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NOTE

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# Poisson brackets formulation for the dynamics of a position-dependent mass particle

*Specially dedicated to Giulia de Souza Ciampone*

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**Abstract** Following the history of mechanics (Dugas in A history of mechanics, Dover, New York, 1988), we read that Poisson's theorem figured in the 1811 edition of the celebrated *Mécanique Analytique*. Indeed, as it can be observed in the classical textbooks, Poisson brackets formulation is one of the cardinal chapters of analytical mechanics. Given the natural motivation to accommodate variable-mass systems at the level of classical analytical mechanics, we will herein direct our attention toward Poisson brackets formulation. To wit, considering the case of a position-dependent mass particle, in which we will assume that the absolute velocity of mass ejection or aggregation is a linear function of the generalized velocity, we will endeavor to provide such position-dependent mass problems with an appropriate Poisson brackets formulation. To our very best knowledge, this means an original contribution to the research field of the analytical mechanics of variable-mass systems. We will start establishing the Poisson brackets definition for the dynamics of a position-dependent mass particle, which will be posited in harmony with the classical mathematical portrait of analytical mechanics. Therefrom, we will demonstrate consequent results which give rise to the required formulation, namely, Jacobi's identity, canonical equations expressed by means of such Poisson brackets and Poisson's theorem. Last, we will apply Poisson's theorem to evaluate the relationship between the conservation laws which are at our disposal in the domain of position-dependent mass problems.

## 1 Motivation

The idea of varying mass is intimately related to situations in which—through the agency of certain standpoints—we conceive of the quantity of matter in such a way that its conservation does not hold true obligatorily. It means that variable-mass problems arise from approaches that view a specific portion of the system. To wit, the mechanics of variable-mass systems encompasses problems which in general comprehend non-material control volumes and points of varying mass. To unfamiliar readers, we suggest, as representative works, [1–3].

Studies of the subject date back to eighteenth century, and, thenceforth, a flow of contributions has established itself (see, e.g., [1,2,4]). We recollect that the mathematical framework of classical mechanics was originally conceived for constant-mass problems. This has driven forward a series of investigations which aim to provide variable-mass systems with suitable formulations. Needless to say, there have been serious efforts in this sense, and, for a brief sample of them, we indicate [3,5–7].

It has been evidenced that the inverse problem of Lagrangian mechanics (see [8,9]) is a fruitful tool to introduce variable-mass systems into the cardinal chapters of analytical mechanics. See, for instance, [7] and references therein cited. An important result in this field is the fact that position-dependent mass particles admit a first integral of energy (see, e.g., [7]). Recently, it was demonstrated that there is a conservation law of

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different form holding in such type of variable-mass problem (see [10]). We clarify that by position-dependent mass particle we mean a particle having its quantity of matter given by a function depending only upon the generalized coordinate.

Given the conservation laws, it is interesting to evaluate if there exists a relation between them. With the initial intent of tackling this issue, we will herein make use of Poisson's theorem. This consequently motivates us to delineate the Poisson brackets formulation of the dynamics of a position-dependent mass particle. Moreover, to our very best knowledge, an investigation of this nature is original within the realm of position-dependent mass problems. Poisson brackets formulation is indeed one of the basic topics of analytical mechanics (see, e.g., [11]); therefore, we humbly believe that our present endeavor can be of some value for the current developments of the theory about variable-mass systems.

Before proceeding, it is appropriate to somehow emphasize the advantage of the aimed formulation with respect to the previously established one. First, we explain that Poisson brackets formulation is related to the Hamiltonian picture. Second, Poisson brackets formulation inherently owns an aspect of elegancy, and, as remarked by Dugas [12, p. 387], Poisson's theorem has become classical and evidently exhibits considerable aesthetic value. This reinforces our motivation to delineate the Poisson brackets formulation of the dynamics of a position-dependent mass particle. Turning our attention to the use of the Poisson brackets formulation within the dynamics of a position-dependent mass particle, we lay stress on the fact that, besides the possibility of being applied to deal with the conservation laws currently at our disposal, there is also the possibility of being used to test the canonical character of a given transformation. The latter use will not be addressed herein, but the reader finds a general explanation of it in [13, p. 216].

## 2 Preliminaries

This Section brings forward elements of the dynamics of a position-dependent mass particle which are important for our discussion. Please consider the following:

In conformity with our previous works (see [7,10]), we assume that the equation of motion of our position-dependent mass problem is

$$m(q)\ddot{q} + \frac{\mathrm{d}V(q)}{\mathrm{d}q} - \alpha\dot{q}^2\frac{\mathrm{d}m(q)}{\mathrm{d}q} = 0, \tag{1}$$

in which $q$ is the generalized coordinate, $m = m(q)$ is position-dependent mass, $V = V(q)$ is the real potential energy, and an overdot represents differentiation with respect to $t$. It is also assumed that the absolute velocity of mass ejection or aggregation is a linear function of the generalized velocity $\dot{q}$, i.e., $k\dot{q}$, where $k =$ const. and $\alpha = k - 1 =$ const.

**Hamiltonian formulation** By virtue of the inverse problem of Lagrangian mechanics, the Hamiltonian of our problem is (see [8])

$$\tilde{H}(q, \tilde{p}) = \frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q, \tag{2}$$

where $\tilde{p}$ is the canonical momentum, i.e.,

$$\tilde{p} = m(q)^{-2\alpha}\dot{q}. \tag{3}$$

**Canonical equations** Having the Hamiltonian $\tilde{H}$ and the canonical momentum $\tilde{p}$, we write the canonical equations (see [8]), i.e.,

$$\dot{q} = \frac{\partial\tilde{H}}{\partial\tilde{p}} \quad\Rightarrow\quad \dot{q} = \frac{\tilde{p}}{m(q)^{-2\alpha}}, \tag{4}$$

$$\dot{\tilde{p}} = -\frac{\partial\tilde{H}}{\partial q} \quad\Rightarrow\quad \dot{\tilde{p}} = -\alpha\frac{\mathrm{d}m(q)}{\mathrm{d}q}\frac{\tilde{p}^2}{m(q)^{1-2\alpha}} - m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}. \tag{5}$$

**Conservation law 1** By reason of the mathematical framework settled by the inverse problem, the conservation law (see [8])

$$\tilde{H}(q, \tilde{p}) = \frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q = \tilde{E} = \text{const}. \tag{6}$$

holds.

**Conservation law 2** As demonstrated in [10], the conservation law

$$f(q)\left(\frac{1}{2}m(q)\dot{q}^2 + V(q)\right) = \text{const.}, \tag{7}$$

where $f = \exp\left[-(\frac{1}{2}+\alpha)\int\left(\frac{2(\mathrm{d}m(q)/\mathrm{d}q)m(q)^{2\alpha}(\tilde{E}-\tilde{V}(q))}{m(q)^{2\alpha+1}(\tilde{E}-\tilde{V}(q))+V(q)}\right)\mathrm{d}q\right]$, $\tilde{V} = \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q$, and $\frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \tilde{V}(q) = \tilde{E} = \text{const.}$; also holds in the position-dependent mass problem in question.

Through the canonical momentum (3), Eq. (7) is translated into

$$I(q,\tilde{p}) = f(q)\left(\frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-4\alpha-1}} + V(q)\right) = \text{const.} \tag{8}$$

## 3 Poisson brackets formulation written for the position-dependent mass problem

For the sake of clarity of purpose, we elucidate that the following formulation is consistent with the classical mathematical frame of analytical mechanics (see [11–16]). It means that the following formulae are specific for our position-dependent mass problem in the sense that they will carry the superscript $\sim$. This is a question of symbology, which facilitates the reader to keep in mind that we are dealing with quantities properly resulting from the inverse problem of Lagrangian mechanics (see [8]).

It is likewise appropriate to explain the reason why we will lead the reader through the content which will be next presented. Namely, we will trace a way which originates at Poisson brackets definition in the realm of our position-dependent mass problem and ends at the corresponding Poisson's theorem. The motivation is to provide the reader who is not familiar with Poisson brackets formulation with an easily checked out demonstration of it. Furthermore, we believe that the next exposition will be helpful for who is not familiar with our previous discussions involving the tilde symbology (see [8]), that is, they will be able to immediately recognize the relation between the well-known results of classical literature and the corresponding ones having tilde symbols.

Given that, please consider:

**Poisson brackets** *For any functions $u$ and $v$ of the type $u = u(q,\tilde{p},t)$ and $v = v(q,\tilde{p},t)$, we define the Poisson brackets of the dynamics of a position-dependent mass particle to be*

$$[u,v]_{q,\tilde{p}} = \left(\frac{\partial u}{\partial q}\frac{\partial v}{\partial \tilde{p}} - \frac{\partial u}{\partial \tilde{p}}\frac{\partial v}{\partial q}\right). \tag{9}$$

*Remark 1* For the sake of a clarifying notation, the subscript $q,\tilde{p}$ is attached to the brackets.

**Identity 1** *Given such Poisson brackets definition for the dynamics of a position-dependent mass particle, we become able to accordingly conceive of the meaning of total derivative. Namely, for a function of the type $w = w(q,\tilde{p},t)$, it follows from the canonical equations $\dot{q} = \partial\tilde{H}/\partial\tilde{p}$ and $\dot{\tilde{p}} = -\partial\tilde{H}/\partial q$ (see Sect. 2) that the total derivative can be expressed as*

$$\frac{\mathrm{d}w}{\mathrm{d}t} = [w,\tilde{H}]_{q,\tilde{p}} + \frac{\partial w}{\partial t}. \tag{10}$$

*Proof* By differentiation,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial q}\dot{q} + \frac{\partial w}{\partial \tilde{p}}\dot{\tilde{p}} + \frac{\partial w}{\partial t}. \tag{11}$$

Substituting the canonical equations $\dot{q} = \partial\tilde{H}/\partial\tilde{p}$ and $\dot{\tilde{p}} = -\partial\tilde{H}/\partial q$ in (11), we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}t} = [w,\tilde{H}]_{q,\tilde{p}} + \frac{\partial w}{\partial t}. \tag{12}$$

The proof is concluded. □

**Identity 2** (Jacobi) *In light of the Poisson brackets definition (9), the identity*

$$[u,[v,w]_{q,\tilde{p}}]_{q,\tilde{p}} + [v,[w,u]_{q,\tilde{p}}]_{q,\tilde{p}} + [w,[u,v]_{q,\tilde{p}}]_{q,\tilde{p}} = 0 \tag{13}$$

*holds.*

*Proof* The identity can be verified by simply using definition (9) and differentiation; therefore, for the sake of brevity, we will here not present such a proof. □

*Remark 2* As it will be confirmed by the eye of the reader, identities 1 and 2 play a fundamental role in the demonstration of the next results.

**Canonical equations expressed by means of Poisson brackets** *The canonical equations (4) and (5) are equivalent to*

$$\dot{q} = [q, \tilde{H}]_{q,\tilde{p}}, \tag{14}$$
$$\dot{\tilde{p}} = [\tilde{p}, \tilde{H}]_{q,\tilde{p}}. \tag{15}$$

*Proof* If $w(q, \tilde{p}, t) = q$, it is immediate that $\partial w/\partial t = 0$. Therefrom, identity 1 yields

$$\frac{\mathrm{d}q}{\mathrm{d}t} = [q, \tilde{H}]_{q,\tilde{p}}, \text{ i.e.,} \tag{16}$$
$$\dot{q} = [q, \tilde{H}]_{q,\tilde{p}}. \tag{17}$$

If $w(q, \tilde{p}, t) = \tilde{p}$, then also $\partial w/\partial t = 0$. Since $\dot{\tilde{p}} = \mathrm{d}\tilde{p}/\mathrm{d}t$, identity 1 gives

$$\dot{\tilde{p}} = [\tilde{p}, \tilde{H}]_{q,\tilde{p}}. \tag{18}$$

The proof is concluded. □

**Theorem** (Poisson) *If $A = A(q, \tilde{p})$ and $B = B(q, \tilde{p})$ are conservation laws of the canonical equations (4) and (5); then, $[A, B]_{q,\tilde{p}}$ is also a conservation law.*

*Proof* Let us consider identity 1. For $w = A(q, \tilde{p})$, we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = [A, \tilde{H}]_{q,\tilde{p}}. \tag{19}$$

Due to the fact that $A =$ const. is a conservation law of the canonical equations (4) and (5), the left-hand side of Eq. (19) vanishes, i.e.,

$$[A, \tilde{H}]_{q,\tilde{p}} = 0. \tag{20}$$

By the same manner, we have

$$[B, \tilde{H}]_{q,\tilde{p}} = 0. \tag{21}$$

Now we write identity 2 in the form of

$$[\tilde{H}, [A, B]_{q,\tilde{p}}]_{q,\tilde{p}} + [A, [B, \tilde{H}]_{q,\tilde{p}}]_{q,\tilde{p}} + [B, [\tilde{H}, A]_{q,\tilde{p}}]_{q,\tilde{p}} = 0. \tag{22}$$

In virtue of the anti-symmetry property, i.e., $[u, v]_{q,\tilde{p}} = -[v, u]_{q,\tilde{p}}$ (see, e.g., [13]), we transform Eq. (22) into

$$-[[A, B]_{q,\tilde{p}}, \tilde{H}]_{q,\tilde{p}} + [A, [B, \tilde{H}]_{q,\tilde{p}}]_{q,\tilde{p}} + [B, -[A, \tilde{H}]_{q,\tilde{p}}]_{q,\tilde{p}} = 0. \tag{23}$$

Substituting Eqs. (20) and (21) into (23), we obtain

$$-[[A, B]_{q,\tilde{p}}, \tilde{H}]_{q,\tilde{p}} + [A, 0]_{q,\tilde{p}} + [B, 0]_{q,\tilde{p}} = 0. \tag{24}$$

It follows from definition (9) that

$$[A, 0]_{q,\tilde{p}} = [B, 0]_{q,\tilde{p}} = 0; \tag{25}$$

hence, Eq. (24) becomes

$$[[A, B]_{q,\tilde{p}}, \tilde{H}]_{q,\tilde{p}} = 0. \tag{26}$$

Making use of the definition for $[A, B]_{q,\tilde{p}}$ (see Eq. 9), we are able to conclude that $[A, B]_{q,\tilde{p}}$ is a function of $q$ and $\tilde{p}$. Consequently,

$$\frac{\partial}{\partial t}[A, B]_{q,\tilde{p}} = 0. \tag{27}$$

Notice that, if $w = [A, B]_{q,\tilde{p}}$, identity 1 furnishes

$$\frac{\mathrm{d}}{\mathrm{d}t}[A, B]_{q,\tilde{p}} = [[A, B]_{q,\tilde{p}}, \tilde{H}]_{q,\tilde{p}} + \frac{\partial}{\partial t}[A, B]_{q,\tilde{p}}, \tag{28}$$

which, owing to Eqs. (26) and (27), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}[A, B]_{q,\tilde{p}} = 0 \quad \Rightarrow \quad [A, B]_{q,\tilde{p}} = \text{const}. \tag{29}$$

The proof is concluded. □

## 4 Application

Now we have plenty of tools to deal with our foremost motivation—videlicet, evaluating the relationship between the conservation laws 1 and 2, which are at our disposal in the domain of position-dependent mass problems. Namely, we are able to apply Poisson's theorem of the preceding Section to the conservation laws 1 and 2, i.e., $\tilde{H}$ and $I$—which signifies to calculate $[I, \tilde{H}]_{q,\tilde{p}}$. Thus, inserting Eqs. (6) and (8) in the definition for $[I, \tilde{H}]_{q,\tilde{p}}$ (see Eq. 9), we produce the expression

$$\begin{aligned} [I, \tilde{H}]_{q,\tilde{p}} = f(q)m(q)^{4\alpha}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\Bigg\{\tilde{p}\left[-(1+2\alpha)\left(\frac{(\tilde{E}-\tilde{V}(q))V(q)}{m(q)^{2\alpha+1}(\tilde{E}-\tilde{V}(q))+V(q)}\right)\right] \\ + \tilde{p}^3\left[-\left(\frac{1}{2}+\alpha\right)\left(\frac{m(q)^{4\alpha+1}(\tilde{E}-\tilde{V}(q))}{m(q)^{2\alpha+1}(\tilde{E}-\tilde{V}(q))+V(q)}\right)+\left(\frac{1}{2}+\alpha\right)m(q)^{2\alpha}\right]\Bigg\}. \end{aligned} \tag{30}$$

As can be observed, $[I, \tilde{H}]_{q,\tilde{p}}$ does not equal a constant—at least not for indiscriminate choices of the terms appearing in the right-hand side of Eq. (30). The crux of the matter is that, for $q$ and $\tilde{p}$ satisfying the canonical equations (4) and (5), $[I, \tilde{H}]_{q,\tilde{p}}$ truly gives rise to a constant. To wit, by using the canonical momentum definition (see Eq. 3) and the conservation law 1 in the form of $\frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \tilde{V}(q) = \tilde{E}$, we find

$$[I, \tilde{H}]_{q,\tilde{p}} = 0. \tag{31}$$

This confirms the fact that Eq. (31) holds true particularly for $q$ and $\tilde{p}$ satisfying the canonical equations (4) and (5)—which is the expectation given by the Poisson's theorem for the dynamics of a position-dependent mass particle.

## 5 Conclusions

We have established the Poisson brackets formulation for the dynamics of a position-dependent mass particle. The development of it was guided through the agency of mathematical arguments, and starting from the foundations laid by the inverse problem of Lagrangian mechanics. This was organized into a sequence of original results, which were demonstrated to specially suit the dynamics of a position-dependent mass particle, namely:

1. The Poisson brackets definition (see Eq. 9);
2. The Jacobi's identity (see Eq. 13);
3. The canonical equations expressed by means of such Poisson brackets (see Eqs. 14 and 15);
4. The Poisson's theorem (see Theorem (Poisson));
5. The application of Poisson's theorem to the conservation laws 1 and 2, i.e., $\tilde{H}$ and $I$. To wit, we have proved that $[I, \tilde{H}]_{q,\tilde{p}}$ equals a constant particularly for $q$ and $\tilde{p}$ satisfying the canonical equations $\dot{q} = \partial\tilde{H}/\partial\tilde{p}$ and $\dot{\tilde{p}} = -\partial\tilde{H}/\partial q$ (see the Section on Application).

Our closing observation is that the Poisson brackets formulation herein proved is harmonious with the classical mathematical frame of analytical mechanics (see [11–16]). This notable aspect was achieved via considering the inverse problem of Lagrangian mechanics.

Very humbly, we expect to have revealed to the readers an original contribution to the important research field of variable-mass systems mechanics.

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