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Second-order work criterion: from material point to boundary value problems

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Abstract Although the concept of the second-order work criterion dates back to the middle of the past century, its physical meaning often continues to be debated. Recent papers have established that a certain class of instabilities, related to the occurrence of an outburst in kinetic energy, could be properly detected by the vanishing of the second-order work. This manuscript attempts to extend the second-order work formalism to boundary value problems. For this purpose, the role of the boundary stiffness tensor (relating external forces and displacement components) is put forward in the occurrence of instability by divergence. Omitting body forces, a global method is then given to compute the second-order work terms directly. The capability of this formalism is finally demonstrated in the context of engineering issues.

1 Introduction

In rate-independent solid mechanics, from a historical perspective, the question of stability has been considered mainly from two different points of view. The first one has focused its analyses on discrete elastic systems ([5,38,43] among many others) with, for instance, the emblematic Ziegler column, while the second one was developed within a continuous mechanics framework usually at a specimen or representative volume element (R.V.E.) scale (e.g., [13] and [2,14,29,33,41]). For continuous systems, strains and stresses are related by a constitutive tensor. More recently (e.g., [7,18,22]), some common conclusions have been exhibited when the constitutive operator is not symmetric and linear combinations of strains are applied as loading conditions. See also Bigoni [4] for a thorough review on the second-order work approach for non-symmetric plasticity.

Let us restrict this paper to divergence instabilities (i.e., suddenly monotonously increasing displacements or strains), to quasi-static loading conditions and to the stability "in the small." The first restriction implies that flutter instabilities (i.e., displacements or strains increasing cyclically; see Bigoni and Noselli [3], for a practical example) are not considered here (see [18,26], for instance, for more general considerations). Secondly, only quasi-static loading paths until reaching the unstable state are taken into account. Of course, as soon as the instability is effective, the post-collapse displacement/strain regime usually becomes dynamic with a burst of kinetic energy [8,23]. However, the pre-collapse behavior (including the bifurcation state) remains in static conditions and it allows a purely static analysis of this class of instabilities. Finally, the third limit of this paper

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is given by the fact that only the stability "in the small" is analyzed, leaving asymptotic ("in the large") stability as introduced by [20] beyond its scope. Stability in the small means that a mechanical state is considered stable as soon as any incremental loading included inside an arbitrarily small ball centered on this state in the loading space leads to an incremental response included in a given finite ball in the response space.

Experiments and theoretical considerations show that the constitutive tensor can be non-symmetric for continuous elastoplastic bodies in the non-associative case (yield surface and plastic potential do not coincide, as in the case of Coulombian frictional materials). In this situation, experiments and theoretical analyses show that some "paradoxical" instabilities can be repeatedly observed [6,10,18,31]. Paradoxical instability here means that the collapse appears strictly before any limit state of the stress-controlled elastoplastic body. Moreover, these instabilities can be described using the second-order work criterion. Indeed, the second-order work criterion (i.e., loss of definite positiveness of the constitutive matrix) constitutes a lower bound of all possible instability diagrams, and is thus considered as the "optimal" criterion [18]. The main results are as follows:

– First, there is a bifurcation domain delimited by the singularities of the symmetric part of the constitutive tensor (corresponding to the loss of definite positiveness) as a lower bound and the singularities of the tensor itself as the upper bound.
– Second, in this bifurcation domain, the negative values of internal second-order work are obtained in the "isotropic cone" (according to linear algebra vocabulary) of the tensor. This "instability cone" (according to a mechanical point of view) gathers the potentially unstable loading directions associated with the mixed loading paths. The instability will become effective for proper control loading variables [36].

According to experiments [8,11] and to numerical computations using finite element or discrete element methods [35,42], once the instability state is reached after a quasi-static loading path, and for a proper control variable, instability becomes effective for some ad hoc perturbations, such as an arbitrarily small additional loading applied at the extremum of the critical load. If so, a burst of kinetic energy is generally observed at this bifurcation point, with an abrupt transition from a static regime of deformation to a dynamic one [23]. Thus a link can be expected between kinetic energy and second-order work items. This is the link that gives the proper mechanical interpretation of the predictive capacity of the second-order work criterion with respect to divergence instabilities. However, these analyses of the stability in the small cannot give any indication about the asymptotic stability of the elastoplastic body. Indeed, the notion of asymptotic stability disappears in elastoplasticity because of the history dependence of this mechanical behavior (except in 1D or for drastic assumptions about the loading path considered; [32]).

So far, most investigations have dealt with the material point scale, or with homogeneous specimens subjected to homogeneous loading paths. Extending the second-order work formalism to more general boundary value problems remains an open, and important, issue. At the same time, this challenge is of paramount importance in view of making this approach efficient for engineering purposes. In this paper, we demonstrate how the second-order work approach can be conveniently extended to boundary value problems. First, the basic second-order work equation is recalled, showing how the increase in kinetic energy is related to the difference between both external and internal second-order works. Omitting body forces, a global method is then given to compute the second-order work terms directly. The capability of this formalism is shown by considering two engineering situations: a laboratory test and a shallow foundation problem.

Throughout this paper, all materials are assumed to be rate independent and are simple materials [27,39,40]. Moreover, vectors will be denoted with a single overbar ($\bar{A}$), and second-order tensors will be denoted with a double overbar ($\overline{\overline{A}}$). Generic terms of any vector $\bar{A}$ are noted A_i. Likewise, generic terms of any second-order tensor $\overline{\overline{A}}$ are noted A_{ij}. Einstein convention on summation of repeated indices (underscript position) is used only when there is no ambiguity. Otherwise, the summation is indicated explicitly using the summation symbol.

2 Kinetic energy and second-order work

Consider a material body of volume V_o and density ρ_o enclosed by boundary (Γ_o) in an initial configuration C_o at time t_o. Following a certain loading history, the body is in a strained configuration C and occupies a volume V of boundary (Γ), in equilibrium under a prescribed external loading. This loading is defined by specific static or kinematic parameters, referred to as the loading parameters [17,28,34].

We query in this paper the conditions in which the kinetic energy of the system is a convex function over time, that is $\ddot{E}_c > 0$, at least over a finite time range of amplitude Δt. Starting from an equilibrium configuration at time t_o, $\dot{E}_c(t_o) = 0$. Thus, if $\forall t \in [t_o,\ t_o + \Delta t[$, $\ddot{E}_c(t) > 0$. Both $E_c(t)$ and $\dot{E}_c(t)$ are thereby strictly positive over $[t_o,\ t_o + \Delta t[$. The kinetic energy of the system increases over the time range $[t_o,\ t_o + \Delta t[$. It will be seen hereafter that the stability analysis cannot be carried out in an asymptotical way for elastoplastic materials, contrary to linear systems. Thus the analysis should be restricted to a finite time range. Herein, the notion of *local* stability (in the small) at time t_o contrasts with that of Lyapunov's *asymptotic* stability (in the large), in that only the time range $[t_o,\ t_o + \Delta t[$ is considered from an equilibrium configuration [24,25].

Adopting a semi-Lagrangian formulation (each material point $\bar{x}$ of the current configuration C corresponds (through bijective mapping) to a material point $\bar{X}$ of the initial configuration C_o), and in absence of body forces, the evolution of each material point of the system is given by the equation

$$\rho_o\ \ddot{u}_i - \frac{\partial \Pi_{ij}}{\partial X_j} = 0, \tag{1}$$

where $\overline{\overline{\Pi}}$ is the first Piola–Kirchoff stress tensor and $\bar{u}$ is the displacement field. The kinetic energy of the whole system reads

$$E_c = \frac{1}{2} \int\limits_{V_o} \rho_o\ \dot{\bar{u}}^2\ \mathrm{d}V_o, \tag{2}$$

where $\dot{\bar{u}}(\bar{X})$ is the Lagrangian velocity field.

A double time differentiation of Eq. (2) yields[1]:

$$\ddot{E}_c = \int\limits_{V_o} \rho_o\ \ddot{\bar{u}}^2\ \mathrm{d}V_o + \int\limits_{V_o} \rho_o\ \dot{\bar{u}} \cdot \dddot{\bar{u}}\ \mathrm{d}V_o. \tag{3}$$

Combining Eq. (3) with Eq. (1) gives:

$$\ddot{E}_c = \int\limits_{V_o} \rho_o\ \ddot{\bar{u}}^2\ \mathrm{d}V_o + \int\limits_{V_o} \dot{u}_i\ \frac{\partial \dot{\Pi}_{ij}}{\partial X_j}\ \mathrm{d}V_o. \tag{4}$$

By virtue of the Green formula, Eq. (4) can be rewritten as:

$$\ddot{E}_c = \int\limits_{V_o} \rho_o\ \ddot{\bar{u}}^2\ \mathrm{d}V_o + \int\limits_{\partial V_o} \dot{u}_i\ \dot{\Pi}_{ij}\ N_j\ \mathrm{d}S_o - \int\limits_{V_o} \dot{\Pi}_{ij}\ \frac{\partial \dot{u}_i}{\partial X_j}\ \mathrm{d}V_o. \tag{5}$$

The result is that the second-order time derivative of the kinetic energy is the sum of three terms:

- The first term $I_2 = \int_{V_o} \rho_o\ \ddot{\bar{u}}^2\ \mathrm{d}V_o$ is an inertial term. This is the quadratic average of the acceleration; this term is therefore always positive.
- The second term $\int_{\partial V_o} \dot{u}_i\ \dot{\Pi}_{ij}\ N_j\ \mathrm{d}S_o = \int_{\partial V_o} \dot{u}_i\ \dot{s}_i\ \mathrm{d}S_o$ is a boundary term involving the loading parameters (the displacements $\bar{u}$ and the current external forces $\bar{f}$ with $\mathrm{d}\bar{f} = \bar{s}\ \mathrm{d}S_o$) acting on the boundary of the initial (reference) configuration of the system. It is hereafter called the external second-order work W_2^{ext}.
- The third term explicitly introduces the second-order work, which is expressed following a semi-Lagrangian formalism [13] as $\int_{V_o} \dot{\Pi}_{ij}\ \frac{\partial \dot{u}_i}{\partial X_j}\ \mathrm{d}V_o = \int_{V_o} \dot{\Pi}_{ij}\ \dot{F}_{ij}\ \mathrm{d}V_o$, where $\overline{\overline{F}}$ is the tangent linear transformation. This term is related to the constitutive behavior of the material and is therefore referred to as the internal second-order work W_2^{int}. It should be noted that at any material point of the system, both the stress rate tensor $\dot{\overline{\overline{\Pi}}}$ and velocity gradient tensor $\dot{\overline{\overline{F}}}$ are related by the constitutive relation $\dot{\Pi}_{ij} = L_{ijkl}\ \dot{F}_{kl}$, where the fourth-order tensor $\overset{4}{L}$ is the tangent constitutive tensor for rate-independent materials.

[1] The advantage of the Lagrangian formulation is that all integrals are expressed with respect to a fixed domain (corresponding to the initial configuration).

It follows that Eq. (5) can be expressed as:

$$\ddot{E}_c = I_2 + W_2^{\text{ext}} - W_2^{\text{int}}. \tag{6}$$

When the loading conditions and the constitutive behavior of the material make it possible that $W_2^{\text{int}} < W_2^{\text{ext}}$ from a given time t_o, then the second-order time derivative of the kinetic energy is strictly positive after time t_o. Starting from an equilibrium configuration at time t_o, $\dot{E}_c(t_o) = 0$, the kinetic energy of the system is a growing function over a certain time range $[t_o,\ t_o + \Delta t[$.

Until now, the second-order work criterion [25] has been applied essentially to the material point scale (or for homogeneous specimens under homogeneous loading conditions). In this manuscript, we extend this approach to any material system. Both strain and stress fields may no longer be homogeneous, and the computation of both internal and external second-order works can be demanding, essentially because the internal second-order work requires determining both stress and strain fields. In the following sections, we propose an approach in which internal and external second-order works are computed based on the mechanical parameters (forces, displacements) acting on the boundary of the system only, without requiring any information inside the system. This is highly advantageous, since these boundary parameters are generally accessible.

3 External and internal second-order works

3.1 External second-order work

Let us consider that the system introduced in Sect. 2 is initially at rest. We associate a Galilean frame with the physical space, in which all subsequent derivations will be expressed. A force or displacement loading can be applied to the boundary (Γ_o) of the system. If incremental displacements are prescribed to the whole boundary, then boundary incremental forces develop as a response to the kinematic loading. We assume hereafter that the loading is directed by either forces or displacements applied to the boundary of the system. Body forces will be neglected. Thus, the boundary is composed of n parts 'k' subjected to either a rigid body velocity $\dot{\vec{u}}^k$ or to an external rate force $\dot{\vec{f}}^k$. By renumbering, any boundary loading is therefore defined from a set of $N = 3n$ components u_i or f_i. When $\dot{u}_i$ is imposed, $\dot{f}_i$ stands as the dual rate force response of the system. Likewise, when $\dot{f}_i$ is imposed, $\dot{u}_i$ stands as the dual velocity response of the system. In these conditions, the external second-order work reads:

$$W_2^{\text{ext}} = \sum_{i=1}^{N} \dot{u}_i\ \dot{f}_i. \tag{7}$$

More broadly, the loading can be defined by selecting p control variables C_i, together with $N - p$ loading conditions L_j. See also Khalil [15] for a thorough review on the theory of control in nonlinear systems. If a purely strain loading is considered (as will be done in this section), the p control variables C_i can be the displacement components: $C_1 = u_1, \ldots C_p = u_p$. The loading conditions L_i are given by $N - p$ independent linear combinations of displacement components, as follows [17,23,28,34]:

$$L_i = A_{i-p,1}\, u_1 + \cdots + A_{i-p,N}\, u_N, \quad \text{for} \quad i = p+1, \ldots N, \tag{8}$$

where $\overline{\overline{A}}$ is a rate-independent matrix of dimension $((N-p)\times N)$. ${}^t\overline{\overline{A}}$ is composed of $N - p$ vectors $\bar{A}^i = {}^t(A_{i,1}, \ldots, A_{i,N})$ of R^N. Thus, the loading program can be defined as follows:

$$\dot{C}_i = \text{const.} \quad (>0) \quad \text{for} \quad i = 1, \ldots p, \tag{9a}$$
$$\dot{L}_i = 0 \quad \text{for} \quad i = p+1, \ldots N. \tag{9b}$$

In an analogous manner, the response can be expressed in terms of N independent linear combinations R_i of force components, as follows:

$$R_i = B_{i,1}\ f_1 + \cdots + B_{i,N}\ f_N \quad \text{for} \quad i = 1, \ldots p, \tag{10a}$$
$$R_i = f_i \quad \text{for} \quad i = p+1, \ldots N, \tag{10b}$$

where $\overline{\overline{B}}$ is a rate-independent matrix of dimension $(p \times N)$. ${}^t\overline{\overline{B}}$ is composed of p vectors $\bar{B}^i = {}^t(B_{i1}, \ldots, B_{iN})$ of R^N. These vectors are chosen so that the following condition holds [10,28]:

$$\sum_{i=1}^{N} \dot{u}_i \ \dot{f}_i = \sum_{i=1}^{p} \dot{C}_i \ \dot{R}_i + \sum_{i=p+1}^{N} \dot{L}_i \ \dot{R}_i. \tag{11}$$

Combining Eqs. (9) and (10) with Eq. (11) yields:

$$\sum_{i=1}^{N} \dot{u}_i \ \dot{f}_i = \sum_{i=1}^{p} \left(B_{i,1} \ \dot{u}_i \ \dot{f}_1 + \cdots + B_{i,N} \ \dot{u}_i \ \dot{f}_N\right) + \sum_{i=p+1}^{N} \left(A_{i-p,1} \ \dot{u}_1 \ \dot{f}_i + \cdots + A_{i-p,N} \ \dot{u}_N \ \dot{f}_i\right), \tag{12}$$

which gives, after some algebraic transformations:

$$\sum_{i=1}^{N} \dot{u}_i \ \dot{f}_i = \sum_{i=1}^{p}\sum_{j=1}^{p} B_{i,j} \ \dot{u}_i \ \dot{f}_j + \sum_{i=1}^{p}\sum_{j=p+1}^{N} B_{i,j} \ \dot{u}_i \ \dot{f}_j + \sum_{j=p+1}^{N}\sum_{i=1}^{p} A_{j-p,i} \ \dot{u}_i \ \dot{f}_j + \sum_{i=p+1}^{N}\sum_{j=p+1}^{N} A_{j-p,i} \ \dot{u}_i \ \dot{f}_j. \tag{13}$$

Noting that $\sum_{i=1}^{N} \dot{u}_i \ \dot{f}_i = \sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{ij} \ \dot{u}_i \ \dot{f}_j$, Eq. (13) is rewritten as:

$$\sum_{i=1}^{p}\sum_{j=1}^{p} \left(B_{i,j} - \delta_{ij}\right) \ \dot{u}_i \ \dot{f}_j + \sum_{i=p+1}^{N}\sum_{j=p+1}^{N} \left(A_{j-p,i} - \delta_{ij}\right) \ \dot{u}_i \ \dot{f}_j + \sum_{i=1}^{p}\sum_{j=p+1}^{N} \left(B_{i,j} + A_{j-p,i}\right) \ \dot{u}_i \ \dot{f}_j = 0. \tag{14}$$

As Eq. (14) must be verified whatever u_i and f_j, the following relations hold:

$$i = 1, \ldots p \text{ and } j = 1, \ldots p \quad B_{i,j} = \delta_{ij}, \tag{15a}$$
$$i = p+1, \ldots N \text{ and } j = p+1, \ldots N \quad A_{j-p,i} = \delta_{ij}, \tag{15b}$$
$$i = 1, \ldots p \text{ and } j = p+1, \ldots N \quad B_{i,j} = -A_{j-p,i}. \tag{15c}$$

Property 1 The column vectors of matrices $\overline{\overline{A}}$ and ${}^t\overline{\overline{B}}$ are orthogonal to one another.

Proof For any pair of vectors $\bar{A}^k = {}^t(A_{k,1}, \ldots, A_{k,N})$ and $\bar{B}^l = {}^t(B_{l1}, \ldots, B_{lN})$, with $k = 1, \ldots N-p$ and $l = 1, \ldots p$, we have:

$${}^t\bar{A}^k \ \bar{B}^l = \sum_{i=1}^{N} A_i^k \ B_i^l = \sum_{i=1}^{N} A_{ki} \ B_{li} = \sum_{i=1}^{p} A_{ki} \ B_{li} + \sum_{i=p+1}^{N} A_{ki} \ B_{li}. \tag{16}$$

Taking advantage of relations (15), Eq. (16) yields:

$${}^t\bar{A}^k \ \bar{B}^l = \sum_{i=1}^{p} A_{ki} \ \delta_{li} - \sum_{i=p+1}^{N} \delta_{k+p,i} \ A_{i-p,l} = A_{kl} - A_{kl} = 0. \tag{17}$$

Any two vectors $\bar{A}^k$ and $\bar{B}^l$, with $k = 1, \ldots \ N-p$ and $l = 1, \ldots p$, are therefore orthogonal, which establishes the property.

Moreover, we have:

$$L_i = \sum_{j=1}^{p} A_{i-p,j} \ u_j + u_i \quad \text{for} \quad i = p+1, \ldots N, \tag{18}$$

$$R_i = f_i - \sum_{j=p+1}^{N} A_{j-p,i} \ f_j \quad \text{for} \quad i = 1, \ldots p. \tag{19}$$

Taking advantage of Eqs. (8), (11), and (19), the external second-order work can be expressed as:

$$W_2^{\text{ext}} = \sum_{i=1}^{p} \dot{u}_i \left(\dot{f}_i - \sum_{j=p+1}^{N} A_{j-p,i} \ \dot{f}_j \right) \tag{20}$$

under the loading conditions $\sum_{j=1}^{p} A_{i-p,j} \ \dot{u}_j + \ \dot{u}_i = 0$, for $i = p+1, \ldots N$.

The physical meaning of Eq. (20) is as follows: When W_2^{ext} is nil, at least one of the p terms $\dot{R}_i$ is negative (say $\dot{R}_\alpha$), and the corresponding response parameter $R_\alpha = f_\alpha - \sum_{j=p+1}^{N} A_{j-p,\alpha} \ f_j$ follows a descending branch ($\dot{R}_\alpha < 0$).

3.2 The system stiffness operator

Let a velocity of loading ${}^t\bar{u} = (\dot{u}_1, \ldots, \dot{u}_N)$ be prescribed to the system. The response of the system is defined by the boundary force rates $(\dot{f}_1, \ldots, \dot{f}_N)$. These force rates (vector $\dot{\bar{f}}$) constitute the quasi-static response of the system to the loading defined by vector $\dot{\bar{u}}$.

Given a mechanical system composed of a material considered as a simple medium in the sense of [27], the principle of determinism implies that the force response $\bar{f}(t)$ at a given time t is a functional of the strain history at this point up to this time. Thus, as an assumption when heterogeneous conditions hold, we can conceive that a functional $\Re$ exists such that:

$$\bar{f}\,(t) = \Re\,(\bar{u}\,(\tau)\,, \ \tau \leq t)\,. \tag{21}$$

This is an extension of the general framework that holds on the material point scale [9]. As soon as plastic irreversibilities occur, the functional $\Re$ is not differentiable, making the global formulation (21) inappropriate. It is more convenient to adopt an incremental formulation, as follows:

$$\overline{\overline{H}}\left(\dot{\bar{f}}, \dot{\bar{u}}, h\right) = \bar{\bar{0}}, \tag{22}$$

where $\overline{\overline{H}}$ is a nonlinear tensorial function of arguments $\dot{\bar{f}}$, $\dot{\bar{u}}$ and h, h being a set of parameters characterizing the previous loading history of the system.

Moreover, by restricting the subject at hand to non-viscous materials, and assuming the tensorial function $\overline{\overline{H}}$ to be sufficiently regular, Eq. (22) is written as

$$\dot{\bar{f}} = G_h\left(\dot{\bar{u}}\right), \tag{23}$$

where G_h is a tensorial function that depends on the previous loading path history through state variables and memory parameters h.

Because of the rate-independency condition, G_h is a homogeneous function of degree 1 (for positive values of the multiplicative parameter):

$$\forall \lambda \in R^+ : \quad G_h\left(\lambda \ \dot{\bar{u}}\right) = \lambda \ G_h\left(\dot{\bar{u}}\right). \tag{24}$$

Euler's identity for homogeneous functions implies that $\dot{\bar{f}} = \frac{\partial G_h}{\partial (\dot{\bar{u}})} \ \dot{\bar{u}}$. Thus the system stiffness matrix $\overline{\overline{\Lambda}}$ can be defined as:

$$\dot{\bar{f}} = \overline{\overline{\Lambda}}\,(\bar{e}_u) \ \dot{\bar{u}}, \tag{25}$$

where $\overline{\overline{\Lambda}}$ is a homogeneous function of degree 0, of $\bar{e}_u = \dot{\bar{u}}/\|\dot{\bar{u}}\|$.

As will be shown in a later section, the system stiffness matrix concept stands as an extension of the constitutive operator that holds on the material point scale or for homogeneous volumes (subjected to a uniform loading). This extended framework is much more general, as it applies to any system, subjected to any kinematically controlled loading. It is worth noting that $\overline{\overline{\Lambda}}$ characterizes the behavior of the system through the accessible variables acting on the system's boundary.

It is immediate that, according to Eq. (7), the external second-order work reads:

$$W_2^{\text{ext}} = {}^t\dot{\bar{u}} \ \overline{\overline{\Lambda}} \ \dot{\bar{u}} = {}^t\dot{\bar{u}} \ \overline{\overline{\Lambda}}^s \ \dot{\bar{u}}. \tag{26}$$

As a result, when the loading path is strain-controlled, the external second-order work is a quadratic form associated with the symmetric part $\overline{\overline{\Lambda}}^s$ of the system stiffness matrix. This property no longer holds when the loading is statically (force) controlled. In this case, the system stiffness matrix cannot be defined, because the response of the system may no longer be quasi-static but is likely to be dynamic.

3.3 The internal second-order work

The internal second-order work reads $W_2^{\text{int}} = \int_{V_o} \dot{\Pi}_{ij}\ \dot{F}_{ij}\ \mathrm{d}V_o$, or equivalently, condensing both (3×3) matrices $\overline{\overline{F}}$ and $\overline{\overline{\Pi}}$ in six component vectors $\bar{F}$ and $\bar{\Pi}$, $W_2^{\text{int}} = \int_{V_o} \dot{\Pi}_i\ \dot{F}_i\ \mathrm{d}V_o$. At each material point, the constitutive relation $\dot{\Pi}_i = K_{ij}\ \dot{F}_j$ (stemming from the constitutive relation $\dot{\Pi}_{ij} = L_{ijkl}\ \dot{F}_{kl}$,) applies, where $\overline{\overline{K}}$ is the constitutive tensor operating at the material point considered.

Thus, the internal second-order work reads:

$$W_2^{\text{int}} = \int_{V_o} {}^t\dot{\bar{F}}\ \overline{\overline{K}}\ \dot{\bar{F}}\ \mathrm{d}V_o = \int_{V_o} {}^t\dot{\bar{F}}\ \overline{\overline{K}}^s\ \dot{\bar{F}}\ \mathrm{d}V_o, \tag{27}$$

where $\overline{\overline{K}}^s$ is the symmetric part of $\overline{\overline{K}}$.

When the loading is kinematically controlled, the mechanical response of the system is quasi-static, without any (prominent) inertial effects. No outburst in kinetic energy occurs, and Eq. (6) yields:

$$W_2^{\text{ext}} - W_2^{\text{int}} = 0. \tag{28}$$

Equation (28) means that both internal and external second-order works, in this kinematic control context, coincide. Thus, according to Eq. (18), the internal second-order work also reads:

$$W_2^{\text{int}} = \sum_{i=1}^{p} \dot{u}_i \left(\dot{f}_i - \sum_{j=p+1}^{N} A_{j-p,i}\ \dot{f}_j \right). \tag{29}$$

We conjecture the following proposition:

Proposition 1 *Starting from an equilibrium configuration, the internal second-order work of a given system subjected to any loading program depends only on the infinitesimal loading path and not on the control mode adopted.*

At a given mechanical state, the incremental response of a material depends on the loading direction, not on the control mode, that can be static or kinematic. This is not true over a finite time range, since inertial effects can occur for a stress control path, modifying therefore the response of the system, and the corresponding value of the internal second-order work. Limiting our analysis to the initiation of failure, the internal second-order work of the system, under a given loading path, can be computed by adopting a kinematical control ($W_2^{int,kc}$), as assumed in the previous section. In this case, Eq. (28) holds, and by virtue of Eq. (26) it follows that:

$$W_2^{\text{int}} = W_2^{\text{int},kc} = W_2^{\text{ext}} = {}^t\dot{\bar{u}}\ \overline{\overline{\Lambda}}^s\ \dot{\bar{u}}. \tag{30}$$

The loading path is defined by the $N - p$ relations

$$\sum_{j=1}^{p} A_{i-p,j}\ \dot{u}_j + \dot{u}_i = 0, \quad \text{for} \quad i = p+1, \ldots N, \tag{31}$$

and is controlled by the p kinematical variables $C_1 = u_1, \ldots C_p = u_p$; the p velocities $\dot{u}_i$ are imposed as constant: $\dot{u}_i = v_i$ (v_i being a real constant).

Equations (31) mean that vector $\dot{\bar{u}}$ is normal to the $N-p$ vectors $\bar{A}^i = {}^t(A_{i-p,1}, \ldots, A_{i-p,N})$. As both matrices $\overline{\overline{A}}$ and ${}^t\overline{\overline{B}}$ are orthogonal to one another, $\dot{\bar{u}}$ can be decomposed on the basis formed by the vectors $\bar{B}^l = {}^t(B_{l1}, \ldots, B_{lN})$, with $l = 1, \ldots p$. Taking Eq. (15) into account finally gives

$$\dot{\bar{u}} = \sum_{l=1}^{p} \alpha_l\ \bar{B}^l, \text{ with } \alpha_l \text{ being any real scalar} \tag{32}$$

or

$$\dot{u}_i = \alpha_i \quad \text{for} \quad i = 1, \ldots p \quad \text{and} \quad \dot{u}_i = -\sum_{l=1}^{p} \alpha_l\ A_{i-p,l} \quad \text{for} \quad i = p+1, \ldots N. \tag{33}$$

Finally, the N kinematical variables can be expressed as a function of both the (constant) parameters v_i and A_{ij} as follows:

$$\dot{u}_i = v_i \quad \text{for} \quad i = 1, \ldots p, \tag{34a}$$

$$\dot{u}_i = -\sum_{l=1}^{p} v_l\ A_{i-p,l} \quad \text{for} \quad i = p+1, \ldots N. \tag{34b}$$

As a symmetric, real matrix, $\overline{\overline{\Lambda}}^s$ is diagonalizable with all eigenvalues being real. If all eigenvalues of $\overline{\overline{\Lambda}}^s$ are strictly positive, W_2^{int} is a strictly positive quadratic form. If $\overline{\overline{\Lambda}}^s$ admits p negative eigenvalues λ^k, let $\bar{B}^k$ be the p-associated eigenvectors. Then, by selecting the $N-p$ vectors $\bar{A}^i$ orthogonal to the p vectors $\bar{B}^k$, if the $N-p$ kinematic loading conditions $\dot{L}^i = 0$ are prescribed, with $L^i = A_{i1}\ u_1 + \cdots + A_{iN}\ u_N$ for $i = p+1, \ldots N$, Eq. (10a) hold. By virtue of Eq. (32), the internal second-order work, given by Eq. (30), reads:

$$W_2^{\text{int}} = \sum_{k=1}^{N}\sum_{l=1}^{N} \Lambda_{kl}^s\ \dot{u}_k\ \dot{u}_l = \sum_{k=1}^{p}\sum_{l=1}^{p} \alpha_k\ \alpha_l\ \Lambda_{ij}^s\ B_{kj}\ B_{li} \tag{35}$$

which gives, as $\Lambda_{ij}^s\ B_{kj} = \Lambda_{ij}^s\ B_j^k = \lambda^k\ B_i^k = \lambda^k\ B_{ki}$:

$$W_2^{\text{int}} = \sum_{k=1}^{p}\sum_{l=1}^{p} \alpha_k\ \alpha_l\ \lambda^k\ B_{ki}\ B_{li}. \tag{36}$$

As the vectors $\bar{B}^k$ are orthogonal to one another (the eigen subspaces of a symmetric, real matrix are orthogonal), then $B_{ki}\ B_{li} = \delta_{kl}\ |B^k|\ |B^l|$, where δ_{kl} is the Kronecker symbol. Finally, Eq. (36) is expressed as:

$$W_2^{\text{int}} = \sum_{k=1}^{p} \lambda^k\ \alpha_k^2\ \left|B^k\right|^2 = \sum_{k=1}^{p} \lambda^k\ v_k^2\ \left|B^k\right|^2. \tag{37}$$

As the p eigenvalues λ^k are negative, W_2^{int} is strictly negative. Thus, in these loading conditions, with this choice of control parameters, both external and internal second-order works are equal (Eq. 30) and negative.

If the same loading conditions are applied ($\dot{L}^i = 0$ are prescribed, with $L^i = A_{i1}\ u_1 + \cdots + A_{iN}\ u_N$ for $i = p+1, \ldots N$), but by changing the control parameters C^i (for $i = 1, \ldots p$) into $C^i = f_i - \sum_{j=p+1}^{N} A_{j-p,i}\ f_j$, with $\dot{C}^i$ being constant (positive), proposition 1 implies that the expression of the internal second-order work is unchanged and is still given by Eq. (37). The internal second-order work is negative.

On the other hand, this choice of control parameters results in both force rates and velocities no longer being related by the boundary operator. Thus, the external second-order work is no longer given by Eq. (26). It is given by Eq. (20): $W_2^{\text{ext}} = \sum_{i=1}^{p} \dot{u}_i\ (\dot{f}_i - \sum_{j=p+1}^{N} A_{j-p,i}\ \dot{f}_j)$.

The control parameters C^i (for $i = 1, \ldots p$) are prescribed. As we consider loading conditions, the terms $\dot{C}^i$ are constant and positive (they would be negative if unloading conditions were considered). The p kinematical terms $\dot{u}_i$ constitute a part of the response of the system. As the terms $\dot{u}_i$ are positive (as a result of the loading conditions), the external second-order work is strictly positive as well.

According to Eq. (6), if $W_2^{\text{ext}} > 0$ and $W_2^{\text{int}} < 0$, the second-order time derivative of the kinetic energy of the system is strictly positive. The response of the system is no longer quasi-static. The system evolves with inertial effects. The kinetic energy increases, with undefined (and possibly unbounded) values for the velocities $\dot{u}_i$ ($i = 1, \ldots p$).

3.4 Practical method

The second-order work formalism developed in the subsections above can be applied to any boundary value problem. Given a system, the question that arises is to determine whether a loading program exists that could lead to negative values of the internal second-order work. If so, an adequate choice of control parameters will lead the system to failure (characterized by an outburst in kinetic energy).

If the dimension (N) of the problem at hand is small enough (typically $N = 3$), the boundary stiffness operator can be specified and its spectral properties can be investigated. This enables checking whether this operator admits negative eigenvalues. However, for larger N-values, this method is certainly not convenient. In this case, the spectral properties of the boundary stiffness operator can be investigated indirectly, using a directional analysis. This method is exactly the same as what is done on the material point scale to characterize the spectral properties of $\overline{\overline{K}}^s$ (see [22,23,25]). Boundary velocities $\dot{u}^i$ are prescribed to the system, with the same norm ($\|\dot{\bar{u}}\| = \text{const}$), in all the directions within the related space. The rate force response $\dot{f}_i$ is determined, and the external second-order work is computed as $W_2^{\text{ext}} = \sum_{i=1}^{N} \dot{u}_i \ \dot{f}_i$. Then the sign of the external second-order work can be explored as a function of the loading direction, and the existence of negative eigenvalues is examined. If the second-order work takes negative values along certain loading directions, at least one negative eigenvalue exists. Then the existence of a proper loading program leading to an increase in kinetic energy is guaranteed.

This approach was presented in a general framework including any p control parameters, with p possibly larger than 1. This situation may arise in complex systems, particularly in civil engineering where the loading can be controlled (that is to say, the evolution of the loading over time can be controlled) by different variables operating at different areas of the structure.

However, for the sake of simplicity, the approach is exemplified in the following section by considering the usual case $p = 1$. Two examples are presented: a homogeneous laboratory test and an engineering boundary value problem.

4 Engineering applications

4.1 The homogeneous triaxial laboratory test

The particularization of this framework to the case of homogeneous material specimens is worthy of interest, because it corresponds to the laboratory specimen scale where (for instance) parallelepiped-like specimens subjected, on each wall, to a prescribed force or displacement directing both stress and strain fields are studied. Investigating this elementary scale can be useful in, for example, the interpretation of the derived results of experimental tests. Both strain and stress fields are homogeneous. The external forces applied to the boundary of the specimen are related to the average stress tensor, and the displacements of each point of the boundary are related to the average strain tensor. Both strain and stress states are fully characterized with both forces and displacements measured on the boundary.

Experimental tests with homogeneous specimens allow a constitutive relation to be developed, expressed on the material point scale, when the internal fields are directly related to accessible boundary variables.

Let a parallelepiped specimen be considered. Each side 'i' admits a normal $\bar{N}^i$ that coincides with the directions $\bar{v}^i$ of a fixed reference frame. The initial area of each side 'i' is denoted S^i and the initial length of each edge is denoted L^i, with $i = 1, \ldots 3$. Index '1' refers to the axial direction, whereas indices '2' and '3' refer to the two lateral directions perpendicular to the axial direction (Fig. 1). When a static condition is assigned to a side 'i', it is convenient to introduce the resultant external force f_i acting on this side. This force is assigned to be normal to the side considered. The uniform external Lagrangian stress vector distribution s_i acting on side 'i' and related to f_i is also introduced: $s_i = f_i/S^i$. The displacement of each side 'i', along the direction $\bar{v}^i$, is denoted U_i. No tangential displacement is assumed to take place. When a kinematic condition is assigned to a side 'i', the resultant external force f_i (or the stress vector distribution s_i) acting on this side corresponds to the external loading that must be applied to ensure the prescribed displacement U_i.

In these conditions, the displacement $\bar{u}$ of any point $M(X_1, \ X_2, \ X_3)$ reads in the frame $\{O, \ \bar{v}^1, \ \bar{v}^2, \ \bar{v}^3\}$:

$$\bar{u} = \sum_{i=1}^{3} \frac{X_i}{L^i} \ U_i \ \bar{v}^i, \tag{38}$$

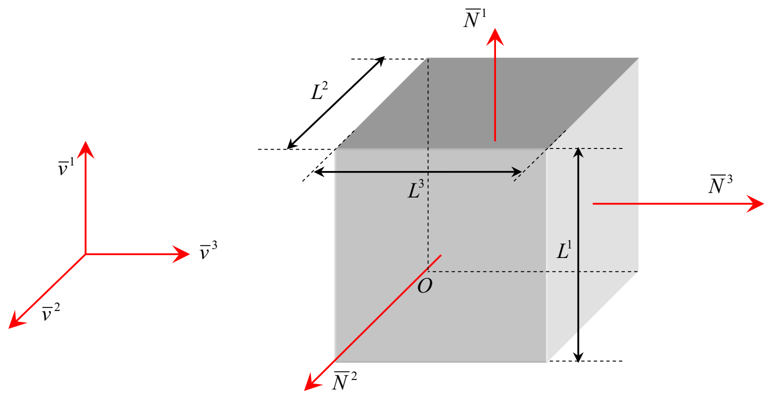

Fig. 1 Parallelepiped specimen and definition of the axes

which gives, as $F_{ij} = \delta_{ij} + \partial u_i/\partial X_j$:

$$\dot{\overline{\overline{F}}} = \begin{bmatrix} \dot{U}_1/L^1 & 0 & 0 \\ 0 & \dot{U}_2/L^2 & 0 \\ 0 & 0 & \dot{U}_3/L^3 \end{bmatrix}. \tag{39}$$

Thus, in homogeneous conditions Eq. (6) is written:

$$\ddot{E}_c = I_2 + \dot{f}_1\, \dot{U}_1 + \dot{f}_2\, \dot{U}_2 + \dot{f}_3\, \dot{U}_3\, - W_2^{\text{int}}. \tag{40}$$

Furthermore, the internal second-order work simplifies in homogeneous conditions as:

$$W_2^{\text{int}} = V_o\, \left(\dot{\Pi}_{11}\, \dot{F}_{11} + \dot{\Pi}_{22}\, \dot{F}_{22} + \dot{\Pi}_{33}\, \dot{F}_{33}\right). \tag{41}$$

Recalling that $V_o = S^1\, L^1 = S^2\, L^2 = S^3\, L^3$ and by virtue of Eq. (39), combining Eqs. (40) and (41) yields:

$$\ddot{E}_c = I_2 + \sum_{i=1}^{3} \left(\dot{f}_i - \dot{\Pi}_{ii}\, S^i\right)\, \dot{U}_i. \tag{42}$$

As already pointed out by Nicot [25], the increase in kinetic energy is related to a conflict between the loading $(\dot{\bar{f}})$ applied to the boundary of the specimen and the internal stress $(\dot{\overline{\overline{\Pi}}})$ that the material can develop in relation with its constitutive properties.

During a quasi-static evolution, no increase in kinetic energy is expected. The different lateral walls are in equilibrium, which gives $\dot{f}_i = \dot{\Pi}_{ii}\, S^i$ (for $i = 1, \ldots 3$). External forces are exactly balanced by the internal forces resulting from the internal stress, which also means that the internal stress can be assessed from the measurement of forces acting on the boundary. In this situation, the system stiffness matrix $\overline{\overline{\Lambda}}$ can be defined. According to Eq. (38), we have:

$$\overline{\overline{K}} = \overline{\overline{S}}\, \overline{\overline{\Lambda}}\, \overline{\overline{L}} \tag{43}$$

with $\overline{\overline{S}} = \begin{bmatrix} 1/S_1 & 0 & 0 \\ 0 & 1/S_2 & 0 \\ 0 & 0 & 1/S_3 \end{bmatrix}$ and $\overline{\overline{L}} = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}$.

The system stiffness tensor $\overline{\overline{\Lambda}}$ is proportional to the standard constitutive tensor $\overline{\overline{K}}$. In particular, both matrices have the same eigen properties.

As an illustration, the following example can be considered. We assume that the constitutive operator $\overline{\overline{K}}$ is known, and that the symmetric part $\overline{\overline{K}}^s$ admits one negative eigenvalue λ. The associated eigen subspace is assumed to be a vectorial line, defined by the vector $\bar{B} = (1,\ -B_2,\ -B_3)$. According to Eqs. (15a–c), $\bar{B}$ can be completed by the vectors $\bar{A}^1 = (B_2,\ 1,\ 0)$ and $\bar{A}^2 = (B_3,\ 0,\ 1)$ to form a base of R^3, with ${}^t\bar{B}\ \bar{A}^1 = 0$ and ${}^t\bar{B}\ \bar{A}^2 = 0$. Then, as developed in Sect. 3, let the following loading program be defined ($N = 3$, $p = 1$):

$$C_1 = U_1, \quad \text{with} \quad \dot{U}_1 = \text{const.} \quad \text{(kinematic control parameter)}, \tag{44a}$$

$$L_2 = \sum_{i=1}^{3} A_i^1\ U_i = B_2\ U_1 + U_2, \quad \text{with} \quad \dot{L}_2 = 0 \quad \text{(loading path)}, \tag{44b}$$

$$L_3 = \sum_{i=1}^{3} A_i^2\ U_i = B_3\ U_1 + U_3, \quad \text{with} \quad \dot{L}_3 = 0 \quad \text{(loading path)}. \tag{44c}$$

This loading program corresponds to the standard proportional strain loading path. Indeed, according to Eq. (39), we have:

$$\dot{F}_{11} = \text{const}, \tag{45a}$$

$$B_2\ \dot{F}_{11} + \dot{F}_{22} = 0, \tag{45b}$$

$$B_3\ \dot{F}_{11} + \dot{F}_{33} = 0. \tag{45c}$$

The response parameters are:

$$R_1 = f_1 - A_1^1\ f_2 - A_1^2\ f_3 = f_1 - B_2\ f_2 - B_3\ f_3, \quad R_2 = f_2 \text{ and } R_3 = f_3$$

so that, according to Eq. (37), the external second-order work is expressed as:

$$W_2^{\text{ext}} = \dot{U}_1\ \left(\dot{f}_1 - B_2\ \dot{f}_2 - B_3\ \dot{f}_3\right). \tag{46}$$

Furthermore, over this quasi-static loading program, the internal second-order work is expressed as $W_2^{\text{int}} = {}^t\dot{\bar{U}}\ \overline{\overline{\Lambda}}^s\ \dot{\bar{U}}$. As $\dot{L}_2 = 0$ and $\dot{L}_3 = 0$, $\dot{\bar{U}}$ is normal to both vectors $\bar{v}_1 = {}^t\left[B_2\ 1\ 0\right]$ and $\bar{v}_2 = {}^t\left[B_3\ 0\ 1\right]$. Thus, $\dot{\bar{U}} = \alpha\ \bar{v}_1 \times \bar{v}_2$, with $\bar{v}_1 \times \bar{v}_2 = \bar{B}$, where $\bar{B}$ is the eigenvector associated with the negative eigenvalue of $\overline{\overline{\Lambda}}^s$. The internal second-order work is therefore negative. As $W_2^{\text{int}} = W_2^{\text{ext}}$, Eq. (46) implies that $R_1 = f_1 - B_2\ f_2 - B_3\ f_3$ follows a descending branch ($\dot{R}_1 < 0$).

The loading control can be changed into a force control, as follows:

$$C_1 = f_1 - B_2\ f_2 - B_3\ f_3, \quad \text{with} \quad \dot{C}_1 > 0 \quad \text{(force control parameter)}, \tag{47a}$$

$$L_2 = \sum_{i=1}^{3} A_i^1\ U_i = B_2\ U_1 + U_2, \quad \text{with} \quad \dot{L}_2 = 0 \quad \text{(loading path)}, \tag{47b}$$

$$L_3 = \sum_{i=1}^{3} A_i^2\ U_i = B_3\ U_1 + U_3, \quad \text{with} \quad \dot{L}_3 = 0 \quad \text{(loading path)}. \tag{47c}$$

The loading path is unchanged, which guarantees that the internal second-order work is unchanged as well. Furthermore, Eq. (46) shows that the external second-order work, due to the force control imposed by the experimentalist, is strictly positive.

As a result, Eq. (6) reveals that an increase (outburst) in kinetic energy should occur. In fact, the response of the specimen is no longer quasi-static, but turns out to be dynamic. This transition characterizes the occurrence of an effective failure [36]. This transition was ascertained from numerical simulations based on a discrete element model (open-source code YADE, [37]). A triaxial loading path was prescribed to a numerical granular specimen made up of an assembly of contacting spheres. For the sake of simplicity, axisymmetric conditions were imposed, and the particular isochoric loading direction was considered: $B_2 = B_3 = 1$. Then, as shown by different authors [12,21,25], when turning the loading conditions from a strain-controlled mode to a stress-controlled mode, an abrupt increase in kinetic energy is observed until the total collapse of the specimen (Fig. 2). As can be seen in Fig. 3, the increase in kinetic energy stems from the difference between both external and internal second-order works. The external second-order work increases with positive values, whereas the internal second-order work decreases (on average) with negative values (Fig. 3) (Table 1).

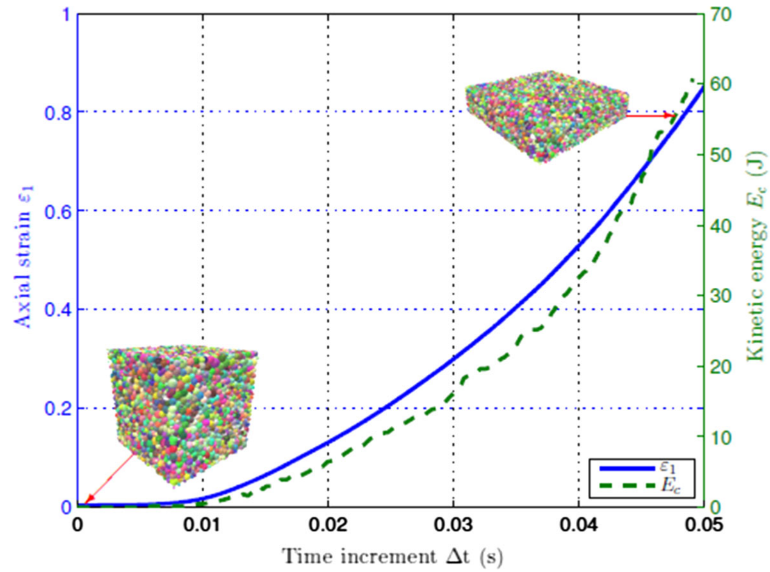

Fig. 2 DEM simulation of an undrained triaxial test: kinetic energy explosion under stress-controlled conditions (after [21])

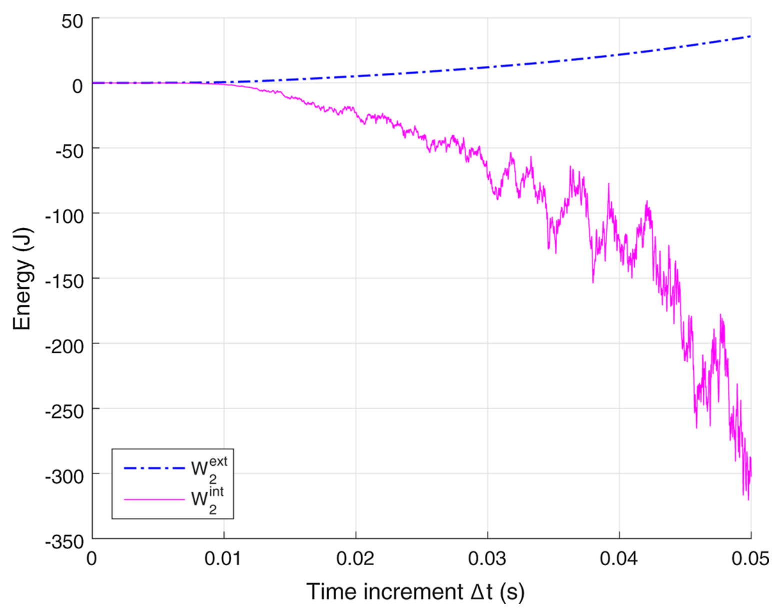

Fig. 3 DEM simulation of an undrained triaxial test: kinetic energy explosion under stress-controlled conditions (after [21])

4.2 The case of a shallow foundation

A non-homogeneous boundary value problem is considered in this section. We analyze the behavior of a soil body loaded by a shallow foundation. The problem is assumed to be two-dimensional and is modeled as described in Fig. 4. A rectangular domain of soil is considered; one part *JK* of length L of the upper side is subjected to a controlled downward vertical displacement denoted U_1, whereas the two deformable adjoining parts (*IJ* and *KL*) are free: $f_5 = f_6 = 0$ (zero tensile force is prescribed). The other three rigid sides (*LM*, *MN* and *IN*) are restricted from undergoing displacement: Horizontal and vertical displacements are nil.

The following loading program is therefore prescribed to the system:

The reaction force applied by the soil to the foundation is denoted f_1. This force evolves continuously with the vertical displacement U_1. According to the loading path applied, the external second-order work takes the straightforward form

Table 1 Definition of the loading program

Boundary section	Boundary condition	Loading program
JK	$C_1 = U_1, \dot{C}_1 > 0$	Kinematic control
IN	$L_2 = U_2, \dot{L}_2 = 0$	Loading path
MN	$L_3 = U_3, \dot{L}_3 = 0$	Loading path
LM	$L_4 = U_4, \dot{L}_4 = 0$	Loading path
IJ	$L_5 = f_5, \dot{L}_5 = 0$	Loading path
KL	$L_6 = f_6, \dot{L}_6 = 0$	Loading path

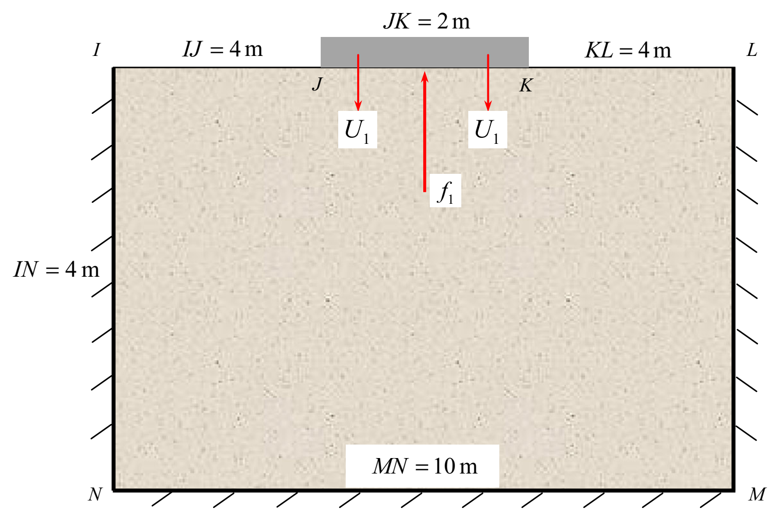

Fig. 4 Simulation of the settlement under a shallow foundation

$$W_2^{\text{ext}} = \dot{U}_1 \; \dot{f}_1. \tag{48}$$

In order to compute the evolution of the external second-order work over the loading path, this problem was simulated by means of a finite element method [16] using the PLASOL elastic–plastic model for soil [1]. A comprehensive review of this method can be found in Prunier et al. [30], and Lignon et al. [19]. The curve $f_1\,(U_1)$ increases until a peak is reached, and then decreases (Fig. 5). After the peak, $\partial f_1/\partial U_1 < 0$. As $\dot{f}_1 = (\partial f_1/\partial U_1)\;\dot{U}_1$, we have $\dot{f}_1 < 0$, which requires that the external second-order work W_2^{ext} be negative (Fig. 6). As the loading path is kinematically controlled, the internal second-order work W_2^{int} equals the external second-order work. W_2^{int} is therefore negative along this loading path, irrespective of the control adopted.

Imagine that after the peak (point C, Figs. 5, 6), the loading turns out to be force-controlled: A rate force $\dot{f}_1$ is imposed on the foundation, with $\dot{f}_1 > 0$. From a practical point of view, this situation arises when additional materials (soil or structure) are deposited above the foundation. The response parameter, U_1, is such that $\dot{U}_1 > 0$. As a result, the external second-order work given by Eq. (48) is strictly positive, whereas the internal second-order work remains unchanged and negative.

As a result, the kinetic energy is strictly positive. The soil fails under the foundation. This was ascertained from a numerical simulation based on a finite element method using the PLASOL elastoplastic model for soil. Given that LAGAMINE software considers static balance equations, omitting inertial terms, the software is no longer able to converge toward a solution. This numerical divergence is in fact related to a transition from a static to a dynamic regime (with a sudden increase in kinetic energy) that cannot be simulated by the software. This absence of convergence demonstrates the occurrence of a failure, as predicted from the second-order work approach.

5 Concluding remarks

This manuscript has revisited the notion of the second-order work, introduced more than half a century ago, by developing a global approach for continuous systems based on the relation between the second-order time

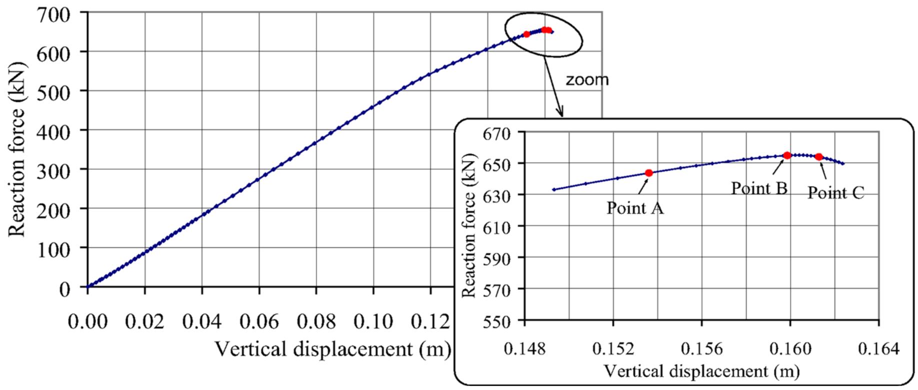

Fig. 5 Reaction force applied to the foundation

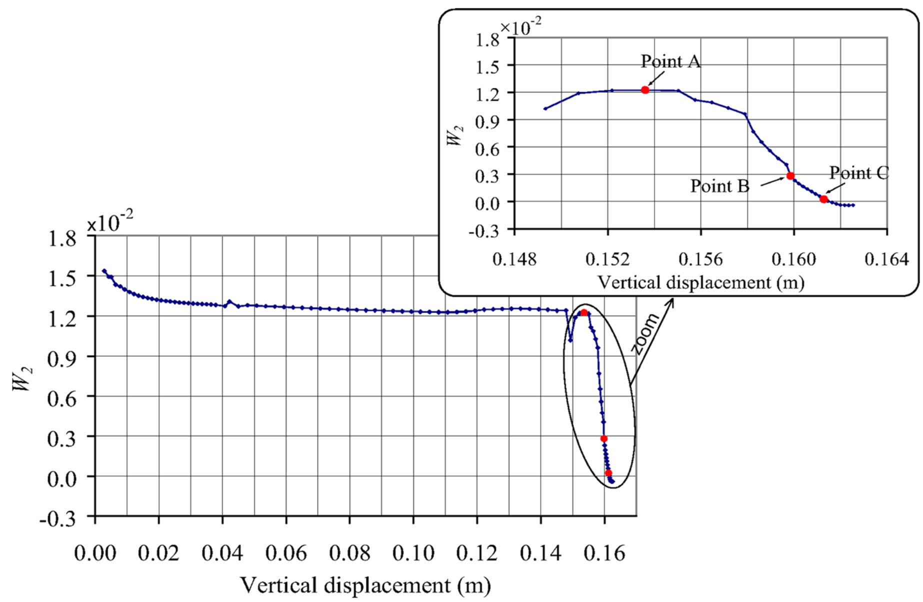

Fig. 6 Normalized external second-order work

derivative of the kinetic energy and the difference between the external second-order work and the internal second-order work.

The external second-order work involves the loading variables acting on the boundary of the system. For continuous systems, in a quite natural way, these variables make a so-called system stiffness tensor emerge, by relating external displacements and forces. In homogenous situations, when the external loading is balanced by the internal stress, this tensor is proportional to the usual constitutive tensor.

The destabilization of a continuous system by divergence can be provoked by adequate loading path and control variables that make the mechanical response of the system follow various critical directions. These directions are defined from the eigenvectors related to the negative eigenvalues of the symmetric part of the system's stiffness tensor.

The advantage of this approach, involving the boundary stiffness tensor, is that only the loading variables acting on the boundary of the system are necessary. No internal information (internal stress or strain fields) is required. Usually, failure analysis for boundary value problems requires computing the internal second-order work, and therefore both internal stress and strain fields. It is notable that the stability analysis of any system can be carried out from only the boundary information (velocity and rate force distribution). From a practical point of view, this method is generally very convenient, since the dimension of the problem (number of boundary variables) is usually not very large.

Finally, the notions of local (in the small) and asymptotic (in the large) instability were distinguished. In the general case of (rate-independent) incrementally nonlinear systems, the approach proposed in this manuscript can only arbitrate on the stability features in the small, namely over a finite, short time range. For most civil engineering applications, the notion of instability in the small is sufficient, since failure affecting soil bodies or structures occurs mainly over small time ranges.

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