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Elasticity solutions for a transversely isotropic functionally graded annular sector plate

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Abstract This paper presents three-dimensional elasticity solutions for an annular sector plate made of transversely isotropic functionally graded material (FGM) subjected to concentrated forces $(X, Y, 0)$ or couples (M_X, M_Y, M_Z) applied at one of its radial edges. The elastic coefficients can vary arbitrarily through the plate thickness. The analysis was based on the assumed forms of displacements for bending of an FGM plate (Mian and Spencer in J Mech Phys Solids 4:2283–2295, 1998), in which the four analytical functions were constructed properly. Appropriate boundary conditions and end conditions similar to those in the classic plate theory were employed to determine the unknown constants contained in the analytical functions so as to accomplish the analysis. When the material coefficients are all constant, the obtained analytical solutions can be degenerated into those for a homogeneous transversely isotropic annular sector plate, which have never been reported before. The solutions may be further reduced to those for a homogeneous isotropic annular sector plate, among which the ones for concentrated couples $(M_X, M_Y, 0)$ are also new to the literature.

1 Introduction

In resent years, functionally graded materials (FGMs) have been developed rapidly, where material properties vary continuously in one or more directions according to a specific profile. Nowadays, FGMs are applied in many fields, such as aviation, aerospace, electronics, chemistry, nuclear energy, and biomedicine, and have shown significant application prospects. A lot of papers have been published on the bending of FGM plates by analytical and numerical methods [1–15]. It is noted that England and Spencer [16] developed a complex

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variable method for the bending analysis of inhomogeneous isotropic and laminated elastic plates. Four complex potentials (analytical functions) were suggested to represent the solutions. By the complex variable method, England [17,18] presented bending solutions for FGM plates under transverse biharmonic or higher-order harmonic loads acting on the plate upper surface. Yang et al. [19] extended the above method to uniformly loaded transversely isotropic FGM annular plates. Yang et al. [20,21] further derived elasticity solutions for FGM rectangular and annular plates subjected to biharmonic loads. Based on the works of Yang et al. [19,20], Huang et al. [22] investigated an infinite transversely isotropic functionally graded sectorial plate subjected to a concentrated force or couple at the tip.

By using the displacement expansions in Refs. [5,17], this paper considers the deformation of an FGM annular sector plate, with its upper and lower surfaces free from tractions but subjected to concentrated forces $(X, Y, 0)$ and couples (M_X, M_Y, M_Z) that are applied at one of its radial edges. The other radial edge is fixed, and the two arc boundaries are free. The elastic coefficients can vary arbitrarily through the plate thickness.

2 Basic equations and boundary conditions

For convenience, we shall use two coordinate systems in the following analysis, i.e., the rectangular Cartesian coordinates and the cylindrical coordinates. The planes Oxy and $Or\theta$ are taken to be identical, and they are coincident with the mid-plane of the plate. In the cylindrical coordinate system, the region of the annular sector plate is: $a \le r \le b$, $0 \le \theta \le \alpha$, $-h/2 \le z \le h/2$, as shown in Fig. 1. Both the concentrated forces $(X, Y, 0)$ and the concentrated couples (M_X, M_Y, M_Z) act on the middle point C on the radial edge between point A $(r = a)$ and point B $(r = b)$. The positive directions of these external forces and couples are coincident with the positive directions of the corresponding axes.

In the Cartesian coordinate system, the equilibrium equations for a three-dimensional (3D) elastic body in the absence of body forces are

$$\sigma_{ij,j} = 0 \tag{1}$$

where $i, j = x, y, z$, and the Einstein convention for repeated indices is adopted. The stress–displacement relations for transversely isotropic materials [23] are

$$\begin{aligned} &\sigma_x = c_{11}u_{,x} + c_{12}v_{,y} + c_{13}w_{,z}, \quad \sigma_y = c_{12}u_{,x} + c_{11}v_{,y} + c_{13}w_{,z}, \quad \tau_{zx} = c_{44}\left(w_{,x} + u_{,z}\right), \\ &\sigma_z = c_{13}u_{,x} + c_{13}v_{,y} + c_{33}w_{,z}, \quad \tau_{yz} = c_{44}\left(v_{,z} + w_{,y}\right), \quad \tau_{xy} = c_{66}\left(u_{,y} + v_{,x}\right) \end{aligned} \tag{2}$$

where u, v, and w are the displacement components, and c_{ij} are the elastic stiffness coefficients with a constraint $2c_{66} = c_{11} - c_{12}$. For FGMs, $c_{ij} = c_{ij}(z)$. If $c_{11} = c_{33}$, $c_{12} = c_{13}$, and $c_{44} = c_{66}$, the material becomes isotropic.

According to the plate theory of Mian and Spencer [5], we take the solutions to Eqs. (1) and (2) as follows:

$$u\,(x, y, z) = \bar{u} + R_1\Delta_{,x} + R_0\bar{w}_{,x} + R_2\nabla^2\bar{w}_{,x},$$

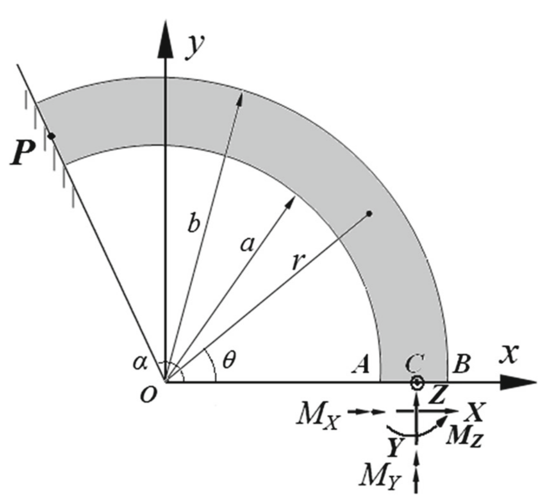

Fig. 1 The annular sector plate, loadings and coordinate systems

$$v(x,y,z) = \bar{v} + R_1\Delta_{,y} + R_0\bar{w}_{,y} + R_2\nabla^2\bar{w}_{,y},$$
$$w(x,y,z) = \bar{w} + T_1\Delta + T_2\nabla^2\bar{w} \tag{3}$$

where R_0, R_1, R_2, T_1, T_2 are functions of z, $\bar{u}$, $\bar{v}$, and $\bar{w}$ are the mid-plane displacement components, and

$$\bar{u} = \bar{u}(x,y), \quad \bar{v} = \bar{v}(x,y), \quad \bar{w} = \bar{w}(x,y), \tag{54.1,2}$$

$$\Delta = \bar{u}_{,x} + \bar{v}_{,y}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{54.3,4}$$

Substituting Eq. (3) into Eq. (2) and then into Eq. (1), and using the stress boundary conditions on the upper and lower surfaces of the plate, i.e., $\tau_{xz}(x,y,\pm h/2) = \tau_{yz}(x,y,\pm h/2) = 0$, $\sigma_z(x,y,-h/2) = 0$, $\sigma_z(x,y,h/2) = p(x,y)$, we can obtain the analytical expressions of $R_0(z), \ldots, T_2(z)$ [24]. Then, the initial 3D problem is converted into a two-dimensional (2D) problem, which involves the determination of the three mid-plane displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$.

According to the complex variable formulations in Ref. [17], Yang et al. [20] obtained, when $p(x,y) = 0$, the following expressions for mid-plane displacements and resultant forces:

$$\bar{w} = \bar{\zeta}\beta(\zeta) + \zeta\overline{\beta}(\zeta) + \alpha(\zeta) + \overline{\alpha}(\zeta), \tag{5.1}$$
$$\bar{u} + i\bar{v} = \frac{\kappa_1+1}{\kappa_1-1}\phi(\zeta) - \zeta\overline{\phi'}(\zeta) - \overline{\psi}(\zeta) - 2\frac{\kappa_2}{\kappa_1}\left[\beta(\zeta) + \zeta\overline{\beta'}(\zeta)\right], \tag{5.2}$$
$$N_x + N_y = a_1\left[\phi'(\zeta) + \overline{\phi'(\zeta)}\right] + 4a_2\left[\beta'(\zeta) + \overline{\beta'(\zeta)}\right], \tag{5.3}$$
$$N_y - N_x + 2iN_{xy} = a_1\left[\bar{\zeta}\phi''(\zeta) + \psi'(\zeta)\right] - a_5\phi'''(\zeta) + 4a_2\bar{\zeta}\beta''(\zeta)$$
$$+2a_6\alpha''(\zeta) - a_7\beta'''(\zeta), \tag{5.4}$$
$$M_x + M_y = -b_1\left[\phi'(\zeta) + \overline{\phi'(\zeta)}\right] + 4b_2\left[\beta'(\zeta) + \overline{\beta'(\zeta)}\right], \tag{5.5}$$
$$M_y - M_x + 2iM_{xy} = a_6\left[\bar{\zeta}\phi''(\zeta) + \psi'(\zeta)\right] - b_5\phi'''(\zeta) + b_6\bar{\zeta}\beta''(\zeta) + b_7\alpha''(\zeta) - b_8\beta'''(\zeta), \tag{5.6}$$
$$Q_{xz} - iQ_{yz} = -(b_1 + a_6)\phi''(\zeta) + (4b_2 - b_6)\beta''(\zeta) \tag{5.7}$$

where $\zeta = x + iy$ and $\bar{\zeta} = x - iy$; the prime denotes derivative with respect to ζ, $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$, and $\psi(\zeta)$ are analytical functions of the complex variable ζ; and $a_1, a_2, a_5, a_6, a_7, b_1, b_2, b_5, b_6, b_7, b_8, \kappa_1$ and κ_2 are constants related to the elastic coefficients [20], and

$$\left(N_x, N_y, N_{xy}\right) = \int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \tau_{xy}\right)\mathrm{d}z, \quad \left(M_x, M_y, M_{xy}\right) = \int_{-h/2}^{h/2}\left(\sigma_x, \sigma_y, \tau_{xy}\right)z\mathrm{d}z,$$
$$\left(Q_{xz}, Q_{yz}\right) = \int_{-h/2}^{h/2}\left(\tau_{xz}, \tau_{yz}\right)\mathrm{d}z. \tag{6}$$

The boundary conditions for the problem studied are as summarized in the following:

(i) The two arc edges ($r = a$ and $r = b$) are free:

$$N_r(a,\theta) = 0, N_{r\theta}(a,\theta) = 0, \quad M_r(a,\theta) = 0, \quad V_r(a,\theta) = 0,$$
$$N_r(b,\theta) = 0, N_{r\theta}(b,\theta) = 0, \quad M_r(b,\theta) = 0, \quad V_r(b,\theta) = 0 \tag{7}$$

where V_r, the effective shear force, is defined as

$$V_r = Q_{rz} + \frac{\partial M_{r\theta}}{r\partial\theta}. \tag{8}$$

(ii) At the radial edge ($\theta = 0$), the following equivalent boundary conditions in the sense of Saint-Venant approximation are employed:

$$-\int_a^b N_{r\theta}(r,0)\,\mathrm{d}r = X, \qquad -\int_a^b N_\theta(r,0)\,\mathrm{d}r = Y, \tag{9.1,2}$$

$$-\int_a^b Q_{\theta z}(r,0)\,\mathrm{d}r + M_{r\theta}(b,0) - M_{r\theta}(a,0) = 0, \tag{9.3}$$

$$\int_a^b M_\theta(r,0)\,\mathrm{d}r = M_X, \tag{10.1}$$

$$\int_a^b \left(r - \frac{a+b}{2}\right) Q_{\theta z}(r,0)\,\mathrm{d}r - \int_a^b M_{r\theta}(r,0)\,\mathrm{d}r - \frac{a-b}{2}\left[M_{r\theta}(a,0) + M_{r\theta}(b,0)\right] = M_Y, \tag{10.2}$$

$$-\int_a^b N_\theta(r,0)\cdot\left(r - \frac{a+b}{2}\right)\mathrm{d}r = M_Z. \tag{10.3}$$

In Eqs. (7)–(10), the internal forces $(N_r, N_\theta, N_{r\theta}, Q_{rz}, Q_{\theta z})$ and internal couples $(M_r, M_\theta, M_{r\theta})$ expressed in the cylindrical coordinate system are

$$(N_r, N_\theta, N_{r\theta}, Q_{rz}, Q_{\theta z}) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta, \tau_{r\theta}, \tau_{rz}, \tau_{\theta z})\,\mathrm{d}z, \quad (M_r, M_\theta, M_{r\theta}) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta, \tau_{r\theta})\,z\mathrm{d}z. \tag{11}$$

By using Eq. (9.2), Eq. (10.3) can be rewritten as

$$-\int_a^b N_\theta(r,0)\,rdr = M_Z + \frac{a+b}{2}Y. \tag{12}$$

$M_{r\theta}(b,0)$ and $M_{r\theta}(a,0)$ in Eqs. (9.3) and (10.2) are the concentrated forces in the z-direction at points B and A, resulting when calculating the effective shear force V_r.

(iii) The radial edge $\theta = \alpha$ is fixed.

3 Analytical solution

The four analytical functions $\phi(\zeta)$, $\beta(\zeta)$, $\psi(\zeta)$, and $\alpha(\zeta)$ are constructed here using the trial-and-error method. Enlightened by the stress solutions for an elastic curved bar [25], we try to find those analytical functions, from which the internal forces and couples are either independent of θ or proportional to $\sin\theta$ and $\cos\theta$. Thus, we may assume

$$\begin{aligned}
\phi(\zeta) &= (A_1 + iA_2)\zeta^2 + (C_1 + iC_2)\ln\zeta + A_0\zeta + C_0\zeta\ln\zeta,\\
\beta(\zeta) &= (B_1 + iB_2)\zeta^2 + (D_1 + iD_2)\ln\zeta + B_0\zeta + D_0\zeta\ln\zeta,\\
\psi(\zeta) &= (F_1 + iF_2)\zeta^{-2} + (E_1 + iE_2)\ln\zeta + F_0\zeta^{-1},\\
\alpha(\zeta) &= (G_1 + iG_2)\zeta^{-1} + (H_1 + iH_2)\zeta\ln\zeta + G_0\ln\zeta
\end{aligned} \tag{13}$$

where $A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2, A_0, B_0, C_0, D_0, F_0$, and G_0 are 22 undetermined real constants.

By virtue of Eqs. (5.3,4) and (13), we have

$$\begin{aligned}
N_r + N_\theta = N_x + N_y &= \left[4(a_1A_1 + 4a_2B_1)r + 2(a_1C_1 + 4a_2D_1)r^{-1}\right]\cos\theta\\
&\quad + \left[-4(a_1A_2 + 4a_2B_2)r + 2(a_1C_2 + 4a_2D_2)r^{-1}\right]\sin\theta\\
&\quad + 2a_1(A_0 + C_0) + 8a_2(B_0 + D_0) + 2(a_1C_0 + 4a_2D_0)\ln r,\\
N_\theta - N_r + 2iN_{r\theta} &= \left(N_y - N_x + 2iN_{xy}\right)e^{2i\theta}\\
&= \left\{\left[a_1(E_1 - C_1) + 2(a_6H_1 - 2a_2D_1)\right]r^{-1} + 2(a_1A_1 + 4a_2B_1)r\right.\\
&\quad \left.-2(a_1F_1 + a_5C_1 - 2a_6G_1 + a_7D_1)r^{-3}\right\}\cos\theta\\
&\quad - \left[2(a_1A_2 + 4a_2B_2)r + (a_1E_2 + 4a_2D_2 + 2a_6H_2 + a_1C_2)r^{-1}\right.\\
&\quad \left.+2(a_1F_2 + a_5C_2 - 2a_6G_2 + a_7D_2)r^{-3}\right]\sin\theta
\end{aligned} \tag{14}$$

$$
\begin{aligned}
&+a_1 C_0 + 4a_2 D_0 + (-a_1 F_0 + a_5 C_0 - 2a_6 G_0 + a_7 D_0)\, r^{-2} \\
&+ \left[2\,(a_1 A_2 + 4a_2 B_2)\, r + (a_1 E_2 - 4a_2 D_2 + 2a_6 H_2 - a_1 C_2)\, r^{-1}\right. \\
&\left.-2\,(a_1 F_2 + a_5 C_2 - 2a_6 G_2 + a_7 D_2)\, r^{-3}\right] i \cos\theta \\
&+ \left\{[a_1\,(E_1 + C_1) + 2\,(a_6 H_1 + 2a_2 D_1)]\, r^{-1} + 2\,(a_1 A_1 + 4a_2 B_1)\, r\right. \\
&\left.+2\,(a_1 F_1 + a_5 C_1 - 2a_6 G_1 + a_7 D_1)\, r^{-3}\right\} i \sin\theta. \qquad (15)
\end{aligned}
$$

From Eqs. (14) and (15), we obtain

$$
\begin{aligned}
&N_{r\theta} = N_{r\theta}^1(r)\sin\theta + N_{r\theta}^2(r)\cos\theta, \quad N_\theta = N_\theta^1(r)\cos\theta + N_\theta^2(r)\sin\theta + N_\theta^0(r), \\
&N_r = N_r^1(r)\cos\theta + N_r^2(r)\sin\theta + N_r^0(r) \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
N_{r\theta}^1(r) &= (a_1 A_1 + 4a_2 B_1)\, r + \frac{1}{2}(a_1 C_1 + 4a_2 D_1 + a_1 E_1 + 2a_6 H_1)\, r^{-1} \\
&\quad + (a_1 F_1 + a_5 C_1 - 2a_6 G_1 + a_7 D_1)\, r^{-3}, \qquad (17) \\
N_{r\theta}^2(r) &= (a_1 A_2 + 4a_2 B_2)\, r + \frac{1}{2}(-a_1 C_2 - 4a_2 D_2 + a_1 E_2 + 2a_6 H_2)\, r^{-1} \\
&\quad - (a_1 F_2 + a_5 C_2 - 2a_6 G_2 + a_7 D_2)\, r^{-3}, \qquad (18) \\
N_\theta^1(r) &= 3\,(a_1 A_1 + 4a_2 B_1)\, r + \frac{1}{2}[a_1\,(C_1 + E_1) + 4a_2 D_1 + 2a_6 H_1]\, r^{-1} \\
&\quad - (a_1 F_1 + a_5 C_1 - 2a_6 G_1 + a_7 D_1)\, r^{-3}, \qquad (19) \\
N_\theta^2(r) &= -3\,(a_1 A_2 + 4a_2 B_2)\, r + \frac{1}{2}(a_1 C_2 + 4a_2 D_2 - a_1 E_2 - 2a_6 H_2)\, r^{-1} \\
&\quad - (a_1 F_2 + a_5 C_2 - 2a_6 G_2 + a_7 D_2)\, r^{-3}, \qquad (20) \\
N_\theta^0(r) &= a_1 A_0 + \frac{3}{2}a_1 C_0 + 4a_2 B_0 + 6a_2 D_0 + (a_1 C_0 + 4a_2 D_0)\ln r \\
&\quad + \frac{1}{2}(-a_1 F_0 + a_5 C_0 - 2a_6 G_0 + a_7 D_0)\, r^{-2}, \qquad (21) \\
N_r^1(r) &= (a_1 A_1 + 4a_2 B_1)\, r + \frac{1}{2}(3a_1 C_1 - a_1 E_1 + 12a_2 D_1 - 2a_6 H_1)\, r^{-1} \\
&\quad + (a_1 F_1 + a_5 C_1 - 2a_6 G_1 + a_7 D_1)\, r^{-3}, \qquad (22) \\
N_r^2(r) &= -(a_1 A_2 + 4a_2 B_2)\, r + \frac{1}{2}(3a_1 C_2 + a_1 E_2 + 12a_2 D_2 + 2a_6 H_2)\, r^{-1} \\
&\quad + (a_1 F_2 + a_5 C_2 - 2a_6 G_2 + a_7 D_2)\, r^{-3}, \qquad (23) \\
N_r^0(r) &= a_1 A_0 + \frac{1}{2}a_1 C_0 + 4a_2 B_0 + 2a_2 D_0 + (a_1 C_0 + 4a_2 D_0)\ln r \\
&\quad + \frac{1}{2}(a_1 F_0 - a_5 C_0 + 2a_6 G_0 - a_7 D_0)\, r^{-2}. \qquad (24)
\end{aligned}
$$

Meanwhile, Eqs. (5.5,6) and (13) yield

$$
\begin{aligned}
M_r + M_\theta = M_x + M_y &= \left[4\,(-b_1 A_1 + 4b_2 B_1)\, r + 2\,(-b_1 C_1 + 4b_2 D_1)\, r^{-1}\right]\cos\theta \\
&\quad + \left[4\,(b_1 A_2 - 4b_2 B_2)\, r - 2\,(b_1 C_2 - 4b_2 D_2)\, r^{-1}\right]\sin\theta \\
&\quad -2b_1\,(A_0 + C_0) + 8b_2\,(B_0 + D_0) + 2\,(-b_1 C_0 + 4b_2 D_0)\ln r, \qquad (25) \\
M_\theta - M_r + 2i M_{r\theta} &= \left(M_y - M_x + 2i M_{xy}\right) e^{2i\theta} \\
&= \left\{[a_6\,(E_1 - C_1) + (b_7 H_1 - b_6 D_1)]\, r^{-1} + 2\,(a_6 A_1 + b_6 B_1)\, r\right. \\
&\quad \left.-2\,(a_6 F_1 + b_5 C_1 - b_7 G_1 + b_8 D_1)\, r^{-3}\right\}\cos\theta \\
&\quad + \left[-2\,(a_6 A_2 + b_6 B_2)\, r - (a_6 E_2 + b_7 H_2 + a_6 C_2 + b_6 D_2)\, r^{-1}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(a_6F_2+b_5C_2-b_7G_2+b_8D_2\right)r^{-3}\big]\sin\theta\\
&+a_6C_0+b_6D_0+\left(-a_6F_0+b_5C_0-b_7G_0+b_8D_0\right)r^{-2}\\
&+\big\{\left[a_6\left(E_1+C_1\right)+\left(b_7H_1+b_6D_1\right)\right]r^{-1}\\
&+2\left(a_6A_1+b_6B_1\right)r+2\left(a_6F_1+b_5C_1-b_7G_1+b_8D_1\right)r^{-3}\big\}\,i\sin\theta\\
&+\big[2\left(a_6A_2+b_6B_2\right)r+\left(a_6E_2+b_7H_2-a_6C_2-b_6D_2\right)r^{-1}\\
&-2\left(a_6F_2+b_5C_2-b_7G_2+b_8D_2\right)r^{-3}\big]\,i\cos\theta. \qquad (26)
\end{aligned}
$$

From Eqs. (25) and (26), we have

$$
\begin{aligned}
M_{r\theta}&=M^1_{r\theta}(r)\sin\theta+M^2_{r\theta}(r)\cos\theta,\quad M_\theta=M^1_\theta(r)\cos\theta+M^2_\theta(r)\sin\theta+M^0_\theta(r),\\
M_r&=M^1_r(r)\cos\theta+M^2_r(r)\sin\theta+M^0_r(r) \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
M^1_{r\theta}(r)&=\left(a_6A_1+b_6B_1\right)r+\frac{1}{2}\left(a_6C_1+b_6D_1+a_6E_1+b_7H_1\right)r^{-1}\\
&\quad+\left(a_6F_1+b_5C_1-b_7G_1+b_8D_1\right)r^{-3}, \qquad (28)\\
M^2_{r\theta}(r)&=\left(a_6A_2+b_6B_2\right)r+\frac{1}{2}\left(-a_6C_2-b_6D_2+a_6E_2+b_7H_2\right)r^{-1}\\
&\quad-\left(a_6F_2+b_5C_2-b_7G_2+b_8D_2\right)r^{-3}, \qquad (29)\\
M^1_\theta(r)&=\left[\left(-2b_1+a_6\right)A_1+\left(8b_2+b_6\right)B_1\right]r+\frac{1}{2}\left[-\left(2b_1+a_6\right)C_1\right.\\
&\quad\left.+\left(8b_2-b_6\right)D_1+a_6E_1+b_7H_1\right]r^{-1}\\
&\quad-\left(b_5C_1+b_8D_1+a_6F_1-b_7G_1\right)r^{-3}, \qquad (30)\\
M^2_\theta(r)&=\left[\left(2b_1-a_6\right)A_2-\left(8b_2+b_6\right)B_2\right]r\\
&\quad-\frac{1}{2}\left[\left(2b_1+a_6\right)C_2+\left(b_6-8b_2\right)D_2+a_6E_2+b_7H_2\right]r^{-1}\\
&\quad-\left(b_5C_2+b_8D_2-b_7G_2+a_6F_2\right)r^{-3}, \qquad (31)\\
M^0_\theta(r)&=-b_1A_0+4b_2B_0+\left(\frac{1}{2}a_6-b_1\right)C_0\\
&\quad+\left(4b_2+\frac{1}{2}b_6\right)D_0+\left(-b_1C_0+4b_2D_0\right)\ln r\\
&\quad+\frac{1}{2}\left(b_5C_0+b_8D_0-a_6F_0-b_7G_0\right)r^{-2}, \qquad (32)\\
M^1_r(r)&=\left[-\left(2b_1+a_6\right)A_1+\left(8b_2-b_6\right)B_1\right]r\\
&\quad+\frac{1}{2}\left[\left(a_6-2b_1\right)C_1+\left(8b_2+b_6\right)D_1-a_6E_1-b_7H_1\right]r^{-1}\\
&\quad+\left(a_6F_1+b_5C_1-b_7G_1+b_8D_1\right)r^{-3}, \qquad (33)\\
M^2_r(r)&=\left[\left(2b_1+a_6\right)A_2+\left(b_6-8b_2\right)B_2\right]r\\
&\quad+\frac{1}{2}\left[\left(a_6-2b_1\right)C_2+\left(8b_2+b_6\right)D_2+a_6E_2+b_7H_2\right]r^{-1}\\
&\quad+\left(b_5C_2+b_8D_2+a_6F_2-b_7G_2\right)r^{-3}, \qquad (34)\\
M^0_r(r)&=-b_1A_0+4b_2B_0-\left(\frac{1}{2}a_6+b_1\right)C_0+\left(4b_2-\frac{1}{2}b_6\right)D_0\\
&\quad+\left(-b_1C_0+4b_2D_0\right)\ln r-\frac{1}{2}\left(b_5C_0+b_8D_0-a_6F_0-b_7G_0\right)r^{-2}. \qquad (35)
\end{aligned}
$$

Also, Eqs. (5.7) and (13) give rise to

$$
Q_{rz}-i\,Q_{\theta z}=\left(Q_{xz}-i\,Q_{yz}\right)e^{i\theta}
$$

$$
\begin{aligned}
&= \left\{2\left[-\left(b_1+a_6\right) A_1+\left(4 b_2-b_6\right) B_1\right]+\left[\left(b_1+a_6\right) C_1-\left(4 b_2-b_6\right) D_1\right] r^{-2}\right\} \cos \theta \\
&\quad +\left[\left(2 b_1 A_2+2 a_6 A_2-8 b_2 B_2+2 b_6 B_2\right)+\left(b_1 C_2+a_6 C_2-4 b_2 D_2+b_6 D_2\right) r^{-2}\right] \sin \theta \\
&\quad +\left\{2\left[-\left(b_1+a_6\right) A_1+\left(4 b_2-b_6\right) B_1\right]-\left[\left(b_1+a_6\right) C_1-\left(4 b_2-b_6\right) D_1\right] r^{-2}\right\} i \sin \theta \\
&\quad +\left[\left(-2 b_1 A_2-2 a_6 A_2+8 b_2 B_2-2 b_6 B_2\right)+\left(b_1 C_2+a_6 C_2-4 b_2 D_2+b_6 D_2\right) r^{-2}\right] i \cos \theta \\
&\quad +\left[-\left(b_1+a_6\right) C_0+\left(4 b_2-b_6\right) D_0\right] r^{-1}. \qquad (36)
\end{aligned}
$$

From Eq. (36), we get

$$
Q_{rz}=Q_{rz}^1(r)\cos\theta+Q_{rz}^2(r)\sin\theta+Q_{rz}^0(r), \quad Q_{\theta z}=Q_{\theta z}^1(r)\sin\theta+Q_{\theta z}^2(r)\cos\theta \qquad (37)
$$

where

$$
Q_{rz}^1(r)=-2\left[\left(b_1+a_6\right)A_1-\left(4b_2-b_6\right)B_1\right]+\left[\left(b_1+a_6\right)C_1-\left(4b_2-b_6\right)D_1\right]r^{-2}, \qquad (38)
$$

$$
Q_{rz}^2(r)=2\left[\left(b_1+a_6\right)A_2-\left(4b_2-b_6\right)B_2\right]+\left[\left(b_1+a_6\right)C_2-\left(4b_2-b_6\right)D_2\right]r^{-2}, \qquad (39)
$$

$$
Q_{rz}^0(r)=\left[-\left(b_1+a_6\right)C_0+\left(4b_2-b_6\right)D_0\right]r^{-1}, \qquad (40)
$$

$$
Q_{\theta z}^1(r)=2\left[\left(b_1+a_6\right)A_1-\left(4b_2-b_6\right)B_1\right]+\left[\left(b_1+a_6\right)C_1-\left(4b_2-b_6\right)D_1\right]r^{-2}, \qquad (41)
$$

$$
Q_{\theta z}^2(r)=2\left[\left(b_1+a_6\right)A_2-\left(4b_2-b_6\right)B_2\right]+\left[-\left(b_1+a_6\right)C_2+\left(4b_2-b_6\right)D_2\right]r^{-2}. \qquad (42)
$$

Then, Eqs. (8), (37.1) and (27.1) lead to

$$
V_r=V_r^1(r)\cos\theta+V_r^2(r)\sin\theta+Q_{rz}^0(r) \qquad (43)
$$

where

$$
\begin{aligned}
V_r^1(r) &= Q_{rz}^1(r)+M_{r\theta}^1(r)r^{-1}=-\left(2b_1+a_6\right)A_1+\left(8b_2-b_6\right)B_1 \\
&\quad +\left[\left(\frac{3}{2}a_6+b_1\right)C_1+\left(\frac{3}{2}b_6-4b_2\right)D_1+\frac{1}{2}a_6E_1+\frac{1}{2}b_7H_1\right]r^{-2} \\
&\quad +\left(a_6F_1+b_5C_1-b_7G_1+b_8D_1\right)r^{-4}, \qquad (44)\\
V_r^2(r) &= Q_{rz}^2(r)-M_{r\theta}^2(r)r^{-1}=\left(2b_1+a_6\right)A_2+\left(b_6-8b_2\right)B_2 \\
&\quad +\left[\left(b_1+\frac{3}{2}a_6\right)C_2+\left(\frac{3}{2}b_6-4b_2\right)D_2-\frac{1}{2}a_6E_2-\frac{1}{2}b_7H_2\right]r^{-2} \\
&\quad +\left(a_6F_2+b_5C_2-b_7G_2+b_8D_2\right)r^{-4}. \qquad (45)
\end{aligned}
$$

Substituting Eqs. (16), (27), (37) and (43) into Eqs. (7), (9), (10.1,2) and (12), we can get the following expressions for the boundary conditions:

$$
N_r^1(a)=0,\quad N_{r\theta}^1(a)=0,\quad M_r^1(a)=0,\quad V_r^1(a)=0, \qquad (46.1\text{–}4)
$$

$$
N_r^2(a)=0,\quad N_{r\theta}^2(a)=0,\quad M_r^2(a)=0,\quad V_r^2(a)=0, \qquad (46.5\text{–}8)
$$

$$
N_r^1(b)=0,\quad N_{r\theta}^1(b)=0,\quad M_r^1(b)=0,\quad V_r^1(b)=0, \qquad (47.1\text{–}4)
$$

$$
N_r^2(b)=0,\quad N_{r\theta}^2(b)=0,\quad M_r^2(b)=0,\quad V_r^2(b)=0, \qquad (47.5\text{–}8)
$$

$$
N_r^0(a)=0,\quad M_r^0(a)=0,\quad Q_{rz}^0(a)=0, \qquad (48.1\text{–}3)
$$

$$
N_r^0(b)=0,\quad M_r^0(b)=0,\quad Q_{rz}^0(b)=0, \qquad (49.1\text{–}3)
$$

and

$$
-\int_a^b N_{r\theta}^2(r)\,\mathrm{d}r=X,\quad -\int_a^b\left[N_\theta^1(r)+N_\theta^0(r)\right]\mathrm{d}r=Y, \qquad (50.1,2)
$$

$$
-\int_a^b Q_{\theta z}^2(r)\,\mathrm{d}r+M_{r\theta}^2(b)-M_{r\theta}^2(a)=0, \qquad (50.3)
$$

$$
\int_a^b\left[M_\theta^1(r)+M_\theta^0(r)\right]\mathrm{d}r=M_X, \qquad (50.4)
$$

$$\int_a^b \left[\left(r-\frac{a+b}{2}\right)Q_{\theta z}^2(r)-M_{r\theta}^2(r)\right]\mathrm{d}r-\frac{a-b}{2}\left[M_{r\theta}^2(a)+M_{r\theta}^2(b)\right]=M_Y,$$
$$-\int_a^b\left[N_\theta^1(r)+N_\theta^0(r)\right]r\mathrm{d}r=M_Z+\frac{a+b}{2}Y. \tag{51}$$

4 Solutions for the algebraic equations

There are totally twenty eight equations in Eqs. (46)–(51), but with only twenty two undetermined constants. Hence, for these equations to have unique solutions, we can assure that there must be redundant equations or linear dependent equations among the twenty eight equations. For example, by substituting Eq. (40) into Eq. (48.3) or (49.3), we get

$$(-b_1+a_6)\,C_0+(4b_2-b_6)\,D_0=0. \tag{52}$$

Substitution of Eq. (52) into Eq. (40), we have $Q_{rz}^0(r)=0$. Thus, from Eq. (37), we obtain

$$Q_{rz}=Q_{rz}^1(r)\cos\theta+Q_{rz}^2(r)\sin\theta. \tag{53}$$

By virtue of Eqs. (19), (21), (32), and (42), and then using Eqs. (17), (24), (35), (39), and (52), respectively, we have

$$\int_a^b N_\theta^0(r)\mathrm{d}r=\left[rN_r^0(r)\right]_a^b,\qquad \int_a^b M_\theta^0(r)\mathrm{d}r=\left[rM_r^0(r)\right]_a^b, \tag{54.1,2}$$
$$\int_a^b rN_\theta^1(r)\mathrm{d}r=\left[r^2N_{r\theta}^1(r)\right]_a^b,\qquad \int_a^b Q_{\theta z}^2(r)\mathrm{d}r=\left[rQ_{rz}^2(r)\right]_a^b. \tag{54.3,4}$$

Substituting Eq. (54.4) into Eq. (50.3) and using Eqs. (46.8) and (47.8), we can prove that Eq. (50.3) is an identity. Substituting Eqs. (48.1) and (49.1) into Eq. (54.1), substituting Eqs. (48.2) and (49.2) into Eq. (54.2) and substituting Eqs. (46.2) and (47.2) into Eq. (54.3), we can find that the definite integrals on the left-hand side of Eqs. (54.1–3) are all zero. Thus, Eqs. (50.2), (51.1) and (51.3) can be simplified as

$$-\int_a^b N_\theta^1(r)\mathrm{d}r=Y, \tag{55}$$
$$\int_a^b M_\theta^1(r)\mathrm{d}r=M_X, \tag{56}$$
$$-\int_a^b rN_\theta^0(r)\mathrm{d}r=M_Z+\frac{a+b}{2}Y. \tag{57}$$

Hence, the left six equations in Eqs. (50) and (51) can be further simplified to five equations, i.e., Eqs. (50.1), (51.2), (55), (56), and (57). Now we have twenty seven equations (Eqs. (46)–(49), (50.1), (51.2), (55), (56) and (57)) to solve 22 undetermined constants. We can divide these twenty seven equations into three groups to solve related problems: The first group contains ten equations, i.e., Eqs. (46.1–4), (47.1–4), (55) and (56) to solve eight undetermined constants, $A_1, B_1, \ldots, H_1$; the second group contains ten equations including Eqs. (46.5–8), (47.5–8), (50.1) and (51.2) to solve eight unknowns, $A_2, B_2, \ldots, H_2$; and the last group consist of the rest seven equations, i.e., Eqs. (48), (49) and (57), which will be used to solve for the rest six undetermined constants, $A_0, B_0, \ldots, F_0$.

For the third group, both Eqs. (48.3) and (49.3) are replaced by Eq. (52), as shown earlier. Hence, the left six equations, Eqs. (48.1,2), (49.1,2), (52), and (57), can be used to solve for six undetermined constants. These six algebraic equations are listed in Eqs. (81)–(86) in “Appendix 1.”

For the first group, by comparing Eq. (17) with Eq. (22), we can obtain

$$N_r^1(r)-N_{r\theta}^1(r)=J_1r^{-1} \tag{58}$$

where

$$J_1=a_1C_1+4a_2D_1-a_1E_1-2a_6H_1 \tag{59}$$

Comparing Eq. (33) with Eq. (34), we have

$$M_r^1(r) - rV_r^1(r) = J_2 r^{-1} \tag{60}$$

where

$$J_2 = (-a_6 + 2b_1) C_1 + (8b_2 - b_6) D_1 - a_6 E_1 - b_7 H_1. \tag{61}$$

In Eqs. (58) and (60), if

$$J_1 = 0, \qquad J_2 = 0, \tag{62}$$

then we can find: (i) By Eq. (58), Eqs. (46.1) and (46.2) are linear dependent, and Eqs. (47.1) and (47.2) are linear dependent; (ii) by Eq. (60), Eqs. (46.3) and (46.4) are linear dependent, and Eqs. (47.3) and (47.4) are linear dependent. Thus, the eight equations, Eqs. (46.1–4) and (47.1–4), can be simplified into six equations, i.e., Eqs. (46.1,3), (47.1,3) and (62). Together with Eqs. (55) and (56), we now have eight equations to solve for eight undetermined constants, $A_1, B_1, \ldots, H_1$. These equations are listed in Eqs. (87)–(94).

For the second group, we can similarly get

$$N_r^2(r) + N_{r\theta}^2(r) = J_3 r^{-1}, \quad M_r^2 + M_{r\theta}^2 - rQ_{rz}^2 = J_4 r^{-1} \tag{63}$$

where

$$J_3 = a_1 C_2 + 4a_2 D_2 + a_1 E_2 + 2a_6 H_2, \tag{64}$$

$$J_4 = -(a_6 + 2b_1) C_2 + (8b_2 - b_6) D_2 + a_6 E_2 + b_7 H_2. \tag{65}$$

If we let

$$J_3 = 0, \quad J_4 = 0, \tag{66}$$

then with the same reasoning, the eight equations, Eqs. (46.5–8) and (47.5–8), can be simplified into four equations, Eqs. (46.5,7) and (47.5,7). Together with Eqs. (50.1), (51.2) and (66), we now have eight equations that can be used to solve for eight undetermined constants $A_2, B_2, \ldots H_2$. These equations are listed in Eqs. (95)–(102).

So far, all undetermined constants in Eq. (13) have been determined with all boundary conditions satisfied. From the above derivation, we know that:

(i) When the annular sector plate is subjected to an in-plane bending moment M_Z at $\theta = 0$ only, we can just retain the undetermined constants $A_0, B_0, \ldots G_0$ in Eq. (13) and let all other constants be zero.
(ii) When the annular sector plate is subjected to an out-of-plane bending moment M_X at $\theta = 0$ only, we can just retain the undetermined constants $A_1, B_1, \ldots H_1$ in Eq. (13) and let all other constants be zero.
(iii) When the annular sector plate is subjected to a concentrated force Y in the y-direction at $\theta = 0$ only, we can just retain the undetermined constants $A_0, B_0, \ldots G_0$ and $A_1, B_1, \cdots H_1$ in Eq. (13) and let the other constants be zero.
(iv) When the annular sector plate is subjected to a concentrated force X in the x-direction or a concentrated couple M_Y in the y-direction at $\theta = 0$, we can just retain the undetermined constants $A_2, B_2 \ldots H_2$ in Eq. (13) and let all other constants be zero.

5 The degenerated stress solutions for a homogeneous plate

In general, it is difficult to obtain the concise and explicit solutions directly by hand to the algebraic equations in "Appendix 1" because of their high-order nature. We thus turn to consider several special cases of a homogeneous plate. According to Yang et al. [20], for a homogeneous plate, we have

$$a_2 = 0, \quad a_6 = 0, \quad a_7 = 0, \quad b_1 = 0, \quad b_5 = 0, \quad \kappa_2 = 0. \tag{67}$$

(i) The plate is subjected to M_Z

In this case, only the constants $A_0, B_0, \ldots G_0$ should be retained in Eq. (13). By using Eq. (67), Eqs. (81)–(86) can be simplified into Eqs. (103)–(108) in "Appendix 2." Then, the constants B_0, D_0, G_0 can be determined from Eqs. (104), (106), and (107), while A_0, C_0, F_0 can be obtained from the other three equations. The expressions are listed in Eq. (125) in "Appendix 3."

Substituting Eq. (125) into Eqs. (24) and (21), and then into Eq. (16), we can obtain the internal forces in the homogeneous transversely isotropic plate induced by M_Z,

$$N_{r\theta}=0,\quad N_r=-\frac{4M_Z}{N}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a}+b^2\ln\frac{r}{b}+a^2\ln\frac{a}{r}\right),$$
$$N_\theta=-\frac{4M_Z}{N}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a}+b^2\ln\frac{r}{b}+a^2\ln\frac{a}{r}+b^2-a^2\right) \tag{68}$$

where N is listed in Eq. (127).

Substituting Eq. (125) into Eqs. (32), (35) and (40), and then into Eqs. (27) and (37), we can obtain

$$M_{r\theta}=0,\qquad M_\theta=0,\qquad M_r=0,\qquad Q_{rz}=0,\qquad Q_{\theta z}=0. \tag{69}$$

For an isotropic plate [20], we have

$$a_1=\frac{2Eh}{1+\nu},\quad a_5=\frac{\nu Eh^3}{3\,(1+\nu)^2}. \tag{70}$$

Substituting Eq. (70) into Eq. (125), we can get the expression for the constants in Eq. (126) for the homogeneous isotropic plate.

It is seen from Eq. (68) that the internal forces are independent of the material constants and hence are suitable for both transversely isotropic and isotropic plates. Furthermore, Eq. (68) agrees well with the integration over the thickness of the stresses in a curved bar as documented in Ref. [25].

(ii) The plate is subjected to M_X

In this case, only constants $A_1, B_1, \ldots H_1$ are kept in Eq. (13). By virtue of Eq. (67), Eqs. (87)–(94) can be simplified into Eqs. (109)–(116). Then, from Eqs. (109), (111), (113), (115) and $Y=0$, we can obtain A_1, C_1, E_1 and F_1, and from (110), (112), (114) and (116), we can get B_1, D_1, G_1, and H_1. All the final results are listed in Eq. (128).

Substituting Eq. (128) into Eqs. (17), (19), (22), (28), (30), (33), (38), and (41), and then into Eqs. (16), (27) and (37), we obtain the internal forces in the plate,

$$N_r=0,\quad N_\theta=0,\quad N_{r\theta}=0,$$
$$M_{r\theta}=\frac{\left[b_6^2r^4-4b_2\,(8b_2-b_6)\left(a^2+b^2\right)r^2+b_6\,(8b_2-b_6)\,a^2b^2\right]M_X}{\left[b_6^2\left(b^2-a^2\right)-(8b_2-b_6)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]r^3}\sin\theta,$$
$$M_\theta=\frac{\left[b_6\,(8b_2+b_6)\,r^4-(8b_2-b_6)^2\left(a^2+b^2\right)r^2-b_6\,(8b_2-b_6)\,a^2b^2\right]M_X}{\left[b_6^2\left(b^2-a^2\right)-(8b_2-b_6)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]r^3}\cos\theta,$$
$$M_r=\frac{b_6\,(8b_2-b_6)\left[r^4-\left(a^2+b^2\right)r^2+a^2b^2\right]M_X}{\left[b_6^2\left(b^2-a^2\right)-(8b_2-b_6)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]r^3}\cos\theta,$$
$$Q_{rz}=\frac{(4b_2-b_6)\left[2b_6r^2+(8b_2-b_6)\left(a^2+b^2\right)\right]M_X}{\left[b_6^2\left(b^2-a^2\right)-(8b_2-b_6)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]r^2}\cos\theta,$$
$$Q_{\theta z}=\frac{(4b_2-b_6)\left[-2b_6r^2+(8b_2-b_6)\left(a^2+b^2\right)\right]M_X}{\left[b_6^2\left(b^2-a^2\right)-(8b_2-b_6)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]r^2}\sin\theta. \tag{71}$$

For isotropic materials [20], we have

$$b_2=-\frac{Eh^3}{12\,(1-\nu)},\qquad b_6=b_7=\frac{Eh^3}{3\,(1+\nu)},\qquad b_8=-\frac{(8+\nu)\,Eh^5}{30\,(1+\nu)\,(1-\nu)}. \tag{72}$$

Substituting Eq. (72) into Eq. (128), we get Eq. (129) for an isotropic plate.

Substituting Eq. (129) into Eqs. (17), (19), (22), (28), (30), (33), (38), and (41), and then into Eqs. (16), (27), and (37), we have the following internal forces in a homogeneous isotropic plate induced by M_X:

$$N_\theta=0,\qquad N_r=0,\qquad N_{r\theta}=0,$$

$$M_{r\theta} = \frac{\left[(1-\nu)^2 r^4 - (3+\nu)(1+\nu)\left(a^2+b^2\right)r^2 - (3+\nu)(1-\nu)a^2b^2\right]M_X}{r^3\left[(1-\nu)^2\left(b^2-a^2\right) - (3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}\sin\theta,$$
$$M_r = \frac{-(1-\nu)(3+\nu)\left[r^4 - \left(a^2+b^2\right)r^2 + a^2b^2\right]M_X}{r^3\left[(1-\nu)^2\left(b^2-a^2\right) - (3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}\cos\theta,$$
$$M_\theta = \frac{\left[-(1-\nu)(1+3\nu)r^4 - (3+\nu)^2\left(a^2+b^2\right)r^2 + (3+\nu)(1-\nu)a^2b^2\right]M_X}{r^3\left[(1-\nu)^2\left(b^2-a^2\right) - (3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}\cos\theta,$$
$$Q_{rz} = \frac{-2\left[2(1-\nu)r^2 - (3+\nu)\left(a^2+b^2\right)\right]M_X}{r^2\left[(1-\nu)^2\left(b^2-a^2\right) - (3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}\cos\theta,$$
$$Q_{\theta z} = \frac{2\left[2(1-\nu)r^2 + (3+\nu)\left(a^2+b^2\right)\right]M_X}{r^2\left[(1-\nu)^2\left(b^2-a^2\right) - (3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}\sin\theta. \tag{73}$$

(iii) The plate is subjected to Y

In this case, constants $A_0, B_0, \ldots G_0$ and $A_1, B_1, \ldots H_1$ should be retained. Among them, B_0, D_0, and G_0 can be solved from Eqs. (104), (106), and (107), and A_0, C_0, and F_0 can be obtained from Eqs. (103), (105), and (108) with $M_Z = 0$. The expressions for $A_0, B_0, \ldots G_0$ are listed in Eq. (130). Substitution of Eq. (130) into Eqs. (24), (21), (32), (35), and (40) leads to

$$N_r^0(r) = \frac{2\left\{\left[b^2\ln\left(\frac{b}{r}\right) - a^2\ln\left(\frac{a}{r}\right)\right]r^2 + \ln\left(\frac{a}{b}\right)a^2b^2\right\}(a+b)Y}{\left[\left(b^2-a^2\right)^2 - 4\left(\ln\frac{b}{a}\right)^2a^2b^2\right]r^2},$$
$$N_\theta^0(r) = -\frac{2\left\{\left[\left(1-\ln\frac{b}{r}\right)b^2 - \left(1-\ln\frac{a}{r}\right)a^2\right]r^2 + \ln\left(\frac{a}{b}\right)a^2b^2\right\}(a+b)Y}{\left[\left(b^2-a^2\right)^2 - 4\left(\ln\frac{a}{b}\right)^2a^2b^2\right]r^2},$$
$$M_\theta^0(r) = 0, \quad M_r^0(r) = 0, \quad Q_{rz}^0(r) = 0. \tag{74}$$

From Eqs. (110), (112), (114), and (116) along with $M_X = 0$, we can obtain $B_1,\ D_1,\ G_1,\ H_1$, and from Eqs. (109), (111), (113), and (115) along with $Y = 0$, we can obtain $A_1,\ C_1,\ E_1,\ F_1$. The results are listed in Eq. (131). Substituting Eq. (131) into Eqs. (22), (19), (17), (28), (30), (33), (38), and (41), we get

$$N_r^1 = \frac{\left(b^2-r^2\right)\left(a^2-r^2\right)Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3}, \qquad N_\theta^1(r) = \frac{\left[3r^4 - \left(a^2+b^2\right)r^2 - a^2b^2\right]Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3},$$
$$N_{r\theta}^1(r) = \frac{\left(b^2-r^2\right)\left(a^2-r^2\right)Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3},$$
$$M_{r\theta}^1(r) = 0, \quad M_\theta^1(r) = 0, \quad M_r^1(r) = 0, \quad Q_{rz}^1(r) = 0, \quad Q_{\theta z}^1(r) = 0. \tag{75}$$

Substitution of Eqs. (74) and (75) into Eqs. (16), (27), and (37) gives rise to

$$N_r(r) = \frac{\left(b^2-r^2\right)\left(a^2-r^2\right)Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3}\cos\theta + \frac{2\left\{\left[b^2\ln\left(\frac{b}{r}\right) - a^2\ln\left(\frac{a}{r}\right)\right]r^2 + \ln\left(\frac{a}{b}\right)a^2b^2\right\}(a+b)Y}{\left[\left(b^2-a^2\right)^2 - 4\left(\ln\frac{b}{a}\right)^2a^2b^2\right]r^2},$$
$$N_\theta(r) = \frac{\left[3r^4 - \left(a^2+b^2\right)r^2 - a^2b^2\right]Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3}\cos\theta - \frac{2\left\{\left[\left(1-\ln\frac{b}{r}\right)b^2 - \left(1-\ln\frac{a}{r}\right)a^2\right]r^2 + \ln\left(\frac{a}{b}\right)a^2b^2\right\}(a+b)Y}{\left[\left(b^2-a^2\right)^2 - 4\left(\ln\frac{a}{b}\right)^2a^2b^2\right]r^2},$$
$$N_{r\theta}(r) = \frac{\left(b^2-r^2\right)\left(a^2-r^2\right)Y}{\left[a^2-b^2-\ln\left(\frac{a}{b}\right)\left(a^2+b^2\right)\right]r^3}\sin\theta,$$

$$M_{r\theta}=0,\qquad M_\theta=0,\qquad M_r=0,\qquad Q_{rz}=0,\qquad Q_{\theta z}=0. \tag{76.1–5}$$

It is noticed that the expressions in Eq. (76) for the internal forces in a homogeneous plate subjected to Y are all independent of the material constants, and the above expressions are valid for both homogeneous transversely isotropic and isotropic plates. It is seen that the expressions in Eqs. (76.1–3) are coincident with the results of Папкович [26].

(iv) The plate is subject to $X=-Q$

In this case, we retain $A_2, B_2, \ldots H_2$ only in Eq. (13). By using Eq. (67), we can simplify Eqs. (95)–(102) into Eqs. (117)–(124). Then we can solve A_2, C_2, E_2, and F_2 from Eqs. (117), (119), (121), and (123) and solve B_2, D_2, G_2, and H_2 from Eqs. (118), (120), (122), and (124) along with $M_Y=0$. The results are listed in Eq. (134). When the material is isotropic, by substituting Eq. (70) into Eq. (134), we get Eq. (135).

Substitution of Eq. (134) or (135) into Eqs. (23), (20), and (18), and then into Eq. (16) leads to

$$\begin{aligned}
N_r &= \frac{Q}{\frac{a^2-b^2}{a^2+b^2}+\ln\frac{b}{a}}\left[\frac{a^2b^2}{\left(a^2+b^2\right)r^3}+\frac{r}{a^2+b^2}-\frac{1}{r}\right]\sin\theta,\\
N_\theta &= \frac{Q}{\frac{a^2-b^2}{a^2+b^2}+\ln\frac{b}{a}}\left[-\frac{a^2b^2}{a^2+b^2}\frac{1}{r^3}+\frac{3r}{a^2+b^2}-\frac{1}{r}\right]\sin\theta,\\
N_{r\theta} &= -\frac{Q}{\frac{a^2-b^2}{a^2+b^2}+\ln\frac{b}{a}}\left[\frac{a^2b^2}{\left(a^2+b^2\right)r^3}+\frac{r}{\left(a^2+b^2\right)}-\frac{1}{r}\right]\cos\theta.
\end{aligned} \tag{77}$$

As we can see, the above internal forces due to the concentrated force Q are also independent of the material constants. They actually coincide with the integrations of the stresses over the thickness of a curved bar under the shear force Q [25].

Substitution of Eq. (134) or (135) into Eqs. (31), (34), (39), and (42), and then into (27) and (37) leads to

$$M_{r\theta}=0,\qquad M_\theta=0,\qquad M_r=0,\qquad Q_{rz}=0,\qquad Q_{\theta z}=0. \tag{78}$$

Equations (77) and (78) are valid for both homogeneous transversely isotropic and isotropic plates.

(v) The plate is subject to M_Y

In this case, only the constants $A_2, B_2, \ldots H_2$ should be retained in Eq. (13). From Eqs. (117), (119), (121), and (123) along with $X=0$, we can get A_2, C_2, E_2, F_2. Meanwhile, from Eqs. (118), (120), (122), and (124), we can obtain B_2, D_2, G_2, H_2. The results are listed in Eq. (136).

Substituting Eq. (136) into Eqs. (18), (20), (23), (29), (31), (34), (39), and (42), we can get the following internal forces in a homogeneous transversely isotropic plate due to M_Y:

$$\begin{aligned}
&N_\theta=0,\qquad N_r=0,\qquad N_{r\theta}=0,\\
&M_{r\theta}(r)=\frac{\left[b_6^2r^4-4b_2\left(8b_2-b_6\right)\left(a^2+b^2\right)r^2+b_6\left(8b_2-b_6\right)a^2b^2\right]abM_Y}{\left(8b_2-b_6\right)\left[\left(4b_2-b_6\right)\left(a^4-b^4\right)+\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)ab\left(a^2+b^2\right)+b_6ab\left(a^2-b^2\right)\right]r^3}\cos\theta,\\
&M_\theta(r)=\frac{\left[-b_6\left(8b_2+b_6\right)r^4+\left(8b_2-b_6\right)^2\left(a^2+b^2\right)r^2+b_6\left(8b_2-b_6\right)a^2b^2\right]abM_Y}{\left(8b_2-b_6\right)\left[\left(4b_2-b_6\right)\left(a^4-b^4\right)+\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)ab\left(a^2+b^2\right)+b_6ab\left(a^2-b^2\right)\right]r^3}\sin\theta,\\
&M_r(r)=\frac{-b_6\left(b^2-r^2\right)\left(a^2-r^2\right)abM_Y}{\left[\left(4b_2-b_6\right)\left(a^4-b^4\right)+\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)ab\left(a^2+b^2\right)+b_6ab\left(a^2-b^2\right)\right]r^3}\sin\theta,\\
&Q_{rz}(r)=-\frac{\left(4b_2-b_6\right)\left[2b_6r^2+\left(8b_2-b_6\right)\left(a^2+b^2\right)\right]abM_Y}{\left(8b_2-b_6\right)\left[\left(4b_2-b_6\right)\left(a^4-b^4\right)+\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)ab\left(a^2+b^2\right)+b_6ab\left(a^2-b^2\right)\right]r^2}\sin\theta,\\
&Q_{\theta z}(r)=\frac{\left(4b_2-b_6\right)\left[-2b_6r^2+\left(8b_2-b_6\right)\left(a^2+b^2\right)\right]abM_Y}{\left(8b_2-b_6\right)\left[\left(4b_2-b_6\right)\left(a^4-b^4\right)+\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)ab\left(a^2+b^2\right)+b_6ab\left(a^2-b^2\right)\right]r^2}\cos\theta.
\end{aligned} \tag{79}$$

When the material is isotropic, substituting Eq. (72) into Eq. (136), we get Eq. (137). Substituting Eq. (137) into Eqs. (18), (20), (23), (29), (31), (34), (39), and (42), and then into Eqs. (16), (27), and (37), we have

$$
\begin{aligned}
& N_\theta = 0, \quad N_r = 0, \quad N_{r\theta} = 0, \\
& M_{r\theta} = \frac{\left[(1-\nu)^2 r^4 - (3+\nu)(1+\nu)\left(a^2+b^2\right) r^2 - (3+\nu)(1-\nu) a^2 b^2\right] abM_Y}{(3+\nu)\left[2\left(a^4-b^4\right)+(1-\nu)\left(ab^3-a^3b\right)+(3+\nu)\ln\left(\frac{b}{a}\right)\left(ab^3+a^3b\right)\right] r^3}\cos\theta, \\
& M_\theta(r) = \frac{\left[(1+3\nu)(1-\nu) r^4 + (3+\nu)^2\left(a^2+b^2\right) r^2 - (1-\nu)(3+\nu) a^2 b^2\right] abM_Y}{(3+\nu)\left[2\left(a^4-b^4\right)+(1-\nu)\left(ab^3-a^3b\right)+(3+\nu)\ln\left(\frac{b}{a}\right)\left(ab^3+a^3b\right)\right] r^3}\sin\theta, \\
& M_r(r) = \frac{(1-\nu)\left(r^2-b^2\right)\left(r^2-a^2\right) abM_Y}{\left[2\left(a^4-b^4\right)+(1-\nu)\left(ab^3-a^3b\right)+(3+\nu)\ln\left(\frac{b}{a}\right)\left(ab^3+a^3b\right)\right] r^3}\sin\theta, \\
& Q_{rz}(r) = \frac{2\left[2(1-\nu) r^2 - (3+\nu)\left(a^2+b^2\right)\right] abM_Y}{(3+\nu)\left[2\left(a^4-b^4\right)+(1-\nu)\left(ab^3-a^3b\right)+(3+\nu)\ln\left(\frac{b}{a}\right)\left(ab^3+a^3b\right)\right] r^2}\sin\theta, \\
& Q_{\theta z}(r) = \frac{2\left[2(1-\nu) r^2 + (3+\nu)\left(a^2+b^2\right)\right] abM_Y}{(3+\nu)\left[2\left(a^4-b^4\right)+(1-\nu)\left(ab^3-a^3b\right)+(3+\nu)\ln\left(\frac{b}{a}\right)\left(ab^3+a^3b\right)\right] r^2}\cos\theta.
\end{aligned} \tag{80}
$$

6 Conclusions

In this paper, we derived 3D analytical elasticity solutions for a functionally graded annular sector plate subjected to concentrated forces $(X, Y, 0)$ and concentrated couples (M_X, M_Y, M_Z) at one radial edge, with the other radial edge fixed and the two arc edges free. The material parameters can vary arbitrarily through the thickness direction. When the material parameters keep unchanged, the results are readily degenerated into those for homogeneous transversely isotropic plates, which are all new to the literature, and can serve as useful supplements to the monograph by Ding et al. [23]. The results can be further reduced to those for homogeneous isotropic plates. Some results coincide with integrations of the stresses over the thickness of a curved bar subjected to concentrated forces X, Y or concentrated couple M_Z, which have been reported in the literature. This agreement, although for some special cases, does verify the correctness of the derivations in our analysis. We also note that the results for a homogeneous isotropic elastic plate subjected to concentrated couples M_X and M_Y are also new to the literature. The obtained results can serve as benchmarks for further approximate or numerical studies.

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Appendix 1: Algebraic equations of a transversely isotropic functionally graded annular sector plate

$$
\begin{aligned}
& 2a_1 A_0 + 8a_2 B_0 + \left[a_1(1+2\ln a) - a_5 a^{-2}\right] C_0 \\
& + \left[4a_2(1+2\ln a) - a_7 a^{-2}\right] D_0 + a_1 a^{-2} F_0 + 2a_6 a^{-2} G_0 = 0,
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
& -2b_1 A_0 + 8b_2 B_0 - \left[a_6 + 2b_1(1+\ln a) + b_5 a^{-2}\right] C_0 \\
& + \left[8b_2(1+\ln a) - b_6 - b_8 a^{-2}\right] D_0 \\
& + a_6 a^{-2} F_0 + b_7 a^{-2} G_0 = 0,
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
& 2a_1 A_0 + 8a_2 B_0 + \left[a_1(1+2\ln b) - a_5 b^{-2}\right] C_0 \\
& + \left[4a_2(1+2\ln b) - a_7 b^{-2}\right] D_0 + a_1 b^{-2} F_0 + 2a_6 b^{-2} G_0 = 0,
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
& -2b_1 A_0 + 8b_2 B_0 - \left[a_6 + 2b_1(1+\ln b) + b_5 b^{-2}\right] C_0 \\
& + \left[8b_2(1+\ln b) - b_6 - b_8 b^{-2}\right] D_0 \\
& + a_6 b^{-2} F_0 + b_7 b^{-2} G_0 = 0,
\end{aligned} \tag{84}
$$

$$
-(b_1 + a_6) C_0 + (4b_2 - b_6) D_0 = 0, \tag{85}
$$

$$
\begin{aligned}
&-a_1\left(b^2-a^2\right)A_0-4a_2\left(b^2-a^2\right)B_0\\
&-\left[a_1\left(b^2-a^2+b^2\ln b-a^2\ln a\right)+a_5\ln\frac{b}{a}\right]C_0\\
&-\left[4a_2\left(b^2-a^2+b^2\ln b-a^2\ln a\right)+a_7\ln\frac{b}{a}\right]D_0\\
&+a_1\ln\left(\frac{b}{a}\right)F_0+2a_6\ln\left(\frac{b}{a}\right)G_0=2M_Z+(a+b)\,Y,
\end{aligned}\tag{86}
$$

$$
\begin{aligned}
&2a_1aA_1+8a_2aB_1+\left(3a_1a^{-1}+2a_5a^{-3}\right)C_1\\
&+2\left(6a_2a^{-1}+a_7a^{-3}\right)D_1-a_1a^{-1}E_1\\
&+2a_1a^{-3}F_1-4a_6a^{-3}G_1-2a_6a^{-1}H_1=0,
\end{aligned}\tag{87}
$$

$$
\begin{aligned}
&-2\left(2b_1+a_6\right)aA_1+2\left(8b_2-b_6\right)aB_1+\left[\left(a_6-2b_1\right)a^{-1}+2b_5a^{-3}\right]C_1\\
&+\left[\left(8b_2+b_6\right)a^{-1}+2b_8a^{-3}\right]D_1\\
&-a_6a^{-1}E_1+2a_6a^{-3}F_1-2b_7a^{-3}G_1-b_7a^{-1}H_1=0,
\end{aligned}\tag{88}
$$

$$
\begin{aligned}
&2a_1bA_1+8a_2bB_1+\left(3a_1b^{-1}+2a_5b^{-3}\right)C_1+2\left(6a_2b^{-1}+a_7b^{-3}\right)D_1-a_1b^{-1}E_1\\
&+2a_1b^{-3}F_1-4a_6b^{-3}G_1-2a_6b^{-1}H_1=0,
\end{aligned}\tag{89}
$$

$$
\begin{aligned}
&-2\left(2b_1+a_6\right)bA_1+2\left(8b_2-b_6\right)bB_1+\left[\left(a_6-2b_1\right)b^{-1}+2b_5b^{-3}\right]C_1\\
&+\left[\left(8b_2+b_6\right)b^{-1}+2b_8b^{-3}\right]D_1\\
&-a_6b^{-1}E_1+2a_6b^{-3}F_1-2b_7b^{-3}G_1-b_7b^{-1}H_1=0,
\end{aligned}\tag{90}
$$

$$
a_1C_1+4a_2D_1-a_1E_1-2a_6H_1=0,\tag{91}
$$

$$
-\left(a_6+2b_1\right)C_1+\left(8b_2-b_6\right)D_1-a_6E_1-b_7H_1=0,\tag{92}
$$

$$
\begin{aligned}
&3a_1\left(b^2-a^2\right)A_1+12a_2\left(b^2-a^2\right)B_1+\left[a_1\ln\left(\frac{b}{a}\right)+a_5\left(b^{-2}-a^{-2}\right)\right]C_1\\
&+\left[4a_2\ln\left(\frac{b}{a}\right)+a_7\left(b^{-2}-a^{-2}\right)\right]D_1\\
&+a_1\ln\left(\frac{b}{a}\right)E_1+a_1\left(b^{-2}-a^{-2}\right)F_1-2a_6\left(b^{-2}-a^{-2}\right)G_1+2a_6\ln\left(\frac{b}{a}\right)H_1=-2Y,
\end{aligned}\tag{93}
$$

$$
\begin{aligned}
&\left(-2b_1+a_6\right)\left(b^2-a^2\right)A_1+\left(8b_2+b_6\right)\left(b^2-a^2\right)B_1-\left[\left(2b_1+a_6\right)\ln\left(\frac{b}{a}\right)-b_5\left(b^{-2}-a^{-2}\right)\right]C_1\\
&+\left[\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)+b_8\left(b^{-2}-a^{-2}\right)\right]D_1+a_6\ln\left(\frac{b}{a}\right)E_1+a_6\left(b^{-2}-a^{-2}\right)F_1\\
&-b_7\left(b^{-2}-a^{-2}\right)G_1+b_7\ln\left(\frac{b}{a}\right)H_1=2M_X,
\end{aligned}\tag{94}
$$

$$
\begin{aligned}
&-2a_1aA_2-8a_2aB_2+\left(3a_1a^{-1}+2a_5a^{-3}\right)C_2+2\left(6a_2a^{-1}+a_7a^{-3}\right)D_2\\
&+a_1a^{-1}E_2+2a_1a^{-3}F_2-4a_6a^{-3}G_2+2a_6a^{-1}H_2=0,
\end{aligned}\tag{95}
$$

$$
\begin{aligned}
&2\left(2b_1+a_6\right)aA_2+2\left(b_6-8b_2\right)aB_2+\left[\left(a_6-2b_1\right)a^{-1}+2b_5a^{-3}\right]C_2\\
&+\left[\left(8b_2+b_6\right)a^{-1}+2b_8a^{-3}\right]D_2\\
&+a_6a^{-1}E_2+2a_6a^{-3}F_2-2b_7a^{-3}G_2+b_7a^{-1}H_2=0,
\end{aligned}\tag{96}
$$

$$
\begin{aligned}
&-2a_1bA_2-8a_2bB_2+\left(3a_1b^{-1}+2a_5b^{-3}\right)C_2+2\left(6a_2b^{-1}+a_7b^{-3}\right)D_2\\
&+a_1b^{-1}E_2+2a_1b^{-3}F_2-4a_6b^{-3}G_2+2a_6b^{-1}H_2=0,
\end{aligned}\tag{97}
$$

$$
\begin{aligned}
&2\left(2b_1+a_6\right)bA_2+2\left(b_6-8b_2\right)bB_2+\left[\left(a_6-2b_1\right)b^{-1}+2b_5b^{-3}\right]C_2\\
&+\left[\left(8b_2+b_6\right)b^{-1}+2b_8b^{-3}\right]D_2\\
&+a_6b^{-1}E_2+2a_6b^{-3}F_2-2b_7b^{-3}G_2+b_7b^{-1}H_2=0,
\end{aligned}\tag{98}
$$

$$a_1\left(b^2-a^2\right)A_2+4a_2\left(b^2-a^2\right)B_2-\left[a_1\ln\left(\frac{b}{a}\right)-a_5\left(b^{-2}-a^{-2}\right)\right]C_2$$
$$-\left[4a_2\ln\left(\frac{b}{a}\right)-a_7\left(b^{-2}-a^{-2}\right)\right]D_2$$
$$+a_1\ln\left(\frac{b}{a}\right)E_2+a_1\left(b^{-2}-a^{-2}\right)F_2-2a_6\left(b^{-2}-a^{-2}\right)G_2+2a_6\ln\left(\frac{b}{a}\right)H_2=-2X, \tag{99}$$

$$\left[-2\left(2b_1+a_6\right)\ln\left(\frac{b}{a}\right)a^3b^3+2b_5\left(a+b\right)\left(a-b\right)^3-\left(2b_1+a_6\right)a^2b^2\left(a^2-b^2\right)\right]C_2$$
$$+\left\{2\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)a^3b^3+\left[\left(8b_2-b_6\right)a^2b^2+2b_8\left(b-a\right)^2\right]\left(a^2-b^2\right)\right\}D_2$$
$$-a_6\left[2ab\ln\left(\frac{b}{a}\right)+\left(a^2-b^2\right)\right]a^2b^2E_2+2a_6\left(a+b\right)\left(a-b\right)^3F_2-2b_7\left(a+b\right)\left(a-b\right)^3G_2$$
$$-b_7\left[2ab\ln\left(\frac{b}{a}\right)+\left(a^2-b^2\right)\right]a^2b^2H_2=4a^3b^3M_Y, \tag{100}$$

$$a_1C_2+4a_2D_2+a_1E_2+2a_6H_2=0, \tag{101}$$
$$-\left(a_6+2b_1\right)C_2+\left(8b_2-b_6\right)D_2+a_6E_2+b_7H_2=0. \tag{102}$$

Appendix 2: Algebraic equations for a transversely isotropic homogeneous annular sector plate

$$2a_1A_0+\left[a_1\left(1+2\ln a\right)-a_5a^{-2}\right]C_0+a_1a^{-2}F_0=0, \tag{103}$$
$$8b_2B_0+\left[8b_2\left(1+\ln a\right)-b_6-b_8a^{-2}\right]D_0+b_7a^{-2}G_0=0, \tag{104}$$
$$2a_1A_0+\left[a_1\left(1+2\ln b\right)-a_5b^{-2}\right]C_0+a_1b^{-2}F_0=0, \tag{105}$$
$$8b_2B_0+\left[8b_2\left(1+\ln b\right)-b_6-b_8b^{-2}\right]D_0+b_7b^{-2}G_0=0, \tag{106}$$
$$D_0=0, \tag{107}$$
$$-a_1\left(b^2-a^2\right)A_0-\left[a_1\left(b^2-a^2+b^2\ln b-a^2\ln a\right)+a_5\ln\frac{b}{a}\right]C_0$$
$$+a_1\ln\left(\frac{b}{a}\right)F_0=2M_Z+\left(a+b\right)Y, \tag{108}$$
$$2a_1aA_1+\left(3a_1a^{-1}+2a_5a^{-3}\right)C_1-a_1a^{-1}E_1+2a_1a^{-3}F_1=0, \tag{109}$$
$$2\left(8b_2-b_6\right)aB_1+\left[\left(8b_2+b_6\right)a^{-1}+2b_8a^{-3}\right]D_1-2b_7a^{-3}G_1-b_7a^{-1}H_1=0, \tag{110}$$
$$2a_1bA_1+\left(3a_1b^{-1}+2a_5b^{-3}\right)C_1-a_1b^{-1}E_1+2a_1b^{-3}F_1=0, \tag{111}$$
$$2\left(8b_2+b_6\right)bB_1+\left[\left(8b_2+b_6\right)b^{-1}+2b_8b^{-3}\right]D_1-2b_7b^{-3}G_1-b_7b^{-1}H_1=0, \tag{112}$$
$$C_1=E_1, \tag{113}$$
$$\left(8b_2-b_6\right)D_1=b_7H_1, \tag{114}$$
$$3a_1\left(b^2-a^2\right)A_1+\left[a_1\ln\left(\frac{b}{a}\right)+a_5\left(b^{-2}-a^{-2}\right)\right]C_1$$
$$+a_1\ln\left(\frac{b}{a}\right)E_1+a_1\left(b^{-2}-a^{-2}\right)F_1=-2Y, \tag{115}$$
$$\left(8b_2+b_6\right)\left(b^2-a^2\right)B_1+\left[2\left(8b_2-b_6\right)\ln\left(\frac{b}{a}\right)+b_8\left(b^{-2}-a^{-2}\right)\right]D_1$$
$$-b_7\left(b^{-2}-a^{-2}\right)G_1=2M_X, \tag{116}$$
$$-2a_1aA_2+\left(3a_1a^{-1}+2a_5a^{-3}\right)C_2+a_1a^{-1}E_2+2a_1a^{-3}F_2=0, \tag{117}$$
$$2\left(b_6-8b_2\right)aB_2+\left[\left(8b_2+b_6\right)a^{-1}+2b_8a^{-3}\right]D_2-2b_7a^{-3}G_2+b_7a^{-1}H_2=0, \tag{118}$$
$$-2a_1bA_2+\left(3a_1b^{-1}+2a_5b^{-3}\right)C_2+a_1b^{-1}E_2+2a_1b^{-3}F_2=0, \tag{119}$$

$$2\left(b_6-8b_2\right)bB_2+\left[\left(8b_2+b_6\right)b^{-1}+2b_8b^{-3}\right]D_2-2b_7b^{-3}G_2+b_7b^{-1}H_2=0, \tag{120}$$

$$a_1\left(b^2-a^2\right)A_2+\left[-a_1\ln\left(\frac{b}{a}\right)+a_5\left(b^{-2}-a^{-2}\right)\right]C_2$$
$$+a_1\ln\left(\frac{b}{a}\right)E_2+a_1\left(b^{-2}-a^{-2}\right)F_2=-2X, \tag{121}$$

$$\left[2b_8\left(a+b\right)\left(a-b\right)^3+\left(8b_2-b_6\right)a^2b^2\left(a^2-b^2+2ab\ln\frac{b}{a}\right)\right]D_2$$
$$-2b_7\left(a+b\right)\left(a-b\right)^3G_2-b_7a^2b^2\left(a^2-b^2+2ab\ln\frac{b}{a}\right)H_2=4a^3b^3M_Y, \tag{122}$$

$$C_2=-E_2, \tag{123}$$

$$\left(8b_2-b_6\right)D_2=-b_7H_2. \tag{124}$$

Appendix 3: Constants for several special cases of a homogeneous plate

(i) Constants for the plate subjected to M_Z; Constants for the transversely isotropic plate:

$$B_0=0,\quad D_0=0,\quad G_0=0,$$
$$A_0=\frac{2\left[\left(1+2\ln b\right)b^2-\left(1+2\ln a\right)a^2\right]M_Z}{a_1\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2a^2b^2\right]},\qquad C_0=-\frac{4\left(b^2-a^2\right)M_Z}{a_1\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2a^2b^2\right]},$$
$$F_0=-\frac{4\left[2a_1\ln\left(\frac{b}{a}\right)a^2b^2+a_5\left(b^2-a^2\right)\right]M_Z}{a_1^2\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2a^2b^2\right]}. \tag{125}$$

Constants for the isotropic plate:

$$B_0=0,\quad D_0=0,\quad G_0=0,$$
$$A_0=\frac{(1+\nu)}{EhN}\left[\left(1+2\ln b\right)b^2-\left(1+2\ln a\right)a^2\right]M_Z,\qquad C_0=-\frac{2\left(1+\nu\right)\left(b^2-a^2\right)M_Z}{EhN},$$
$$F_0=-\frac{M_Z}{3EhN}\left[12\left(1+\upsilon\right)\ln\left(\frac{b}{a}\right)a^2b^2+\nu\left(b^2-a^2\right)h^2\right] \tag{126}$$

where

$$N=\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2a^2b^2. \tag{127}$$

(ii) Constants for the plate subjected to M_X;
Constants for the transversely isotropic plate:

$$A_1=0,\quad C_1=0,\quad E_1=0,\quad F_1=0,$$
$$B_1=\frac{b_6M_X}{b_6^2\left(b^2-a^2\right)-\left(8b_2-b_6\right)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)},$$
$$D_1=-\frac{\left(8b_2-b_6\right)\left(a^2+b^2\right)M_X}{b_6^2\left(b^2-a^2\right)-\left(8b_2-b_6\right)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)},$$
$$G_1=-\frac{\left(8b_2-b_6\right)\left[b_6a^2b^2+b_8\left(a^2+b^2\right)\right]M_X}{b_6^2b_7\left(b^2-a^2\right)-b_7\left(8b_2-b_6\right)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)},$$
$$H_1=-\frac{\left(8b_2-b_6\right)^2\left(a^2+b^2\right)M_X}{b_6^2b_7\left(b^2-a^2\right)-b_7\left(8b_2-b_6\right)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)}. \tag{128}$$

Constants for the isotropic plate:

$$
\begin{aligned}
&A_1 = 0, \quad C_1 = 0, \quad E_1 = 0, \quad F_1 = 0,\\
&B_1 = \frac{3\,(1+\nu)\,(1-\nu)^2\, M_X}{Eh^3\left[(1-\nu)^2\left(b^2-a^2\right)-(3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]},\\
&D_1 = \frac{3\left(a^2+b^2\right)(3+\nu)\left(1-\nu^2\right) M_X}{Eh^3\left[(1-\nu)^2\left(b^2-a^2\right)-(3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]},\\
&G_1 = \frac{3\,(3+\nu)\left[10\left(1-\nu^2\right)a^2b^2-(8+\nu)\,(1+\nu)\left(a^2+b^2\right)h^2\right] M_X}{10Eh^3\left[(1-\nu)^2\left(b^2-a^2\right)-(3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]},\\
&H_1 = -\frac{3\,(1+\nu)\,(3+\nu)^2\left(a^2+b^2\right) M_X}{Eh^3\left[(1-\nu)^2\left(b^2-a^2\right)-(3+\nu)^2\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}.
\end{aligned}
\tag{129}
$$

(iii) Constants for the plate subjected to Y;
Constants for the transversely isotropic plate:

$$
\begin{aligned}
&B_0 = 0, \quad D_0 = 0, \quad G_0 = 0,\\
&A_0 = \frac{(a+b)\left[(1+2\ln b)\,b^2-(1+2\ln a)\,a^2\right] Y}{a_1\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]}, \quad C_0 = -\frac{2\,(a+b)\left(b^2-a^2\right) Y}{a_1\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]},\\
&F_0 = -\frac{2\,(a+b)\left[2a_1\ln\left(\frac{b}{a}\right)a^2b^2+a_5\left(b^2-a^2\right)\right] Y}{a_1^2\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]},
\end{aligned}
\tag{130}
$$

$$
\begin{aligned}
&B_1 = 0, \quad D_1 = 0, \quad G_1 = 0, \quad H_1 = 0,\\
&A_1 = \frac{Y}{a_1\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}, \quad C_1 = E_1 = -\frac{\left(a^2+b^2\right) Y}{a_1\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]},\\
&F_1 = \frac{\left[a_1a^2b^2+a_5\left(a^2+b^2\right)\right] Y}{a_1^2\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}.
\end{aligned}
\tag{131}
$$

Constants for the isotropic plate:

$$
\begin{aligned}
&B_0 = 0, \quad D_0 = 0, \quad G_0 = 0,\\
&A_0 = \frac{(1+\nu)\,(a+b)\left(b^2-a^2+2b^2\ln b-2a^2\ln a\right) Y}{2Eh\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]},\\
&C_0 = -\frac{(1+\nu)\,(a+b)\left(b^2-a^2\right) Y}{Eh\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]},\\
&F_0 = -\frac{(a+b)\left[4\,(1+\nu)\ln\left(\frac{b}{a}\right)a^2b^2+\frac{\nu}{3}\left(b^2-a^2\right)h^2\right] Y}{2Eh\left[\left(b^2-a^2\right)^2-4\left(\ln\frac{b}{a}\right)^2 a^2b^2\right]},
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
&B_1 = 0, \quad D_1 = 0, \quad G_1 = 0, \quad H_1 = 0,\\
&A_1 = \frac{(1+\nu)\,Y}{2Eh\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}, \quad C_1 = E_1 = -\frac{(1+\nu)\left(a^2+b^2\right) Y}{2Eh\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]},\\
&F_1 = \frac{\left[6\,(1+\nu)\,a^2b^2+\nu\left(a^2+b^2\right)h^2\right] Y}{12Eh\left[a^2-b^2+\ln\left(\frac{b}{a}\right)\left(a^2+b^2\right)\right]}.
\end{aligned}
\tag{133}
$$

(iv) Constants for the plate subjected to $X = -Q$;
Constants for the transversely isotropic plate:

$$B_2 = 0, \quad D_2 = 0, \quad G_2 = 0, \quad H_2 = 0,$$
$$A_2 = -\frac{Q}{a_1\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]}, \qquad C_2 = -\frac{\left(a^2 + b^2\right)Q}{a_1\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]},$$
$$F_2 = \frac{\left[a_1 a^2 b^2 + a_5\left(a^2 + b^2\right)\right]Q}{a_1^2\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]}. \tag{134}$$

Constants for the isotropic plate:

$$B_2 = 0, \quad D_2 = 0, \quad G_2 = 0, \quad H_2 = 0,$$
$$A_2 = -\frac{(1+\nu)Q}{2Eh\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]}, \quad C_2 = -\frac{(1+\nu)\left(a^2 + b^2\right)Q}{2Eh\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]},$$
$$F_2 = \frac{\left[\nu\left(a^2 + b^2\right)h^2 + 6(1+\nu)a^2 b^2\right]Q}{12Eh\left[a^2 - b^2 + \ln\left(\frac{b}{a}\right)\left(a^2 + b^2\right)\right]}. \tag{135}$$

(v) Constants for the plate subjected to M_Y;
Constants for the transversely isotropic plate:

$$A_2 = 0, \quad C_2 = 0, \quad E_2 = 0, \quad F_2 = 0,$$
$$B_2 = \frac{b_6 ab M_Y}{(8b_2 - b_6)\left[(4b_2 - b_6)\left(a^4 - b^4\right) + (8b_2 - b_6)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) + b_6 ab\left(a^2 - b^2\right)\right]},$$
$$D_2 = \frac{ab\left(a^2 + b^2\right)M_Y}{(4b_2 - b_6)\left(a^4 - b^4\right) + (8b_2 - b_6)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) + b_6 ab\left(a^2 - b^2\right)},$$
$$G_2 = \frac{\left[b_6 a^2 b^2 + b_8\left(a^2 + b^2\right)\right]ab M_Y}{b_7\left[(4b_2 - b_6)\left(a^4 - b^4\right) + (8b_2 - b_6)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) + b_6 ab\left(a^2 - b^2\right)\right]},$$
$$H_2 = -\frac{(8b_2 - b_6)ab\left(a^2 + b^2\right)M_Y}{b_7\left[(4b_2 - b_6)\left(a^4 - b^4\right) + (8b_2 - b_6)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) + b_6 ab\left(a^2 - b^2\right)\right]}. \tag{136}$$

Constants for the isotropic plate:

$$A_2 = 0, \quad C_2 = 0, \quad E_2 = 0, \quad F_2 = 0,$$
$$B_2 = \frac{3(1-\nu)^2(1+\nu)ab M_Y}{(3+\nu)Eh^3\left[2\left(a^4 - b^4\right) + (3+\nu)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) - (1-\nu)ab\left(a^2 - b^2\right)\right]},$$
$$D_2 = \frac{-3\left(1-\nu^2\right)\left(a^2 + b^2\right)ab M_Y}{Eh^3\left[2\left(a^4 - b^4\right) + (3+\nu)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) - (1-\nu)ab\left(a^2 - b^2\right)\right]},$$
$$G_2 = \frac{3(1+\nu)\left[(8+\nu)\left(a^2 + b^2\right)h^2 - 10(1-\nu)a^2 b^2\right]ab M_Y}{10Eh^3\left[2\left(a^4 - b^4\right) + (3+\nu)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) - (1-\nu)ab\left(a^2 - b^2\right)\right]},$$
$$H_2 = \frac{-3(1+\nu)(3+\nu)\left(a^2 + b^2\right)ab M_Y}{Eh^3\left[2\left(a^4 - b^4\right) + (3+\nu)\ln\left(\frac{b}{a}\right)ab\left(a^2 + b^2\right) - (1-\nu)ab\left(a^2 - b^2\right)\right]}. \tag{137}$$

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