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ORIGINAL PAPER

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On a constitutive equation of heat conduction with fractional derivatives of complex order

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Abstract We study the heat conduction with a general form of a constitutive equation containing fractional derivatives of real and complex order. Using the entropy inequality in a weak form, we derive sufficient conditions on the coefficients of a constitutive equation that guarantee that the second law of thermodynamics is satisfied. This equation, in special cases, reduces to known ones. Moreover, we present a solution of a temperature distribution problem in a semi-infinite rod with the proposed constitutive equation.

Mathematics Subject Classification 26A33 · 74A20

1 Introduction

The aim of the paper is twofold. First, we generalize a heat conduction law, that is, a constitutive equation for the heat flux vector, involving the complex order fractional Caputo-type derivative along the lines proposed in [4], and then, by the use of the weak form of the second law of thermodynamics, called Clausius inequality, we find a necessary condition for the coefficients of that constitutive equation. Note that the strong form of the second law of thermodynamics, known as the Clausius–Duhem inequality (cf. [15]), could not be applied to the proposed constitutive equation. Our method is based on a new form of the complex order fractional derivative derived by the use of a method proposed in [14]. The second aim is to apply the proposed model to the heat conduction in a rigid spatially one-dimensional body. The restrictions of the second law of thermodynamics guarantee the solvability for the corresponding equations determining the heat flux vector and the temperature.

2 Preliminaries

First, we explain our generalization of the classical Fourier law for the heat flux vector given as

$$q(x,t) = -K_1 \frac{\partial T}{\partial x}, \quad t \in [t_0, \infty), \quad x \in (0, L) \tag{1}$$

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in order to achieve a better agreement with experiments. In this paper, for the sake of simplicity, we consider a one-dimensional body; q is the heat flux vector, K_1 is the thermal conductivity, and T is the absolute temperature at x and time instant t. Also, $t_0 \in \mathbb{R}$ is the initial time instant, and $L \leq \infty$ is the length of the body. Using the arguments of the kinetic theory of gases, Müller [25] stated that the modifications of (1) can be classified as a differential type and a rate type:

$$q\left(x,t\right)=-K_1\frac{\partial T\left(x,t\right)}{\partial x}+K_2\frac{\partial}{\partial t}\frac{\partial T\left(x,t\right)}{\partial x}, \tag{2.1}$$

$$q\left(x,t\right)+\sigma\frac{\partial}{\partial t}q\left(x,t\right)=-K_1\frac{\partial T\left(x,t\right)}{\partial x},\quad t\in(t_0,\infty), x\in(0,L), \tag{2.2}$$

respectively[1]. The constant σ is called the relaxation time. We assume, because of (2.1) and the analysis that follows, that

$$\frac{\partial T(x,t)}{\partial x},\frac{\partial^2 T(x,t)}{\partial x^2},\frac{\partial T(x,t)}{\partial t},\frac{\partial^2 T(x,t)}{\partial t^2},\frac{\partial(\partial T(x,t)/\partial x)}{\partial t},\frac{\partial(\partial^2 T(x,t)/\partial^2 x)}{\partial t}$$

are real valued, continuous, and bounded with respect to $x \in [0, L]$ for every $t \geq t_0$, that they are equal to zero in $(-\infty, t_0)$ and belong to $L^1_{\mathrm{loc}}[t_0,\infty)$, for every $x \in (0, L)$ (If $L = \infty$, then $(0, L) = (0,\infty)$). So, the continuity and boundedness with respect to x and local integrability with respect to t hold for q and $\partial q/\partial t$, as well. It could be shown (see [9, p. 96]) that (2.1) follows from the kinetic theory of gases when the kinetic energy of a molecule consists of translational, rotational, and internal degrees of freedom.

Equation (2.2) is known as the Cattaneo equation [8]. It follows that

$$q\left(x,t\right)=-\frac{K_1}{\sigma}\int_{t_0}^{t}\exp\left(-\frac{t-\tau}{\sigma}\right)\frac{\partial T\left(x,\tau\right)}{\partial x}\mathrm{d}\tau,\quad t>t_0,\ x\in(0,L). \tag{3}$$

Equation (3) expresses the heat flux vector in terms of the history of the temperature gradient (see [16,17,29]). In general, one can assume instead of (3) the following relation for the heat flux vector (cf. [16], p. 43):

$$q\left(x,t\right)=-\int_{t_0}^{t}Q\left(t-\tau\right)\frac{\partial T\left(x,\tau\right)}{\partial x}\mathrm{d}\tau,\quad t>t_0,\ x\in(0,L) \tag{4}$$

where $Q(t)$ is a positive, decreasing relaxation function that tends to zero as $t \to \infty$.

The energy balance for a rigid body at rest, in a simplified version, reads

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{\partial q}{\partial x},\qquad u=cT+u_0 \tag{5}$$

where ρ is the density of the body, u is the specific internal energy, c is the specific heat, and u_0 does not depend on T. The equations in (2.1) combined with (5) give (see [26]):

$$\frac{\partial T\left(x,t\right)}{\partial t}=\frac{K_1}{\rho c}\frac{\partial^2 T\left(x,t\right)}{\partial x^2}-\frac{K_2}{\rho c}\frac{\partial}{\partial t}\frac{\partial^2 T\left(x,t\right)}{\partial x^2}, \tag{6.1}$$

$$\frac{\sigma}{K_1}\frac{\partial^2 T\left(x,t\right)}{\partial t^2}+\frac{\partial T\left(x,t\right)}{\partial t}=\frac{K_1}{\rho c}\frac{\partial^2 T\left(x,t\right)}{\partial x^2},\quad t>t_0,\ x\in(0,L), \tag{6.2}$$

respectively. Equation (6.2) is of hyperbolic type and has a finite speed of propagation of thermal disturbances. Recall ([18]) that the Caputo fractional derivative of real order $\alpha \in (0, 1)$ is defined as

$${}_{t_0}^{c}\mathrm{D}_t^{\alpha}q(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\frac{1}{(t-\tau)^{\alpha}}\frac{\partial q(x,\tau)}{\partial\tau}\mathrm{d}\tau=\left(\frac{\tau^{-\alpha}}{\Gamma\left(1-\alpha\right)}*\frac{\partial q(x,\tau)}{\partial\tau}\right)(x,t),\quad t>t_0,\ x\in(0,L)$$

where $*$ denotes the convolution with respect to t; in general, for $f, g \in L^1_{\mathrm{loc}}(\mathbb{R})$ being equal zero in $(-\infty, t_0)$,

$$\left(f*g\right)\left(t\right)=\int_{t_0}^{t}f\left(t-\tau\right)g\left(\tau\right)\mathrm{d}\tau,\quad t>t_0.$$

[1] The interesting question of replacing the partial derivative $\partial q/\partial t$ with the material time derivative Dq/Dt in (2.2) was discussed in [27].

It is a locally integrable function, as well.

The Cattaneo constitutive Eq. (2.2) can be considered as a special case of a constitutive equation with distributed order fractional derivatives for the heat flux vector,

$$\int_0^1 \phi_q(\gamma)\, {}_{t_0}^{c}\mathrm{D}_t^{\gamma} q(t,x)\,\mathrm{d}\gamma = -K_1 \frac{\partial T(x,t)}{\partial x}, \quad t \geq t_0, \quad x \in (0,L) \tag{7}$$

where $\phi_q(\gamma)$ is a constitutive function, or distribution, see [1]. In the special case when $\phi_q(\gamma) = \delta(\gamma)$, where δ is the Dirac distribution, (7) becomes the classical Fourier constitutive Eq. (1). If $\phi_q(\gamma) = \delta(\gamma) + \tau\delta(\gamma - 1)$, Eq. (7) becomes the Cattaneo equation (cf. [26,31] for the properties of such equations).

Heat conduction equations involving real order fractional derivatives were studied in detail in [11,29] with selection of concrete problems. A model presented in [2] is a generalization of the one treated in [11, pp. 7280]. We refer to Section 2.1 of [29] for different possibilities of the use of the generalized telegraph equation modeling the heat conduction. Note that a distributed order fractional heat conduction equation is treated in [3].

Up to our knowledge, complex order fractional derivatives in Mechanics are used by Macris, see [20–22]. In our recent paper [4], with a new form of the complex fractional derivative in a constitutive equation, we derived restrictions following from the dissipation inequality and studied creep and stress relaxation for viscoelasticity. In the cases when the Lagrangian density contains fractional derivatives of complex order, our more recent papers [5,6] are related to waves and variational problems, respectively.

The physical interpretation of the parameters of the model will be examined by experiments. This part of our work is in progress. In the case of viscoelasticity with complex order derivatives presented in [21] it is stated that the complex order derivatives modulate the phase and amplitude of harmonic components of a time-dependent function in a more complex way than the real order derivatives. In the model that we propose here [see (12)], there are five constants characterizing it. Since we are using the complex order derivative in the context of heat conduction problems, it is expected that the modulation of phase and amplitude in harmonic components of the solution of the heat conduction equation will be also more complex than in the case of real order derivatives. In any case, it is an interesting open problem to understand fully the physical meaning of the parameters in the model proposed here. However, before attempting to interpret the constants in the model, we derive in this paper the restrictions on them that follow from fundamental principles of Physics. In order to obtain restrictions that follow from the Entropy inequality, we use here a new representation of the complex order derivative. Then, as we mentioned, we apply these restrictions to an equation related to the heat conduction in a rigid spatially one-dimensional body. Note that in our approach the fractional derivatives of complex order must have real part different from zero. Complex derivatives of purely imaginary order are studied in [19] with appropriate assumptions implying the existence of a fractional derivative of purely imaginary order. Such assumptions are different from the ones used in this paper.

Concerning a constitutive equation of the form (4) it was observed in [32] that the extension of the "memory" involved in a problem implies a more wavelike solution of the heat conduction equation. In our case, the solution depends on both real and imaginary part of the complex fractional derivatives of the heat flux vector appearing in the constitutive equation. From the results of [4] we know that the complex part of the fractional derivative brings a wavelike behavior. This makes the use of complex fractional derivatives useful and challenging, in general.

3 Generalized heat conduction law

As it is postulated, a heat conduction equation, that is, an equation connecting $q(x,t)$ and $\frac{\partial T(x,t)}{\partial x}$, $t > t_0$, $x \in (0,L)$, must satisfy the restrictions that follow from the second law of thermodynamics.

Let r denote the heat supply per unit mass, and let η be the entropy density. Then, the local form of the Clausius–Duhem inequality reads

$$\rho(x,t)\frac{\partial \eta(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\frac{q(x,t)}{T(x,t)}\right] - \frac{\rho(x,t)\,r(x,t)}{T(x,t)} \geq 0, \quad t > t_0, \ x \in (0,L).$$

There exists a weaker form of the second law of thermodynamics (see [15], p. 11), called the Clausius inequality,

$$\int_{t_1}^{t_2}\left[-\frac{\partial}{\partial x}\left[\frac{q(x,t)}{T(x,t)}\right] + \frac{\rho(x,t)\,r(x,t)}{T(x,t)}\right]\mathrm{d}t \leq 0, \tag{8}$$

that involves cyclic processes between equilibrium states. Thus, (8) states that in any cyclic process (called D-cyclic process in [33, p. 585]) between times t_1 and t_2, where $t_0 \leq t_1 < t_2$, which starts from equilibrium at a material point x, the inequality (8) must hold.

Following [13, p. 25] and [33, p. 586], the energy Eq. (5) with $r = 0$ and the periodicity condition imply that Eq. (8) can be written as

$$\int_{t_1}^{t_2} \frac{q\,(x,t)\,\frac{\partial T(x,t)}{\partial x}}{T^2\,(x,t)}\mathrm{d}t \;\leq 0, \quad x \in (0,L). \tag{9}$$

It was shown in [24] that (9) holds if and only if

$$\int_{s_1}^{s_2} q\,(x,t)\,\frac{\partial T\,(x,t)}{\partial x}\mathrm{d}t \leq 0, \quad x \in (0,L), \tag{10}$$

for arbitrary $t_1 \leq s_1 < s_2 \leq t_2$.

Motivated by the results of [4,5], we propose in this paper a completely different constitutive equation for the heat flux vector:[2]

$$q\,(x,t) + \widehat{a}\,{}_{t_0}^{\,c}\mathrm{D}_t^{\alpha} q\,(x,t) + 2\widehat{b}\,{}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q\,(x,t) = -K_1 \frac{\partial T\,(x,t)}{\partial x}, \quad t > t_0, \quad x \in (0,L)\,. \tag{11}$$

Here ${}_{t_0}^{\,c}\mathrm{D}_t^{\alpha}$, $\alpha \in (0,1)$ is given by (7), and ${}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta}$ is a combination of complex order fractional derivatives defined as

$${}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q(x,t) = \frac{1}{2}\left[\mathbb{T}^{i\beta c}{}_{t_0}\mathrm{D}_t^{\alpha+i\beta} q(x,t) + \mathbb{T}^{-i\beta c}{}_{t_0}\mathrm{D}_t^{\alpha-i\beta} q(x,t)\right]$$

or

$${}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q(x,t) = \frac{1}{2}\left[\mathbb{T}^{i\beta}\left(\frac{t^{-\alpha-i\beta}}{\Gamma\,(1-\alpha-i\beta)} * \frac{\partial q(x,t)}{\partial t}\right) + \mathbb{T}^{-i\beta}\left(\frac{t^{-\alpha-1+i\beta}}{\Gamma\,(1-\alpha+i\beta)} * \frac{\partial q(x,t)}{\partial t}\right)\right] \tag{12}$$

where $t > t_0, 0 < \alpha < 1, \beta > 0, i = \sqrt{-1}$, and $\mathbb{T}$ is a constant having the dimension of time, and $\widehat{a}, \widehat{b} \geq 0$ are constants having dimension $(\text{time})^{1/\alpha}$. Both constants $\widehat{a}$ and $\mathbb{T}$ can be interpreted as relaxation times. Since q is a real valued function of x and t, the derivative ${}_{t_0}\bar{\mathrm{D}}_t^{\alpha,\beta}$ is also real valued. Equation (11) is a generalization of the classical Fourier law (1) and Cattaneo conduction law (2.1). Physically, it shows a thermal inertia of fractional type, i.e., thermal inertia with memory effects.

As our main result in this paper, we give the restrictions on coefficients in (11) which are a consequence of (10).

Introducing the dimensionless quantities in a usual way (and (0, 1) instead of $(0, L)$), we obtain, instead of (11),

$$q\,(x,t) + a\,{}_{t_0}^{\,c}\mathrm{D}_t^{\alpha} q\,(x,t) + 2b\,{}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q\,(x,t) = -\frac{\partial T\,(x,t)}{\partial x}, \quad t > t_0, \quad x \in (0,1) \tag{13}$$

where ${}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q\,(x,t)$ (in the dimensionless form) is

$${}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta} q(x,t) = \frac{1}{2}\left[{}_{t_0}^{\,c}\mathrm{D}_t^{\alpha+i\beta} q(x,t) + {}_{t_0}^{\,c}\mathrm{D}_t^{\alpha-i\beta} q(x,t)\right], \quad t > t_0, \quad x \in (0,1). \tag{14}$$

Equation (13) can be written in an equivalent form as

$$q\,(x,t) + \left(\Phi_{\alpha,\beta}\,(\tau) * \frac{\partial q(x,\tau)}{\partial \tau}\right)(x,t) = -\frac{\partial T\,(x,t)}{\partial x}, \quad t > t_0, \quad x \in (0,1) \tag{15}$$

with

$$\Phi_{\alpha,\beta}\,(\tau) = a\frac{\tau^{-\alpha}}{\Gamma\,(1-\alpha)} + b\left\{\frac{\tau^{-\alpha-i\beta}}{\Gamma\,(1-\alpha-i\beta)} + \frac{\tau^{-\alpha+i\beta}}{\Gamma\,(1-\alpha+i\beta)}\right\}, \quad \tau \geq 0. \tag{16}$$

[2] We studied in [4,5] the stress strain relation for the viscoelastic body of the Kelvin–Voigt type $\sigma\,(t,x) = E\left(1 + \widehat{a}\,{}_{t_0}^{\,c}\mathrm{D}_t^{\alpha}\varepsilon\,(t,x) + 2\,\widehat{b}\,{}_{t_0}^{\,c}\bar{\mathrm{D}}_t^{\alpha,\beta}\varepsilon\,(t,x)\right)$ and derived restrictions on this constitutive equation which follow from the dissipation inequality $\int_0^{t_1}\sigma\,(x,t)\,\frac{\mathrm{d}\varepsilon(x,t)}{\mathrm{d}t}\mathrm{d}t \geq 0$.

Now, the heat conduction Eq. (15) can be written as

$$q(x,t)+\int_0^t\left\{\left[a\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}+b\left(\frac{(t-\tau)^{-\alpha-i\beta}}{\Gamma(1-\alpha-i\beta)}+\frac{(t-\tau)^{-\alpha+i\beta}}{\Gamma(1-\alpha+i\beta)}\right)\right]\frac{\partial q(x,\tau)}{\partial\tau}\mathrm{d}\tau\right\}$$
$$=-\frac{\partial T(x,t)}{\partial x},\quad t>t_0, x\in(0,1). \tag{17}$$

Remark 1 (i) We note that the second law of thermodynamics interpreted as (8) for (17) can be proved by a method given in our book [1] and papers [4,5]. Here, we give another simpler proof which can be used in different situations, as well.

(ii) Equation (17) is the generalized heat conduction law that we propose in this paper. As a motivation for the generalization of (1), we quote a statement from [16, p. 44]: *It would be a miracle if for some real conductor the relaxation kernel could be rigorously represented by an exponential kernel with a single time of relaxation, as it is required by Cattaneo's model.* Our generalization of (2.2) given by (17) follows the idea of the quoted statement. It contains the Cattaneo equation as a special case $\alpha=1$, $\beta=0$, $b=0$.

4 New representation of the complex order fractional derivative

Recall that $\frac{\partial q(x,t)}{\partial t}\in L^1([t_0,t_1])$ $t_1>t_0$, for each $x\in(0,1)$. First, we state a new representation for the complex order fractional derivative given by (14).

Proposition 1 *Let* $F\in L^1([t_0,t_1])$*,* $t\in[t_0,t_1]$*. Then*

$${}_{t_0}^{c}\bar{\mathrm{D}}_t^{\alpha,\beta}F(t)=\frac{1}{2}\left[{}_{t_0}^{c}\mathrm{D}_t^{\alpha+i\beta}F(t)+{}_{t_0}^{c}\mathrm{D}_t^{\alpha-i\beta}F(t)\right],\quad\alpha\in(0,1),\quad\beta\in\mathbb{R}$$

can be represented as

$${}_{t_0}^{c}\bar{\mathrm{D}}_t^{\alpha,\beta}F(t)=\frac{1}{\pi}\int_0^\infty G(\xi)\left(\int_{t_0}^t F^{(1)}(\tau)\exp(-(t-\tau)/\xi)\,\mathrm{d}\tau\right)\mathrm{d}\xi \tag{18}$$

where

$$G(\xi)=\frac{\sin(\pi\alpha)\cosh(\pi\beta)\cos(\beta\ln\xi)+\cos(\pi\alpha)\sinh(\pi\beta)\sin(\beta\ln\xi)}{\xi^{\alpha+1}},\ \xi>0.$$

Proof We start with the definition of the Gamma function of complex argument $\alpha+i\beta$,

$$\Gamma(\alpha+i\beta)=\int_0^\infty u^{\alpha+i\beta-1}\exp(-u)\,\mathrm{d}u.$$

The integral converges for $\alpha>0$. Fix $t>0$. By the substitution $u=t/\xi$, as in [14], one obtains

$$\Gamma(\alpha+i\beta)=\int_0^\infty t^{\alpha+i\beta}\left(\frac{1}{\xi}\right)^{\alpha+i\beta+1}\exp(-t/\xi)\,\mathrm{d}\xi.$$

This implies, as in the real case [14],

$$\frac{1}{t^{\alpha+i\beta}}=\frac{1}{\Gamma(\alpha+i\beta)}\int_0^\infty\left(\frac{1}{\xi}\right)^{\alpha+i\beta+1}\exp(-t/\xi)\,\mathrm{d}\xi,\quad t>0,$$

so that

$$\begin{aligned}{}_{t_0}^{c}\mathrm{D}_t^{\alpha+i\beta}F(t)&=\frac{1}{\Gamma(1-\alpha-i\beta)}\int_{t_0}^t\frac{F^{(1)}(\tau)}{(t-\tau)^{\alpha+i\beta}}\mathrm{d}\tau\\&=\frac{1}{\Gamma(1-\alpha-i\beta)}\int_{t_0}^t F^{(1)}(\tau)\left(\int_0^\infty\frac{1}{\Gamma(\alpha+i\beta)}\left(\frac{1}{\xi}\right)^{\alpha+i\beta+1}\exp(-(t-\tau)/\xi)\,\mathrm{d}\xi\right)\mathrm{d}\tau\\&=\int_0^\infty\frac{1}{g(\xi)}\left(\int_{t_0}^t F^{(1)}(\tau)\exp(-(t-\tau)/\xi)\,\mathrm{d}\tau\right)\mathrm{d}\xi,\quad t>t_0.\end{aligned}$$

Since $\Gamma(u)\Gamma(1-u) = \pi/\sin(\pi u)$, $\Re u < 1$, with $u = \alpha + i\beta$ and $u = \alpha - i\beta$, we obtain

$$
\begin{aligned}
g(\xi) &= \frac{1}{\Gamma(1-\alpha-i\beta)\,\Gamma(\alpha+i\beta)\,\xi^{(\alpha+i\beta+1)}} \\
&= \frac{\sin(\pi\alpha)\cosh(\pi\beta)\cos(\beta\ln\xi) + \cos(\pi\alpha)\sinh(\pi\beta)\sin(\beta\ln\xi)}{\pi\xi^{(\alpha+1)}} \\
&\quad + i\,\frac{\cos(\pi\alpha)\sinh(\pi\beta)\cos(\beta\ln\xi) - \sin(\pi\alpha)\cosh(\pi\beta)\sin(\beta\ln\xi)}{\pi\xi^{(\alpha+1)}}, \quad \xi > 0.
\end{aligned} \tag{19}
$$

Similarly, for ${}_{t_0}^{c}\mathrm{D}_t^{\alpha-i\beta}F$ the same type of equation is obtained with the complex conjugate $\overline{g}$ instead of g. Adding ${}_{t_0}^{c}\mathrm{D}_t^{\alpha+i\beta}F$ to ${}_{t_0}^{c}\mathrm{D}_t^{\alpha-i\beta}F$ we obtain (18) with $G = 2\,\Re g > 0$.

Remark 2 If $\beta = 0$, then the definition of the Caputo derivative of real order $\alpha \in (0,1)$ follows from (19):

$$
{}_{t_0}^{c}\mathrm{D}_t^{\alpha}F(t) = \frac{1}{\pi}\int_0^{\infty}\frac{\sin(\pi\alpha)}{\pi\xi^{(\alpha+1)}}\left(\int_{t_0}^{t}F^{(1)}(\tau)\exp\left(-(t-\tau)/\xi\right)\mathrm{d}\tau\right)\mathrm{d}\xi, \quad t > t_0. \tag{20}
$$

It is presented in [14] in a slightly different form.

Remark 3 It could be easily shown that the right Caputo fractional derivative of order $\alpha \in (0,1)$ defined by ${}_{t}^{c}\mathrm{D}_T^{\alpha}F(t) = \frac{-1}{\Gamma(1-\alpha)}\int_t^T \frac{F(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau$, for a function $F \in L^1([t_0, T])$ $T \geq t_0$, can be represented as

$$
{}_{t}^{c}\mathrm{D}_T^{\alpha}F(t) = -\frac{1}{\pi}\int_0^{\infty}\frac{\sin(\pi\alpha)}{\pi\xi^{(\alpha+1)}}\left(\int_t^T F^{(1)}(\tau)\exp\left(-(\tau-t)/\xi\right)\mathrm{d}\tau\right)\mathrm{d}\xi.
$$

For the right Caputo derivative of complex order, we obtain ($t_0 \leq t \leq T$)

$$
\begin{aligned}
{}_{t_0}^{c}\mathrm{D}_t^{\alpha+i\beta}F(t) &= \frac{-1}{\Gamma(1-\alpha-i\beta)}\int_t^T \frac{F^{(1)}(\tau)}{(\tau-t)^{\alpha+i\beta}}\mathrm{d}\tau \\
&= -\int_0^{\infty}\frac{1}{g(\xi)}\left(\int_t^T F^{(1)}(\tau)\exp\left(-(\tau-t)/\xi\right)\mathrm{d}\tau\right)\mathrm{d}\xi, \\
&\quad \alpha \in (0,1), \quad \beta \in \mathbb{R}
\end{aligned}
$$

(function g is given by (19)).

5 Clausius inequality

We now state the main result related to the heat flux vector given by (17). We shall prove the consequences of (10).

Theorem 1 *The heat flux vector given by* (17)*, or equivalently* (13)*, satisfies* (10) *if*

$$
a > 2b\cosh\beta\pi\sqrt{1 + \left(\cot(\alpha\pi)\tanh(\beta\pi)\right)^2}, \quad \alpha \in (0,1), \quad \beta \in \mathbb{R}. \tag{21}
$$

Proof For the sake of simplicity, we took in the previous section $t_0 = 0$, $t_2 = d$. Multiplying (13) with q and integrating with respect to time, we obtain

$$
\int_0^d q^2(t)\,\mathrm{d}t + \int_0^d \left(a\,{}_{t_0}^{c}\mathrm{D}_t^{\alpha}q(x,t) + 2b\,{}_{t_0}^{c}\bar{\mathrm{D}}_t^{\alpha,\beta}q(x,t)\right)q(t)\,\mathrm{d}t = -\int_0^d q(t)\frac{\partial T(x,t)}{\partial x}\mathrm{d}t, \quad x \in (0,1).
$$

A sufficient condition for (10) to hold, that is, $\int_0^d q(x,t)\frac{\partial T(x,t)}{\partial x}\mathrm{d}t < 0$, $x \in (0,1)$, is

$$
\int_0^d \left[a\,{}_{t_0}^{c}\mathrm{D}_t^{\alpha}q(x,t) + 2b\,{}_{t_0}^{c}\bar{\mathrm{D}}_t^{\alpha,\beta}q(x,t)\right]q(t)\,\mathrm{d}t \geq 0. \tag{22}
$$

Using (19) and (20), the inequality (22) becomes ($x \in (0, 1)$)

$$\int_0^\infty \left[\frac{\sin(\pi\alpha)}{\pi\xi^{(\alpha+1)}} \{a + 2b [\cosh(\pi\beta)\cos(\beta \ln\xi) + \cot(\pi\alpha)\sinh(\pi\beta)\sin(\beta\ln\xi)]\} \right.$$
$$\left. \times \int_0^d q(x,t) \left(\int_0^t q^{(1)}(\tau) \exp(-(t-\tau)/\xi)\, \mathrm{d}\tau \right) \mathrm{d}t \right] \mathrm{d}\xi \geq 0. \tag{23}$$

Let $M(x,t,\xi) = \int_0^t \frac{\partial}{\partial\tau} q(x,\tau) \exp(-(t-\tau)/\xi)\, \mathrm{d}\tau$ $x \in (0,1), t > 0, \xi > 0$. Then (for $x \in (0,1), t > 0, \xi > 0$),

$$\frac{\partial M(x,t,\xi)}{\partial t} = \frac{\partial}{\partial t} q(x,t) - \frac{1}{\xi} M(x,t,\xi),$$

so that

$$\frac{\partial q(t,x)}{\partial t} = \frac{\partial M(x,t,\xi)}{\partial t} + \frac{1}{\xi} M(x,t,\xi), \quad M(0,\xi) = 0.$$

Condition $M(x,0,\xi) = 0$ is satisfied since $\frac{\partial q(x,t)}{\partial t} \in L^1([0,d]), x \in (0,1)$. Also,

$$q(x,t) = M(x,t,\xi) + \frac{1}{\xi} \int_0^t M(x,\tau,\xi)\, \mathrm{d}\tau, \ \ t \in [0,d],$$

and (23) becomes

$$\int_0^d \left[a\, {}_{t_0}^{\ c}\mathrm{D}_t^{\alpha} q(x,t) + 2b\, {}_{t_0}^{\ c}\bar{\mathrm{D}}_t^{\alpha,\beta} q(x,t) \right] q(x,t)\, \mathrm{d}t$$
$$= \int_0^\infty \frac{1}{\pi\xi^{(\alpha+1)}} S(\xi) \left\{ \frac{1}{2\xi} M^2(x,d,\xi) + \int_0^d M^2(x,t,\xi)\, \mathrm{d}t \right\} \mathrm{d}\xi \tag{24}$$

where

$$S(\xi) = a\sin(\pi\alpha) + b[\sin(\pi\alpha)\cosh(\pi\beta)\cos(\beta\ln\xi) + \cos(\pi\alpha)\sinh(\pi\beta)\sin(\beta\ln\xi)], \quad \xi > 0.$$

The sign of the expression in (24) depends on the sign of $S(\xi), \xi \in [0, \infty)$. Condition $S(\xi) > 0$ leads to

$$a > 2b\cosh\beta\pi\sqrt{1 + (\cot\alpha\pi\, \tanh\beta\pi)^2}, \quad \alpha \in (0,1), \quad \beta \in \mathbb{R}.$$

This proves the theorem.

Remark 4 In the case of a real order fractional derivative, we have $b = 0$ so that $a > 0$ guarantees that

$$q(x,t) + a\, {}_{t_0}^{\ c}\mathrm{D}_t^{\alpha} q(x,t) = -\frac{\partial T(x,t)}{\partial x}, \quad t > 0, \quad \alpha \in (0,1) \tag{25}$$

satisfies the Clausius inequality. Recently, in [23], the constitutive Eq. (25) was treated, and it was shown that (in our notation) q given by (25) satisfies the second law of thermodynamics for sufficiently small a. To prove this result, the authors of [23] used the strong form of the entropy inequality.

6 Thermo-mechanical problem

We consider a rigid semi-infinite rod, that is $L = \infty$, as well as $t_0 = 0$. Suppose that for $t < 0$ and $x > 0$ the temperature in the rod is constant, i.e., $T = \theta_\infty$. Moreover, suppose that for $t \geq 0$ the end $x = 0$ is kept at the constant temperature $T = \theta_\infty + T_0$, $T_0 > 0$. Thus, we have to solve

$$\frac{\partial T(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x}, \tag{26.1}$$

$$q(x,t) + a\, {}_{t_0}\mathrm{D}_t^{\alpha} q(x,t) + 2b\, {}_{t_0}\bar{\mathrm{D}}_t^{\gamma} q(x,t) = -\frac{\partial T(x,t)}{\partial x}, \quad t > 0, x > 0, \tag{26.2}$$

$$T(0,t) = T_0 \;\; t > 0, T(x,0) = 0, \quad t = 0, \quad x > 0, \quad \lim_{x\to\infty} T(x,t) = 0 \tag{26.3}$$

where we used T for $T - \theta_\infty$. A simplified problem (26.1) for $a = b = 0$ is the classical one treated in [12].

Applying the Laplace transform

$$\mathcal{L}(f)(s) = \overline{f}(s) = \int_0^\infty \exp(-st)\, f(t)\, \mathrm{d}t, \;\; s \in \mathbb{C}$$

to (26.1,2) we obtain, for $\Re s > 0$, $(x > 0)$

$$s\overline{T}(x,s) = -\frac{\partial \overline{q}(x,s)}{\partial x},$$

$$\overline{q}(x,s)\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right] = -\frac{\partial \overline{T}(x,s)}{\partial x},$$

so that

$$\frac{\partial^2 \overline{T}(x,s)}{\partial x^2} = s\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right] T(x,s). \tag{27}$$

With $T(x,t)$, we can determine $q(x,t)$ as

$$q(x,t) = \mathcal{L}^{-1}\left(\frac{-\frac{\partial \overline{T}(x,s)}{\partial x}}{1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)}\right), \quad t > 0, \quad x > 0. \tag{28}$$

From the boundary condition (26.3) we obtain $\overline{T}_0 = \int_0^\infty \exp(-st)\, T_0 \mathrm{d}t = \frac{T_0}{s}$. Solving (27) and using the value $\overline{T}_0$ and (26.3), we obtain, for $\Re s > 0$, $t > 0$,

$$\hat{T}(x,s) = \frac{T_0}{s} \exp\left[-xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]\right\}^{\frac{1}{2}}\right]$$

or

$$\hat{T}(x,s) = \frac{T_0}{s} \exp\left(-xs^{1/2} - xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right).$$

This implies

$$T(x,t) = T_0 \mathrm{erfc}\left(x\frac{1}{2\sqrt{t}}\right) * G(x,t), t > 0, \;\; x > 0 \tag{29}$$

where we used the fact that $\mathcal{L}^{-1}\left(\frac{T_0}{s}\exp\left(-xs^{1/2}\right)\right)(x,t) = T_0 \mathrm{erfc}\left(x\frac{1}{2\sqrt{t}}\right)$, where $\mathrm{erfc} z = 1 - \mathrm{erf} z$ is the complementary error function (see [12]) and where

$$G(x,t) = \mathcal{L}^{-1}\left(\exp\left(-xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right)\right),$$
$$t > 0, \;\; x > 0.$$

To determine G, we need the following proposition.

Proposition 2 *Let*

$$M(s) = s^{1/2}\left(1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right)^{1/2} - 1, \quad \Re s \geq 0, \quad |\Im s| > 0.$$

Then, $M(0) = 0$, and M does not have any zero if the condition (21) is satisfied.

Proof It is clear that $s = 0$ is the zero and the branch point of M. Consider

$$\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\} = 0.$$

Then

$$as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right) = s^{\alpha}\left[a + b\left(s^{i\beta} + s^{-i\beta}\right)\right] = 0. \tag{30}$$

Clearly, $s = 0$ is the zero and the branch point. Moreover, if other zeros exist, they are solutions of

$$a + b\left(s^{i\beta} + s^{-i\beta}\right) = 0. \tag{31}$$

Let us show that such zeros do not exist. The zeros of (30) are complex conjugate so we consider only the upper part of the complex plane $\Im s > 0$ in analyzing (31). Let $s = r\exp(i\theta)$, $\theta \in [0, \pi]$. So, (31) leads to

$$a + 2b\cosh\theta\beta\cos\beta\ln r = 0, \quad 2b\sinh\theta\beta\sin\beta\ln r = 0. \tag{32.1,2}$$

Condition (32.2) is satisfied for $\theta = 0$ or $\beta\ln r = n\pi, n = 1, 2, \ldots$ In the first case $\theta = 0$, from (32.1) one obtains $a - 2b = 0$. This contradicts to (21). In the second case $\beta\ln r = n\pi, n = 1, 2, \ldots$, (32.1) leads to $a - 2b\cosh\theta\beta = 0$. However, $\sqrt{1 + (\cot\alpha\pi\tanh\beta\pi)^2} > 1$, so the condition (21) implies that

$$a > 2b\cosh\pi\beta.$$

This is a contradiction, again. So, there are no zeros of (31).

Next, we determine G. Let $c_0 > 0$. Let $t > 0$, $x > 0$. The inversion reads

$$\begin{aligned} G(x,t) &= \mathcal{L}^{-1}\left(\exp\left(-xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right)\right)(x,t) \\ &= \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \exp\left(st - xs^{1/2}\left\{\left[\left\{1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right\}\right]^{1/2} - 1\right\}\right) \mathrm{d}s. \end{aligned}$$

We use the contour shown in Fig. 1 to avoid the branch point at the origin. The branch cut runs along the negative real axis from origin to infinity. Since

$$\left|\exp\left(-xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right)\right| < a/|s|^{-1/2}$$

we conclude (see Lemma 2.2 of [10])[3] that the integrals over parts of the contour BC, CD, KP, and PA tend to zero when $R \to \infty$.

On the arc EQ, we have $s = \varepsilon\exp(i\varphi)$, $\varphi \in (\pi, -\pi)$. Therefore, taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{EQ}\exp\left(st - xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right)\mathrm{d}s \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\pi}^{-\pi}\left(\exp\varepsilon\exp(i\varphi)t - x\varepsilon^{1/2}\exp(i\varphi/2)\right. \\ &\quad\times\left.\left\{\left[1 + \varepsilon^{\alpha}\exp(i\alpha\varphi)\left\{a + b\left((\varepsilon\exp(i\varphi))^{i\beta} + (\varepsilon\exp(i\varphi))^{-i\beta}\right)\right\}\right]^{1/2} - 1\right\} i\varepsilon\exp(i\varphi)\,d\varphi\right) = 0. \end{aligned}$$

[3] Lemma 2.2 of [10] reads: Let R_0, C, and ν be positive constants. If $|f(s)| < CR^{-\nu}$, $s = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, $R > R_0$, then, $\int_{BCE} f(s)\exp(st) \to 0$ and $\int_{KPA} f(s)\exp(st) \to 0$, for $t > 0$.

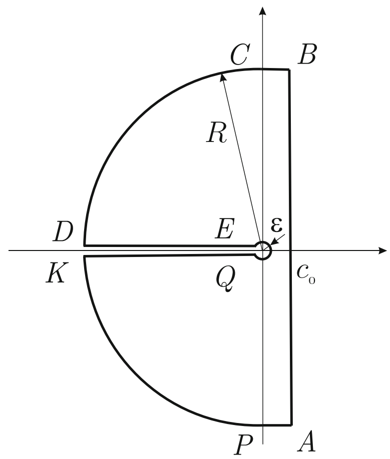

Fig. 1 Integration path for the determination of G

We consider now the parts of the contour DE and QK. Let $s = p\exp(i\pi)$, $p \in (\infty, \varepsilon)$ on DE and $s = p\exp(-i\pi)$, (ε, ∞) on KQ. We obtain the inversion ($t > 0$, $x \in (0, 1)$),

$$\begin{aligned} G(x,t) &= \mathcal{L}^{-1}\left(\exp\left(-xs^{1/2}\left\{\left[1 + as^{\alpha} + b\left(s^{\alpha+i\beta} + s^{\alpha-i\beta}\right)\right]^{1/2} - 1\right\}\right)\right)(x,t) \\ &= -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\infty}^{\varepsilon}\exp\left(-pt - xp^{1/2}\exp(i\pi/2)\left\{\left[1 + p^{\alpha}\exp(i\alpha\pi)\right.\right.\right. \\ &\quad\times\left.\left.\left.\left[a + b\left((p\exp(i\pi))^{i\beta} + (p\exp(i\pi))^{-i\beta}\right)\right]\right]^{1/2} - 1\right\}\right)\mathrm{d}p, \\ &\quad - -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\exp\left(-pt - xp^{1/2}\exp(-i\pi/2)\left\{\left[1 + p^{\alpha}\exp(-i\alpha\pi)\right.\right.\right. \\ &\quad\times\left.\left.\left.\left[a + b\left((p\exp(-i\pi))^{i\beta} + (p\exp(-i\pi))^{-i\beta}\right)\right]\right]^{1/2} - 1\right\}\right)\mathrm{d}p. \end{aligned}$$

This implies

$$\begin{aligned} G(x,t) &= \frac{1}{\pi}\,\mathrm{Im}\int_{0}^{\infty}\exp\left(-pt - ixp^{1/2}\left\{\left[1 + p^{\alpha}\exp(i\pi\alpha)\right.\right.\right. \\ &\quad\times\left.\left.\left.\{a + b\left[2(\cosh\beta\pi)(\cos\beta\ln p) - 2i(\sinh\beta\pi)\sin\beta\ln p\right]\}\right]^{1/2} - 1\right\}\right)\mathrm{d}p. \end{aligned} \tag{33}$$

Equations (29) and (33) determine the solution.

7 Conclusions

The main results of this paper are a new form of the constitutive equation for the heat flux and restrictions on coefficients obtained from the weak form of the Entropy inequality as well as the solution of the thermo-mechanical problem by the use of restrictions. In particular, we did the following:

1. We formulated a constitutive equation for the heat flux q given by (13) containing real and complex order fractional derivatives. It is of the Cattaneo type (2.2).
2. We determined the restrictions on the coefficients in the constitutive equation that follow from the entropy inequality in the weak form (10). In proving (10), we used a new representation of the symmetrized fractional derivative (14) given by (18). The restriction (21) also guarantees the solvability of (13) for the heat flux q in the form (28). Let us note that in our case the memory function given by (16) depends on both real and imaginary part of the derivatives of the heat flux vector in (13).

3. We presented an equation of heat conduction in a rigid conductor. The influence of the parameters in the model on the solution may be examined along the lines presented in [4], where we use results of [7]. On the basis of results presented in [4] we expect that the complex order part of the fractional derivative provides a non-monotonic temperature distribution in a rod. Also from the general property of complex order derivatives, presented in [21, p. 1459], we conclude that the modulation of phase and amplitude of harmonic components of a solution will be more complicated as compared with the case of real order derivatives.

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