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ORIGINAL PAPER

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On the stability of the permanent rotations of a charged rigid body-gyrostat

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Abstract We consider the motion of a charged rigid body about a fixed point carrying a rotor that is attached along one of the principal axes of the body. This motion occurs under the action of the resultant of the uniform gravity field and the homogeneous magnetic field. The equations of motion are formulated, and they are presented by means of the Hamiltonian function in the framework of the Lie–Poisson system. These equations of motion have six equilibrium solutions. The sufficient conditions for instability for these equilibria are studied by utilizing the linear approximation method, while the sufficient conditions for stability are presented by means of the energy-Casimir method. For certain configuration of the body, the regions of Lyapunov stability and instability are determined in the plane of some parameters. Furthermore, we clarify that the regions of Lyapunov stability are a portion of the regions of linear stability.

1 Introduction

Stability problems of rigid bodies have received extensive attention in the latter decades, essentially due to an increasing interest in spaceflight. Numerous investigations of such problems have been performed and can be found in the literature. In 1956, Rumiantsev presented the sufficient conditions for the stability of a rigid body rotating about a principal axis through a fixed point of the body, with no requirement that the other two principal moments of inertia with respect to the fixed point be equal [1]. His results have been extended to accommodate, for example, Newtonian force fields [2], arbitrary potential fields of forces [3], Euler case [4], and other stability problems related to the motion of the rigid body under the influence of a uniform force. In many of these investigations, Rumiantsev and his followers made extensive employment of the methods developed by Lyapunov [5] in 1892 for determining the stability of the system of differential equations. Routh [6] considered the general stability problem of a heavy unsymmetrical rotating top, using a linear analysis to examine its stability. Unfortunately, only the instability results of Routh's analysis were extendible to the complete nonlinear system. This fact comes from Lyapunov's theorem on the stability in the first approximation (see [7], p. 227). By the direct method due to Lyapunov, Rumiantsev [8] investigated the stability of certain motions of a heavy gyrostat with a fixed point. In the case where the mass center of the gyrostat is taken to be the fixed point, he obtained the sufficient conditions for both stability and instability of these motions. The same problem was independently considered by Kan and Fowler [9] and also was considered by Crespo da Silva [10]. The results of all these studies were equivalent. Various different cases were also examined by Rumiantsev [8,11], Anchev [12,13], Kolensnikov [14], and others.

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A gyrostat is a mechanical system consisting of an invariable part S_1 and the other body S_2 variable or rigid but not rigidly connected with S_1 in order that the motion of S_2 with respect to S_1 keeps the distribution of the mass within the mechanical system S unchanged. It is also named in the literature as *dual-spin* body due to the motion of the two bodies S_1 and S_2 [15]. It is well known that the notions of a gyrostat were firstly presented by Volterra in order to study the motion of the Earth's polar axis and explain variations in the Earth's latitude by means of the internal motion that does not change the planet's distribution of mass [16]. Also, it has various applications in diverse branches of science such as Astrodynamics. For instance, it is used as a control device in spacecrafts for stabilizing their rotations (see, e.g., [17–21]). Moreover, the majority of the problems concerning the rigid body-gyrostat can be summarized in the following:

(i) *The first problem*: The stability of the equilibria in a rigid body-gyrostat either with a fixed point or in orbit (see, e.g., [12,15,22–27]).
(ii) *The second problem*: The periodic solutions, bifurcation or chaos in several problems of a rigid body gyrostat (see, e.g., [28–30]).
(iii) *The third problem*: The integrability and the construction of the first integrals of motion. The majority of these problems were collected in [31], and some new integrable problems have been added by many authors (see, e.g., [32–35]).

Iñarrea et al. [23] studied the stability of the permanent rotations of a heavy gyrostat with a fixed point in the presence of a constant gyrostatic momentum resulting from a rotor that rotates with a constant angular velocity around an axis passing through the center of mass of the gyrostat. They presented the necessary and sufficient conditions for those permanent rotations to be stable by employing the energy-Casimir method. The present work deals with this problem in the presence of a homogeneous magnetic field, and therefore, the current work is an extension of the problem that has been discussed in [23].

2 Equations of motion

We consider the motion of a charged gyrostat about a fixed point O, composed of a rigid body attaching an axisymmetric rotor that is aligned along one of the principal axes of the body and rotates with a constant angular velocity. We assume that the gyrostat is subjected to a uniform gravity field and a homogenous magnetic field $\vec{H}$. To describe the motion, we assume that $OXYZ$ and $Oxyz$ are two Cartesian coordinate systems fixed in the space and in the body, respectively. Let $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ be the angular velocity of the body and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vector in the direction of OZ-axis (see Fig. 1). The two vectors $\vec{\omega}$ and $\vec{\gamma}$ are referred to the body system which is taken as the system of principal axes of inertia at the fixed point O. Assume the tensor of inertia of the gyrostat in the body's system is $\mathbb{I} = \text{diag}(A, B, C)$. The vector $\vec{\gamma}$ can be expressed in terms of Eulerian angles as outlined in [15],

$$\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta), \tag{1}$$

where ψ is the angle of precession about the z-axis, θ is the angle of nutation between z- and Z-axes, and φ is the angle of proper rotation. Now, we introduce down the equations of the motion. The total angular momentum of the gyrostat admits the form

$$\vec{G} = \vec{M} + \vec{K}, \tag{2}$$

where $\vec{M} = \mathbb{I}\vec{\omega}$ is the angular momentum of the whole gyrostat when the rotor is at relative rest and $\vec{K} = (0, 0, k)$ is the gyrostatic momentum, that is, the relative angular momentum of the rotor with respect to the body. The torque due to the gravity field is given by

$$\vec{\mathcal{M}}_O^1 = \vec{r}_0 \times (-mg\vec{\gamma}) = mg\vec{\gamma} \times \vec{r}_0 \tag{3}$$

where m and g are the total mass of the gyrostat and the acceleration due to the gravity while $\vec{r}_0 = (x_0, y_0, z_0)$ is the position vector for the center of mass with respect to the fixed point O. For simplicity, we assume the center of mass lies on the Oz-axis, and so, we have $x_0 = y_0 = 0$. Let the magnetic field be a constant acting in the direction of OZ-axis and thus $\vec{H} = \nu\vec{\gamma}$, where ν is a constant characterizing the magnitude of the magnetic field. Let p be an element of the body that moves with velocity $\vec{v}(p)$ and is carrying a charge dq. Also, assume that $\vec{r}$ be a position vector of this element. This element is subjected to Lorenz forces $d\vec{F} = dq(\vec{v} \times \vec{H}) = \nu\sigma \mathrm{d}V[(\vec{\omega} \times \vec{r}) \times \vec{\gamma}] = \nu\sigma \mathrm{d}V[(\vec{\omega} \cdot \vec{\gamma})\vec{r} - (\vec{r} \cdot \vec{\gamma})\vec{\omega}]$ where $\mathrm{d}V$ is the volume element of

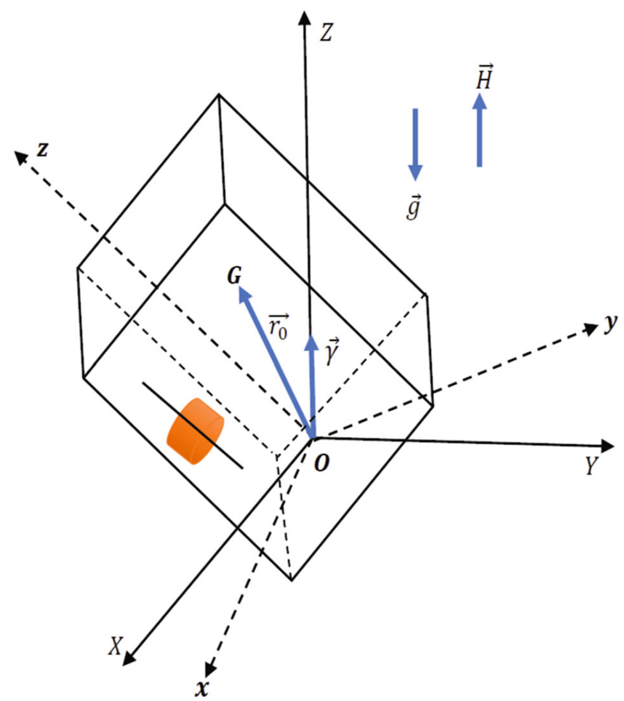


Fig. 1 The charged gyrostat and references frames

the gyrostat and σ is the charge distribution.[1] Thus, the torque due to the magnetic field can be expressed in the form

$$\vec{\mathcal{M}}_O^2 = \int_V \vec{r} \times d\vec{F} = \vec{\omega} \times \sigma \nu \int_V \vec{r}(\vec{r} \cdot \vec{\gamma}) \mathrm{d}V = \vec{\omega} \times A\vec{\gamma} \tag{4}$$

where A is a constant 3×3 matrix. For simplicity, we postulate $A = \mathrm{diag}(a, b, c)$ where a, b, and c are arbitrary parameters. Taking all obtained results into account and according to the angular momentum's theorem about the fixed point O, we have

$$\frac{\mathrm{d}\vec{G}}{\mathrm{d}t} = \vec{\mathcal{M}}_O$$

where $\vec{\mathcal{M}}_O$ is the total torque about the point O. Then, the equations of the motion with respect to coordinates $Oxyz$ that are fixed in the body take the form

$$\dot{\vec{M}} = -\vec{\omega} \times (\vec{M} + \vec{\mu}) + mg\vec{\gamma} \times \vec{r_0} \tag{5}$$

where

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3) = \vec{K} + A\vec{\gamma} \tag{6}$$

Expression (6) represents the torque due to gyroscopic forces that are velocity-dependent forces. Since to the vector $\vec{\gamma}$ is a constant unit vector in space, we have

$$\dot{\vec{\gamma}} = \vec{\gamma} \times \vec{\omega}. \tag{7}$$

Although the variables employed in Eqs. (5) and (7) are not canonical, the problem under consideration can be described by means of the Hamiltonian function in the framework of Lie–Poisson systems. In this case, the Hamiltonian function admits the form

$$\mathcal{H} = \frac{1}{2}\left[\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}\right] + mgz_0\gamma_3. \tag{8}$$

[1] Here MKS units are used. In Gaussian units dq should be divided by the velocity of the light c (e.g., [36]). We also assume that the velocity and acceleration are sufficiently small to neglect both relativistic effects and classical radiation damping.

Following [37], the equations of motion (5) and (7) become a Hamiltonian-Poisson system generating the matrix $\Pi_{\vec{\mu}}$

$$\Pi_{\vec{\mu}} = \begin{pmatrix} 0 & -M_3-\mu_3 & M_2+\mu_2 & 0 & -\gamma_3 & \gamma_2 \\ M_3+\mu_3 & 0 & -M_1-\mu_1 & \gamma_3 & 0 & -\gamma_1 \\ -M_2-\mu_2 & M_1+\mu_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

provided that the Jacobi-identity holds, i.e.,

$$\Pi_{\vec{\mu}}^{li}\partial_l\Pi_{\vec{\mu}}^{jk} + \Pi_{\vec{\mu}}^{lj}\partial_l\Pi_{\vec{\mu}}^{ki} + \Pi_{\vec{\mu}}^{lk}\partial_l\Pi_{\vec{\mu}}^{ij} = 0, \quad i, j, k = 1, 2, \ldots, 6. \tag{10}$$

Or, equivalently, the vector $\vec{\mu}(\vec{\gamma})$ satisfies the equation

$$\vec{\gamma}\cdot\nabla_{\vec{\gamma}}\times\vec{\mu} = 0. \tag{11}$$

It is easy to prove that the vector $\vec{\mu}$ satisfies the condition (11). The equations of motion (5) and (7) can be expressed as

$$\dot{\vec{X}} = \Pi_{\vec{\mu}}\vec{\nabla}H \tag{12}$$

where $\vec{X} = (M_1, M_2, M_3, \gamma_1, \gamma_2, \gamma_3)$ and $\vec{\nabla}H$ is the naive gradient of H. This system has three general integrals of motion bearing the name Casimirs. They are the Hamiltonian $\mathcal{H}$ itself and

$$C_1 := \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \tag{13}$$

$$C_2 := (M_1 + \frac{a}{2}\gamma_1)\gamma_1 + (M_2 + \frac{b}{2}\gamma_2)\gamma_2 + (M_3 + k + \frac{c}{2}\gamma_3)\gamma_3 = c_0 \tag{14}$$

where c_0 is an arbitrary constant. The second Casimir equation (14) is named in the literature as a cyclic integral due to its correspondences to the cyclic variable ψ.

3 Equilibria

The permanent rotations have a significant interest in numerous fields of application, and they are obtained as equilibrium solutions [15,38]. Thus, we set $\dot{\vec{M}} = \dot{\vec{\gamma}} = \vec{0}$. Equations (5) and (7) take the form

$$\vec{\omega}\times(\vec{M} + \vec{K} + A\vec{\gamma}) - mg\vec{\gamma}\times\vec{r_0} = \vec{0}, \tag{15}$$

$$\vec{\gamma}\times\vec{\omega} = 0. \tag{16}$$

Equation (16) indicates that the vectors $\vec{\omega}$ and $\vec{\gamma}$ are parallel, and so $\vec{\omega} = \omega\vec{\gamma}$. Inserting the last expression in Eq. (15), we have

$$\vec{\gamma}\times\left[\omega^2\mathbb{I}\vec{\gamma} + \omega(\vec{K} + A\vec{\gamma}) - mg\vec{r_0}\right] = \vec{0}. \tag{17}$$

Equation (17) can be expressed in a scalar form as

$$\gamma_2[\omega^2(C-B)\gamma_3 + k\omega + \omega\gamma_3(c-b) - mgz_0] = 0, \tag{18}$$

$$\gamma_1[\omega^2(A-C)\gamma_3 - k\omega + \omega(a-c)\gamma_3 + mgz_0] = 0, \tag{19}$$

$$\gamma_1\gamma_2\omega[\omega(B-A) + (b-a)] = 0. \tag{20}$$

Equation (20) holds in four possible cases. They are ($\gamma_1 = 0, \gamma_2 = 0$) or ($\gamma_1 = 0, \gamma_2 \neq 0$) or ($\gamma_1 \neq 0, \gamma_2 = 0$) and ($\gamma_1 \neq \gamma_2 \neq 0$). Let us study each case individually. The equilibrium solutions are introduced in terms of the components of angular momentum vector $\vec{M}$ and the components of the vector $\vec{\gamma}$.

- For the first case in which $\gamma_1 = 0, \gamma_2 = 0$, Eqs. (18)–(20) hold identically. Casimir (13) gives $\gamma_3 = \pm 1$. Thus, we obtain two equilibrium solutions $E_1^{\pm} = (0, 0, C\omega, 0, 0, \pm 1)$. For the equilibrium solution E_1^{+}, the angle θ among the two axes Oz and OZ equals zero. Consequently, this equilibrium solution indicates the motion of the body in the upward direction (in other words, the center of the mass of the body lies above the fixed point O). Similarly, the equilibrium solution E_1^{-} refers to the motion of the body around the vertical in the downward direction, i.e., the center of mass lies down the fixed point O.
- For $\gamma_1 = 0$ and $\gamma_2 \neq 0$, the two Eqs. (19) and (20) are identically satisfied. Taking into account expression (1) and Casimir (13), we obtain $\gamma_2 = \sin\theta$ and $\gamma_3 = \cos\theta$. Inserting all obtained results in Eq. (18), we get

$$\omega^2 (C - B)\cos\theta + \omega[k + (c - b)cos\theta] - mgz_0 = 0. \tag{21}$$

  Thus, the equilibrium solution $E_2 = (0, B\omega\sin\theta, C\omega\cos\theta, 0, \sin\theta, \cos\theta)$ exists if the angular velocity ω satisfies condition (21). In a similar way, for the case in which $\gamma_1 \neq 0$ and $\gamma_2 = 0$, the equilibrium solution $E_3 = (A\omega\sin\theta, 0, B\omega\cos\theta, \sin\theta, 0, \cos\theta)$ exists if the condition

$$\omega\cos\theta[\omega(A - C) + a - c] - k\omega + mgz_0 = 0 \tag{22}$$

  is satisfied.
- For the case in which $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, the two equations (18) and (19) give

$$\gamma_3 = \frac{k\omega - mgz_0}{\omega((B - C)\omega + b - c)} \quad \text{and} \quad \gamma_3 = \frac{k\omega - mgz_0}{\omega((A - C)\omega + a - c)} \tag{23}$$

  provided that

$$\left|\frac{k\omega - mgz_0}{\omega((B - C)\omega + b - c)}\right| < 1 \quad \text{and} \quad \left|\frac{k\omega - mgz_0}{\omega((A - C)\omega + a - c)}\right| < 1. \tag{24}$$

  The two expressions (23) are identical if $\omega = \omega_0 = \frac{b-a}{A-B}$ provided that $A \neq B$ and $a \neq b$. It is clear that this value of ω satisfies Eq. (20). Thus, the equilibrium solution $E_4 = (A\omega_0\gamma_1, B\omega_0\gamma_2, C\omega_0\gamma_3, \gamma_1, \gamma_2, \gamma_3)$ with $\omega_0 = \frac{b-a}{A-B}$ and $\gamma_3 = \frac{k\omega - mgz_0}{\omega((B-C)\omega + b - c)}$ exists if the condition $\left|\frac{k\omega - mgz_0}{\omega((B-C)\omega+b-c)}\right| < 1$ is verified. From another side, if $A = B$, Eq. (20) is satisfied if $a = b$ and the two equations (18) and (19) are identical and they give $\gamma_3 = \frac{k\omega - mgz_0}{\omega[\omega(A-C)+a-c]}$. This is verified only if $\left|\frac{k\omega - mgz_0}{\omega[\omega(A-C)+a-c]}\right| < 1$. Thus, we obtain the permanent rotation $E_5 = (A\omega\gamma_1, B\omega\gamma_2, C\omega\gamma_3, \gamma_1, \gamma_2, \gamma_3)$.

The above results can be collected and summarized in the following:

Theorem 1 *The mechanical system* (5) *and* (7) *describing the motion of a charged gyrostat has at least six equilibria. They are:*

(i) $E_1^{\pm} = (0, 0, C\omega, 0, 0, \pm 1)$;
(ii) $E_2 = (0, B\omega\sin\theta, C\omega\cos\theta, 0, \sin\theta, \cos\theta)$ *exists if* ω *satisfies the condition* $\omega^2(C - B)\cos\theta + \omega[k + (c - b)cos\theta] + mgz_0 = 0$;
(iii) $E_3 = (A\omega\sin\theta, 0, B\omega\cos\theta, \sin\theta, 0, \cos\theta)$ *exists if* ω *satisfies the condition* $\omega\cos\theta[\omega(A - C) + a - c] - k\omega - mgz_0 = 0$;
(iv) *If* $A \neq B$ *and* $a \neq b$, *the permanent rotation* $E_4 = (A\omega_0\gamma_1, B\omega_0\gamma_2, C\omega_0\gamma_3, \gamma_1, \gamma_2, \gamma_3)$ *with* $\omega_0 = \frac{b-a}{A-B}$ *and* $\gamma_3 = \frac{k\omega - mgz_0}{\omega((B-C)\omega+b-c)}$ *exists if and only if the condition* $\left|\frac{k\omega - mgz_0}{\omega((B-C)\omega+b-c)}\right| < 1$ *is verified. On the other side, if* $A = B$ *and* $a = b$, *the permanent rotation* $E_5 = (A\omega\gamma_1, B\omega\gamma_2, C\omega\gamma_3, \gamma_1, \gamma_2, \gamma_3)$ *exists if the condition* $\left|\frac{k\omega - mgz_0}{\omega[\omega(A-C)+a-c]}\right| < 1$ *holds,*

4 Stability analysis

In this Section, we study the stability of the permanent rotations presented in Theorem 1. Applying a linear approximation method, we obtain the necessary conditions for stability. This means, according to the Lyapunov theorem (see, e.g., [39]), that the equilibrium solutions that are unstable in linear approximation remain unstable when the nonlinear terms are taken into account while the stable equilibrium solutions need further investigation when the nonlinear terms are taken into account. Thus, the linear approximation method gives the necessary conditions for the stability and the sufficient conditions for instability. Consequently, in order to obtain the sufficient conditions for the stability of these permanent rotations, we utilize the energy-Casimir method. This method was applied in various works (see, e.g., [22–25]). This method is epitomized in the following.

Theorem 2 (Generalized energy-Casimir method) *Let $(M, \{.,.\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field X_h. If there is a set of conserved quantities $C_1, C_2, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + C_2 + \cdots + C_n)(m) = 0$$

and

$$\mathbf{d}^2(h + C_1 + C_2 + \cdots + C_n)(m)|_{W\times W}$$

is definite for W defined by

$$W = \ker\mathbf{d}C_1(m) \sqcap \ker\mathbf{d}C_2(m) \sqcap \cdots \sqcap \ker\mathbf{d}C_n(m), \tag{25}$$

then m is stable. If $W = \{0\}$, m is always stable.

4.1 Necessary conditions for stability

We are going to demonstrate the necessary conditions for stability (or sufficient conditions for instability) for the permanent rotations that are introduced in Theorem 1. We evaluate the tangent flow of Eqs. (5) and (7) at the equilibrium solution $E = (M_{10}, M_{20}, M_{30}, \gamma_{10}, \gamma_{20}, \gamma_{30})$ that is denoted

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathcal{J}_E \mathbf{z}$$

where $\mathcal{J}_E$ is the Jacobian matrix and is given by

$$\mathcal{J}_E = \begin{pmatrix} 0 & J_{12} & J_{13} & 0 & \frac{bM_{30}}{C} + mgz_0 & -\frac{cM_{20}}{B} \\ \frac{k+c\gamma_{30}}{A} + \frac{(C-A)M_{30}}{AC} & 0 & J_{23} & -mgz_0 - \frac{aM_{30}}{C} & 0 & \frac{cM_{10}}{A} \\ \frac{(A-B)M_{20}}{AB} - \frac{b\gamma_{20}}{A} & \frac{(A-B)M_{10}}{BA} + \frac{a\gamma_{10}}{B} & 0 & \frac{aM_{20}}{B} & -\frac{bM_{10}}{A} & 0 \\ 0 & -\frac{\gamma_{30}}{B} & \frac{\gamma_{20}}{C} & 0 & \frac{M_{30}}{C} & -\frac{M_{20}}{B} \\ \frac{\gamma_{30}}{A} & 0 & -\frac{\gamma_{10}}{C} & -\frac{M_{30}}{C} & 0 & \frac{M_{10}}{A} \\ -\frac{\gamma_{20}}{A} & \frac{\gamma_{10}}{B} & 0 & \frac{M_{20}}{B} & -\frac{M_{10}}{A} & 0 \end{pmatrix} \tag{26}$$

where

$$J_{12} = -\frac{k + c\gamma_{30}}{B} + \frac{(B-C)M_{30}}{BC}, \quad J_{13} = \frac{b\gamma_{20}}{C} + \frac{(B-C)M_{20}}{BC}, J_{23} = \frac{(C-A)M_{10}}{AC} - \frac{a\gamma_{10}}{C}.$$

In order to study the linear stability, we determine the eigenvalues of the matrix $\mathcal{J}_E$ corresponding to the equilibrium solutions. These eigenvalues are the roots of the characteristic equation

$$\det(\mathcal{J}_E - \lambda I_6) = 0 \tag{27}$$

where I_6 is the 6×6 identity matrix and $\mathcal{J}_E$ is the Jacobian matrix that is given by (26) while λ denotes the eigenvalue. The characteristic equation (27) corresponding to the equilibrium solutions E_i, $i = 1, 2, 3, 4, 5$ takes the form

$$\lambda^2(\lambda^4 + P_i\lambda^2 + Q_i) = 0, \quad i = 1, 2, \ldots, 5 \tag{28}$$

where P_i and Q_i are calculated for each equilibrium solution E_i individually. Thus, we have:

- For the equilibrium solution $E_1^{\pm}$, we have

$$\begin{aligned} P_1 &= \frac{1}{AB}[((2A - C)A - C(B - C))\omega^2 - (A(K \pm a) + B(K \pm b) - 2CK)\omega \\ &\quad + K^2 \mp mgz_0(A + B)], \\ Q_1 &= \frac{1}{AB}[(B - C)\omega^2 - (K \mp b)\omega \pm mgz_0][(A - C)\omega^2 - (K \mp a)\omega \pm mgz_0], \end{aligned} \tag{29}$$

  where $K = k \pm c$.
- For the permanent rotation E_2, the two expressions P_2 and Q_2 are given by

$$\begin{aligned} P_2 &= \frac{\omega^2}{ABC}[(B - C)\cos^2\theta(-C^2 + (2A + B)C + B(A - B) + B^3 - (A + C)B^2 + 2ACB] \\ &\quad + \frac{k^2}{AB} + \frac{b^2}{AC} - \frac{2k\omega\cos\theta(A + B - C)}{BA} + \frac{\cos^2\theta(Cc^2 - Bb^2)}{ABC} + \frac{\omega}{ACB}[B(2bB - (b + c)C \\ &\quad - A(a + b)) + (B(-2bB^2 + ((b - c)C + A(a + b))B + (2c + A(a - b + 2c))c))\cos^2\theta], \\ Q_2 &= \frac{3\omega\sin^2\theta((B - A)\omega + b - a)}{ABC}[(B - C)(3\omega^2(B - C)\cos^2\theta - 4k\omega\cos\theta + B\omega^2) + k^2 \\ &\quad + (b - c)((3(B - C)\omega + b - c)\cos^2\theta - 2k\cos\theta + B\omega)]. \end{aligned} \tag{30}$$

  Notice that the weight of the gyrostat is eliminated by utilizing the condition (21).
- The two expressions P_3 and Q_3 corresponding to the permanent rotation E_3 admit the form

$$\begin{aligned} P_3 &= \omega^2[A(A^2 - A(B + C) + 2BC) - (A - C)(C^2 - (A + 2B)C + A(A - B))\cos^2\theta] \\ &\quad - 2Ck\omega(A + B - C)\cos\theta + k^2C - 2cCk\cos\theta + [[2cC^2 - (A(c - a) - B(a - b - 2c))C\omega \\ &\quad + (2Aa + B(a + b)A)] - Aa^2 - Cc^2]\cos^2\theta + A[a^2 + ((2A - B - C)a - Bb - cC)a], \\ Q_3 &= \omega\sin^2\theta(\omega(A - B) + a - b)[(3(A - C)^2\omega^2 + 3(a - c)(A - C)\omega + (a - c)^2) \\ &\quad - 2k(2\omega(A - C) + a - c)\cos\theta + \omega^2(A^2 - C^2) + A\omega(a - c) + k^2]. \end{aligned} \tag{31}$$

  We should observe that the weight of the gyrostat is removed by using the condition (22).
- Taking into account the permanent solution E_4, the two expressions P_4 and Q_4 are given by

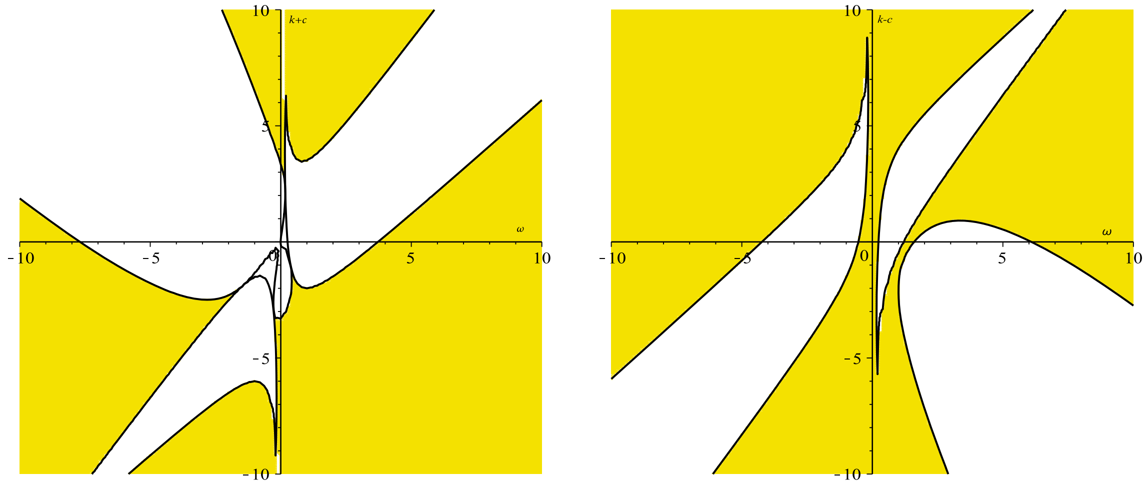
$$\begin{aligned} P_4 &= \frac{\gamma_{30}^2}{AB}[C\omega^2(C - A - B) - \omega(A(a + c) + B(b + c) - 2cC) + c^2] - \frac{\gamma_{30}}{AB}[k\omega(2C - A - B) \\ &\quad + mgz_0(A - B) + 2ck] + 2\omega^2 + \frac{\omega\gamma_{10}^2}{CB}[a(2A - C) - B(b + a) - cC] - \frac{\gamma_{20}^2\omega}{AC}[A(a + b) \\ &\quad + C(b + c) - 2bB] + \frac{a^2\gamma_{10}^2}{CB} + \frac{b^2\gamma_{20}^2}{AC} + \frac{k^2}{AB}, \\ Q_4 &= \frac{\gamma_{10}^2\gamma_{20}^2(b - a)^2}{ABC(A - B)^2}[a(C - B) + b(A - C) + c(B - A)] \end{aligned} \tag{32}$$

  where γ_{10}, γ_{20}, and γ_{30} are given by

$$\gamma_{30} = \frac{k\omega + mgz_0}{\omega((B - C)\omega + b - c}, \quad \gamma_{10} = \sqrt{1 - \gamma_{30}^2}\sin\varphi, \quad \gamma_{20} = \sqrt{1 - \gamma_{30}^2}\cos\varphi. \tag{33}$$

- The two expressions P_5 and Q_5 associated to the permanent rotation E_5 take the form

$$\begin{aligned} P_5 &= \frac{1}{CA^2\omega^2((A - C)\omega + a - c)^2}[m^2g^2z_0^2A(3A^2C\omega^2 - (4C^2\omega^2 - C\omega(a - 3c) - a^2)) \\ &\quad - 2mgz_0k\omega(C\omega(A - C)(A\omega - a) - a(Aa - cC)) + (A - C)^2A^2C\omega^6 + (A - C)AC \end{aligned}$$



(a) **(b)**

Fig. 2 The regions of linear stability and instability corresponding to the equilibrium positions $E_1^{\pm}$ in the plane of the two parameters $k + c, \omega$ when the other constants admit the value $A = 3$, $B = 2.5$, $C = 1.5$, $mgz_0 = 1$, $a = 1$, $b = -4$. **a** E_1^+ and **b** E_1^-

$$+C(a+c))\omega^5 + A(a^2A^2 - 3(a^2 - c^2 + \frac{2ac}{3})AC + C^2(3a^2 - 2c^2) + A(a-c)(C(c^2$$
$$-3a^2k^2)((a-3c)A + 2a^2A)\omega^3 + \omega^2((a-c-k)(a+k-c)A + Ck^2)a^2),$$
$$Q_5 = 0. \tag{34}$$

Theorem 3 *The necessary conditions for the permanent rotation E_i involving in theorem 1 to be linearly stable (spectrally stable) are*

$$P_i \geq 0, \quad Q_i \geq 0, \quad P_i^2 - 4Q_i \geq 0, \quad i = 1, 2, \ldots, 4 \tag{35}$$

where P_i and Q_i are given by expressions (29), (30), (31), and (32). While the permanent rotation E_5 is linearly stable if $P_5 \geq 0$. Or, equivalently, the sufficient condition for the instability of those permanent rotations can be obtained if one of the conditions (35) is not verified.

Now, we are going to determine the regions where the necessary conditions for stability (35) are satisfied in the plane of certain parameters while the other parameters take constant values. In Fig. 2, the three curves $P_1 = 0$, $Q_1 = 0$, $P_1^2 - 4Q_1 = 0$ that are characterized by the solid lines divide the plane of the parameters $k \pm c$ and ω into some regions. The regions of linear stability have a yellow color while the uncolored regions represent the regions of instability. A similar conclusion can be performed for other Figs. 3 and 4.

4.2 Sufficient conditions for stability

In this Subsection, we are going to delimit the sufficient conditions for the permanent rotations E_i to be Lyapunov stable employing the energy-Casimir method. We apply this method to the permanent rotations $E_1^{\pm}$ and E_2 with more details, and due to similar computations, the final results for the other permanent rotations are introduced without details.

To apply the energy-Casimir method for $E_1^{\pm}$, we introduce the augmented Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}\right) + mgz_0\gamma_3 + \delta\left[(M_1 + \frac{a}{2}\gamma_1)\gamma_1 + (M_2 + \frac{b}{2}\gamma_2)\gamma_2 + (M_3 + k + \frac{c}{2}\gamma_3)\gamma_3\right]$$
$$+\rho(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \tag{36}$$

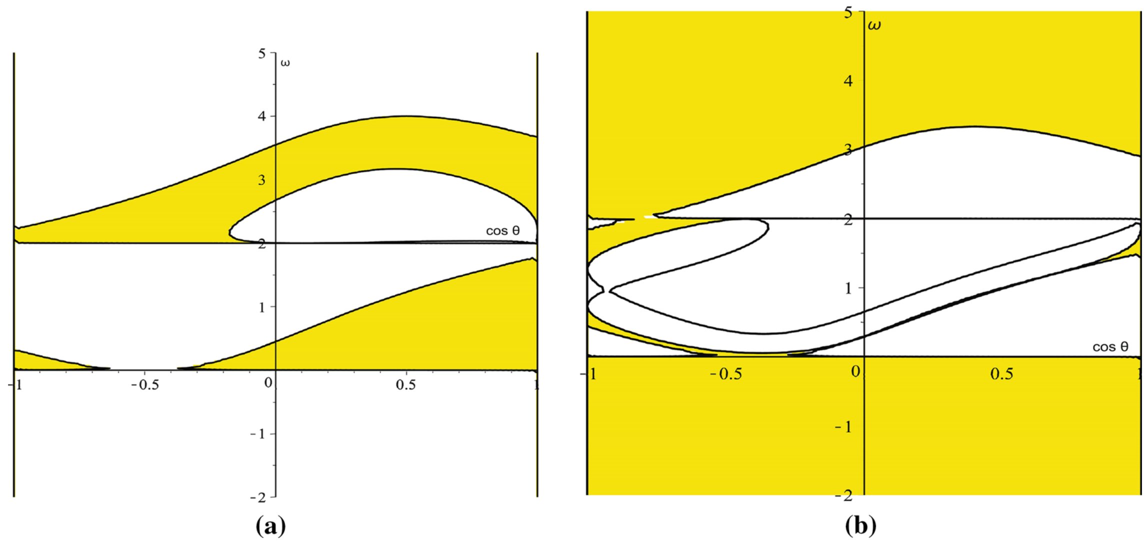


Fig. 3 The regions of linear stability and instability corresponding to the equilibrium positions $E_{2,3}$ in the plane of the two parameters γ_3, ω when the other constants admit the values $A = 3, B = 2.5, C = 1.5, a = 1, b = 2, c = 6, k = 2$. **a** E_2 and **b** E_3

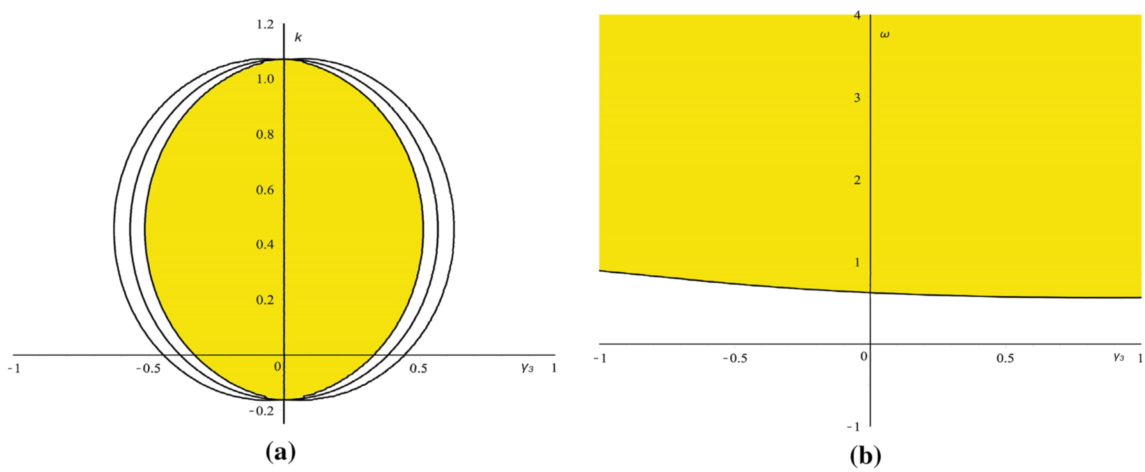


Fig. 4 The regions of linear stability and instability corresponding to the equilibrium positions $E_{2,3}$ in the plane of the two parameters γ_3, k, and γ_3, ω, respectively, when the other constants admit the values $B = 3, A = 2.5, C = 1.5, m = 1, g = 1, z0 = 1, a = 2, b = 1, c = 3$. **a** E_4 and **b** E_5

where δ and ρ are arbitrary constants that are determined in such a way guaranteeing that $E_1^{\pm}$ is a critical point of $\mathcal{H}$, i.e.,

$$\frac{\partial \mathcal{H}}{\partial M_j}|_{E_1^{\pm}} = 0, \quad \frac{\partial \mathcal{H}}{\partial \gamma_j}|_{E_1^{\pm}} = 0, \quad j = 1, 2, 3, \tag{37}$$

which give the values of δ, ρ as

$$\delta = \mp\omega, \quad \rho = \frac{1}{2}[C\omega^2 + (k \pm c)\omega \mp mgz_0]. \tag{38}$$

Now, we determine the space W,

$$W = \ker \mathrm{d}C_1(E_1^{\pm}) \bigcap \ker \mathrm{d}C_2(E_1^{\pm}), \tag{39}$$

where C_1, C_2 are two Casimirs (13) and (14). On the other side, we have

$$\mathrm{d}C_1(E_1^{\pm}) = \pm 2d\gamma_3, \quad \mathrm{d}C_2(E_1^{\pm}) = \pm dM_3 + (C\omega + k \pm c)d\gamma_3. \tag{40}$$

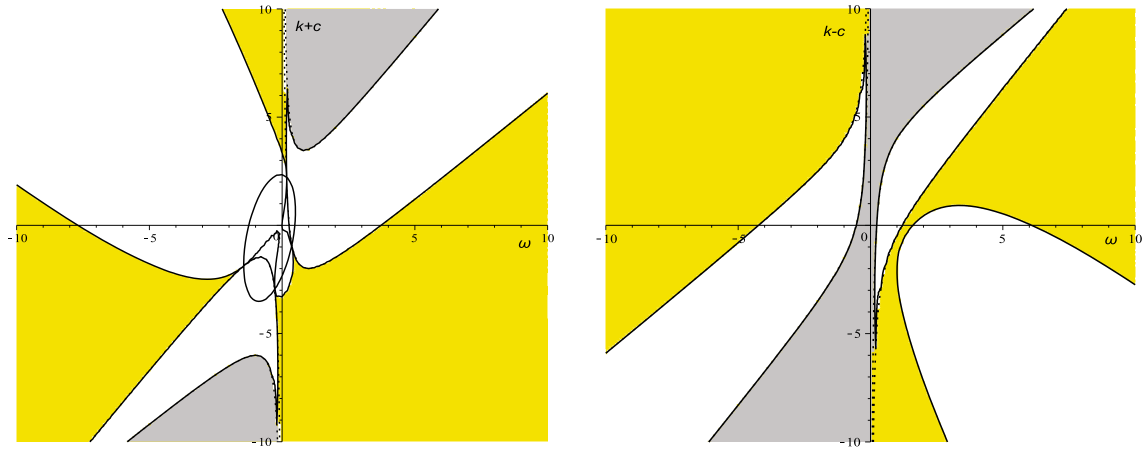


Fig. 5 The regions of Lyapunov stability and instability corresponding to the equilibrium positions $E_1^{\pm}$ in the plane of the two parameters ω, $k \pm c$ when the other constants admit the values $B = 3, A = 2.5, C = 1.5, m = 1, g = 1, z0 = 1, a = 2, b = 1, c = 3$. **a** E_1^{+} and **b** E_1^{-}

Thus, the space W is spanned by the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5$ where $\vec{e}_i$, $i = 1, 2, \ldots, 6$ are the canonical basis of $\mathbb{R}^6$. The Hessian matrix corresponding to the augmented Hamiltonian (36) in the reduced space W admits the form

$$\text{Hess}|_{W\times W} = \begin{pmatrix} \frac{1}{A} & 0 & 0 & \mp\omega \\ 0 & \frac{1}{B} & \mp\omega & 0 \\ 0 & \mp\omega & C\omega^2 + (k \pm c \mp b)\omega \mp mgz_0 & 0 \\ \mp\omega & 0 & 0 & C\omega^2 + (k \pm c \mp a)\omega \mp mgz_0 \end{pmatrix}. \tag{41}$$

Employing the Sylvester criterion to demonstrate the definiteness of the Hessian matrix (41), we compute its principal minors

$$\begin{aligned} \Delta_1 &= \frac{1}{A}, \quad \Delta_2 = \frac{1}{AB}, \\ \Delta_3 &= -\frac{(B-C)\omega^2 - \omega(k \pm c \mp b) \pm mgz_0}{AB}, \\ \Delta_4 &= -\Delta_3[(A-C)\omega^2 - (k \pm c \mp a)\omega \pm mgz_0]. \end{aligned} \tag{42}$$

It is clear that Δ_1 and Δ_2 are always positive while Δ_3 and Δ_4 are always positive together if and only if

$$(B-C)\omega^2 - \omega(k \pm c \mp b) \pm mgz_0 < 0, \quad (A-C)\omega^2 - (k \pm c \mp a)\omega \pm mgz_0 < 0. \tag{43}$$

Taking into account all obtained results concerning the permanent rotation $E_1^{\pm}$, we can present the following

Theorem 4 *The necessary and sufficient conditions for the permanent rotation $E_1^{\pm}$ to be Lyapunov stable are*

$$(B-C)\omega^2 - \omega(k \pm c \mp b) \pm mgz_0 < 0, \quad (A-C)\omega^2 - (k \pm c + a)\omega \pm mgz_0 < 0. \tag{44}$$

Two conditions (44) are represented by a gray region in the plane of the two parameters ω and $k \pm c$ as it is outlined in Fig. 5. This region specifies the regions of Lyapunov stability. Moreover, two Figs. 2 and 5 clarify that the region of Lyapunov stability for the permanent rotation $E_1^{\pm}$ is a portion of the yellow region that defines the linear stability.

Now, we consider a special case corresponding to the axisymmetric gyrostat $B = A$ and $b = a$, and the stability conditions (44) for E_1^{+} (similarly for E_1^{-}) reduce to a single condition

$$(A-C)\omega^2 - \omega(k + c - a) + mgz_0 < 0. \tag{45}$$

For the uncharged gyrostat $c = a = 0$, condition (45) reduces to

$$(A - C)\omega^2 - k\omega + mgz_0 < 0, \tag{46}$$

which is identified with the condition obtained in [23]. Furthermore, this condition is different from the classical condition [15,40]

$$(C\omega + k)^2 \geq mgAz_0. \tag{47}$$

Condition (46) is more restrictive, in the sense that if condition (46) is verified also condition (47) is satisfied, but not necessarily in the reverse way. This is due to

$$(C\omega + k)^2 - 4A((C - A)\omega^2 + k\omega) = (k + (C - 2A)\omega)^2 \geq 0. \tag{48}$$

We now are going to study the Lyapunov stability for the permanent rotation E_2 utilizing the energy-Casimir method. To do this, we consider the augmented Hamiltonian (36) and determine the values of the two constants δ, ρ which make E_2 a critical point for the augmented Hamiltonian (36), i.e.,

$$\frac{\partial \mathcal{H}}{\partial M_j}|_{E_2} = 0, \quad \frac{\partial \mathcal{H}}{\partial \gamma_j}|_{E2} = 0, \quad j = 1, 2, 3. \tag{49}$$

Using condition (21), Eq. (49) gives

$$\delta = -\omega, \quad \rho = \frac{1}{2}(B\omega + b)\omega. \tag{50}$$

Let us now specify the space W that is defined as

$$W = \ker \mathrm{d}C_1(E_2) \bigcap \ker \mathrm{d}C_2(E_2) \tag{51}$$

where C_1 and C_2 are given by (13) and (14), respectively. On another side, we have

$$\begin{aligned}
\mathrm{d}C_1 &= d\gamma_3 + \frac{\sin\theta}{\cos\theta} d\gamma_2, \\
\mathrm{d}C_2 &= dM_3 + \frac{\sin\theta}{\cos\theta} dM_2 + \frac{\sin\theta}{\cos^2\theta}[((B - C)\omega + b - c)\cos\theta - k]d\gamma_2.
\end{aligned} \tag{52}$$

After some manipulations, the space W is spanned by the vectors

$$\vec{e}_1, \vec{e}_4, \cos\theta\vec{e}_2 - \sin\theta\vec{e}_3, \sin\theta[((B - C)\omega + b - c)\cos\theta - k]\vec{e}_3 - \cos^2\theta\vec{e}_5 + \sin\theta\cos\theta\vec{e}_6$$

where $\vec{e}_i, i = 1, 2, \ldots, 6$ are the canonical basis of $\mathbb{R}^6$. The Hessian matrix associated to the augmented Hamiltonian in the reduced space W takes the form

$$\mathrm{Hess}|_{W\times W} = \begin{pmatrix} \frac{1}{A} & 0 & 0 & -\omega \\ 0 & \frac{\cos^2\theta}{B} + \frac{\sin^2\theta}{C} & H_{23} & 0 \\ 0 & H_{23} & H_{33} & 0 \\ -\omega & 0 & 0 & \omega(B\omega + b - a) \end{pmatrix} \tag{53}$$

where

$$\begin{aligned}
H_{23} &= \frac{C\omega\cos^3\theta + ((B - 2C)\omega + b - c)\cos\theta\sin^2\theta - k\sin^2\theta}{C}, \\
H_{33} &= \frac{1}{C}[BC\omega^2\cos^4\theta + ((B^2 - 3BC + 3C^2)\omega^2 + (b - c)(2B - 3C) + (b - c)^2)\sin^2\theta\cos^2\theta \\
&\quad -2k((B - 2C)\omega + b - c)\cos\theta\sin^2\theta + k^2\sin^2\theta].
\end{aligned} \tag{54}$$

The Sylvester criterion is used to determine the definiteness of the Hessian matrix (53), and so, we write down the principal minors as

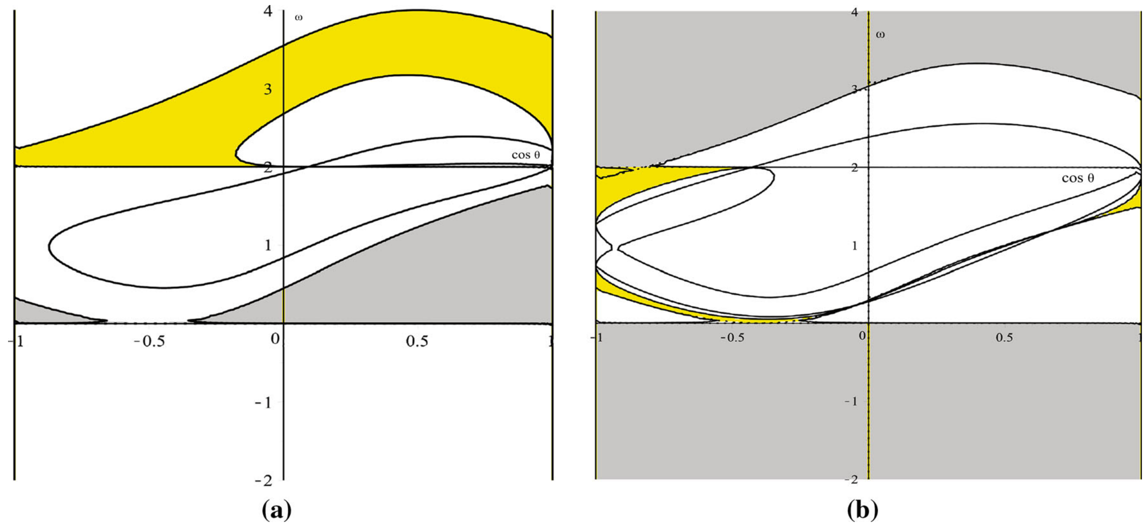


Fig. 6 The regions of linear stability and instability corresponding to the equilibrium positions $E_{2,3}$ in the plane of the two parameters γ_3, ω when the other constants admit the values $A = 3$, $B = 2.5$, $C = 1.5$, $a = 1$, $b = 2$, $c = 6$, $k = 2$. **a** E_2 and **b** E_3

$$\begin{aligned}
\Delta_1 &= \frac{1}{A}, \quad \Delta_2 = \frac{1}{A}\left(\frac{\cos^2\theta}{B} + \frac{\sin^2\theta}{C}\right),\\
\Delta_3 &= \frac{\sin^2\theta\cos^2\theta}{ABC}[((4B^2 - 7BC + 3C^2)\omega^2 + (b-c)(4B-3C)\omega + (b-c)^2)\cos^2\theta\\
&\quad -2k\cos\theta(2(B-C)\omega + b - c) + B\omega((B-C)\omega + b - c)\sin^2\theta + k^2],\\
\Delta_4 &= \omega[(B-A)\omega - a + b]\Delta_3. \qquad (55)
\end{aligned}$$

It is clear that Δ_1 and Δ_2 are always positive for any values of the angle $\theta \in]0, 2\pi[$. The principal minor Δ_3 is positive if

$$\begin{aligned}
&((4B^2 - 7BC + 3C^2)\omega^2 + (b-c)(4B-3C)\omega + (b-c)^2)\cos^2\theta - 2k\cos\theta(2(B-C)\omega + b - c)\\
&\quad + B\omega((B-C)\omega + b - c)\sin^2\theta + k^2 > 0, \qquad (56)
\end{aligned}$$

and Δ_4 is positive only if $\omega[(A-B)\omega + a - b] < 0$. Thus, we can formulate the following theorem:

Theorem 5 *The necessary and sufficient conditions for the permanent rotation E_2 to be Lyapunov stable are $\omega[(B-A)\omega - a + b] > 0$ and*

$$\begin{aligned}
&((4B^2 - 7BC + 3C^2)\omega^2 + (b-c)(4B-3C)\omega + (b-c)^2)\cos^2\theta - 2k\cos\theta(2(B-C)\omega + b - c)\\
&\quad + B\omega((B-C)\omega + b - c)\sin^2\theta + k^2 > 0.
\end{aligned}$$

In an analogous way, we can study the Lyapunov stability for the permanent rotation E_3, and this result is summarized in the following theorem:

Theorem 6 *The necessary and sufficient conditions for the permanent rotation E_3 to be Lyapunov stable are $\omega[(A-B)\omega + a - b] > 0$,*

$$\begin{aligned}
&[(4A^2 - 7AC + 3C^2)\omega^2 + (a-c)(4A-3C) + (a-c)^2]\cos^2\theta - 2k\cos\theta[2\omega(A-C) + a - c]\\
&\quad + A\omega[(A-C)\omega + a - c]\sin^2\theta + k^2 > 0. \qquad (57)
\end{aligned}$$

It is clear that the two Figs. 3 and 6 illustrate the regions of the Lyapunov stability for the permanent rotations E_2 and E_3 that are characterized by a gray color which are a part of the regions of linear stability that are colored by yellow color.

For the two permanent rotations E_4 and E_5, the energy-Casimir method does not give us any information about the sufficient conditions of stability because the quadratic form is semidefinite.

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