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ORIGINAL PAPER

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Coefficients of nonlinear thermal expansion for fiber-reinforced composites

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Abstract In the present study, a micromechanics model is proposed to predict the coefficients of nonlinear thermal expansion (CTEs) of fiber-reinforced composites. The influence of fiber aspect ratio on the CTEs is also investigated. It is noted that the parameters of fiber aspect ratio have a significant effect on both the longitudinal CTEs and transverse CTEs. The CTEs of composites are also very sensitive to the different fiber volume fractions. Moreover, the Young's modulus and Poisson's ratio of composites are taken into account in the present analysis. The theoretical derivations are applicable for the composites under mechanical or thermal environment conditions. The present model offers a direct prediction of CTEs and can account for the effects of fiber aspect ratio and volume fractions.

1 Introduction

The fiber-reinforced composites are widely employed in modern engineering structures. This is because they have some advantages, such as a high stiffness to weight ratio, an excellent durability and a design flexibility. To date, many mechanics models have been developed to study the thermal properties of fiber-reinforced composites [1–4]. Bian and Zhao [5] proposed a continuum model to study the mechanical properties of carbon nanotubes, in which a thermal expansion coefficient of carbon nanotubes is proposed and is defined as a continuous variation. Zhao et al. [6] developed a new continuum theory incorporating interatomic potentials and finite temperature to study the bifurcation strain and force of single-walled carbon nanotube (SWNT) under the action of tension. Dong [7] studied the transverse coefficients of thermal expansion (CTEs) for unidirectional carbon fiber composites by finite element analysis with a representative unit cell. Gusev [8] derived an exact self-consistent solution of n-layered composite sphere model based on the homogeneous solutions, which could be applied to predict effective thermal expansion coefficients of composites. The study of Karadeniz and Kumlutas [9] developed a model for the microstructure of composites using a representative unit cell of finite element method, and thermal expansion coefficients of composites have been estimated.

Some analysis methods for effective coefficients of thermal expansion of composites have been suggested based on the micromechanics [10–12]. The earlier investigators, Turner [11] and Kerner [12] proposed models for predicting CTEs of composites, which can be used to define a lower and an upper bound, respectively. A model has been developed by Rosen and Hashin [13] using thermoelastic energy principles. This model can determine the effective thermal expansion coefficients of anisotropic composites having any number of anisotropic phases. Schapery [14] presented a model that was based on the simple planar model with alternative fibers and matrix strips. A study of Schneider [15] was based on a cylinder assemblage model, which has

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been cited in German literature. Islam et al. [16] used the finite element method to investigate the linear thermal expansion coefficients of unidirectional fiber composites. A combined experimental and numerical methodology for the evaluation of fiber properties was presented in the study by Rupnowski et al. [17], on the basis of the composite macro-data.

The aim of the present paper is to investigate the CTEs of unidirectional fiber-reinforced composites, with an emphasis on the influence of fiber aspect ratio on the CTEs. The self-consistent and Mori–Tanaka approaches have been extended in the present investigation, and a new model of isotropic composites which includes two-phase materials has been developed. Moreover, analytical expressions are also derived for predicting the CTEs of fiber-reinforced composites without the filler aspect ratio and any particle inclusions. It is shown that the fiber aspect ratio has a significant effect on both the longitudinal CTEs and transverse CTEs. The CTEs of composites at different fiber volume fractions are very sensitive to the aspect ratio of fiber.

2 Model development

2.1 Prediction of CTEs

The representative element of composites is subjected to uniform stress $\overline{\boldsymbol{\sigma}}$. Local stress and strain that caused in representative element are denoted by $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively. In the same way, it is assumed that the composite is subjected to a uniform temperature change ΔT. Local stress and strain that caused in representative element with ΔT are denoted by $\boldsymbol{\sigma}^T$ and $\boldsymbol{\varepsilon}^T$, respectively. Local stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^T$ conform to the balance equation, and $\langle\boldsymbol{\sigma}\rangle = \overline{\boldsymbol{\sigma}}$, $\left\langle\boldsymbol{\sigma}^T\right\rangle = 0$,

$$\begin{aligned}\frac{1}{V}\int_V \boldsymbol{\sigma}\boldsymbol{\varepsilon}^T \mathrm{d}V &= \frac{1}{V}\int_V \sigma_{ij}u_{i,j}^T \mathrm{d}V = \frac{1}{V}\int_V \left(\sigma_{ij}u_i^T\right)_{,j} \mathrm{d}V \\ &= \frac{1}{V}\int_{\partial V} \sigma_{ij}u_i^T n_j \mathrm{d}S = \overline{\sigma}_{ij}\left\langle\varepsilon_{ij}^T\right\rangle, \end{aligned} \tag{1}$$

$$\begin{aligned}\frac{1}{V}\int_V \boldsymbol{\sigma}^T\boldsymbol{\varepsilon} \mathrm{d}V &= \frac{1}{V}\int_V \sigma_{ij}^T u_{i,j} \mathrm{d}V = \frac{1}{V}\int_V \left(\sigma_{ij}^T u_i\right)_{,j} \mathrm{d}V \\ &= \frac{1}{V}\int_{\partial V} \sigma_{ij}^T u_i n_j \mathrm{d}S = 0, \end{aligned} \tag{2}$$

where $\boldsymbol{u}^T$ is a relative displacement.

Here, Eq. (1) can be simplified as follows:

$$\begin{aligned}\frac{1}{V}\int_V \boldsymbol{\sigma}\boldsymbol{\varepsilon}^T \mathrm{d}V &= \frac{1}{V}\int_V \sigma\left(\boldsymbol{H}\boldsymbol{\sigma}^T + \boldsymbol{\alpha}\Delta T\right)\mathrm{d}V \\ &= \frac{1}{V}\int_V \left(\boldsymbol{\varepsilon}\boldsymbol{\sigma}^T + \boldsymbol{\sigma}\boldsymbol{\alpha}\Delta T\right)\mathrm{d}V = \langle\boldsymbol{\sigma}\boldsymbol{\alpha}\rangle\,\Delta T = \overline{\boldsymbol{\sigma}}\left\langle\boldsymbol{\varepsilon}^T\right\rangle. \end{aligned}$$

According to the definition of the coefficient of thermal expansion,

$$\left\langle\boldsymbol{\varepsilon}^T\right\rangle = \boldsymbol{\alpha}^{\mathrm{c}}\Delta T. \tag{3}$$

For the composites with two-phase materials, the thermal expansion coefficients of composite, matrix and filler are denoted by $\boldsymbol{\alpha}^{\mathrm{c}}$, $\boldsymbol{\alpha}_{\mathrm{m}}$ and $\boldsymbol{\alpha}_{\mathrm{f}}$, respectively, and using Eqs. (1)–(3), we have

$$c_{\mathrm{f}}\boldsymbol{\alpha}_{\mathrm{f}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{f}} + c_{\mathrm{m}}\boldsymbol{\alpha}_{\mathrm{m}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{m}} = \overline{\boldsymbol{\sigma}}\boldsymbol{\alpha}^{\mathrm{c}} = \boldsymbol{\alpha}^{\mathrm{c}}\overline{\boldsymbol{\sigma}}, \tag{4}$$

where c_{m} and c_{f} are volume fractions of matrix and filler, $\langle\boldsymbol{\sigma}\rangle_{\mathrm{m}}$ and $\langle\boldsymbol{\sigma}\rangle_{\mathrm{f}}$ are stresses of matrix and filler, respectively.

For the general two-phase composites, we have

$$\left.\begin{aligned}\overline{\boldsymbol{\sigma}} &= \langle\boldsymbol{\sigma}\rangle = c_{\mathrm{f}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{f}} + c_{\mathrm{m}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{m}} \\ \overline{\boldsymbol{\varepsilon}} &= \langle\boldsymbol{\varepsilon}\rangle = c_{\mathrm{f}}\,\langle\boldsymbol{\varepsilon}\rangle_{\mathrm{f}} + c_{\mathrm{m}}\,\langle\boldsymbol{\varepsilon}\rangle_{\mathrm{m}} \\ \overline{\boldsymbol{\varepsilon}} &= \overline{H}\overline{\boldsymbol{\sigma}},\ \langle\boldsymbol{\varepsilon}\rangle_{\mathrm{f}} = H_{\mathrm{f}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{f}}\,,\ \langle\boldsymbol{\varepsilon}\rangle_{\mathrm{m}} = H_{\mathrm{m}}\,\langle\boldsymbol{\sigma}\rangle_{\mathrm{m}}\end{aligned}\right\} \tag{5}$$

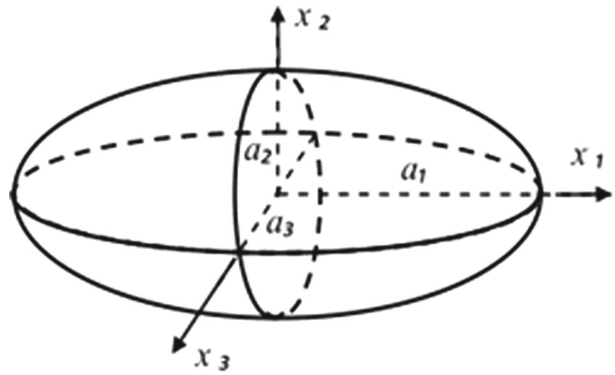

Fig. 1 Coordinate system of unidirectional ellipsoid inclusion along x_1 direction in the matrix with aspect ratio $\lambda = a_1/a_2\,(a_1 \neq a_2 = a_3)$

where $\overline{\boldsymbol{H}}, \boldsymbol{H}_{\mathrm{m}}$ and $\boldsymbol{H}_{\mathrm{f}}$ are the flexibility tensors of composite, matrix and filler, respectively, and $\overline{\boldsymbol{H}} = \left(\frac{1}{3\overline{K}}, \frac{1}{2\overline{G}}\right)$.

From Eq. (5), the following equations can be defined:

$$\langle \boldsymbol{\sigma} \rangle_{\mathrm{m}} = \frac{1}{c_{\mathrm{m}}} \left(\boldsymbol{H}_{\mathrm{m}} - \boldsymbol{H}_{\mathrm{f}}\right)^{-1} \left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{f}}\right) \overline{\boldsymbol{\sigma}}, \tag{6}$$

$$\langle \boldsymbol{\sigma} \rangle_{\mathrm{f}} = \frac{1}{c_{\mathrm{f}}} \left(\boldsymbol{H}_{\mathrm{f}} - \boldsymbol{H}_{\mathrm{m}}\right)^{-1} \left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{m}}\right) \overline{\boldsymbol{\sigma}}. \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (4), the effective thermal expansion coefficient of composites can be obtained as follows:

$$\boldsymbol{\alpha}^{\mathrm{c}} = \boldsymbol{\alpha}_{\mathrm{f}} \left(\boldsymbol{H}_{\mathrm{f}} - \boldsymbol{H}_{\mathrm{m}}\right)^{-1} \left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{m}}\right) + \boldsymbol{\alpha}_{\mathrm{m}} \left(\boldsymbol{H}_{\mathrm{m}} - \boldsymbol{H}_{\mathrm{f}}\right)^{-1} \left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{f}}\right). \tag{8}$$

Simplifying Eq. (8), then

$$\boldsymbol{\alpha}^{\mathrm{c}} = \boldsymbol{\alpha}_{\mathrm{m}} + \left(\boldsymbol{\alpha}_{\mathrm{f}} - \boldsymbol{\alpha}_{\mathrm{m}}\right) \left(\boldsymbol{H}_{\mathrm{f}} - \boldsymbol{H}_{\mathrm{m}}\right)^{-1} \left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{m}}\right). \tag{9}$$

For unidirectional ellipsoid inclusion ($a_1 \neq a_2 = a_3$) with x_1 to be the symmetry axis of rotation, plane 2–3 is isotropic, as shown in Fig. 1. Then, the composite is a transversely isotopic material.

Therefore, Eq. (9) can be expressed as follows:

$$\alpha_{ij}^{\mathrm{c}} = \alpha_{ij}^{\mathrm{m}} + \left(\alpha_{kl}^{\mathrm{f}} - \alpha_{kl}^{\mathrm{m}}\right) \left(H_{klmn}^{\mathrm{f}} - H_{klmn}^{\mathrm{m}}\right)^{-1} \left(\overline{H}_{mnij} - H_{mnij}^{\mathrm{m}}\right). \tag{10}$$

Since the effective modulus tensor of transversely isotopic composite can be denoted by $\overline{\boldsymbol{L}} = \left(2\overline{k}, \overline{l}, \overline{l}, \overline{n}, 2\overline{m}, 2\overline{p}\right)$, the elastic constants $\overline{k}, \overline{l}, \overline{n}, \overline{m}, \overline{p}$ are Hill's notation [18]. Thus, we get the flexibility tensor as follows:

$$\overline{\boldsymbol{H}} = \left(\frac{\overline{n}}{2\overline{k}\overline{n} - 2\overline{l}^2}, -\frac{\overline{l}}{2\overline{k}\overline{n} - 2\overline{l}^2}, -\frac{\overline{l}}{2\overline{k}\overline{n} - 2\overline{l}^2}, \frac{\overline{k}}{\overline{k}\overline{n} - \overline{l}^2}, \frac{1}{2\overline{m}}, \frac{1}{2\overline{p}}\right). \tag{11}$$

The flexibility tensor can be denoted by the bulk and shear modulus,

$$H_{ijkl} = \frac{1}{9K}\delta_{ij}\delta_{kl} + \frac{1}{4G}\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \frac{2}{3}\delta_{ij}\delta_{kl}\right), \tag{12}$$

where K and G are bulk modulus and shear modulus, respectively. The parameter δ is the Kronecker delta.

Then, we can obtain the following formula,

$$\left(H_{ijkl}^{\mathrm{f}} - H_{ijkl}^{\mathrm{m}}\right)^{-1} = \frac{1}{1/K_{\mathrm{f}} - 1/K_{\mathrm{m}}}\delta_{ij}\delta_{kl} + \frac{1}{1/G_{\mathrm{f}} - 1/G_{\mathrm{m}}}\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \frac{2}{3}\delta_{ij}\delta_{kl}\right). \tag{13}$$

By the expression of thermal expansion coefficient and flexibility of isotropic material, we get

$$\left(\alpha_{kl}^{\mathrm{f}} - \alpha_{kl}^{\mathrm{m}}\right) \left(H_{klmn}^{\mathrm{f}} - H_{klmn}^{\mathrm{m}}\right)^{-1} = \frac{3\left(\alpha_{\mathrm{f}} - \alpha_{\mathrm{m}}\right)}{1/K_{\mathrm{f}} - 1/K_{\mathrm{m}}}\delta_{mn}. \tag{14}$$

By Eq. (14), the expression, i.e., Eq. (9), of thermal expansion coefficient can be further simplified into a unified form as follows:

$$\alpha_{ij}^{\mathrm{c}} = \alpha_{\mathrm{m}}\delta_{ij} + \frac{\alpha_{\mathrm{f}} - \alpha_{\mathrm{m}}}{1/K_{\mathrm{f}} - 1/K_{\mathrm{m}}}\left(3\overline{H}_{mnij} - \frac{1}{K_{\mathrm{m}}}\delta_{ij}\right). \tag{15}$$

Here, considering a unidirectional ellipsoid inclusion and macroscopically isotropic composite, the longitudinal CTEs and transverse CTEs are defined by α_{11}^{c} and α_{22}^{c}, respectively. Then, combining Eqs. (11) and (15), the longitudinal CTE α_{11}^{c} and transverse CTE α_{22}^{c} are obtained as:

$$\alpha_{11}^{\mathrm{c}} = \alpha_{\mathrm{m}} + \frac{\alpha_{\mathrm{f}} - \alpha_{\mathrm{m}}}{1/K_{\mathrm{f}} - 1/K_{\mathrm{m}}}\left[\frac{3\left(\overline{k} - \overline{l}\right)}{\overline{k}\overline{n} - \overline{l}^2} - \frac{1}{K_{\mathrm{m}}}\right], \tag{16}$$

$$\alpha_{22}^{\mathrm{c}} = \alpha_{\mathrm{m}} + \frac{\alpha_{\mathrm{f}} - \alpha_{\mathrm{m}}}{1/K_{\mathrm{f}} - 1/K_{\mathrm{m}}}\left[\frac{3\left(\overline{n} - \overline{l}\right)}{2\overline{k}\overline{n} - 2\overline{l}^2} - \frac{1}{K_{\mathrm{m}}}\right]. \tag{17}$$

2.2 Elastic constants considering fiber aspect ratio

The effective elastic modulus of hybrid inclusions and matrix can be predicted by different micromechanics methods. In this paper, it is assumed that the composites are transversely isotropic, and the elastic modulus of composites is estimated. According to the extension of Mori–Tanaka approach, the basic formula of the equivalent modulus of two-phase composites is

$$\overline{\boldsymbol{L}} = \boldsymbol{L}_{\mathrm{m}} + c_{\mathrm{f}}\left[(\boldsymbol{L}_{\mathrm{f}} - \boldsymbol{L}_{\mathrm{m}})^{-1} + c_{\mathrm{m}}\boldsymbol{Q}\right]^{-1}. \tag{18}$$

Here, the subscripts m and f denote the matrix and effective fibers, respectively, and tensor $\boldsymbol{Q} = \boldsymbol{S}\boldsymbol{L}_{\mathrm{m}}^{-1}$. The tensor S is the Eshelby tensor. The tensor $\boldsymbol{L}_i$ $(i = m, f)$ and the components of tensor $\boldsymbol{Q}$ can be denoted as follows:

$$\boldsymbol{L}_i = \left(2K_i + \frac{2}{3}G_i, K_i - \frac{2}{3}G_i, K_i - \frac{2}{3}G_i, K_i + \frac{4}{3}G_i, 2G_i, 2G_i\right), \tag{19}$$

$$\boldsymbol{Q} = \left(Q_{2222} + Q_{2233}, Q_{1122}, Q_{1122}, Q_{1111}, 2Q_{2323}, 2Q_{1212}\right). \tag{20}$$

If we assume tensor $\boldsymbol{R} = (a, b, c, d, e, f)$, based on the formula of inverse tensor, we get $\boldsymbol{R}^{-1} = \left(\frac{d}{Z}, -\frac{b}{Z}, -\frac{c}{Z}, \frac{a}{Z}, \frac{1}{e}, \frac{1}{f}\right), Z = ad - 2bc$.

In the process of analysis, we assume

$$\boldsymbol{W} = \left[(\boldsymbol{L}_{\mathrm{f}} - \boldsymbol{L}_{\mathrm{m}})^{-1} + c_{\mathrm{m}}\boldsymbol{Q}\right]^{-1} = (y_1, y_2, y_3, y_4, y_5, y_6), \tag{21}$$

where the tensor $\boldsymbol{W}$ is a shorthand symbol, and $y_j (j = 1 - 6)$ are the components of the tensor $\boldsymbol{W}$. Based on the principle of inverse tensor, we obtain

$$(\boldsymbol{L}_{\mathrm{f}} - \boldsymbol{L}_{\mathrm{m}})^{-1} = \left(\frac{2}{9K} + \frac{1}{6G}, \frac{1}{9K} - \frac{1}{6G}, \frac{1}{9K} - \frac{1}{6G}, \frac{1}{9K} + \frac{1}{3G}, \frac{1}{2G}, \frac{1}{2G}\right), \tag{22}$$

where $\widetilde{K} = K_{\mathrm{f}} - K_{\mathrm{m}}, \widetilde{G} = G_{\mathrm{f}} - G_{\mathrm{m}}$.

In the same way,

$$y_1 = \frac{1}{\xi}\left(\frac{1}{9K} + \frac{1}{3G} + c_{\mathrm{m}}Q_{1111}\right), \quad y_2 = y_3 = \frac{1}{\xi}\left(\frac{1}{6G} - \frac{1}{9K} - c_{\mathrm{m}}Q_{1122}\right), \tag{23}$$

$$y_4 = \frac{1}{\xi}\left(\frac{2}{9K} + \frac{1}{6G} + c_{\mathrm{m}}\left(Q_{2222} + Q_{2233}\right)\right), \tag{24}$$

$$y_5 = \frac{1}{1/2G + 2c_{\mathrm{m}}Q_{2323}}, \quad y_6 = \frac{1}{1/2G + 2c_{\mathrm{m}}Q_{1212}}. \tag{25}$$

According to the above definition, we have:

$$\xi = A \cdot B - 2C^2, \tag{26}$$

$$A = \left[\frac{2}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} + \frac{1}{6G_{\mathrm{f}} - 6G_{\mathrm{m}}} + c_{\mathrm{m}}\left(Q_{2222} + Q_{2233}\right)\right], \tag{27}$$

$$B = \left(\frac{1}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} + \frac{1}{3G_{\mathrm{f}} - 3G_{\mathrm{m}}} + c_{\mathrm{m}} Q_{1111}\right), \tag{28}$$

$$C = \left(\frac{1}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} - \frac{1}{6G_{\mathrm{f}} - 6G_{\mathrm{m}}} + c_{\mathrm{m}} Q_{1122}\right), \tag{29}$$

where ξ, A, B and C are shorthand symbols, respectively.

Substituting Eqs. (19) and (21)–(25) into Eq. (18), the expression of elastic constants can be obtained as

$$\overline{k} = K_{\mathrm{m}} + \frac{1}{3}G_{\mathrm{m}} + \frac{c_{\mathrm{f}}}{2\xi}\left(\frac{1}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} + \frac{1}{3G_{\mathrm{f}} - 3G_{\mathrm{m}}} + c_{\mathrm{m}} Q_{1111}\right), \tag{30}$$

$$\overline{l} = K_{\mathrm{m}} - \frac{2}{3}G_{\mathrm{m}} + \frac{c_{\mathrm{f}}}{\xi}\left(\frac{1}{6G_{\mathrm{f}} - 6G_{\mathrm{m}}} - \frac{1}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} - c_{\mathrm{m}} Q_{1122}\right), \tag{31}$$

$$\overline{n} = K_{\mathrm{m}} + \frac{4}{3}G_{\mathrm{m}} + \frac{c_{\mathrm{f}}}{\xi}\left[\frac{1}{6G_{\mathrm{f}} - 6G_{\mathrm{m}}} + \frac{2}{9K_{\mathrm{f}} - 9K_{\mathrm{m}}} + c_{\mathrm{m}}\left(Q_{2222} + Q_{2233}\right)\right], \tag{32}$$

$$\overline{m} = G_{\mathrm{m}} + \frac{c_{\mathrm{f}}}{1/(G_{\mathrm{f}} - G_{\mathrm{m}}) + 4c_{\mathrm{m}} Q_{2323}}, \tag{33}$$

$$\overline{p} = G_{\mathrm{m}} + \frac{c_{\mathrm{f}}}{1/(G_{\mathrm{f}} - G_{\mathrm{m}}) + 4c_{\mathrm{m}} Q_{1212}}. \tag{34}$$

The components of the tensor $\boldsymbol{Q}$ can be derived by $\boldsymbol{Q} = \boldsymbol{S}\boldsymbol{L}_{\mathrm{m}}^{-1}$ and are given as follows (aspect ratio $\lambda = a_1/a_2$),

$$Q_{1111} = \frac{1}{2G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left[(1-2\gamma_{\mathrm{m}}) + \frac{\lambda^2}{\lambda^2-1}\right] + \frac{1}{4G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left[4(\gamma_{\mathrm{m}}-1) - \frac{3}{\lambda^2-1}\right]\chi, \tag{35}$$

$$Q_{2222} = Q_{3333} = \frac{3}{16G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\frac{\lambda^2}{\lambda^2-1} + \frac{1}{32G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left[4(1-4\gamma_{\mathrm{m}}) - \frac{9}{\lambda^2-1}\right]\chi, \tag{36}$$

$$Q_{1212} = Q_{1313} = -\frac{1}{4G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left(\gamma_{\mathrm{m}} + \frac{1}{\lambda^2-1}\right) + \frac{1}{8G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left[(1+\gamma_{\mathrm{m}}) + \frac{3}{\lambda^2-1}\right]\chi, \tag{37}$$

$$Q_{2323} = \frac{1}{16G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\frac{\lambda^2}{\lambda^2-1} + \frac{1}{32G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left[4(1-2\gamma_{\mathrm{m}}) - \frac{3}{\lambda^2-1}\right]\chi, \tag{38}$$

$$\begin{aligned} Q_{1122} &= Q_{1133} = Q_{2211} = Q_{3311} \\ &= -\frac{1}{4G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\frac{\lambda^2}{\lambda^2-1} + \frac{1}{8G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left(\frac{1+2\lambda^2}{\lambda^2-1}\right)\chi, \end{aligned} \tag{39}$$

$$Q_{2233} = Q_{3322} = \frac{1}{16G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\frac{\lambda^2}{\lambda^2-1} + \frac{1}{32G_{\mathrm{m}}(1-\gamma_{\mathrm{m}})}\left(\frac{1-4\lambda^2}{\lambda^2-1}\right)\chi, \tag{40}$$

where γ_{m} is the Poisson's ratio of the matrix, then

$$\lambda < 1:\ \chi = \frac{\lambda}{\left(1-\lambda^2\right)^{3/2}}\left[\arccos(\lambda) - \lambda\left(1-\lambda^2\right)^{1/2}\right]$$

$$\lambda > 1:\ \chi = \frac{\lambda}{\left(\lambda^2-1\right)^{3/2}}\left[\lambda\left(\lambda^2-1\right)^{1/2} - \arccos h(\lambda)\right]$$

2.3 Case study of related composites

In the case without fiber aspect ratio, for the unidirectional fiber-reinforced composites, the tensor $\boldsymbol{Q}$ can be denoted as follows:

$$\boldsymbol{Q} = \left(\frac{3}{8G_{\mathrm{m}} + 6K_{\mathrm{m}}}, 0, 0, 0, \frac{7G_{\mathrm{m}} + 3K_{\mathrm{m}}}{4G_{\mathrm{m}}\left(4G_{\mathrm{m}} + 3K_{\mathrm{m}}\right)}, \frac{1}{4G_{\mathrm{m}}}\right). \tag{41}$$

Substituting Eqs. (19) and (41) into Eq. (18), the expression of elastic constants can be obtained as follows:

$$\overline{k^*} = K_{\mathrm{m}} + \frac{1}{3}G_{\mathrm{m}} + \frac{\left(4G_{\mathrm{m}} + 3K_{\mathrm{m}}\right)\left(G_{\mathrm{f}} - G_{\mathrm{m}} + 3K_{\mathrm{f}} - 3K_{\mathrm{m}}\right)}{3\left[\left(3 + c_{\mathrm{f}}\right)G_{\mathrm{m}} + c_{\mathrm{m}}G_{\mathrm{f}} + 3\left(c_{\mathrm{f}}K_{\mathrm{m}} + c_{\mathrm{m}}K_{\mathrm{f}}\right)\right]}c_{\mathrm{f}}, \tag{42}$$

$$\overline{l^*} = K_{\mathrm{m}} - \frac{2}{3}G_{\mathrm{m}} + \frac{\left(4G_{\mathrm{m}} + 3K_{\mathrm{m}}\right)\left(2G_{\mathrm{m}} - 2G_{\mathrm{f}} + 3K_{\mathrm{f}} - 3K_{\mathrm{m}}\right)}{3\left[\left(3 + c_{\mathrm{f}}\right)G_{\mathrm{m}} + c_{\mathrm{m}}G_{\mathrm{f}} + 3\left(c_{\mathrm{f}}K_{\mathrm{m}} + c_{\mathrm{m}}K_{\mathrm{f}}\right)\right]}c_{\mathrm{f}}, \tag{43}$$

$$\overline{n^*} = K_{\mathrm{m}} + \frac{4}{3}G_{\mathrm{m}} + \frac{F + 3\left[3K_{\mathrm{m}}\left(K_{\mathrm{f}} - K_{\mathrm{m}}\right) + G_{\mathrm{f}}\left(9c_{\mathrm{f}}K_{\mathrm{m}} + 9c_{\mathrm{m}}K_{\mathrm{f}} - 5K_{\mathrm{m}}\right)\right]}{3\left[\left(3 + c_{\mathrm{f}}\right)G_{\mathrm{m}} + c_{\mathrm{m}}G_{\mathrm{f}} + 3\left(c_{\mathrm{f}}K_{\mathrm{m}} + c_{\mathrm{m}}K_{\mathrm{f}}\right)\right]}c_{\mathrm{f}}, \tag{44}$$

$$\overline{m^*} = G_{\mathrm{m}} + \frac{2G_{\mathrm{m}}\left(G_{\mathrm{f}} - G_{\mathrm{m}}\right)\left(4G_{\mathrm{m}} + 3K_{\mathrm{m}}\right)}{2G_{\mathrm{m}}\left(4G_{\mathrm{m}} + 3K_{\mathrm{m}}\right) + c_{\mathrm{m}}\left(G_{\mathrm{f}} - G_{\mathrm{m}}\right)\left(7G_{\mathrm{m}} + 3K_{\mathrm{m}}\right)}c_{\mathrm{f}}, \tag{45}$$

$$\overline{p^*} = G_{\mathrm{m}} + \frac{2c_{\mathrm{f}}G_{\mathrm{m}}\left(G_{\mathrm{f}} - G_{\mathrm{m}}\right)}{2G_{\mathrm{m}} + c_{\mathrm{m}}\left(G_{\mathrm{f}} - G_{\mathrm{m}}\right)}, \tag{46}$$

where $F = -16G_{\mathrm{m}}^2 + 16G_{\mathrm{m}}G_{\mathrm{f}} + 3G_{\mathrm{m}}\left[K_{\mathrm{m}} - 5K_{\mathrm{f}} + 9c_{\mathrm{f}}\left(K_{\mathrm{f}} - K_{\mathrm{m}}\right)\right]$.

For the particle-reinforced composites, the details of Mori–Tanaka scheme can be found in [19] and the bulk modulus can be written as

$$\overline{K} = K_{\mathrm{m}}\left[1 + \frac{c_{\mathrm{f}}(K_{\mathrm{f}} - K_{\mathrm{m}})}{K_{\mathrm{m}} + \alpha(1 - c_{\mathrm{f}})(K_{\mathrm{f}} - K_{\mathrm{m}})}\right], \tag{47}$$

where $\overline{K}$ is the bulk modulus of composites with $\alpha = \frac{3K_{\mathrm{m}}}{3K_{\mathrm{m}} + 4G_{\mathrm{m}}}$.

The tensor $\overline{\boldsymbol{H}}$ can be denoted as $\overline{\boldsymbol{H}} = \left(\frac{1}{3\overline{K}}, \frac{1}{2\overline{G}}\right)$; thus, we get

$$\left(\overline{\boldsymbol{H}} - \boldsymbol{H}_{\mathrm{m}}\right) = \left(\frac{1}{3\overline{K}} - \frac{1}{3K_{\mathrm{m}}}, \frac{1}{2\overline{G}} - \frac{1}{2G_{\mathrm{m}}}\right). \tag{48}$$

Substituting Eqs. (47) and (48) into Eq. (9), the CTE expression of particle-reinforced composites can be obtained as

$$\alpha^{\mathrm{c}} = c_{\mathrm{f}}\alpha_{\mathrm{f}} + c_{\mathrm{m}}\alpha_{\mathrm{m}} + \frac{4c_{\mathrm{f}}c_{\mathrm{m}}\left(K_{\mathrm{f}} - K_{\mathrm{m}}\right)\left(\alpha_{\mathrm{f}} - a_{\mathrm{m}}\right)G_{\mathrm{m}}}{3K_{\mathrm{f}}K_{\mathrm{m}} + 4G_{\mathrm{m}}\left(c_{\mathrm{f}}K_{\mathrm{f}} + c_{\mathrm{m}}K_{\mathrm{m}}\right)}. \tag{49}$$

2.4 Fibers of spatial and planar arbitrary orientations

Through the above analysis, the thermal expansion coefficients of unidirectional fiber-reinforced composites have been defined. The effective thermal expansion coefficients of composites with spatial arbitrary orientation can be also predicted using the same methodology as unidirectional fiber-reinforced composites [20].

For the fibers of spatial arbitrary orientation, the mechanical and thermal properties of composites are isotropic. Differentially, for the planar arbitrary orientation fibers, the mechanical and thermal properties of composites are transversely isotropic. Therefore, we have the following formulas,

$$\left\langle\alpha_{2d}^{\mathrm{c}}\right\rangle_{11} = \alpha_{22}^{\mathrm{c}}, \left\langle\alpha_{2d}^{\mathrm{c}}\right\rangle_{22} = \frac{1}{2}\left(\alpha_{11}^{\mathrm{c}} + \alpha_{22}^{\mathrm{c}}\right), \tag{50}$$

$$\alpha_{3d}^{\mathrm{c}} = \frac{1}{3}\alpha_{11}^{\mathrm{c}} + \frac{2}{3}\alpha_{22}^{\mathrm{c}}, \tag{51}$$

where $\left\langle\alpha_{2d}^{\mathrm{c}}\right\rangle_{11}$ and $\left\langle\alpha_{2d}^{\mathrm{c}}\right\rangle_{22}$ are longitudinal CTE and transverse CTE of planar arbitrary orientation fiber-reinforced composites, respectively. α_{3d}^{c} is the thermal expansion coefficient of composites reinforced by spatial arbitrary orientation fibers.

3 Results and discussion

3.1 Some existing models

For a comparison, some of existing micromechanical models for predicting the CTEs of composites from the literature are reviewed. A simple summary of some theories for CTEs of composites is as follows.

3.1.1 Law of mixtures [10]

$$\alpha^{\mathrm{c}} = c_{\mathrm{m}}\alpha_{\mathrm{m}} + c_{\mathrm{f}}\alpha_{\mathrm{f}} \tag{52}$$

3.1.2 Turner model [11]

$$\alpha^{\mathrm{c}} = \frac{c_{\mathrm{m}}\alpha_{\mathrm{m}}K_{\mathrm{m}} + c_{\mathrm{f}}\alpha_{\mathrm{f}}K_{\mathrm{f}}}{c_{\mathrm{m}}K_{\mathrm{m}} + c_{\mathrm{f}}K_{\mathrm{f}}}, \tag{53}$$

where K_{m} and K_{f} are the bulk moduli of the matrix and fibers, respectively.

3.1.3 Kerner model [12]

$$\alpha^{\mathrm{c}} = c_{\mathrm{m}}\alpha_{\mathrm{m}} + c_{\mathrm{f}}\alpha_{\mathrm{f}} + \frac{(\alpha_{\mathrm{f}} - \alpha_{\mathrm{m}})(K_{\mathrm{f}} - K_{\mathrm{m}})}{c_{\mathrm{m}}K_{\mathrm{m}} + c_{\mathrm{f}}K_{\mathrm{f}} + 3K_{\mathrm{m}}K_{\mathrm{f}}/4G_{\mathrm{m}}}c_{\mathrm{m}}c_{\mathrm{f}}, \tag{54}$$

where G_{m} is the shear modulus of the matrix.

3.1.4 Schapery model [14]

$$\alpha^{\mathrm{c}}_{11} = \frac{c_{\mathrm{m}}\alpha_{\mathrm{m}}E_{\mathrm{m}} + c_{\mathrm{f}}\alpha_{\mathrm{f}}E_{\mathrm{f}}}{c_{\mathrm{m}}E_{\mathrm{m}} + c_{\mathrm{f}}E_{\mathrm{f}}}, \tag{55}$$

$$\alpha^{\mathrm{c}}_{22} = (1+\gamma_{\mathrm{m}})\,c_{\mathrm{m}}\alpha_{\mathrm{m}} + (1+\gamma_{\mathrm{f}})\,c_{\mathrm{f}}\alpha_{\mathrm{f}} - \alpha^{\mathrm{c}}_{11}\,(c_{\mathrm{m}}\gamma_{\mathrm{m}} + c_{\mathrm{f}}\gamma_{\mathrm{f}})\,, \tag{56}$$

where E_{f} and E_{m} are elastic moduli; γ_{f} and γ_{m} are the Poisson's ratios of fiber and matrix, respectively.

3.1.5 Schneider model [15]

$$\alpha^{\mathrm{c}}_{11} = \frac{c_{\mathrm{m}}\alpha_{\mathrm{m}}E_{\mathrm{m}} + c_{\mathrm{f}}\alpha_{\mathrm{f}}E_{\mathrm{f}}}{c_{\mathrm{m}}E_{\mathrm{m}} + c_{\mathrm{f}}E_{\mathrm{f}}} \tag{57}$$

$$\alpha^{\mathrm{c}}_{22} = \alpha_{\mathrm{m}} - (\alpha_{\mathrm{m}} - \alpha_{\mathrm{f}})\left[\frac{2\,(1+\gamma_{\mathrm{m}})\left(\gamma_{\mathrm{m}}^2 - 1\right)}{2\gamma_{\mathrm{m}}^2 + \gamma_{\mathrm{m}} - 1 - (1+\gamma_{\mathrm{m}})/bc_{\mathrm{f}}} - \frac{\gamma_{\mathrm{m}}E_{\mathrm{f}}/E_{\mathrm{m}}}{E_{\mathrm{f}}/E_{\mathrm{m}} + (1-bc_{\mathrm{f}})/bc_{\mathrm{f}}}\right], \tag{58}$$

where the parameter b is correction factor, and $b = 1$ for the fibers which are ideally aligned as straight and parallel.

3.2 Numerical result predictions

The CTEs of composites with the epoxy matrix are numerically predicted in the present investigation. The material property data of epoxy matrix and fillers for Figs. 2, 3, 4, 5 and 6 are given in Table 1, whereas Table 2 presents the isotropic phase properties of particle-reinforced composites for Fig. 7.

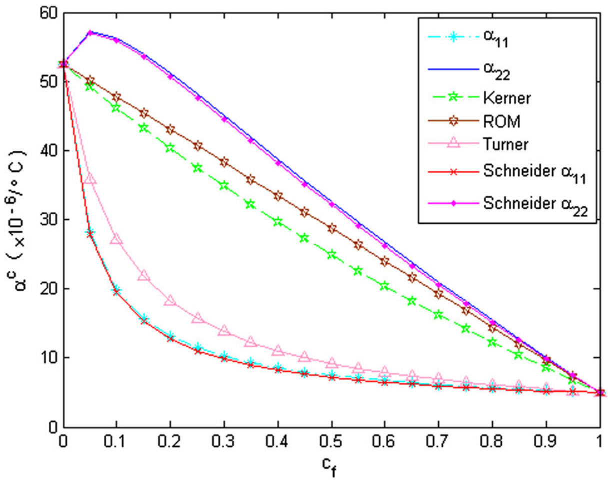

Fig. 2 The CTEs of composite changed with fiber volume fractions without the variation of aspect ratio

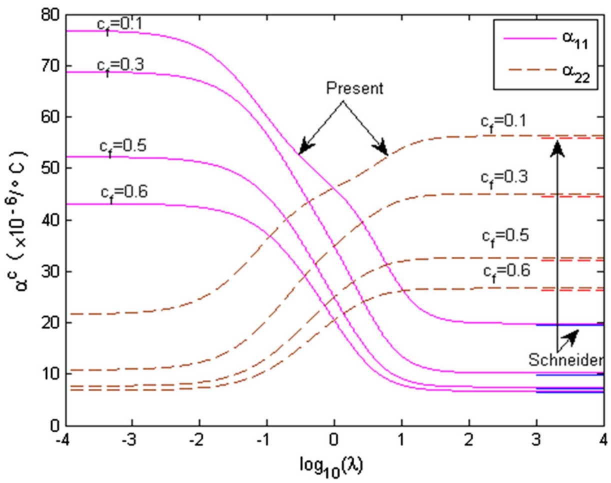

Fig. 3 Effect of fiber aspect ratio $10^{-4} < \lambda < 10^{4}$ on the thermal expansion coefficients of composites

Figure 2 presents a comparison between the present predictions and other theories given in the literature in the case of not considering the effect of fiber aspect ratio. Note that the predictions are obtained from Eqs. (16), (17) and (42)–(44). The results show that both the longitudinal CTE and the transverse CTE according to the present study are closer to Schneider's predictions. Under the same conditions, the data from ROM model lie between them and can only offer an approximate prediction of transverse CTE. Moreover, it can be also found from Fig. 2 that the works by Turner and Kerner provide the upper and lower bounds for the predictions of longitudinal CTE and transverse CTE, respectively.

Figures 3 and 4 mainly describe the change of longitudinal and transverse CTEs with fiber volume fractions and aspect ratio using Eqs. (16), (17) and (30)–(32). The results from the present model and Schneider's method are shown in Fig. 3, and Fig. 3 also presents the effect of aspect ratio ($10^{-4} < \lambda < 10^{4}$) on the CTEs. The overall pictures show that the longitudinal CTE decreases and the transverse CTE increases with the increase of fiber aspect ratio, and at $\lambda = 1$ the longitudinal CTE is equal to transverse CTE, which is in the situation of spherical inclusion. Moreover, both longitudinal CTE and transverse CTE decrease with the increase of fiber volume fractions, and the Schneider's results provide an upper bound for CTEs. However, as shown in Fig. 3, when $\lambda \to \infty$, it is observed that the variation of CTEs with fiber aspect ratios is insignificant and remains unchanged. The present analyses are in a good agreement with the results given by studies [20,21].

To investigate the influences of fiber aspect ratio ($\lambda > 1$) on the CTEs of composites, the variation of CTEs with fiber volume fractions is shown in Fig. 4. As shown in Fig. 4, the longitudinal CTE α_{11}^{c} decreases with the increase of fiber aspect ratio. On the contrary, the transverse CTE α_{22}^{c} increases with the increase of fiber

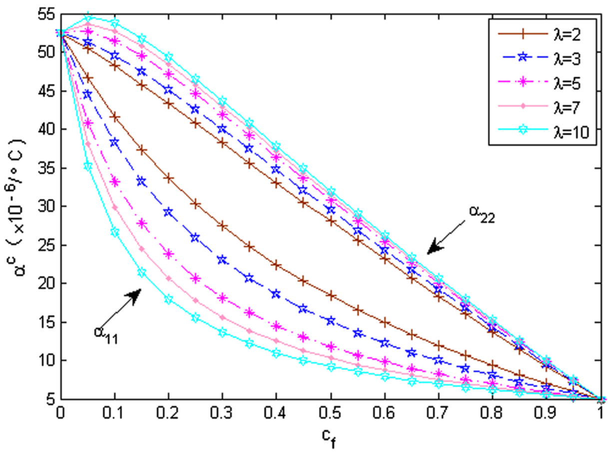

Fig. 4 The variation of longitudinal CTE and transverse CTE with fiber volume fractions in the case of aspect ratio $\lambda > 1$

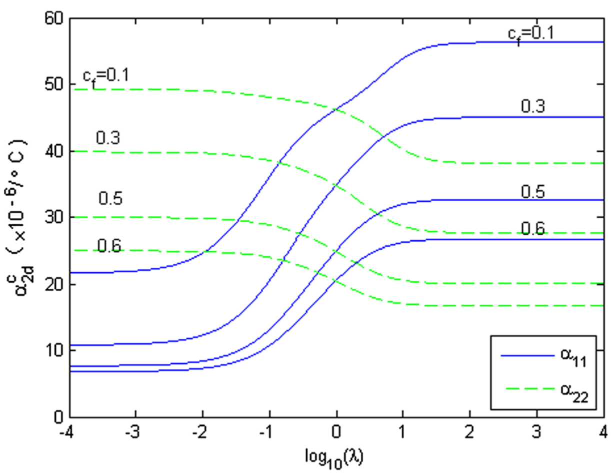

Fig. 5 Variation of coefficients of thermal expansion with aspect ratio for planar arbitrary orientation fiber-reinforced composites

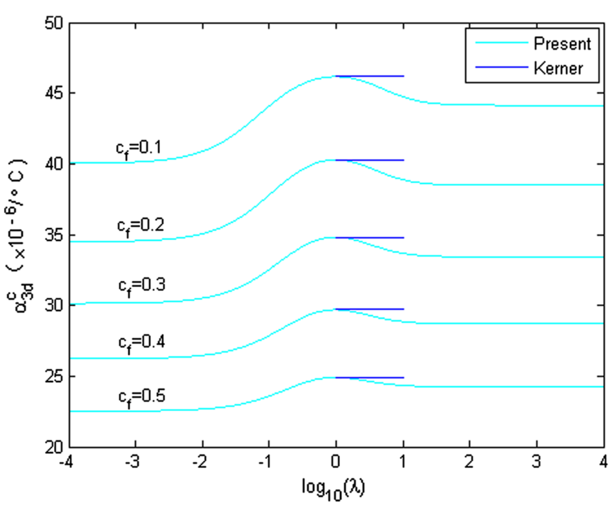

Fig. 6 Variation of coefficients of thermal expansion with aspect ratio for spatial arbitrary orientation fiber-reinforced composites

Table 1 Material properties of composite consisting of isotropic glass fibers and isotropic epoxy matrix [10]

Material	E (GPa)	γ	$\alpha\ \left(\times 10^{-6}/^{\circ}\mathrm{C}\right)$
Epoxy	3.5	0.35	52.5
Glass fiber	72	0.2	5

Table 2 Isotropic phase properties of particle-reinforced composites [8]

Material	E (GPa)	γ	$\alpha\ \left(\times 10^{-6}/\mathrm{K}\right)$
Epoxy	2.8	0.4	80
Silica particles	70	0.2	4

E Young's modulus, γ Poisson's ratio, α coefficient of nonlinear thermal expansion

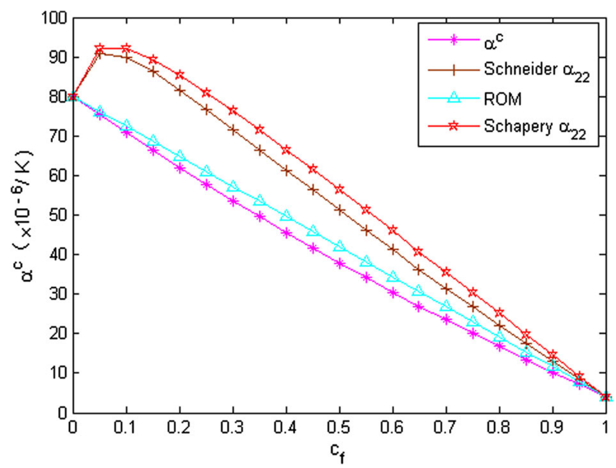

Fig. 7 Effect of silica particle volume fractions on thermal expansion coefficient of particle-reinforced composite

aspect ratio. Furthermore, it is also found that the effect of fiber aspect ratio on the transverse CTE α_{22}^{c} is more significant than the effect on the longitudinal CTE α_{11}^{c} when the fiber aspect ratio is in a range $\lambda > 1$. Both longitudinal CTE and transverse CTE decrease with the increase of fiber volume fractions in this case.

For the composites reinforced by fibers of planar arbitrary orientation, the change of CTEs with the fiber aspect ratio ($10^{-4} < \lambda < 10^{4}$) is plotted in Fig. 5 using Eq. (50). It can be found from Fig. 5 that the longitudinal CTE increases, but the transverse CTE decreases with increase fiber aspect ratio. As compared with the curves shown in Fig. 3, the influence of fiber aspect ratio on the longitudinal CTE and transverse CTE is completely opposite.

For the composites reinforced by fibers of spatial arbitrary orientation, Fig. 6 shows how CTEs of composites change with the aspect ratio using Eq. (51). As plotted in Fig. 6, the CTE of composites has the maximum value at $\lambda = 1$ and gently decreases with the change of fiber aspect ratio λ in other cases. It is also found from Fig. 6 that the Turner prediction results can serve as the upper bound for the present work. The present model for the CTEs of composites is not sensitive to the fiber aspect ratio as compared to the results shown in Fig. 3. The reason for this difference may be due to that the composite with spatial arbitrary orientation fiber is isotropic in the present model, whereas the unidirectional fiber and planar arbitrary orientation fibers are transversely isotropic.

As shown in Fig. 7, the present investigation is related to the thermal expansion coefficient of particle-reinforced composites. It is seen that the numerical results defined by Eq. (49) are closer to the results obtained from ROM model. Moreover, it can be also found from Fig. 7 that the results from other theories by Schapery [14] and Schneider [15] are higher than the present predictions.

4 Concluding remarks

A micromechanical model is proposed for predicting the coefficients of nonlinear thermal expansion of epoxy matrix composites. Two cases for the CTEs of fiber-reinforced composites are considered. The effect of filler aspect ratio on the CTEs is investigated. The analysis results illustrate that the theoretical derivations are applicable for the composites under mechanical or thermal environment conditions and have the advantage to obtain the desired responses of composites.

In this study, it is noted that the CTEs depend on both the fiber aspect ratio and volume fractions of reinforcements. The model proposed offers a direct prediction of CTEs, and the predictions from the model agree well with the results of those theories in the literature. Therefore, the present analysis provides a helpful tool for optimizing the material design and manufacture of advanced composites.

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