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NOTE

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A Riccati-type solution of Euler-Poisson equations of rigid body rotation over the fixed point

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Abstract A new approach is developed here for resolving the Poisson equations in case the components of angular velocity of rigid body rotation can be considered as functions of the time parameter t only. A fundamental solution is presented by the analytical formulae in dependence on two time-dependent, real-valued coefficients. Such coefficients are proved to be the solutions of a mutual system of 2 *Riccati* ordinary differential equations (which has no analytical solution in the general case). All in all, the cases of analytical resolving of Poisson equation are quite rare (according to the cases of exact resolving of the aforementioned system of *Riccati* ODEs). So, the system of Euler–Poisson equations is proved to have analytical solutions (in quadratures) only in classical simplifying cases: (1) *Lagrange's* case or (2) *Kovalevskaya's* case or (3) *Euler's* case or other well-known but particular cases (where the existence of particular solutions depends on the choice of the appropriate initial conditions).

Mathematics Subject Classification 70E40 (integrable cases of motion)

1 Introduction, equations of motion

Euler–Poisson equations, describing the dynamics of rigid body rotation, are known to be one of the famous problems in classical mechanics.

In accordance with [1–3], Euler equations describe the rotation of a rigid body in a frame of reference fixed in the rotating body for the case of rotation over the fixed point as below (*at given initial conditions*):

$$\begin{cases} I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3 = P\,(\gamma_2 c_0 - \gamma_3 b_0)\,, \\ I_2 \frac{d\Omega_2}{dt} + (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1 = P\,(\gamma_3 a_0 - \gamma_1 c_0)\,, \\ I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2 = P\,(\gamma_1 b_0 - \gamma_2 a_0)\,, \end{cases} \tag{1.1}$$

– where $I_i \neq 0$ are the principal moments of inertia ($i = 1, 2, 3$) and Ω_i are the components of the *angular velocity vector* along the proper principal axis; γ_i are the components of the weight of mass P and a_0, b_0, c_0 are the appropriate coordinates of the center of masses in a frame of reference fixed in the rotating body (*in regard to the absolute system of coordinates* X, Y, Z).

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Poisson equations for the components of the weight in a frame of reference fixed in the rotating body (*in regard to the absolute system of coordinates* X, Y, Z) should be presented as below [4–6]:

$$\begin{cases} \frac{d\gamma_1}{dt} = \Omega_3\gamma_2 - \Omega_2\gamma_3, \\ \frac{d\gamma_2}{dt} = \Omega_1\gamma_3 - \Omega_3\gamma_1, \\ \frac{d\gamma_3}{dt} = \Omega_2\gamma_1 - \Omega_1\gamma_2; \end{cases} \tag{1.2}$$

– besides, we should present the invariants (*first integrals of motion*) as below:

$$\begin{cases} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_1 \cdot \Omega_1 \cdot \gamma_1 + I_2 \cdot \Omega_2 \cdot \gamma_2 + I_3 \cdot \Omega_3 \cdot \gamma_3 = \text{const} = C_0, \\ \frac{1}{2}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + P(a_0\gamma_1 + b_0\gamma_2 + c_0\gamma_3) = \text{const} = C_1. \end{cases} \tag{1.3}$$

2 Derivation of the invariants (*first integrals*) of motion

Let us recall how to derive the invariants (1.3). From (1.1), (1.2), we obtain

$$\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right)}{P} = \Omega_1 \cdot (\gamma_2 c_0 - \gamma_3 b_0) + \Omega_2 \cdot (\gamma_3 a_0 - \gamma_1 c_0) + \Omega_3 \cdot (\gamma_1 b_0 - \gamma_2 a_0)$$

$$\Rightarrow -\frac{\frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right)}{P} = a_0 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3) + b_0 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1) + c_0 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2)$$

$$\Rightarrow \frac{d}{dt}\left(I_1\frac{\Omega_1^2}{2} + I_2\frac{\Omega_2^2}{2} + I_3\frac{\Omega_3^2}{2}\right) + a_0 \cdot P \cdot \frac{d\gamma_1}{dt} + b_0 \cdot P \cdot \frac{d\gamma_2}{dt} + c_0 \cdot P \cdot \frac{d\gamma_3}{dt} = 0.$$

So, we have obtained the third integral of (1.3) in [5]. To obtain the second integral of (1.3) in [5], we should multiply each of the equations of (1.1) by γ_i accordingly, but also each of the equations of (1.2) by $(I_i \cdot \Omega_i)$ accordingly, then we should sum them up as below:

$$\begin{cases} \left(I_1 \cdot \gamma_1 \cdot \frac{d\Omega_1}{dt} + \gamma_1 \cdot (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3\right) + \left(I_1 \cdot \Omega_1 \frac{d\gamma_1}{dt}\right) = \gamma_1 \cdot P\,(\gamma_2 c_0 - \gamma_3 b_0) + I_1 \cdot \Omega_1 \cdot (\Omega_3\gamma_2 - \Omega_2\gamma_3), \\ \left(I_2 \cdot \gamma_2 \cdot \frac{d\Omega_2}{dt} + \gamma_2 \cdot (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1\right) + \left(I_2 \cdot \Omega_2 \frac{d\gamma_2}{dt}\right) = \gamma_2 \cdot P\,(\gamma_3 a_0 - \gamma_1 c_0) + I_2 \cdot \Omega_2 \cdot (\Omega_1\gamma_3 - \Omega_3\gamma_1), \\ \left(I_3 \cdot \gamma_3 \cdot \frac{d\Omega_3}{dt} + \gamma_3 \cdot (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2\right) + \left(I_3 \cdot \Omega_3 \frac{d\gamma_3}{dt}\right) = \gamma_3 \cdot P\,(\gamma_1 b_0 - \gamma_2 a_0) + I_3 \cdot \Omega_3 \cdot (\Omega_2\gamma_1 - \Omega_1\gamma_2). \end{cases}$$

Having done this, we should sum up all the three equations above:

$$I_1 \cdot \frac{d}{dt}(\Omega_1 \cdot \gamma_1) + I_2 \cdot \frac{d}{dt}(\Omega_2 \cdot \gamma_2) + I_3 \cdot \frac{d}{dt}(\Omega_3 \cdot \gamma_3) = 0.$$

So, we have obtained the second integral of (1.3) in [5].

The first integral of (1.3) is trivial, but belongs to Poisson equations only: To obtain it, we should multiply each of equations of (1.2) by γ_i accordingly, then sum them up (*the constant of integration is chosen equal to* 1*, due to trigonometric sense of the solution in the absolute system of coordinates via Euler angles*):

$$\frac{1}{2}\frac{d}{dt}(\gamma_1^2) + \frac{1}{2}\frac{d}{dt}(\gamma_2^2) + \frac{1}{2}\frac{d}{dt}(\gamma_3^2) = 0.$$

As we can see, two of three proper additional invariants above are obtained by using all the six EP-equations (including Poisson equations).

But, nevertheless, the system of equations (1.1) and (1.2) is supposed *not to be equivalent* to the system of equations (1.1) along with invariants (1.3) [Dr. Hamad H. Yehya, personal communications] for some particular cases, as it was suggested earlier in [5]. The rather complex case, which describes the motion of the constrained rigid body around a fixed point, was considered in the comprehensive article [7].

So, to solve the system of equations (1.1) and (1.2), we should first solve the Poisson equations (1.2).

3 Presentation of the solution of Poisson equations

The system (1.2) has *the analytical* way to present the general solution [8,9] (in regard to the time parameter t):

$$\gamma_1 = \frac{(\sigma - \gamma_3)\cdot(\xi - \eta^{-1})}{2}, \quad \gamma_2 = -\frac{(\sigma - \gamma_3)\cdot i\cdot(\xi + \eta^{-1})}{2},$$
$$\gamma_3 = \sigma \cdot \frac{\left(1 + \frac{\eta}{\xi}\right)}{\left(1 - \frac{\eta}{\xi}\right)}, \tag{3.1}$$

– where σ is some arbitrary (real) constant, given by the initial conditions ($\sigma = 1$); Ω_i are functions of the time parameter t only—we consider here only the case for the first approximation, which means that $\Omega_i \neq \Omega_i(\{\gamma_i\}, t)$.

For auxiliary functions ξ(t), η(t) of *complex* value, we could obtain the appropriate *Riccati* equations as [8,9]

$$\xi' = \left(\frac{\Omega_2 + i\cdot\Omega_1}{2}\right)\cdot\xi^2 - i\cdot\Omega_3\cdot\xi + \left(\frac{\Omega_2 - i\cdot\Omega_1}{2}\right), \tag{3.2}$$
$$\eta' = \frac{(\Omega_2 + i\cdot\Omega_1)}{2}\cdot\eta^2 - i\cdot\Omega_3\cdot\eta + \frac{(\Omega_2 - i\cdot\Omega_1)}{2}; \tag{3.3}$$

– besides, we should note that

$$\eta^{-1} = -\bar{\xi}; \tag{*}$$

– that is why all the components γ_i (3.1) are *real* functions in any case.

Previously, the solution (3.1) was also presented in the form [9]

$$\gamma_1 = -\sigma\cdot\left(\frac{2a}{1 + (a^2 + b^2)}\right), \quad \gamma_2 = -\sigma\cdot\left(\frac{2b}{1 + (a^2 + b^2)}\right),$$
$$\gamma_3 = \sigma\cdot\left(\frac{1 - (a^2 + b^2)}{1 + (a^2 + b^2)}\right), \tag{3.4}$$

– where the real-valued coefficients $a(t)$, $b(t)$ (3.4) are solutions of the mutual system of 2 *Riccati* ordinary differential equations:

$$\begin{cases} a' = \frac{\Omega_2}{2}\cdot a^2 - (\Omega_1\cdot b)\cdot a - \frac{\Omega_2}{2}(b^2 - 1) + \Omega_3\cdot b, \\ b' = -\frac{\Omega_1}{2}\cdot b^2 + (\Omega_2\cdot a)\cdot b + \frac{\Omega_1}{2}\cdot(a^2 - 1) - \Omega_3\cdot a. \end{cases} \tag{3.5}$$

Indeed, let us present the auxiliary functions $\xi(t)$, $\eta(t)$ (3.2), (3.3) of *complex* value as

$$\eta = a + b\cdot i\ , \quad \xi = c + d\cdot i, \tag{3.6}$$

– then due to (*), we obtain the nonlinear dependence of the coefficients c, d on the coefficients a, b:

$$\eta^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\cdot i = -\bar{\xi} = -(c - d\cdot i),$$
$$\Rightarrow \begin{cases} c = -\left(\frac{a}{a^2+b^2}\right), \\ d\cdot i = -\left(\frac{b}{a^2+b^2}\right)\cdot i. \end{cases} \tag{3.7}$$

Analyzing the expressions (3.7) above, it is explicitly obvious that we should explore only the function $\eta(t)$ and appropriate dynamics of the coefficients $a(t)$, $b(t)$.

So, we obtain from the *Riccati* equation (3.3) for $\eta\ (t)$ (3.6) the mutual system of 2 *Riccati* ordinary differential equations (3.5) [8], which has no analytical solution in the general case [8,9].

Just to confirm the *Riccati*-type of Eq. (3.5): Indeed, if we multiply the first of Eq. (3.5) by Ω_2, the second Eq. by Ω_1, then summarize them properly, we should obtain

$$\Omega_2 \cdot a' \ - \ \frac{1}{2}((\Omega_1)^2 + (\Omega_2)^2) \cdot a^2 + \Omega_1 \cdot \Omega_3 \cdot a + \frac{(\Omega_1)^2}{2}$$
$$= -\Omega_1 \cdot b' \ - \ \frac{1}{2}((\Omega_1)^2 + (\Omega_2)^2) \cdot b^2 + \Omega_2 \cdot \Omega_3 \cdot b + \ \frac{(\Omega_2)^2}{2}. \quad (3.8)$$

Equation (3.8) above is the classical *Riccati* ODE. It describes the evolution of the function $a(t)$ in dependence on the function $b(t)$ along with the functions $\{\Omega_i\}$ in regard to the time parameter t; such a *Riccati* ODE has no analytical solution in the general case [8] and could be presented as below:

$$a' \ = \ A \cdot a^2 + B \cdot a + \ D,$$
$$A = \frac{1}{2}\frac{((\Omega_1)^2 + (\Omega_2)^2)}{\Omega_2}, \quad B = -\left(\frac{\Omega_1 \cdot \Omega_3}{\Omega_2}\right),$$
$$D = -\frac{\Omega_1}{\Omega_2} \cdot b' \ - \ \frac{1}{2}\frac{((\Omega_1)^2 + (\Omega_2)^2)}{\Omega_2} \cdot b^2 + \Omega_3 \cdot b + \ \frac{\Omega_2}{2} \ - \ \frac{(\Omega_1)^2}{2\Omega_2}. \quad (3.9)$$

A lot of important partial solutions of (3.9) have been considered properly [9], but in each case we should restrict the choosing of the appropriate functions $\{\Omega_i\}$ (it means that the $\{\Omega_i\}$ depend on each other). In the current research, we wish to avoid such a restricting dependence.

4 Discussion

In our development, we have derived the three proper additional invariants (1.3), two of which are obtained by using all the six Euler–Poisson equations (including Poisson equations).

But, nevertheless, the system of equations (1.1), (1.2) is supposed *not to be equivalent* to the system (1.1) along with invariants (1.3) [Dr. Hamad H. Yehya, personal communications], the as it was suggested earlier in [5]. If you solve the dynamical Eq. (1.1) using the only the integrals (1.3) without the Poisson equations (1.2), some untrue solutions of Euler–Poisson equations may result.

For example, the wrong analysis was made by the authors in [10]. In the first section, the authors begin as follows: "To verify this let us consider the problem in the absolute system of coordinates." But to consider it in the absolute system of coordinates, the authors should transform the proper components of the previously presented solution from one (rotating) coordinate system to another (absolute) Cartesian coordinate system via Euler's angles, as it has been done in [5]. If they obtain the results like this, they will not obviously be a "sub-case of the Euler's case: steady rotation."

The main mistake by the authors is that the point of gravity application does not coincide with the point of application of the angular momentum vector. So, the principle moments of inertia of the rigid body (according to Steiner's theorem [1]) should be changed if the vector of gravity is translated to the point of application of the angular momentum vector. Thus, the vector of gravity force is not to be collinear to the new vector of angular momentum (with the new coefficients of the principle moments of inertia, forming it). In fact, initial solution suggests that the vector of gravity is to be parallel (not collinear!) to the constant *vertical* vector of the angular momentum during the motion of rigid body rotations.

So, the authors should better state in [10] that the system of equations (1.1), (1.2) is supposed *not to be equivalent* to the system, (1.1) along with the invariants (1.3) (for the particular case under their consideration).

Last, but not least, we have come to the conclusion that, to solve system of equations (1.1), (1.2), we should first solve the Poisson equations (1.2).

This was the motivation to develop a new approach how to resolve Poisson equations (1.2) in case if $\{\Omega_i\}$ are functions of the time parameter t only.

Also, some remarkable articles should be cited, which concern the problem under consideration [11–13].

5 Conclusion

The main conclusion is that the system of Euler–Poisson equations of rigid body rotation is supposed *not to be equivalent* to the system of Euler equations (1.1) along with the well-known invariants (1.3).

A new approach is developed here for resolving of the Poisson equations in case the components of angular velocity of rigid body rotation can be considered as functions of the time parameter t only. A fundamental solution is presented by the analytical formulae in dependence on two time-dependent, real-valued coefficients. Such coefficients are proved to be the solutions of a mutual system of 2 *Riccati* ordinary differential equations (which has no analytical solution in the general case). All in all, the cases of analytical resolving of Poisson equation are quite rare (according to the cases of exact resolving of the aforementioned system of *Riccati* ODEs). So, the system of Euler–Poisson equations is proved to have the analytical solutions (in quadratures) only in classical simplifying cases: (1) *Lagrange's* case, or (2) *Kovalevskaya's* case or (3) *Euler's* case or other well-known but particular cases (where the existence of particular solutions depends on the choosing of the appropriate initial conditions).

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