tag{42}$$

By Eqs. (42), considering conditions (3) and (4), when $\gamma\neq-1$, we obtain

$$\begin{aligned}&\dot{q}_k-\frac{\partial H}{\partial p_k}=0,-\dot{p}_k-\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\dot{p}_i\frac{\partial H}{\partial p_i}-\frac{\partial H}{\partial q_k}\\&-\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_kp_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right)+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\frac{\mathrm{d}H}{\mathrm{d}t}=0,\quad(k=1,2,\ldots,n).\end{aligned} \tag{43}$$

Making use of the total differential (6) of $H\left(t,q_k,p_k\right)$, Eq. (43) can be expressed as

$$\begin{aligned}&\dot{q}_k-\frac{\partial H}{\partial p_k}=0,-\dot{p}_k-\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_kp_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right)-\frac{\partial H}{\partial q_k}\\&+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\frac{\partial H}{\partial t}+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}=0,\quad(k=1,2,\ldots,n).\end{aligned} \tag{44}$$

Equations (43) or (44) are the differential equations of motion for dynamical systems with power-law Hamiltonian.

When $\gamma=0$, Eq. (44) reduces to the classical Hamilton canonical equation [2]

$$\dot{q}_k-\frac{\partial H}{\partial p_k}=0,\ \dot{p}_k+\frac{\partial H}{\partial q_k}=0,\quad(k=1,2,\ldots,n). \tag{45}$$

3.2 Noether symmetry

Under the infinitesimal transformations of (8), the action (41) is transformed to

$$\begin{aligned}\Delta A&=\delta A+\dot{A}\Delta t\\&=\int_a^b\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\delta\dot{q}_k+\dot{q}_k\delta p_k-\frac{\partial H}{\partial q_k}\delta q_k-\frac{\partial H}{\partial p_k}\delta p_k\right)\right]\mathrm{d}t+\left(p_s\dot{q}_s-H\right)^{1+\gamma}\Delta t\\&=\int_a^b\left\{(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\delta p_k+(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left[-\frac{\partial H}{\partial q_k}+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}\right.\right.\\&\quad\cdot p_k\frac{\partial H}{\partial t}-\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_kp_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right)+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}-\left.\dot{p}_k\right]\delta q_k\\&\quad\left.+\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\delta q_k+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\Delta t\right)\right]\right\}\mathrm{d}t\end{aligned} \tag{46}$$

and

$$\Delta A=\int_a^b\left[\Delta\left(p_s\dot{q}_s-H\right)^{1+\gamma}+\left(p_s\dot{q}_s-H\right)^{1+\gamma}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t$$

$$= \int_a^b (1+\gamma)(p_s\dot{q}_s - H)^\gamma \Big[p_k \Delta\dot{q}_k + \dot{q}_k \Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k - \frac{\partial H}{\partial p_k}\Delta p_k + (1+\gamma)^{-1}\left(p_l \frac{\partial H}{\partial P_l} - H\right)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Big]\mathrm{d}t. \tag{47}$$

Considering Eq. (12), and from Eq. (46), we obtain

$$\Delta A = \int_a^b \Bigg\{ (1+\gamma)(p_s\dot{q}_s - H)^\gamma \left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + (1+\gamma)(p_s\dot{q}_s - H)^\gamma \Bigg[-\frac{\partial H}{\partial q_k} + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} \cdot p_k \frac{\partial H}{\partial t} - \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k p_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \dot{p}_k \Bigg]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)(p_s\dot{q}_s - H)^\gamma\left(p_k\bar{\xi}_k^\alpha + (1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right)\xi_0^\alpha\right)\right]\Bigg\} dt. \tag{48}$$

Equations (47) and (48) are the basic formulae for the variation in the action (41).

Now, we give the definitions and criteria of Noether symmetry for dynamical systems with power-law Hamiltonian.

Definition 3 For a dynamical system with power-law Hamiltonian (43), the transformations (8) are called the Noether symmetric transformations if and only if

$$\Delta A = 0 \tag{49}$$

for each of the infinitesimal transformations.

Criterion 5 For the infinitesimal transformations (8), if the condition

$$(1+\gamma)(p_s\dot{q}_s - H)^\gamma \Big[p_k \Delta\dot{q}_k + \dot{q}_k \Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k - \frac{\partial H}{\partial p_k}\Delta p_k + (1+\gamma)^{-1}\left(p_l \frac{\partial H}{\partial P_l} - H\right)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Big] = 0 \tag{50}$$

is satisfied, then the transformations (8) are the Noether symmetric transformations for dynamical systems with power-law Hamiltonian.

Criterion 6 For the infinitesimal transformations (9), if the condition

$$(1+\gamma)(p_s\dot{q}_s - H)^\gamma \left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^\alpha + (1+\gamma)(p_s\dot{q}_s - H)^\gamma \Bigg[-\frac{\partial H}{\partial q_k} + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} \cdot p_k \frac{\partial H}{\partial t} - \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k p_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \dot{p}_k \Bigg]\bar{\xi}_k^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)(p_s\dot{q}_s - H)^\gamma\left(p_k\bar{\xi}_k^\alpha + (1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right)\xi_0^\alpha\right)\right] = 0, \quad (\alpha = 1, 2, \ldots, r) \tag{51}$$

is satisfied, then the transformations (9) are the Noether symmetric transformations for dynamical systems with power-law Hamiltonian.

Considering Eq. (18) and the former of Eq. (44), condition (50) can be expressed as

$$(1+\gamma)(p_s\dot{q}_s - H)^\gamma \left[p_k\dot{\xi}_k^\alpha - \frac{\partial H}{\partial t}\xi_0^\alpha - \frac{\partial H}{\partial q_k}\xi_k^\alpha + \left((1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right) - p_k\frac{\partial H}{\partial P_k}\right)\dot{\xi}_0^\alpha\right] = 0. \tag{52}$$

Equation (52) is also the criterion of the Noether symmetric transformations for dynamical systems with power-law Hamiltonian. When $\alpha = 1$, Eq. (52) is transformed to

$$(1+\gamma)(p_s\dot{q}_s - H)^\gamma \left[p_k\dot{\xi}_k - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_k}\xi_k + \left((1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right) - p_k\frac{\partial H}{\partial p_k}\right)\dot{\xi}_0\right] = 0. \tag{53}$$

Using Criteria 5 and 6, one can find the Noether symmetry for dynamical systems with power-law Hamiltonian.

Definition 4 For a dynamical system with power-law Hamiltonian (43), the transformations (8) are called the Noether quasi-symmetric transformations if and only if

$$\Delta A = -\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\,\mathrm{d}t \tag{54}$$

for each of the infinitesimal transformations.

Criterion 7 For the infinitesimal transformations (8), if the condition

$$\begin{aligned}(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\Bigg[p_k\Delta\dot{q}_k + \dot{q}_k\Delta p_k - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q_k}\Delta q_k \\ -\frac{\partial H}{\partial p_k}\Delta p_k + (1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right)\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\Bigg] = -\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\end{aligned} \tag{55}$$

is satisfied, then the transformations (8) are the Noether quasi-symmetric transformations for dynamical systems with power-law Hamiltonian.

Criterion 8 For the infinitesimal transformations (9), if the condition

$$\begin{aligned}&(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^{\alpha} + (1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\Bigg[-\frac{\partial H}{\partial q_k} + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} \\ &\cdot p_k\frac{\partial H}{\partial t} - \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \dot{p}_k\Bigg]\bar{\xi}_k^{\alpha} \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha} + (1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right)\xi_0^{\alpha}\right)\right] = -\frac{\mathrm{d}}{\mathrm{d}t}G^{\alpha}, \quad (\alpha = 1, 2, \ldots, r)\end{aligned} \tag{56}$$

is satisfied, then the transformations (9) are the Noether quasi-symmetric transformations for dynamical systems with power-law Hamiltonian.

Considering Eqs. (12) and (18), and the former of Eq. (44), condition (55) can be expressed as

$$(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\left[p_k\dot{\xi}_k^{\alpha} - \frac{\partial H}{\partial t}\xi_0^{\alpha} - \frac{\partial H}{\partial q_k}\xi_k^{\alpha} + \left((1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right) - p_k\frac{\partial H}{\partial P_k}\right)\dot{\xi}_0^{\alpha}\right] = -\dot{G}^{\alpha}. \tag{57}$$

Equation (57) is also the criterion of the Noether quasi-symmetric transformations for dynamical systems with power-law Hamiltonian. When $\alpha = 1$, Eq. (57) gives the Noether identity for dynamical systems with power-law Hamiltonian,

$$(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\left[p_k\dot{\xi}_k - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_k}\xi_k + \left((1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l} - H\right) - p_k\frac{\partial H}{\partial p_k}\right)\dot{\xi}_0\right] = -\dot{G}. \tag{58}$$

Using Criteria 7 and 8, one can find the Noether quasi-symmetry for dynamical systems with power-law Hamiltonian.

3.3 Noether's theorem

Under the infinitesimal transformations (9), from Eqs. (48) and (49), we have

$$\begin{aligned}&(1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^{\alpha} + (1+\gamma)(p_s\dot{q}_s - H)^{\gamma}\Bigg[-\frac{\partial H}{\partial q_k} + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} \\ &\cdot p_k\frac{\partial H}{\partial t} - \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k p_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right) + \gamma\left(p_l\frac{\partial H}{\partial p_l} - H\right)^{-1} p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \dot{p}_k\Bigg]\bar{\xi}_k^{\alpha}\end{aligned}$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)\right]=0,\quad(\alpha=1,2,\ldots,r).\tag{59}$$

Substituting Eq. (44) into Eq. (59), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)\right]=0.\tag{60}$$

Integrating Eq. (60), we get the Noether conserved quantity

$$I^{\alpha}=(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)=\text{const.},\quad(\alpha=1,2,\ldots,r).\tag{61}$$

Therefore, we have

Theorem 3 *For the dynamical system with power-law Hamiltonian (43), if the infinitesimal transformations (9) are the Noether symmetric transformations in the sense of Definition 3, then the system admits r linearly independent Noether conserved quantities (61).*

Under the infinitesimal transformations (9), from Eqs. (48) and (54), we have

$$\begin{aligned}&(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(\dot{q}_k-\frac{\partial H}{\partial p_k}\right)\bar{\eta}_k^{\alpha}+(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left[-\frac{\partial H}{\partial q_k}+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}\right.\\&\left.\cdot p_k\frac{\partial H}{\partial t}-\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_kp_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial p_i}\right)+\gamma\left(p_l\frac{\partial H}{\partial p_l}-H\right)^{-1}p_k\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}-\dot{p}_k\right]\bar{\xi}_k^{\alpha}\\&+\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)+G^{\alpha}\right]=0,\quad(\alpha=1,2,\ldots,r).\end{aligned}\tag{62}$$

Substituting Eq. (44) into Eq. (62), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)+G^{\alpha}\right]=0.\tag{63}$$

Integrating Eq. (63), we get the Noether conserved quantity

$$I^{\alpha}=(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left(p_k\bar{\xi}_k^{\alpha}+(1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial P_l}-H\right)\xi_0^{\alpha}\right)+G^{\alpha}=\text{const.},\quad(\alpha=1,2,\ldots,r).\tag{64}$$

Therefore, we have

Theorem 4 *For the dynamical system with power-law Hamiltonian (43), if the infinitesimal transformations (9) are the Noether quasi-symmetric transformations in the sense of Definition 4, then the system admits r linearly independent Noether conserved quantities (64).*

Theorems 3 and 4 are called Noether's theorem for dynamical systems with power-law Hamiltonian (43). The theorem shows that one can obtain a conserved quantity of the system if one can find a Noether symmetry transformation or a Noether quasi-symmetry transformation.

If the Hamiltonian H of the dynamical system is not dependent on time t, i.e., $H=H\left(q_k,p_k\right)$*, and letting*

$$\xi_0=1,\,\xi_k=0,\quad(k=1,2,\ldots,n),\tag{65}$$

by Theorem 3, we obtain

$$I=(1+\gamma)\left(p_s\dot{q}_s-H\right)^{\gamma}\left((1+\gamma)^{-1}\left(p_l\frac{\partial H}{\partial p_l}-H\right)-p_k\frac{\partial H}{\partial p_k}\right)=\text{const.}\tag{66}$$

Equation (66) is called the generalized energy integral for the dynamical system with power-law Hamiltonian (43).

3.4 Example

Consider the dynamical system with power-law Hamiltonian whose action is defined by [27]

$$A = \int_a^b (p\dot{q} - H)^{1+\gamma} \mathrm{d}t \tag{67}$$

where $H = \frac{1}{2}p^2 + \frac{1}{6}q^6$ is the standard Hamiltonian. By Eq. (45), we have

$$\dot{q} - p = 0,\ \ \dot{p} + q^5 - 2\gamma p^2 \left(\frac{1}{2}p^2 - \frac{1}{6}q^6 - \gamma p^2\right)^{-1} = 0. \tag{68}$$

When $\gamma = 0$, the problem is transformed to the well-known Emden equation [2]. And Eq. (68) reduces to the standard Hamilton canonical equation

$$\dot{q} - p = 0, \dot{p} + q^5 = 0. \tag{69}$$

Let us study the Noether symmetry and conserved quantity. The generalized Noether identity (59) gives

$$(1+\gamma)\left(\frac{1}{2}p^2 - \frac{1}{6}q^6\right)^{\gamma} \left[p\dot{\xi} - q^5\xi + \left(\left(\frac{1}{2}p^2 - \frac{1}{6}q^6\right)(1+\gamma)^{-1} - p^2\right)\dot{\xi}_0\right] = -\dot{G}. \tag{70}$$

Equation (70) has the following solutions:

$$\xi_0 = -1,\ \ \xi = 0,\ \ G = 0. \tag{71}$$

The generator (71) corresponds to the Noether symmetry for the dynamical system. By the Noether theorem we have obtained, the system has the following conserved quantity:

$$I = (1+\gamma)\left(\frac{1}{2}p^2 - \frac{1}{6}q^6\right)^{\gamma} p^2 - \left(\frac{1}{2}p^2 - \frac{1}{6}q^6\right)^{1+\gamma} = \text{const}. \tag{72}$$

4 Conclusions

The non-standard Hamiltonians, possessing some properties that standard Hamiltonians do not have, can be used to describe the nonlinear dynamics, etc. Based on Refs. [27,39], Noether's theorem for dynamical systems with exponential Hamiltonian is studied. An action for dynamical systems with power-law Hamiltonian is given, and its differential equation of motion is derived, and its Noether's theorem is given. From Noether's theorem, we can find the generalized energy integrals for the dynamical systems with non-standard Hamiltonians. It is worth mentioning that when $\gamma = 0$, the power-law Hamiltonian reduces to the standard Hamiltonian. Therefore, the standard Hamiltonian can be viewed as a special example of the power-law Hamiltonian.

However, there are much more kinds of non-standard Lagrangians or non-standard Hamiltonians such as logarithm form. It is of great interest to explore their roles in the nonlinear dynamics, etc. and to apply Noether's theorem to non-standard Lagrangians, with which much more work is under progress. It is worth pointing out that the exponential Hamiltonian and the power-law Hamiltonian in this paper can be considered as an uncomplicated, simple form in a more generalized exponential Lagrangian or Hamiltonian $\alpha \exp(L)$ and power-law ones $L + \eta L^{1+\gamma}$ [27], which we can make a further research on. However, we leave this for a future work. In different dimensional spaces, we can set different parameters that guarantee the correct physical dimensionalities for all terms, but we let the parameter equal to one for simplicity in this paper. Here, we have revealed an intrinsic relationship between symmetry and conserved quantity. The method here is of universal significance and can be further used to study Lie symmetry and Mei symmetry for dynamical systems with non-standard Hamiltonians.

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