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ORIGINAL PAPER

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Frictional contact between an elastic strip and a semi-infinite punch with rounded edge

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Abstract This paper concerns the contact problem for an elastic strip with rigidly fixed bottom line. The upper strip line is pressed by the semi-infinite punch with rounded edge under uniformly distributed normal and tangential loads. The friction forces are taken into account in the contact area. The exact analytic solution is obtained by using the Wiener–Hopf method. A factorization of the functional equation coefficient is performed in the form of infinite products. We have found the distributions of the contact stress and of the tangential and normal stresses on the bottom strip line. Moreover, for the stress-free part of the upper strip line the normal displacement is calculated. The stress distribution inside the strip is derived in quadratures. Contours of principal shear stress are built, and the location of its maximum value is established in dependence on the rounding parameter of the punch edge and the friction coefficient.

1 Introduction

To solve the contact problems of the theory of elasticity for a strip, the asymptotic methods [1–3] and the method of orthogonal polynomials [4,5] are applied in the case of finite punches, while one can use the Wiener–Hopf technique [6] if the contact area is infinite. Approximate asymptotic solutions are obtained for a narrow or a wide strip, compared to the punch size, with no friction between a punch and a strip [1,2,7–9]. General schemes to solve the contact problems (without numerical results) are proposed in works [3,10], where asymptotic methods have been employed, and in works [4,5] with making use of the method of orthogonal polynomials. These contact problems concern the cases of complete adhesion or slip with the friction between a punch and a strip. The solution to the problem of frictional contact between an elastic strip and the punch with both the straight horizontal basis and of a parabolic shape is considered in [11]. Moreover, in this work the asymptotic solutions are derived, when a strip width exceeds the length of the contact area.

The contact interaction between the semi-infinite punch, with the straight horizontal basis, and an elastic strip was discussed in [7,9,12–14], where friction in the contact area has not been taken into account, while with it is given in [15]. Applying the Wiener–Hopf method, the exact solutions of these problems were found. Making use of the generalized Wiener–Hopf method, the problems for a finite punch and for both smooth and sliding contact were reduced to the infinite systems of algebraic equations [16]. The case of the adhesive and slip contact between the infinite/finite straight basis punch and an elastic strip is considered in [17,18].

In the presented manuscript, we have derived the analytic solution to the contact problem between the semi-infinite, rounded punch and a strip. The normal and shear stresses satisfy the Amonton's (Coulomb) law of sliding friction. The influence of the rounded edges of a punch with straight basis on the contact stress distribution in an elastic half-space was investigated in [19,20].

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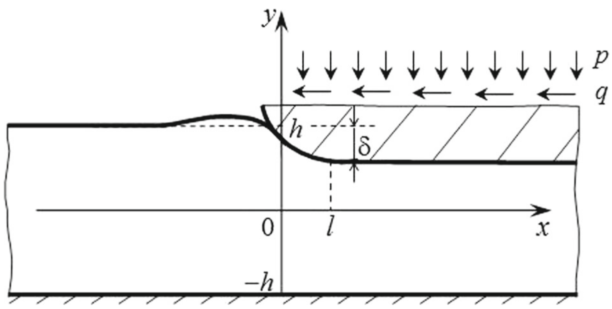

Fig. 1 Interaction between a punch and an elastic strip

2 Statement of the problem

Let us discuss the plane deformation of an elastic strip $-\infty < x < \infty,\ -h \le y \le h$ with a width $2h$ (Fig. 1). The bottom strip line, $y = -h,\ -\infty < x < \infty$, is considered to be rigidly fixed, while the semi-infinite punch is pressed by the uniformly distributed normal $p\ (p > 0)$ and tangential q loads into the upper strip line $y = h$ at the interval $0 \le x < \infty$. The remaining part, $-\infty < x < 0$, of the upper strip line, $y = h$, is free of loading. The punch basis is considered to be straight at the interval $l < x < \infty$ and rounded with a radius R for $x \le l$. The curvilinear part of the punch basis, which comes into contact with the elastic strip, has the size l. This size is unknown and needs to be determined. It is assumed that the punch moves uniformly along a strip, so that the complete slip is established between the contact surfaces. The normal and shear stresses satisfy the Amonton's (Coulomb) law of sliding friction in the contact area, which corresponds to the interval $0 \le x < \infty$ of the upper strip line $y = h$. The loading intensities p and q are connected, namely $q = \mu_0 p$, where $|\mu_0|$ is a friction coefficient. Herewith, the magnitude μ_0 gets positive values ($\mu_0 > 0$), if the punch moves to the left ($q > 0$), while it takes negative values ($\mu_0 < 0$), when the punch moves to the right ($q < 0$). In the presented work, we consider the quasi-static setting, namely we neglect the dynamic effects and assume that a punch velocity is small in comparison with the propagation velocities of elastic waves in the strip.

As we discussed above, the boundary conditions look as follows:

$$\begin{aligned} &u_y\big|_{y=h} = f_0(x)H(l-x) - \delta, \quad \tau_{xy}\big|_{y=h} = \mu_0\, \sigma_y\big|_{y=h}, \quad 0 < x < \infty, \\ &\sigma_y\big|_{y=h} = 0, \quad \tau_{xy}\big|_{y=h} = 0, \quad -\infty < x < 0, \\ &u_x\big|_{y=-h} = 0, \quad u_y\big|_{y=-h} = 0, \quad -\infty < x < \infty, \end{aligned} \tag{1}$$

where $y = f_0(x) \equiv \frac{1}{2R}(x-l)^2,\ x \le l$ is an equation of the rounded punch edge, $H(x)$ is the Heaviside function, and δ is a strip subsidence under the straight part of the punch.

At infinity, under the punch the stress–strain state is set such that it is unchanged along the strip, and thus,

$$\sigma_y = -p, \quad \tau_{xy} = -q, \quad \varepsilon_x = 0, \quad \varepsilon_y = \frac{\partial u_y}{\partial y} = -\frac{\delta}{2h}, \quad x \to \infty. \tag{2}$$

From Hooke's law, one can determine the strip subsidence

$$\delta = \frac{1-2\nu}{1-\nu}\frac{ph}{G}, \tag{3}$$

with ν is the Poisson's ratio, G is the shear modulus, and

$$\sigma_x = -\frac{\nu}{1-\nu}\,p, \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} = -\frac{q}{G}, \quad x \to \infty. \tag{4}$$

By integrating the last equalities in (4), (2), with help of the fifth and sixth conditions in (1), the following result is found:

$$u_x = -\frac{q}{G}(y+h), \quad u_y = -\frac{1-2\nu}{1-\nu}\frac{p}{2G}(y+h), \quad x \to \infty. \tag{5}$$

3 Integral equation

We will introduce the unknown function of the normal contact stress

$$\sigma(x) = \frac{1}{2G}\,\sigma_y\big|_{y=h}\,, \quad 0 < x < \infty, \quad \sigma(x) = 0, \quad -\infty < x \le 0 \tag{6}$$

and consider the basic mixed boundary value problem for a strip with boundary conditions:

$$\frac{1}{2G}\,\sigma_y\big|_{y=h} = \sigma(x), \quad \frac{1}{2G}\,\tau_{xy}\big|_{y=h} = \mu_0\sigma(x), \quad u_x\big|_{y=-h} = 0, \quad u_y\big|_{y=-h} = 0. \tag{7}$$

One can express the general solution of the equilibrium equations for the strip through the Fourier integrals in the following form [21]:

$$\begin{aligned}
u_x &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} u(\mu, y)e^{-i\mu x}\mathrm{d}\mu, \quad u_y = \int_{-\infty}^{\infty} v(\mu, y)e^{-i\mu x}\mathrm{d}\mu,\\
\frac{1}{2G}\sigma_x &= \int_{-\infty}^{\infty} \sigma_1(\mu, y)e^{-i\mu x}\mathrm{d}\mu, \quad \frac{1}{2G}\sigma_y = \int_{-\infty}^{\infty} \sigma_2(\mu, y)e^{-i\mu x}\mathrm{d}\mu,\\
\frac{1}{2G}\tau_{xy} &= \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \tau(\mu, y)e^{-i\mu x}\mathrm{d}\mu,\\
\mu^2 u(\mu, y) &= \mu A(\mu)\cosh(\mu y) + \mu B(\mu)\sinh(\mu y) + C(\mu)\Big[(3-4\nu)\cosh(\mu y) + \mu y\sinh(\mu y)\Big]\\
&\quad - D(\mu)\Big[(3-4\nu)\sinh(\mu y) + \mu y\cosh(\mu y)\Big],\\
v(\mu, y) &= A(\mu)\sinh(\mu y) + B(\mu)\cosh(\mu y) + C(\mu)y\cosh(\mu y) - D(\mu)y\sinh(\mu y),\\
\sigma_1(\mu, y) &= -\mu A(\mu)\cosh(\mu y) - \mu B(\mu)\sinh(\mu y) - C(\mu)\Big[(3-2\nu)\cosh(\mu y) + \mu y\sinh(\mu y)\Big]\\
&\quad + D(\mu)\Big[(3-2\nu)\sinh(\mu y) + \mu y\cosh(\mu y)\Big],\\
\sigma_2(\mu, y) &= \mu A(\mu)\cosh(\mu y) + \mu B(\mu)\sinh(\mu y) + C(\mu)\Big[(1-2\nu)\cosh(\mu y) + \mu y\sinh(\mu y)\Big]\\
&\quad - D(\mu)\Big[(1-2\nu)\sinh(\mu y) + \mu y\cosh(\mu y)\Big],\\
\mu\tau(\mu, y) &= \mu A(\mu)\sinh(\mu y) + \mu B(\mu)\cosh(\mu y) + C(\mu)\Big[2(1-\nu)\sinh(\mu y) + \mu y\cosh(\mu y)\Big]\\
&\quad - D(\mu)\Big[2(1-\nu)\cosh(\mu y) + \mu y\sinh(\mu y)\Big],
\end{aligned} \tag{8}$$

where $A(\mu)$, $B(\mu)$, $C(\mu)$, $D(\mu)$ are arbitrary functions.

Applying the boundary conditions (7) to the solution (8), we derive the system of linear algebraic equations

$$\sigma_2(\mu, h) = \tilde{\sigma}(\mu), \quad \mu\tau(\mu, h) = i\mu_0\tilde{\sigma}(\mu), \quad u(\mu, -h) = 0, \quad v(\mu, -h) = 0 \tag{9}$$

for the unknown quantities $A(\mu)$, $B(\mu)$, $C(\mu)$, $D(\mu)$, where $\tilde{\sigma}(\mu)$ denotes the Fourier transform of the function $\sigma(x)$:

$$\tilde{\sigma}(\mu) = \frac{1}{2\pi}\int_0^{\infty} \sigma(r)e^{i\mu r}\mathrm{d}r. \tag{10}$$

By solving the system of equations (9), it is found that

$$\begin{aligned}
\mu A(\mu)\Delta(2\mu h) &= \Big\{2(1-\nu)(3-4\nu)\cosh(3\mu h) + \Big[(2\mu h)^2 + 2(1-\nu)(3-4\nu)\Big]\cosh(\mu h)\\
&\quad + 2\mu h\Big[(3-4\nu)(\cosh(2\mu h)+1) + 2(1-\nu)\Big]\sinh(\mu h)\Big\}\tilde{\sigma}(\mu)\\
&\quad - i\mu_0\Big\{(1-2\nu)(3-4\nu)\sinh(3\mu h) + \Big[(2\mu h)^2 + (1-2\nu)(3-4\nu)\Big]\sinh(\mu h)\\
&\quad + 2\mu h\Big[(3-4\nu)(\cosh(2\mu h)+1) - 2(1-\nu)\Big]\cosh(\mu h)\Big\}\tilde{\sigma}(\mu),\\
\mu B(\mu)\Delta(2\mu h) &= \Big\{2(1-\nu)(3-4\nu)\sinh(3\mu h) - \Big[(2\mu h)^2 + 2(1-\nu)(3-4\nu)\Big]\sinh(\mu h)\\
&\quad + 2\mu h\Big[(3-4\nu)(\cosh(2\mu h)-1) - 2(1-\nu)\Big]\cosh(\mu h)\Big\}\tilde{\sigma}(\mu)\\
&\quad - i\mu_0\Big\{(1-2\nu)(3-4\nu)\cosh(3\mu h) - \Big[(2\mu h)^2 + (1-2\nu)(3-4\nu)\Big]\cosh(\mu h)\\
&\quad + 2\mu h\Big[(3-4\nu)(\cosh(2\mu h)-1) + 2(1-\nu)\Big]\sinh(\mu h)\Big\}\tilde{\sigma}(\mu),\\
C(\mu)\Delta(2\mu h) &= -\Big[(3-4\nu)\cosh(3\mu h) + \cosh(\mu h) + 4\mu h\sinh(\mu h)\Big]\tilde{\sigma}(\mu)\\
&\quad + i\mu_0\Big[(3-4\nu)\sinh(3\mu h) - \sinh(\mu h) + 4\mu h\cosh(\mu h)\Big]\tilde{\sigma}(\mu),\\
D(\mu)\Delta(2\mu h) &= \Big[(3-4\nu)\sinh(3\mu h) - \sinh(\mu h) - 4\mu h\cosh(\mu h)\Big]\tilde{\sigma}(\mu)\\
&\quad - i\mu_0\Big[(3-4\nu)\cosh(3\mu h) + \cosh(\mu h) - 4\mu h\sinh(\mu h)\Big]\tilde{\sigma}(\mu),\\
\Delta(2\mu h) &= (3-4\nu)(\cosh(4\mu h)-1) + 2(2\mu h)^2 + 8(1-\nu)^2.
\end{aligned} \tag{11}$$

The solutions (8), (11) of the basic mixed boundary value problem (7) satisfy all the boundary conditions (1) except the first one, and it will be used as the solution presentation of the considered contact problem. The normal displacement of the upper strip line can be derived from Eqs. (8), (11), specifically:

$$u_y\big|_{y=h} = 2h\int_{-\infty}^{\infty} K(2\mu h)\tilde{\sigma}(\mu)e^{-i\mu x}\mathrm{d}\mu, \quad K(2\mu h) = \frac{\lambda(2\mu h)}{2\mu h\Delta(2\mu h)},$$

$$\lambda(2\mu h) = 2(1-\nu)\Big[(3-4\nu)\sinh(4\mu h) - 4\mu h\Big] - i\mu_0\Big[(3-4\nu)(1-2\nu)\sinh^2(2\mu h) - (2\mu h)^2\Big]. \tag{12}$$

Substituting the expression for the normal displacement (12) into the first boundary condition (1), one can obtain the following integral equation:

$$2h\int_{-\infty}^{\infty} K(2\mu h)\tilde{\sigma}(\mu)e^{-i\mu x}\mathrm{d}\mu = f_0(x)H(l-x) - \delta, \quad 0 < x < \infty. \tag{13}$$

Based on the asymptotic behavior, namely

$$\sigma(x) \sim -\frac{p}{2G}, \quad x \to \infty, \tag{14}$$

the unknown function is presented as follows:

$$\sigma(x) = -\frac{p}{2G} + \sigma_*(x), \quad \sigma_*(\infty) = 0. \tag{15}$$

This means that instead of Eq. (10), we have

$$\tilde{\sigma}(\mu) = -\frac{p}{4G}\left(\delta(\mu) + \frac{i}{\pi\mu}\right) + \frac{1}{2\pi}\int_0^{\infty}\sigma_*(r)e^{i\mu r}\mathrm{d}r, \tag{16}$$

with the Dirac delta function $\delta(\mu)$.

Let us now apply the following transformation of variables:

$$x = 2h\xi, \quad r = 2h\eta, \quad \tau = 2\mu h, \quad a = l/(2h) \tag{17}$$

and introduce the new unknown function

$$\varphi(\xi) = -\sqrt{2\pi}\frac{2h}{\delta}\sigma_*(2h\xi), \quad 0 < \xi < \infty. \tag{18}$$

As a result, Eq. (13) is transformed to the integral equation at the semi-infinite interval with the difference kernel:

$$\begin{aligned}
&\int_0^\infty k(\xi-\eta)\varphi(\eta)\mathrm{d}\eta = f(\xi), \quad 0 < \xi < \infty,\\
&k(\xi-\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\tau)e^{-i\tau(\xi-\eta)}\mathrm{d}\tau, \quad K(\tau) = \frac{\lambda(\tau)}{\tau\Delta(\tau)},\\
&f(\xi) = -\frac{\sqrt{2\pi}}{2K(0)}\left(4\alpha(\xi-a)^2H(a-\xi) - K(0) - \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{K(\tau)}{\tau}e^{-i\tau\xi}\mathrm{d}\tau\right),\\
&K(0) = \frac{1-2\nu}{1-\nu}, \quad \alpha = \frac{hG}{pR}.
\end{aligned} \tag{19}$$

Making use of the residue theory for the last integral in equalities (19), the right-hand side of the integral equation (19) is rewritten as

$$f(\xi) = -\frac{\sqrt{2\pi}}{K(0)}\left(2\alpha(\xi-a)^2H(a-\xi) - \sum_{k=1}^{\infty}\frac{\lambda(is_k)}{s_k^2\Delta'(is_k)}e^{-s_k\xi}\right), \quad 0 < \xi < \infty, \tag{20}$$

where $s_k,\ k = 1, 2, \ldots$, are the roots of the equation $\Delta(is) = 0$ for the half-plane $\Re s > 0$.

4 Solution of the integral equation

Let us find the exact analytic solution of the integral equation (19) by using the Wiener–Hopf method [6].

To this end, we extend the integral equation (19) onto the entire real axis, assuming that $\varphi(\eta) = 0$ when $\eta < 0$ and apply the integral Fourier transformation. Further, we introduce the yet unknown functions of complex variable

$$\begin{aligned}
\Phi^+(z) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \varphi(\xi)e^{iz\xi}\mathrm{d}\xi,\\
\Phi^-(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{iz\xi}\mathrm{d}\xi\int_0^\infty k(\xi-\eta)\varphi(\eta)\mathrm{d}\eta,
\end{aligned} \tag{21}$$

which are analytical in the half-planes $\Im z > c^+$ and $\Im z < c^-$ ($c^+ < 0,\ c^- > 0$), respectively.

Applying the convolution theorem for the integral Fourier transformation, the integral equation (19) is reduced to the functional equation

$$K(z)\Phi^+(z) - \Phi^-(z) = F^+(z), \quad c^+ < \Im z < c^-. \tag{22}$$

The right-hand side of equation (22)

$$F^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi) e^{iz\xi} d\xi$$
$$= \frac{1}{K(0)} \left\{ 2\alpha \left[\frac{a^2}{iz} + \frac{2a}{(iz)^2} + \frac{2}{(iz)^3}(1 - e^{iza}) \right] + \sum_{k=1}^\infty \frac{\lambda(is_k)}{s_k^2 \Delta'(is_k)} \frac{1}{s_k - iz} \right\} \tag{23}$$

is an analytical function in the half-plane $\Im z > c^+$. One can factorize the coefficient $K(z)$ in the functional equation (22) as follows:

$$K(z) = K(0) K^+(z) K^-(z), \tag{24}$$

where the functions $K^+(z),\ K^-(z)$ are chosen to be analytical and do not have zero values in the half-planes $\Im z > c^+,\ \Im z < c^-$, respectively. The factorization (24) is performed in the form of infinite products [6]

$$K^+(z) = \prod_{k=1}^\infty \left(1 + \frac{iz}{\zeta_k'}\right) \left(1 - \frac{iz}{s_k}\right)^{-1}, \quad K^-(z) = \prod_{k=1}^\infty \left(1 + \frac{iz}{\zeta_k}\right) \left(1 + \frac{iz}{s_k}\right)^{-1}, \tag{25}$$

where ζ_k and $\zeta_k',\ k = 1, 2, \ldots$, are the roots of the equation $\lambda(is) = 0$, lying in the half-planes, $\Re s > 0$, and $\Re s < 0$, respectively.

Let us divide Eq. (22) by $K^-(z)$ and represent its right-hand side as a difference of two functions

$$\frac{F^+(z)}{K^-(z)} = f^+(z) - f^-(z), \quad c^+ < \Im z < c^-, \tag{26}$$

which are analytical in the half-planes $\Im z > c^+$ and $\Im z < c^-$, respectively. One can find an explicit form of the function $f^+(z)$ by using the Cauchy type integral, specifically:

$$f^+(z) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{F^+(\zeta) d\zeta}{K^-(\zeta)(\zeta - z)} = f_1^+(z) + f_2^+(z), \quad \Im z > 0,$$
$$f_1^+(z) = \frac{\alpha}{\pi i} \int_{-\infty}^\infty \frac{\zeta \Delta(\zeta) K^+(\zeta)}{\lambda(\zeta)(\zeta - z)} \left[\frac{a^2}{i\zeta} + \frac{2a}{(i\zeta)^2} + \frac{2}{(i\zeta)^3}(1 - e^{i\zeta a}) \right] d\zeta,$$
$$f_2^+(z) = \frac{1}{K(0)} \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{d\zeta}{K^-(\zeta)(\zeta - z)} \sum_{k=1}^\infty \frac{\lambda(is_k)}{s_k^2 \Delta'(is_k)} \frac{1}{s_k - i\zeta}. \tag{27}$$

To find out $f_1^+(z)$, we calculate residues of the integrand in $f_1^+(z)$ at simple poles $\zeta = i\zeta_k,\ k = 1, 2, \ldots,\ \zeta = z$ in the half-plane $\Im \zeta > 0$, where $e^{i\zeta a} \to 0$ when $|\zeta| \to \infty$. This results in:

$$f_1^+(z) = -2\alpha \left\{ \sum_{k=1}^\infty \frac{\zeta_k \Delta(i\zeta_k) K^+(i\zeta_k)}{\lambda'(i\zeta_k)(\zeta_k + iz)} \left[\frac{a^2}{\zeta_k} - \frac{2a}{\zeta_k^2} + \frac{2}{\zeta_k^3}(1 - e^{-\zeta_k a}) \right] \right.$$
$$\left. - \frac{1}{K(0) K^-(z)} \left[\frac{a^2}{iz} + \frac{2a}{(iz)^2} + \frac{2}{(iz)^3}(1 - e^{iza}) \right] \right\}.$$

We can calculate the sum of a part of the expression above, which does not have a multiplier $e^{-\zeta_k a}$, by translating the series with use of the residues theory in the half-plane $\Im \tau > c,\ 0 < c < \zeta_1$, into the integral along the line

$\Im\tau = c$. The last integral is calculated through a residue in the single pole $\tau = 0$ in the half-plane $\Im\tau < c$. Hence,

$$K(0)\sum_{k=1}^{\infty}\frac{\zeta_k\Delta(i\zeta_k)K^+(i\zeta_k)}{\lambda'(i\zeta_k)(\zeta_k+iz)}\left(\frac{a^2}{\zeta_k}-\frac{2a}{\zeta_k^2}+\frac{2}{\zeta_k^3}\right)=\frac{1}{K^-(z)}\left(\frac{a^2}{iz}+\frac{2a}{(iz)^2}+\frac{2}{(iz)^3}\right)$$
$$-\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{1}{K^-(\tau)(\tau-z)}\left(\frac{a^2}{i\tau}+\frac{2a}{(i\tau)^2}+\frac{2}{(i\tau)^3}\right)\mathrm{d}\tau$$
$$=\frac{1}{K^-(z)}\left(\frac{a^2}{iz}+\frac{2a}{(iz)^2}+\frac{2}{(iz)^3}\right)-\frac{a_1}{iz}-\frac{a_2}{(iz)^2}-\frac{2}{(iz)^3}. \tag{28}$$

Here we have introduced the following notations:

$$a_1=a^2+2ab_1+2b_2,\quad a_2=2(a+b_1),$$
$$\frac{1}{K^-(z)}=1+b_1iz+b_2(iz)^2+O((iz)^3),\quad z\to 0,$$
$$b_1=\sum_{k=1}^{\infty}t_k,\quad b_2=\sum_{k=1}^{\infty}t_k\left(\sum_{m=k+1}^{\infty}t_m-\frac{1}{\zeta_k}\right),\quad t_k=\frac{1}{s_k}-\frac{1}{\zeta_k}.$$

The integral for the function $f_2^+(z)$ in (27) is transformed to the series, using the residues in the points $\zeta=-is_k,\ k=1,2,\ldots,$ in the half-plane $\Im\zeta<0$,

$$f_2^+(z)=\frac{1}{K(0)}\sum_{k=1}^{\infty}\frac{\lambda(is_k)}{s_k^2\Delta'(is_k)K^-(-is_k)}\frac{1}{s_k-iz},$$

which is summed similar to (28). It is found that

$$f_2^+(z)=\frac{K^+(z)-1}{iz}.$$

Having the explicit form of the functions $f_1^+(z)$ and $f_2^+(z)$, after straightforward calculation, one can derive the function $f^+(z)$, which is then given by

$$f^+(z)=2\alpha\left[2\sum_{k=1}^{\infty}\frac{\Delta(i\zeta_k)K^+(i\zeta_k)}{\zeta_k^2\lambda'(i\zeta_k)(\zeta_k+iz)}e^{-\zeta_k a}-\frac{1}{K(0)}\left(\frac{2}{K^-(z)(iz)^3}e^{iza}\right.\right.$$
$$\left.\left.-\frac{a_1}{iz}-\frac{a_2}{(iz)^2}-\frac{2}{(iz)^3}\right)\right]+\frac{K^+(z)-1}{iz}. \tag{29}$$

According to the factorization (24) together with the representation (26), we can rewrite Eq. (22) as follows:

$$K(0)K^+(z)\Phi^+(z)-f^+(z)=\frac{\Phi^-(z)}{K^-(z)}-f^-(z),\quad c^+<\Im z<c^-. \tag{30}$$

The right-hand and the left-hand sides of Eq. (30) continue each other analytically on the entire complex plane and represent some arbitrary rational function $P(z)$:

$$K(0)K^+(z)\Phi^+(z)-f^+(z)=P(z),\quad \frac{\Phi^-(z)}{K^-(z)}-f^-(z)=P(z). \tag{31}$$

In order to determine this function, we will find its behavior at infinity, based on the first equality in (31).

Using the asymptotic behavior of the roots of equations $\lambda(is) = 0,\ \Delta(is) = 0$ from the first and the second quadrants

$$\zeta_n^{(0)} = \pi\left(n + \frac{1}{4}\right) + i\ln(qn) + o(1), \quad \zeta_n'^{(0)} = -\pi\left(n - \frac{1}{4}\right) + i\ln(qn) + o(1),$$
$$s_n^{(0)} = \pi n + i\ln\left(\frac{2\pi n}{\sqrt{3-4\nu}}\right) + o(1), \quad n \to \infty, \quad q = 2\pi\sqrt{\frac{\mu_0}{(3-4\nu)[2(1-\nu) + i\mu_0(1-2\nu)]}}, \tag{32}$$

the asymptotic estimation [22] is obtained for the first infinite product (25):

$$K^+(z) = O(z^{-1/2+\gamma}), \quad |z| \to \infty; \quad \gamma = \frac{1}{\pi}\arctan\frac{\mu_0(1-2\nu)}{2(1-\nu)}. \tag{33}$$

On the other hand, in view of the estimations $\Phi^+(z) = o(1),\ f^+(z) = O(z^{-1}),\ |z| \to \infty$, it is found the asymptotic behavior of the function $P(z)$ from the first equality in (31), specifically: $P(z) = o(z^{-1/2+\gamma}),\ |z| \to \infty$. From the discussion above, we conclude that $P(z) \equiv 0$. Hence, by making use of the relations (31), the solution of the functional equation (22) is derived:

$$\Phi^+(z) = \frac{f^+(z)}{K(0)K^+(z)}, \quad \Phi^-(z) = f^-(z)K^-(z). \tag{34}$$

From the asymptotic behavior of the solution,

$$\Phi^+(z) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \varphi(\xi)e^{iz\xi}\mathrm{d}\xi = O(z^{-\frac{1}{2}-\gamma}), \quad |z| \to \infty, \tag{35}$$

according to the Watson lemma [23]

$$\int_0^\infty \xi^{\rho-1}\varphi_1(\xi)e^{iz\xi}\mathrm{d}\xi \sim \Gamma(\rho)\varphi_1(0)(-iz)^{-\rho}, \quad |z| \to \infty, \quad \rho > 0, \quad 0 < \arg z < \pi, \tag{36}$$

with $\varphi(\xi) = \xi^{\rho-1}\varphi_1(\xi),\ \rho = \frac{1}{2} + \gamma$, it follows first that

$$\varphi(\xi) = O(\xi^{-\frac{1}{2}+\gamma}), \quad \xi \to +0. \tag{37}$$

Second, normal contact stress $\sigma_y\big|_{y=h}$, which depends on $\varphi(\xi)$ (see expressions (6), (15), (18)), diverges on the edge of the contact area:

$$\sigma_y\big|_{y=h} = O(x^{-\frac{1}{2}+\gamma}), \quad x \to +0. \tag{38}$$

In view of the stress limitation at the point $x = 0,\ y = h$, the following condition should be imposed:

$$\lim_{|z|\to\infty} zf^+(z) = 0, \tag{39}$$

which can be rewritten using (29):

$$K(0) - 2\alpha\left(2K(0)\sum_{k=1}^\infty \frac{\Delta(i\zeta_k)K^+(i\zeta_k)}{\zeta_k^2\lambda'(i\zeta_k)}e^{-\zeta_k a} + a_1\right) = 0. \tag{40}$$

Equation (40) connects the relative size, $a = l/(2h)$, of the curvilinear part of the punch basis, which comes into a contact with an elastic strip, and the parameter $\alpha = hG/(pR)$ of the considered problem. In view of the condition (40), an indicator in asymptotic formula (35) is decreased by 1, while it is increased by 1 in formulas (37), (38).

Applying the inverse Fourier transformation to the first equality in (21) and taking the formulas in (34) into account, the solution of the integral equation (19) is obtained:

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \Phi^+(\tau)e^{-i\xi\tau}\mathrm{d}\tau = \frac{1}{K(0)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{f^+(z)}{K^+(z)}e^{-i\xi\tau}\mathrm{d}\tau. \tag{41}$$

5 Determination of stress and displacement

The contact stress is found by making use of formulas (6), (41) and expressions (15), (18). Transforming the integral in Eq. (41), according to the residue theory, one can obtain the following expression (with $\xi > 0$):

$$\frac{1}{p}\sigma_y\bigg|_{y=h} = -1 - \sum_{k=1}^{\infty} \frac{\zeta_k' \Delta(i\zeta_k') K^-(i\zeta_k') e^{\zeta_k'\xi}}{\lambda'(i\zeta_k')} \Bigg\{ 4\alpha \Bigg[\frac{H(\xi - a) e^{-\zeta_k' a}}{\zeta_k'^3 K^-(i\zeta_k')} - \left(\frac{1}{\zeta_k'^3} - \frac{a_2}{2\zeta_k'^2} + \frac{a_1}{2\zeta_k'} \right)$$
$$+ K(0) \sum_{n=1}^{\infty} \frac{\Delta(i\zeta_n) K^+(i\zeta_n) e^{-\zeta_n a}}{\zeta_n^2 \lambda'(i\zeta_n)(\zeta_n - \zeta_k')} \Bigg] + \frac{K(0)}{\zeta_k'} \Bigg\} - 4\alpha \Bigg\{ \frac{1}{K(0)} \Bigg[\frac{\nu(1-4\nu)}{3(1-\nu)(1-2\nu)}$$
$$- \frac{(\xi - a)^2}{2} + \frac{\mu_0}{2} \frac{1-4\nu}{1-2\nu} \left(\xi - a - \frac{\mu_0}{2} \frac{1-4\nu}{1-2\nu} \right) \Bigg] - \sum_{k=1}^{\infty} \frac{\Delta(i\zeta_k) e^{\zeta_k(\xi - a)}}{\zeta_k^2 \lambda'(i\zeta_k)} \Bigg\} H(a - \xi). \quad (42)$$

Starting with formula (12) and considering Eqs. (16)–(18), the normal displacement of points, situated on the unloaded part of the upper strip line, is transformed into the form

$$u_y\big|_{y=h} = -\frac{\delta}{2} \left[1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\Phi^+(\tau) + \frac{i}{K(0)\tau} \right) K(\tau) e^{-i\tau\xi} \mathrm{d}\tau \right]. \quad (43)$$

Substituting in the last integral the solution $\Phi^+(\tau)$ from the first equality in (34) and performing the transformations, the result is found

$$\frac{1}{\delta} u_y\bigg|_{y=h} = \frac{1}{K(0)} \sum_{k=1}^{\infty} \frac{\lambda(is_k) e^{s_k \xi}}{s_k \Delta'(is_k) K^+(is_k)} \Bigg\{ 4\alpha \Bigg[\sum_{n=1}^{\infty} \frac{\Delta(i\zeta_n) K^+(i\zeta_n) e^{-\zeta_n a}}{\zeta_n^2 \lambda'(i\zeta_n)(s_k - \zeta_n)}$$
$$+ \frac{1}{K(0)} \left(\frac{1}{s_k^3} - \frac{a_2}{2s_k^2} + \frac{a_1}{2s_k} \right) \Bigg] - \frac{1}{s_k} \Bigg\}, \quad \xi < 0. \quad (44)$$

In view of Eqs. (8), (11) the stresses in any point inside the strip are written as following combinations:

$$\frac{1}{2G}(\sigma_x + \sigma_y) = \int_{-\infty}^{\infty} \Big\{ \Big[D(\mu) - C(\mu) \Big] e^{\mu y} - \Big[C(\mu) + D(\mu) \Big] e^{-\mu y} \Big\} e^{-i\mu x} \mathrm{d}\mu,$$
$$\frac{1}{2G}(\sigma_y - \sigma_x + 2i\tau_{xy}) = 2 \int_{-\infty}^{\infty} \Big\{ \mu \Big[A(\mu) + B(\mu) \Big] e^{\mu y} + \Big[2(1-\nu) + \mu y \Big] \Big[C(\mu) - D(\mu) \Big] e^{-\mu y} \Big\} e^{-i\mu x} \mathrm{d}\mu.$$
$$(45)$$

According to Eqs. (11), (16)–(18), (21), (34), it is found that

$$\frac{1}{p}(\sigma_x + \sigma_y) = -\frac{1}{2(1-\nu)} + \frac{1}{\pi} \int_{-\infty}^{\infty} M_1(\tau, \zeta) e^{-i\xi\tau} \mathrm{d}\tau, \quad \frac{1}{p}(\sigma_y - \sigma_x + 2i\tau_{xy}) = -\frac{1}{2} K(0) - i\mu_0$$
$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} M_2(\tau, \zeta) e^{-i\xi\tau} \mathrm{d}\tau, \quad M_j(\tau, \zeta) = \frac{\alpha_j(\tau, \zeta)}{\Delta(\tau) K^+(\tau)} \left(\frac{1}{i\tau} - f_1^+(\tau) \right), \quad j = 1, 2, \quad \zeta = \frac{y}{2h},$$

$$\alpha_1(\tau, \zeta) = (3 - 4\nu) \Big[\cosh\Big((\zeta + 3/2)\tau \Big) - i\mu_0 \sinh\Big((\zeta + 3/2)\tau \Big) \Big]$$
$$+ (1 - 2i\mu_0\tau) \cosh\Big((\zeta - 1/2)\tau \Big) - (2\tau + i\mu_0) \sinh\Big((\zeta - 1/2)\tau \Big),$$
$$\alpha_2(\tau, \zeta) = (3 - 4\nu) \Big[(1 - i\mu_0)(1/2 - \zeta)\tau + i\mu_0 \Big] e^{(\zeta + 3/2)\tau} + \Big\{ (3 - 4\nu)^2 (1 + i\mu_0) - 1 + i\mu_0$$
$$+ \Big[1/2 - \zeta + i\mu_0(\zeta + 3/2) \Big] \tau + 2(1 - i\mu_0)(\zeta + 1/2)\tau^2 \Big\} e^{(\zeta - 1/2)\tau}. \quad (46)$$

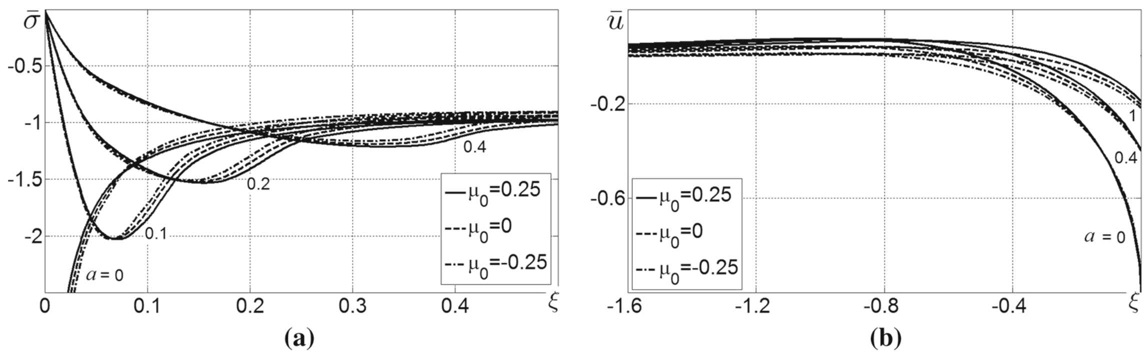

Fig. 2 Distributions of normal contact stress (**a**) and normal displacement (**b**) ($y = h$)

Table 1 Points with the highest values of normal displacement

a	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	
	0.929	0.897	0.868	0.843	0.819	0.799	0.779	0.763	0.748	$\mu_0 = 0.25$
$-\xi$	0.954	0.924	0.897	0.872	0.85	0.831	0.814	0.798	0.785	$\mu_0 = 0$
	0.998	0.969	0.944	0.921	0.902	0.885	0.870	0.858	0.846	$\mu_0 = -0.25$
	0.747	0.743	0.737	0.729	0.720	0.710	0.699	0.688	0.677	$\mu_0 = 0.25$
$\bar u_{\max}$	0.433	0.428	0.423	0.417	0.410	0.403	0.395	0.387	0.379	$\mu_0 = 0$
	0.129	0.126	0.123	0.119	0.115	0.111	0.106	0.102	0.097	$\mu_0 = -0.25$

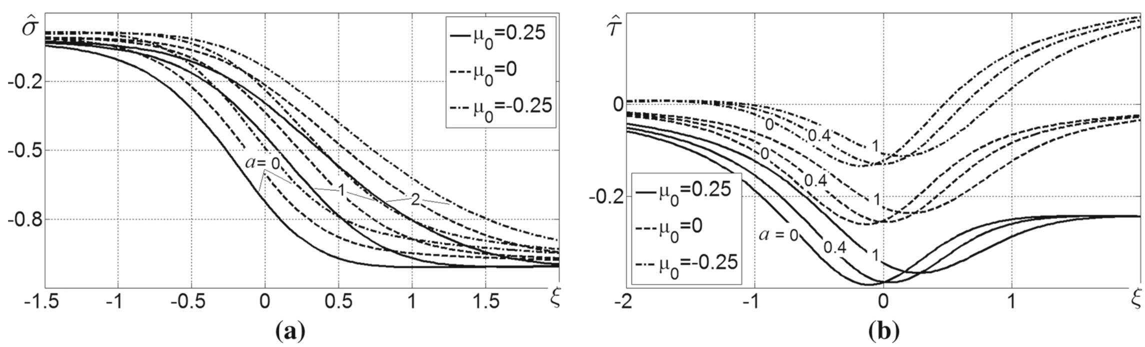

Fig. 3 Distributions of normal (**a**) and shear (**b**) stresses ($y = -h$)

It should be noted that integrals in equalities (46) converge exponentially, when $-h \le y < h$, while they converge slowly with a power velocity for $y = h$. In the last case, it is necessary to transform integrals by the residue theory to the series, as it is made for the contact stress (42). Such the series have an exponential convergence for all points inside the strip, except the single point $x = 0$.

6 Results

For a numerical analysis, in all our calculations we have been using the Poisson's ratio $\nu = 1/3$. We have also considered two types of contacts: (i) the case of absence of friction, $\mu_0 = 0$, and (ii) the case with friction, i.e., the friction coefficient $|\mu_0| = 0.25$. It should be noted that $\mu_0 = 0.25$ if a punch moves to the left ($q > 0$), while $\mu_0 = -0.25$ if the punch moves to the right ($q < 0$). Finally, in these calculations, we are varying the relative size $a = l/(2h)$ of the curvilinear part of the punch basis which comes into contact with an elastic strip.

In Fig. 2a, the distribution of the dimensionless normal contact stress $\bar\sigma = \frac{1}{p}\,\sigma_y\big|_{y=h}$, calculated from the formula (42) ($\xi = x/(2h)$), is shown. In Fig. 2b, the distribution of the dimensionless normal displacement $\bar u = \frac{1}{\delta}\,u_y\big|_{y=h}$ of the unloaded upper strip line, calculated with the help of the formula (44) ($\xi = x/(2h)$), is illustrated. With a decrease in the parameter a, contact stress is approaching its limit distribution at $a = 0$, which corresponds to the punch with a straight basis without rounding [14, 15]. The additional calculations

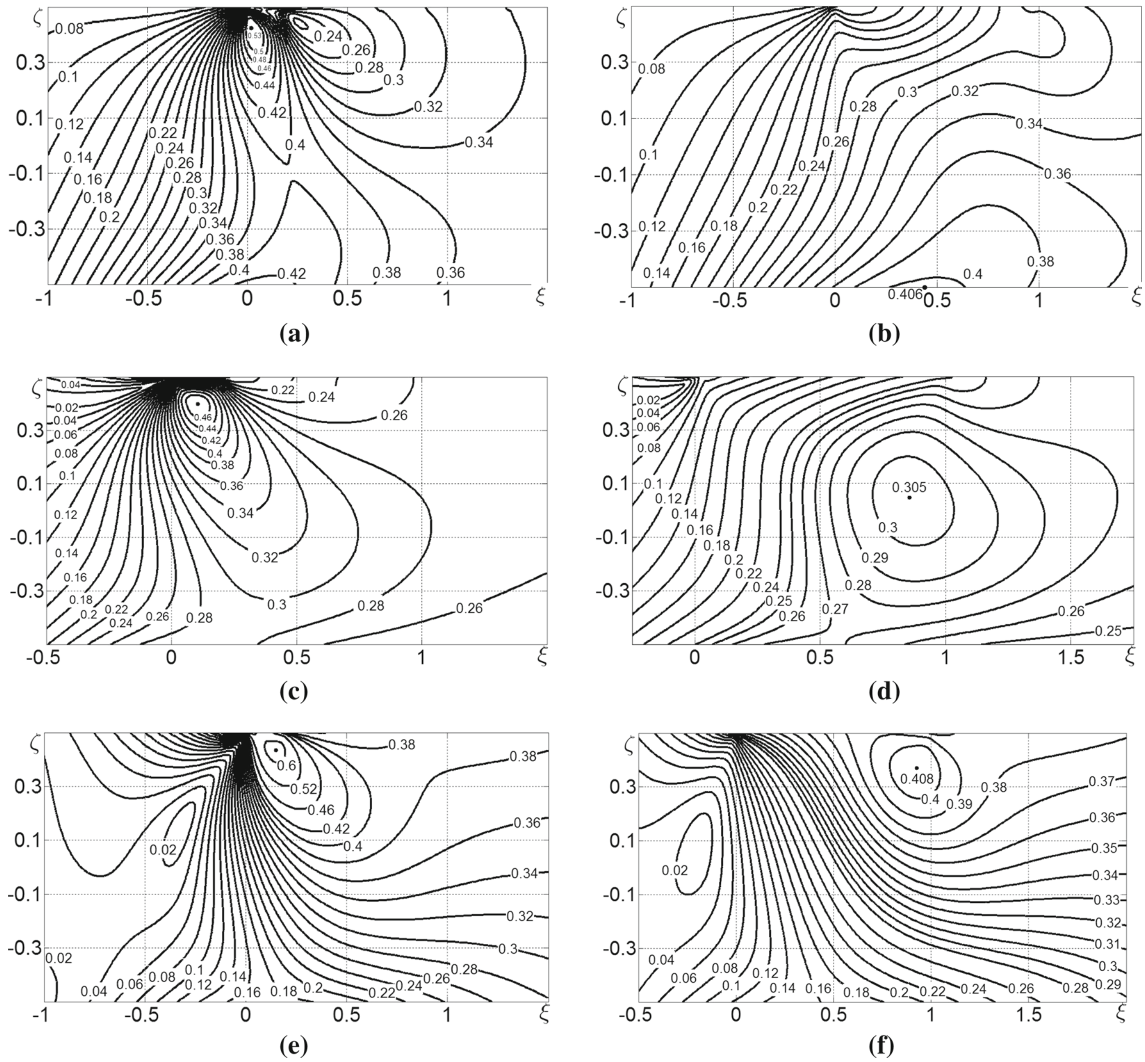

Fig. 4 Contours of principal shear stress (**a** $a = 0.2$, $\mu_0 = 0.25$; **b** $a = 1$, $\mu_0 = 0.25$; **c** $a = 0.2$, $\mu_0 = 0$; **d** $a = 1$, $\mu_0 = 0$; **e** $a = 0.2$, $\mu_0 = -0.25$; **f** $a = 1$, $\mu_0 = -0.25$)

reveal that the contact stress is very similar to the case of the parabolic punch with finite contact area $0 \leq x \leq 2l$ [16], when the parameter a (at $a > 1$) is increasing. It is noticed in Fig. 2b that the deformed upper strip line rises at a certain distance from the edge of the contact area, forming an apex. In Table 1 the maximum values of displacement $\bar{u}_{\max} = 10\frac{1}{\delta}\, u_y\big|_{y=h}$ and the corresponding dimensionless coordinates ξ, which determine the apex height and its position, are presented. The friction significantly affects on the apex height, increasing it if the punch moves to the left and decreasing in the opposite case.

In Fig. 3, we present distributions of the dimensionless normal (**a**) $\hat{\sigma} = \frac{1}{p}\, \sigma_y\big|_{y=-h}$ and shear (**b**) $\hat{\tau} = \frac{1}{p}\, \tau_{xy}\big|_{y=-h}$ stresses along the fixed strip line for different values of the parameter a. The influence of friction on these stress distributions is much higher than in the contact area (Fig. 2a). The normal stress on the fixed strip line is smoothly varying from zero far from the edge of the punch to $-p$ at a certain distance from the edge under the punch. The largest pressure is transmitted on the bottom strip line in the case of moving of an elastic material under the punch (solid lines, $\mu_0 = 0.25$) as well as the smallest one in the opposite case (see dash–dot lines, $\mu_0 = -0.25$). The pressure difference along the fixed strip line between the limit values 0 and p becomes more slow with increasing the parameter of rounding a. It leads to a pressure decline near the punch edge. One can also observe from Fig. 3b that the high shear stress zone occurs on the bottom strip line near the punch edge in the case of smooth contact ($\mu_0 = 0$). It characterizes a material extrusion from under

Table 2 Points with the highest values of principal shear stress

a	0.2	0.4	0.6	0.8	1	1.5	2	2.5	3	
	0.028	0.165	0.252	0.342	0.438	0.717	1.124	1.676	2.244	$\mu_0 = 0.25$
ξ	0.105	0.266	0.464	0.672	0.857	1.310	1.783	2.265	2.750	$\mu_0 = 0$
	0.149	0.329	0.525	0.726	0.927	1.426	1.894	2.174	2.652	$\mu_0 = -0.25$
	0.418	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	−0.5	$\mu_0 = 0.25$
ζ	0.398	0.298	0.192	0.096	0.048	0.034	0.067	0.103	0.127	$\mu_0 = 0$
	0.435	0.393	0.370	0.363	0.370	0.410	0.448	0.5	0.5	$\mu_0 = -0.25$
	0.528	0.424	0.419	0.413	0.406	0.386	0.373	0.364	0.359	$\mu_0 = 0.25$
max $\bar{\tau}_{\max}$	0.479	0.370	0.332	0.315	0.305	0.289	0.279	0.272	0.268	$\mu_0 = 0$
	0.629	0.495	0.445	0.421	0.408	0.393	0.386	0.382	0.378	$\mu_0 = -0.25$
	2.829	0.947	0.487	0.299	0.203	0.098	0.057	0.038	0.027	$\mu_0 = 0.25$
α	2.841	0.940	0.480	0.294	0.199	0.096	0.056	0.037	0.026	$\mu_0 = 0$
	2.864	0.934	0.474	0.289	0.195	0.094	0.055	0.036	0.025	$\mu_0 = -0.25$

the punch. This material extrusion effect also remains, when the friction is present. The shear stress values increase comparing to such the limit ones $q = \mu_0 p$ in the vicinity of the punch edge, when an elastic material moves under the punch ($\mu_0 = 0.25$). And the shear stress changes a sign, when a material moves in opposite direction ($\mu_0 = -0.25$). A presence of rounded edge leads to increasing of the shear stress on the bottom strip line under the punch ($\xi > 0$) and to its decreasing outside the punch ($\xi < 0$).

Finally, in Fig. 4 contours of principal shear stress $\tau_{\max} = \frac{1}{2}|\sigma_y - \sigma_x + 2i\tau_{xy}|$, normalized to the magnitude p, are depicted. To be more specific, the friction is taken into account in the figures a, b ($\mu_0 = 0.25$) and in the figures e, f ($\mu_0 = -0.25$), while it is neglected in the figures c, d ($\mu_0 = 0$). The parameter a is chosen to be equal 0.2 and 1. We present in Table 2 the values of the magnitude max $\bar{\tau}_{\max}$, the dimensionless coordinates ξ, ζ of the corresponding points and the parameter $\alpha = hG/(pR)$ as well. These results are obtained by varying the parameter a and considering the certain case of presence of friction with $|\mu_0| = 0.25$ and the case of friction absence ($|\mu_0 = 0|$) as well. If the parameter a is changing and $|\mu_0| = 0.25$, the highest values of $\bar{\tau}_{\max}$ increase by 20–40% in comparison with those one for the smooth contact. The high value zone of $\bar{\tau}_{\max}$ is located near the punch edge, when the rounding is small and there is no friction. With increase in the rounding, this zone moves gradually to the strip midline ($y = 0$) under a point $x = l,\ y = h$ of punch basis, where the rounded part turns to the straight one. Under the friction influence and for $q > 0$ ($\mu_0 = 0.25$) the point with the highest value of $\bar{\tau}_{\max}$ shifts on the bottom strip line $x = -h,\ \zeta = -0.5$ and situates in front of the curvilinear part of the punch basis between the points $x = 0$ and $x = l$. This can already be observed for a small value of the rounding, $a = 0.4$. In the case $q < 0$ ($\mu_0 = -0.25$), the mentioned point situates near or directly (at $a \leq 2.5$) on the upper strip line slightly to the left from the point $x = l,\ y = h$. So, for real systems (if no friction is present and even when the punch edge rounding is small) a destruction of material (as a result of appearance of a plastic flow) can be expected on the fixed strip line and on the small depth inside the strip (when $q > 0$) or in the contact area, if $q < 0$ and the punch edge rounding is large. Only for the insignificant level of friction and for sharp enough punch edge, the plastic flow might occur in the middle of the strip.

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