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ORIGINAL PAPER

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Description of nonlinear thermal effects by means of a two-component Cosserat continuum

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Abstract A physical object under consideration is a conventional material that has elastic and thermodynamic properties. To describe thermal processes in the material, we use a mechanical model different from the models that are usually used in the kinetic theory and statistical physics. Our method of thermal processes modeling is based on an idea to introduce a continuum with an internal structure and to consider mechanical quantities associated with the additional degrees of freedom as analogies of thermodynamic quantities. In this way, we suggest mechanical interpretations of temperature and entropy, which can be a foundation for the description of thermal processes within the framework of continuum mechanics and by using the methods of continuum mechanics.

1 Introduction

Now thermodynamics covers a wide range of issues including gas dynamics, thermoelasticity, thermoviscoelasticity, thermoelectric and thermomagnetic effects, phase transitions, and chemical reactions. In fact, it is a set of unrelated branches of science, which differ from each other not only in used mathematical methods but also in interpretations of basic concepts. In discussing the mathematical methods, we refer to the theory of thermodynamic potentials, which underlies chemical and electrochemical thermodynamics. We also refer to the methods of continuum mechanics, which allows us to study thermal processes both in elastic bodies and in bodies with complicated rheology. In addition, we refer to the methods of crystal lattice dynamics, which are a basis for the description of transport phenomena in solids from a microscopic viewpoint, and also the statistical technology including classical statistics and quantum statistics.

The purely phenomenological approaches are used for the description of thermal processes both in non-equilibrium thermodynamics and in continuum mechanics. The detailed consideration of the methods of non-equilibrium thermodynamics and possibilities of their use for solving various problems can be found in the works by the originators of this science, namely Prigogine and De Groot—see [1–3]. In order to clearly convey the essence of non-equilibrium thermodynamics, it is appropriate to quote from [3]. "Over the past ten years the macroscopic theory of irreversible processes has become a complete theory. It is based on two grounds which were established earlier than the theory was formulated. First, the introduction of non-equilibrium thermodynamic functions made it possible to establish the concept of entropy flow and the occurrence of entropy, and then to constitute the entropy balance equation on the basis of these concepts.

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Secondly, the thermodynamics of irreversible processes is based on Onsager's reciprocal relations, i.e., on the macroscopic equalities which are the result of microscopic reversibility." The assertion that the macroscopic theory of irreversible processes has become a complete theory refers only to the linear theory. As for nonlinear non-equilibrium thermodynamics, it cannot be called a complete theory—see, e.g., [1,2,4,5]. The book [6] deserves a special reference. This book combines the macroscopic and statistical approaches, that is a rather rare phenomenon. We also refer to the book [7] on the so-called extended non-equilibrium thermodynamics, which is constructed based on a generalized Fourier law (the law contains the time derivative of heat flux).

Methods of non-equilibrium thermodynamics and methods of continuum mechanics are very similar. However, because of some historical reasons, these two sciences have evolved independently of each other. Within the framework of continuum mechanics, there are different approaches to the description of thermal processes. Truesdell's method, which is based on the combined use of the first and second laws of thermodynamics, is mostly used—see [8,9]. The essence of this method is the following. The second law of thermodynamics is written in the form of the Clausius–Duhem inequality. With the help of the energy balance equation, some of the thermal terms, namely the rate of heat supplied directly to the volume of a medium and the divergence of a heat flux vector, are eliminated from the Clausius–Duhem inequality. As a result, the so-called reduced dissipation inequality, which must hold for all conceivable processes in a medium, is obtained. Since the reduced dissipation inequality includes neither external mechanical forces nor the heat supply from an external source, this inequality imposes certain restrictions on the constitutive equations. The applications of Truesdell's method for constructing various thermo-mechanical models can be found, e.g., in [10–13]. Truesdell's method was considerably developed in the works by Zhilin—see [14,15]. Eventually, Zhilin refused the Clausius–Duhem inequality and suggested a fundamentally new approach, involving the more restrictive formulation of the second law of thermodynamics—see [16,17]. The methods similar to those used in non-equilibrium thermodynamics are applied in continuum mechanics by Nowacki [18] and some other authors—see, e.g., [19–21]. The approach based on the combination of non-equilibrium thermodynamics and Truesdell's method is developed [22–24]. Critical reviews of the basic concepts and fundamental principles of thermodynamics can be found in [25,26]. In non-equilibrium thermodynamics and continuum mechanics, there is no mechanical interpretation of nature and physical meaning of temperature, entropy, and other thermodynamic quantities. The concept of temperature as the average kinetic energy of the chaotic motion of molecules and the probabilistic interpretation of entropy, adopted in the kinetic theory and statistical physics, do not contradict non-equilibrium thermodynamics and continuum mechanics. However, the use of such concepts is problematic when deriving the balance equations, and it is almost impossible to derive the equations that describe the diffusion-type transport processes. The absence of mechanical interpretations of thermodynamic quantities in non-equilibrium thermodynamics and continuum mechanics significantly reduces the possibility to use intuitive thinking when constructing new theories. This circumstance is not very important in the case of linear problems, but it hinders the development of nonlinear theories.

Appreciating the achievements of statistical physics in the field of thermodynamics of gases and crystalline solids, we note that the study of thermal phenomena in liquids and amorphous solids, as well as the analysis of coupled problems of thermoelasticity and thermoviscoelasticity, is still beyond the scope of this science. In addition, there are some questions that are difficult to answer by using the interpretations adopted in the kinetic theory and statistical physics. One question is: How to explain in terms of the kinetic theory the thermal radiation that propagates in vacuum over long distances and causes heating of bodies? It is generally accepted that the thermal radiation is a result of the electromagnetic radiation that acts on a body and sets its atoms in thermal motion. But it is unclear why this process occurs at frequencies lying in the infrared range and does not occur at other frequencies, regardless of atomic masses and characteristics of interatomic bonds. Another question is: How to determine the temperature of nano-objects in terms of the kinetic theory? The kinetic theory considers absolute temperature to be proportional to the average kinetic energy of the chaotic microscopic motion of atoms and molecules. In the case of a macroscopic size body, it is not easy but it is possible to separate the mechanical (macroscopic) motion of the body, relating to deformation processes, and the thermal (microscopic) motion of its particles [27]. But it is unclear how to separate the mechanical and thermal motion in the case when an object is composed of only several tens of atoms. All the problems described above force us to ponder the question: Does the interpretation of temperature as the average kinetic energy of the chaotic motion of atoms and molecules reflect some physical reality? To clarify this issue, first of all, we quote Maxwell's remark toward the kinetic theory created by him—see [28, p. 378]: "…If the properties of such a system of bodies are found to correspond to those of gases, an important physical analogy will be established, which may lead to more accurate knowledge of the properties of matter. If experiments on gases are inconsistent with the hypothesis of these propositions, then our theory, though consistent with itself, is proved to be incapable of explaining the phenomena of gases. In either case, it is necessary to follow out the consequences of the hypothesis." We believe that the same can be said with respect to any other theory. In order to appreciate the usefulness of a theory, we have to examine whether consequences of the theory are in agreement with the experimental data.

Next, to clarify the question whether the interpretation of temperature as the average kinetic energy of the chaotic motion of atoms and molecules is a reflection of some physical reality, we discuss some well-known facts. First of all, we note that temperature cannot be measured directly. In order to measure temperature, we have to measure a physical quantity, a change of which is a sign of the change in temperature. Then, we have to calculate the value of temperature taking into account the fixed points of the temperature scale and using a formula that relates the change in the chosen physical quantity and the change in temperature. Thus, there is no reason to believe that measuring temperature we measure the average kinetic energy of the chaotic motion of atoms and molecules. It is generally accepted that the Brownian motion (a chaotic motion of microscopic particles of solid matter which are suspended in a liquid or gas) is a visual experimental confirmation of the molecular-kinetic theory. Indeed, the Brownian motion is a consequence of the thermal motion of molecules. Observation of the Brownian motion convinces us that the thermal motion of molecules becomes more intense with increasing temperature. But this is not a sufficient reason to identify temperature with the average kinetic energy of the chaotic motion of atoms or molecules. It is well known, for example, that all materials possess electrical resistance and a change in temperature of a material leads to the change in its electrical resistance. However, nobody concludes from this fact that temperature is electrical resistance. Similar examples can be found in large numbers. Thus, the known facts do not give us sufficient reasons to take for granted the fact that temperature is the average kinetic energy of the chaotic motion of atoms and molecules. Hence, the interpretation of temperature adopted in the kinetic theory is rather a mathematical model than a physical reality. That is why any alternative model of thermal processes, the mathematical description of which is reduced to the known equations, is of interest from a theoretical point of view.

Our purpose is to suggest a mechanical interpretation of temperature that can be a basis for the description of thermal processes within the framework of continuum mechanics and by using the methods of continuum mechanics. The main idea is to introduce a continuum with an internal structure and additional degrees of freedom. We believe that characteristics of motion and interactions associated with the internal structure can be treated as mechanical analogies of temperature and other thermodynamic quantities. The suggested model of thermal processes is based on internal rotational degrees of freedom. There are the kinetic theories that include rotational degrees of freedom (see, e.g., [29–31]) as well as the kinetic theories that take into account internal degrees of freedom (see, e.g., [32]). The suggested model of thermal processes is not similar to these models. It is based on quite different ideas and approaches.

2 Different views on the nature of heat: historical remarks

Starting from antiquity, there exist different viewpoints on the nature of heat—see [33–36]. According to one point of view, heat is a state of a body. For example, Roger Bacon (1214–1292) and Johannes Kepler (1571–1630) adhered to this opinion. In accordance with another point of view, heat is a substance. Galileo Galilei (1564–1642) formulated the hypothesis of existence of the imponderable fluid accounting for heat. Afterward, this imponderable fluid was called the caloric fluid. The caloric fluid was considered to be dispersed all over the matter and to be capable of penetrating into bodies. Combining with solids, it can transform them into liquids, and combining with liquids it can transform them into gases. Antoine Laurent de Lavoisier (1743–1794), Pierre Simon de Laplace (1749–1827) and Jean Baptiste Joseph Fourier (1768–1830) were adherents of the caloric fluid theory. The success and popularity of the caloric fluid in XVII–XVIII centuries was caused by the fact that predictions of the theory were verified by the experiments carried out at that time. The caloric fluid theory was recognized to be erroneous only in XIX century when, due to the works by Julius Robert Mayer (1814–1878), James Prescott Joule (1818–1889), Hermann Helmholtz (1821–1894) and William Thomson, Lord Kelvin (1824–1907), the principle of equivalence of heat and energy became firmly established and the heat conservation law, which had dominated earlier, was completely replaced by the energy balance equation (the first law of thermodynamics).

Robert Boyle (1627–1691) assumed heat to be associated with the molecular motion. In fact, his assumption was the start of the kinetic theory, which was further developed by Rudolf Clausius (1822–1888) and James Clerk Maxwell (1831–1879). In 1857, Clausius derived the basic formula of the kinetic theory of gases, by which the gas pressure is equal to two-thirds of the average kinetic energy of molecules per unit volume.

The comparison of this formula for pressure with the ideal gas law has led to the identification of temperature with the average kinetic energy of the translational motion of molecules. In 1859, Maxwell suggested the formula for the velocity distribution, which was later named after him. In 1866, Ludwig Boltzmann (1844–1906) generalized the Maxwellian distribution for the case when gas particles are subjected to external forces. The formula suggested by him was later called the Maxwell–Boltzmann distribution. A new stage in the development of statistical thermodynamics began with the works by Josiah Willard Gibbs (1839–1903). In contrast to Maxwell and Boltzmann, who have taken the velocity space as a starting point, Gibbs constructed statistical thermodynamics based on the concept of ensembles. From a mathematical point of view, the use of a probabilistic approach is a significant progress. It would seem that, due to the rapid development of the kinetic theory and statistical mechanics in XIX–XX centuries, the kinetic theory of heat had to become a basis for classical thermodynamics instead of the caloric fluid theory, which was dominant in classical thermodynamics formerly. However, this did not happen. Starting from the middle of XIX century, there are no mechanical interpretations of temperature and other thermodynamic quantities in classical thermodynamics. Any mechanical models of heat processes are also absent in non-equilibrium thermodynamics and continuum mechanics. This refers to the Prigogine non-equilibrium thermodynamics, Truesdell's method in continuum mechanics, and some other approaches which have arisen from classical thermodynamics. Certainly, in statistical physics the methods of the description of non-equilibrium processes are based on the mechanical models (see, e.g., [37–40]). However, these models and methods are not consistent with continuum mechanics.

Besides the caloric fluid theory and the kinetic theory of gas, a number of different mechanical models of thermal processes were suggested by outstanding scientists of past centuries. Some of the models are relevant to the subject of our study, and we briefly discuss them. Leonhard Euler (1707–1783) gave his view on the nature of heat in [41]. Unfortunately, this manuscript is accessible only to the readers who know Latin. A detailed exposition of some parts of the manuscript as well as the translation of a few quotations can be found in [42,43]. According to [42,43], Euler represented particles of a combustible material as the shells containing within themselves some peculiar matter, quickly rotating and very elastic. He supposed that the motion resource of the rotating matter becomes free when the shells are destroyed because of some reason. Euler believed that the release of the motion resource leads to visible results, namely, to the appearance of flames around the burning body. Thus, based on [42,43] we may conclude that Euler considered thermal processes to be associated with the rotational motion of particles constituting an internal structure of a substance. However, there are not ruled out other interpretations of Euler's text. Mikhail V. Lomonosov (1711–1765) held similar views—see [44]. He was convinced that heat consists in rotational motion of particles constituting materials. Benjamin Thompson (1753–1814) carried out experiments which showed the failure of the caloric fluid hypothesis, and he reached a conclusion that all thermal effects should be considered as a phenomenon of motion [33,45]. Thompson's experiments were partially repeated, confirmed and expanded by Humphry Davy (1778–1829). Davy believed that the thermal effects are associated with a rotational motion [33,46]. He reasoned as follows. When a body is heated, it expands, i.e., the matter particles move pushing each other. It is very likely that matter particles are always in motion. While the temperature remains constant a body is not expanding or compressing, i.e., the matter particles do not change their positions. Hence, at the constant temperature the motion of matter particles has to be rotational. There were some other views on the nature of heat. Thomas Young (1773–1829) considered heat to be the oscillations of some particles (not particles of matter). He believed that the thermal oscillations propagate through empty space as waves [33,47]. In his opinion, the only difference between light and heat is the fact that the thermal oscillations are slower than the light oscillations. Augustin Louis Cauchy (1789–1857) created several models of light-bearing ether that include not only the transverse oscillations but also longitudinal ones. In fact, he created several theories of crystal optics with the longitudinal oscillations. Cauchy supposed the longitudinal oscillations to have the heat nature [34,48]. He believed that the existence of longitudinal oscillations would be verified by experiment.

Our approach to model thermal processes is based on an idea to combine Euler's concept of the rotational character of motion that causes thermal effects and Cauchy's concept of the interdependence between the longitudinal oscillations of light-bearing ether and heat processes.

3 A subject of study and basic ideas of the suggested approach

The physical object under consideration is a conventional isotropic homogeneous material without microstructure, inclusions, etc. This material has elastic and thermodynamic properties. In order to describe thermal processes in such a material by means of some mechanical model without using statistical methods, we introduce the continuum possessing internal rotational degrees of freedom. The internal degrees of freedom are used for modeling thermal processes. Motion associated with the internal degrees of freedom has no relation to the real motion of the material particles. Characteristics of motion associated with the internal degrees of freedom, as well as characteristics of interactions associated with the internal degrees of freedom, should be considered as analogies of thermodynamic quantities.

The main ideas of the description of thermoelastic and thermoviscoelastic processes by means of the mechanical model with internal rotational degrees of freedom (to be exact, the model of one-rotor gyrostat continuum) were first stated in [49] and developed in [50–52]. These ideas consist in the following:

- We use the one-rotor gyrostat continuum for modeling solids, liquids and gases. This continuum is considered to be elastic. The interaction of carrier bodies of the gyrostats is attributed to the mechanical processes. The interaction of rotors of gyrostats models thermal processes. The interaction of the carrier bodies and the rotors provides the interplay of mechanical and thermal processes.
- The gyrostats (which model material particles) are considered to be embedded into some medium occupying the whole infinite space. This medium represents the physical vacuum, the field, the ether or something like that. In the papers cited above, it is called the thermal ether. In what follows, it will be called 'the ether' for short.
- We assume that all gyrostats interact with the ether by means of elastic moments associated with the rotational degrees of freedom. Due to the fact that the ether fills the whole infinite space and interacts with all gyrostats, it plays a double role in our model.
- On the one hand, we assume that all interactions of gyrostats with each other are performed by the instrumentality of the ether. To be exact, the carrier bodies of different gyrostats interact through the agency of the ether, and the rotors belonging to different gyrostats interact also via the ether. From a mathematical point of view, this means that the constitutive equations for all quantities characterizing the stress state of the one-rotor gyrostat continuum depend not only on the properties of the carrier bodies and the rotors of the gyrostats, but also on the elastic properties and the stress–strain state of the ether filling the space between the gyrostats.
- On the other hand, we assume that the ether provides a dissipation of the gyrostats' energy. Since the gyrostats interact with the ether, their motion causes appearance of waves in the ether. As a result, a certain part of the gyrostats' energy is spent on formation of the waves. Since the ether is considered to be infinite, waves carrying away the gyrostats energy do not come back. The result is the dissipation of the gyrostats energy into the ether.
- The dissipation of the gyrostats' energy into the ether becomes apparent in the material medium in the form of the heat conduction and the internal damping. The heat conduction mechanism is supposed to be provided only by the moment interactions between the rotors and the ether. The internal damping mechanism can be provided in different ways, both due to the kinematic connection between the rotors and the carrier bodies and thanks to the interaction of the carrier bodies with the ether.

Now we explain the physical meaning of the suggested model. For modeling a medium with a combination of various physical properties, we consider atoms as complex particles with several internal rotors. According to the concepts of modern physics, atoms have a very complex internal structure. For example, they can be in different energy states and possess the ability to radiate and absorb the energy quanta and elementary particles. These facts should be taken into account when the properties of a single atom or a molecule consisting of several atoms are studied. When modeling a medium which consists of millions of atoms, many properties of atoms can be ignored or taken into account integrally, and it should be done so. For example, when modeling crystal lattices, very simple models of atoms are used; namely, atoms are assumed to be mass points or infinitesimal rigid bodies. Let us suppose that we want to model a medium possessing some mechanical, thermal, electric, and magnetic properties. In order to construct such a model, we can use different approaches. One of them is to consider atoms as complex particles with internal structure and internal degrees of freedom.

There are two different types of particles with an internal structure: particles with internal translational degrees of freedom (deformable particles) and particles with internal rotational degrees of freedom (multi-spin particles). Continua consisting of particles of the first type are called micromorphic continua. Continua consisting of particles of the second type are called micropolar continua. A peculiarity of micropolar continua is the fact that each particle of such a continuum has three translational and a number of rotational degrees of freedom. The number of rotational degrees of freedom of a multi-spin particle depends on the number of its rotors and the number of independent variables required to determine the orientation of each rotor relative to the carrier body of a multi-spin particle. All additional strains and stresses in the micropolar continua are related to the rotational degrees of freedom. In principle, both deformable particles and multi-spin ones can

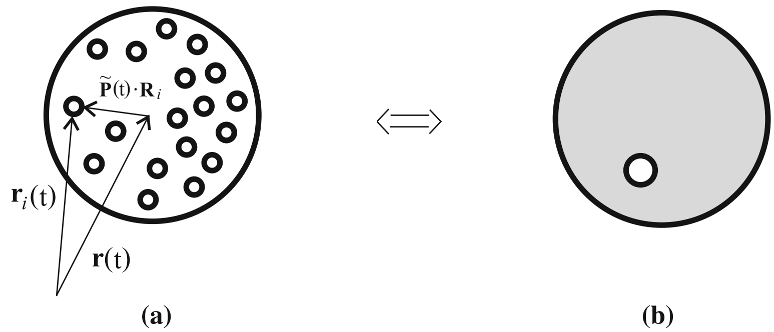

Fig. 1 A quasi-rigid body and its approximate model (the one-rotor gyrostat), which are equivalent in a first approximation

be used to model atoms, and, consequently, both the micromorphic and micropolar continua can be used to model a medium with some non-mechanical properties. Let us assume the possibility of large strains of the micromorphic continuum that are associated with the internal degrees of freedom, as well as the possibility of large relative velocities and accelerations. Then, the problem we are confronted with is to keep atoms as a whole and conserve their characteristic sizes. Such obstacles do not arise in the case of micropolar continua. Therefore, in view of the fact that under certain conditions any physical processes demonstrate nonlinear behavior, it can be argued that the micropolar continua are better suited for modeling media that possess not only the mechanical properties.

In what follows, we consider an atom to be a multi-spin particle like a quasi-rigid body (see Fig. 1a). The quasi-rigid body is a rigid body in the sense that the distances between any two points of this particle are kept unchanged under arbitrary motions of the quasi-rigid body. However, unlike the standard rigid body, the quasi-rigid body contains several rotors inside. Each rotor can rotate independently, and the rotation of rotors does not change the inertia tensor of the quasi-rigid body. In fact, the quasi-rigid body is a multi-rotor gyrostat that consists of a carrier body and a number of rotors rotating independently relative to the carrier body. When modeling atoms by the multi-rotor gyrostats, the motion of carrier bodies characterizes the motion of atoms as rigid bodies. It is the motion of atoms as rigid bodies that causes mechanical strains and mechanical stresses in the material medium. The rotors simulate elementary particles constituting the atoms. Pursuant to this model, the motion of rotors simulates the change in the internal state of atoms. In our opinion, the internal state of atoms determines all the physical processes occurring in the material medium, namely electrical, magnetic, and thermal. Therefore, using the multi-rotor gyrostat continuum we can simulate all the physical processes in the material medium.

As noted above, simple models of atoms can be used for constructing continuous models of matter, and it should be done so. In many cases, atoms can be modeled by mass points or infinitesimal rigid bodies. However, if we want to model a material medium with several physical properties (not only mechanical but also thermal, electric and magnetic), then the simplest models of atom, namely a mass point or a rigid body, are not suitable. Consideration of a multi-rotor gyrostat (instead of a mass point or an infinitesimal rigid body) as a model of atoms significantly complicates a mathematical formulation of the problem and increases the number of parameters. Therefore, we try to simplify this model of atoms, but so that a simpler model retains key features of the multi-rotor gyrostat.

Let us consider the quasi-rigid body shown in Fig. 1a. The motion of the quasi-rigid body as a whole is defined by the position vector of its mass center $\mathbf{r}(t)$ and the rotation tensor of its carrier body $\tilde{\mathbf{P}}(t)$. The mass center velocity and the angular velocity of the carrier body are given by the formulas

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \qquad \tilde{\boldsymbol{\omega}}(t) = -\frac{1}{2}\left(\frac{\mathrm{d}\tilde{\mathbf{P}}}{\mathrm{d}t}\cdot\tilde{\mathbf{P}}\right)_{\times} \tag{1}$$

where $(\ )_{\times}$ denotes the vector invariant of a tensor that is defined for an arbitrary dyad as $(\mathbf{ab})_{\times} = \mathbf{a}\times\mathbf{b}$. Let us consider a rotor with a number i. The position of the rotor with respect to the mass center of the quasi-rigid body in the reference configurations is defined by the position vector $\mathbf{R}_i$. According to the fundamental theorem of the rigid body kinematics, vector $\mathbf{r}_i$ that defines the rotor position in the actual configuration is calculated by the formula

$$\mathbf{r}_i(t) = \mathbf{r}(t) + \tilde{\mathbf{P}}(t)\cdot\mathbf{R}_i. \tag{2}$$

The rotation of this rotor is defined by the rotation tensor $\mathbf{P}_i(t)$; the translational and angular velocities are calculated by the formulas

$$\mathbf{v}_i(t) = \mathbf{v}(t) + \tilde{\boldsymbol{\omega}}(t) \times \tilde{\mathbf{P}}(t) \cdot \mathbf{R}_i, \qquad \boldsymbol{\omega}_i(t) = -\frac{1}{2}\left(\frac{\mathrm{d}\mathbf{P}_i}{\mathrm{d}t} \cdot \mathbf{P}_i^T\right)_\times. \tag{3}$$

The dynamical structures of the rotor i, namely the kinetic energy, the linear momentum and the angular momentum calculated with respect to some fixed point Q, have the form

$$K_i = m_i\left(\frac{1}{2}\,\mathbf{v}_i \cdot \mathbf{v}_i + \frac{1}{2}\,J_i\,\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i\right), \qquad \mathbf{K}_{1i} = m_i\,\mathbf{v}_i, \qquad \mathbf{K}_{2i}^Q = m_i\big[(\mathbf{r}_i - \mathbf{r}_Q) \times \mathbf{v}_i + J_i\,\boldsymbol{\omega}_i\big] \tag{4}$$

where m_i and J_i are the mass and the moment of inertia of the rotor i. For simplicity sake, the rotors are assumed to be isotropic, and the carrier body is assumed to be inertialess. According to the axioms of additivity, the dynamical structures of the quasi-rigid body can be represented as

$$K = \sum_{i=1}^{N} K_i, \qquad \mathbf{K}_1 = \sum_{i=1}^{N} \mathbf{K}_{1i}, \qquad \mathbf{K}_2^Q = \sum_{i=1}^{N} \mathbf{K}_{2i}^Q \tag{5}$$

where N is the number of rotors. Taking into account Eqs. (1)–(5), it is easy to show that the kinetic energy of the quasi-rigid body K can be represented as a quadratic form of vectors $\mathbf{v}$, $\tilde{\boldsymbol{\omega}}$, $\boldsymbol{\omega}_i$, and vectors $\mathbf{K}_1$, $\mathbf{K}_2^Q$ can also be expressed in terms of these quantities (see, e.g., [49]). Let us represent the angular velocity of the rotor i in the form

$$\boldsymbol{\omega}_i(t) = \boldsymbol{\omega}(t) + \hat{\boldsymbol{\omega}}_i(t), \qquad \boldsymbol{\omega}(t) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\omega}_i(t). \tag{6}$$

Here $\boldsymbol{\omega}(t)$ is the average angular velocity of the rotors; $\hat{\boldsymbol{\omega}}_i(t)$ is the deviation of the angular velocity of the given rotor from the average angular velocity. Now we assume that

$$|\hat{\boldsymbol{\omega}}_i(t)| \ll |\boldsymbol{\omega}(t)| \qquad \Rightarrow \qquad \boldsymbol{\omega}_i(t) \approx \boldsymbol{\omega}(t). \tag{7}$$

It is easy to show that the approximate expressions for the kinetic energy, the linear momentum, and the angular momentum of a quasi-rigid body obtained under assumption (7) coincide with those of the one-rotor gyrostat shown in Fig. 1b. Thus, the quasi-rigid body (or what is the same, the multi-rotor gyrostat) and the one-rotor gyrostat are equivalent in a first approximation. The simplified model (the one-rotor gyrostat) is quite suitable for the purposes of this study. That is why modeling the material medium we use the one-rotor gyrostats to model the atoms.

In a continuum theory, we use the physical characteristics averaged over a representative volume that contains billions of atoms. It is important to note that the dynamic properties of a representative volume of the continuous medium have no qualitative difference from the dynamic properties of particles in the representative volume. Within the spatial description, it is customary to refer all physical quantities to the representative volume fixed in space. During its evolution, the representative volume is occupied by different particles, each having its own mass, tensor of inertia, translational and angular velocities. The formulation of the main ideas for defining the inertial and kinematic characteristics of the representative volume within the spatial description can be found in [53].

The derivation of equations of a one-rotor gyrostat continuum can be found in [49]. The concepts of temperature, entropy, and heat flow are introduced in the context of the suggested model. In [49], it is proved that the mathematical description of the suggested model can be reduced (in special cases) to well-known equations such as the heat conduction equation, the self-diffusion equation, and the equations of coupled thermoelasticity. A more complicated mathematical model, which describes both the volume and shear viscosities, is constructed in [51,52]. On the basis of the suggested theory, the dependence of the acoustic wave attenuation factor on the angular frequency is obtained. This dependence is in close agreement with the classical dependence in the low-frequency range. In the hypersonic frequency range, it agrees with the dependence obtained by using the phonon theory. In [52], the physical nature of heat conduction and internal damping is discussed. This discussion is based on the theoretical considerations (the concept of thermal ether) and the analysis of two model problems, which were solved in [50,51]. In [52], the volume viscosity, the shear viscosity, and the heat flux relaxation constant are determined by using the known values of the sound velocity and the acoustic wave attenuation factor. The obtained values of the heat flux relaxation constant are compared with the values derived from the phonon theory.

In [49–52], the mathematical description of the suggested model is constructed within the framework of linear theory. Thus, the physically and geometrically linear theory of thermoviscoelasticity is considered in the cited papers. Now, our purpose is to carry out the further development of the theory in the context of the same mechanical model. In what follows, we consider nonlinear thermodynamic effects. However, for simplicity sake, we ignore the internal damping.

4 The one-rotor gyrostat continuum as a model of an elastic medium with thermodynamic properties

Now we give a brief description of our model and summarize the set of equations that describes the model in the linear approximation [52]. We consider the one-rotor gyrostat continuum. The one-rotor gyrostat is a particle that consists of the carrier body and the rotor (see Fig. 2). The rotor can rotate independently of the carrier body rotation, but it cannot execute translatory motion relative to the carrier body. Thus, the one-rotor gyrostat has nine degrees of freedom, three translational ones and six rotational ones. Free space between the gyrostats is filled up by the ether. The ether is shown in Fig. 2 as the body points in the space between the gyrostats. With respect to the ether, we make the following assumptions:

(i) The ether particles are much smaller than elementary particles of the conventional substance. The structure of the ether particles coincides with the structure of the rotors that belong to the gyroststs. The last assertion is based on Kelvin's idea that atoms (as well as all known elementary particles) are the vortex rings that consist of more "elementary" particles (particles of the ether)—see [54]. According to Kelvin's idea, the only difference between the ponderable matter and ether is that the density of ponderable matter is much larger than the density of free ether.

(ii) Following Kelvin's ideas, we assume that the ether is a medium that is less dense than the conventional substance. The ether particles fill the space between elementary particles of the conventional substance, and the elementary particles interact with each other via the ether particles.

(iii) The interactions of ether particles with each other and the interactions of ether particles with the elementary particles of the conventional substance are based only on the rotational degrees of freedom and the principle of moment interactions. There are no interactions between these particles by means of forces. Thus, from a continuum mechanics point of view, the model of ether is the special case of the Cosserat continuum. A nonlinear model of ether is presented in Appendix F. A linearized model of ether is presented in Appendix G.

(iv) The ether is an infinite medium, i.e., it occupies the whole space. The ether is assumed to be an elastic medium. However, due to its infinite extent the ether carries away the energy of rotational motion of material particles located in it. When the particles interact with the ether, their motion disturbs the ether and causes appearance of waves in it. Since the ether is infinite, the waves cannot be reflected from the boundaries, and hence, they cannot come back. Thus, the part of the material particles energy, which is spent on formation of the waves in the ether, is irretrievably lost.

Thus, initially we consider a two-component medium that consists of the one-rotor gyrostat continuum simulating the conventional substance and the body-point continuum simulating the ether. This two-component

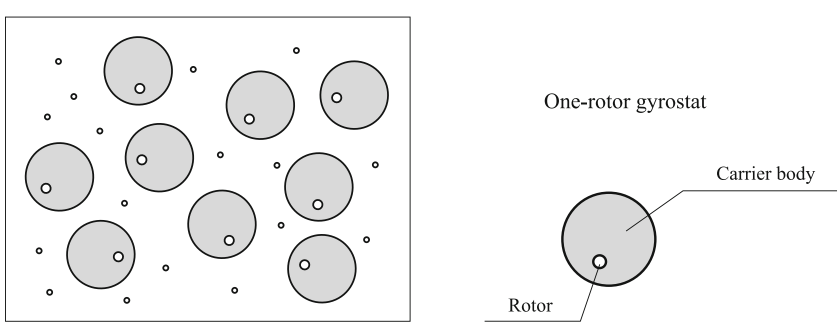

Fig. 2 An elementary volume of the continuum of one-rotor gyrostats together with the continuum of body points in the space between the gyrostats (on the *left-hand side*) and the one-rotor gyrostat (on the *right-hand side*)

medium is assumed to be conservative. When we study only the one-rotor gyrostat continuum, we should consider the ether to be an external factor with respect to the continuum under study. Interacting with the material particles via the rotational degrees of freedom, the boundless ether creates the moment of viscous damping acting on the material particles. Thus, eliminating the ether we obtain a non-conservative model of the one-rotor gyrostat continuum.

Further we consider only the continuum of one-rotor gyrostats. We start with a linear theory. Let vector $\mathbf{r}$ be a position of some point in space if the spatial description is used, or a position of some point of the material medium in the reference configuration if the material description is used. It does not matter because, from a mathematical point of view, there is no difference between the spatial and material descriptions in a linear theory. We introduce the following notations: $\rho(\mathbf{r}, t)$ is the mass density; $\mathbf{I} = I\mathbf{E}$ and $\mathbf{J} = J\mathbf{E}$ are the mass densities of inertia tensors of the carrier bodies and the rotors, respectively, where $\mathbf{E}$ is the unit tensor; $\mathbf{u}(\mathbf{r}, t)$ is the displacement vector; $\mathbf{v}(\mathbf{r}, t)$ is the velocity vector; $\tilde{\mathbf{P}}(\mathbf{r}, t)$ and $\tilde{\boldsymbol{\omega}}(\mathbf{r}, t)$ are the rotation tensor and the angular velocity vector of the carrier bodies; and $\mathbf{P}(\mathbf{r}, t)$ and $\boldsymbol{\omega}(\mathbf{r}, t)$ are the rotation tensor and the angular velocity vector of the rotors. When the material medium is not disturbed, tensors $\tilde{\mathbf{P}}(\mathbf{r}, t)$ and $\mathbf{P}(\mathbf{r}, t)$ are assumed to be equal to the unit tensor. Upon the linearization near the undisturbed position, tensors $\tilde{\mathbf{P}}(\mathbf{r}, t)$ and $\mathbf{P}(\mathbf{r}, t)$ are written as

$$\tilde{\mathbf{P}}(\mathbf{r}, t) = \mathbf{E} + \boldsymbol{\varphi}(\mathbf{r}, t) \times \mathbf{E}, \qquad \mathbf{P}(\mathbf{r}, t) = \mathbf{E} + \boldsymbol{\theta}(\mathbf{r}, t) \times \mathbf{E} \tag{8}$$

where $\boldsymbol{\varphi}(\mathbf{r}, t)$, $\boldsymbol{\theta}(\mathbf{r}, t)$ are the rotation vectors of the carrier bodies and the rotors, respectively. Since $|\boldsymbol{\varphi}(\mathbf{r}, t)|$ and $|\boldsymbol{\theta}(\mathbf{r}, t)|$ are assumed to be small, the kinematic relations take the form

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}, \qquad \tilde{\boldsymbol{\omega}} = \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}, \qquad \boldsymbol{\omega} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}. \tag{9}$$

The balance equations of the linear momentum for the gyrostats and of the angular momentum for the carrier bodies of gyrostats are

$$\nabla \cdot \boldsymbol{\tau} + \rho_* \mathbf{f} = \rho_* \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \qquad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho_* \mathbf{m} = \rho_* I \frac{\mathrm{d}\tilde{\boldsymbol{\omega}}}{\mathrm{d}t}. \tag{10}$$

Here ∇ is the gradient operator; $\boldsymbol{\tau}$ is the stress tensor; $\boldsymbol{\mu}$ is the moment stress tensor modeling the interaction of the carrier bodies of gyrostats; ρ_* is some reference value of ρ, and it does not depend on time; $\mathbf{f}$ is the mass density of external forces; and $\mathbf{m}$ is the mass density of external moments acting on the carrier bodies of gyrostats.

Since the one-rotor gyrostat has nine degrees of freedom, three translational ones and six rotational ones, the balance equations (10) should be added by the balance equation of the angular momentum for the rotors of gyrostats. This equation has the form

$$\nabla \cdot \mathbf{T} + \rho_* \mathbf{L} = \rho_* J \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{11}$$

where $\mathbf{T}$ is the moment stress tensor modeling the interaction of the rotors of gyrostats and $\mathbf{L}$ is the mass density of external moments acting on the rotors.

The energy balance equation for some part of the one-rotor gyrostat continuum that occupies a control volume V is written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{(V)} \rho_* (K + U)\, \mathrm{d}V = \int_{(V)} \rho_* (\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \tilde{\boldsymbol{\omega}} + \mathbf{L} \cdot \boldsymbol{\omega})\, \mathrm{d}V + \int_{(S)} \big(\boldsymbol{\tau}_n \cdot \mathbf{v} + \boldsymbol{\mu}_n \cdot \tilde{\boldsymbol{\omega}} + \mathbf{T}_n \cdot \boldsymbol{\omega}\big) \mathrm{d}S. \tag{12}$$

Here K is the kinetic energy density per unit mass; U is the internal energy density per unit mass; $\boldsymbol{\tau}_n = \mathbf{n} \cdot \boldsymbol{\tau}$, $\boldsymbol{\mu}_n = \mathbf{n} \cdot \boldsymbol{\mu}$, and $\mathbf{T}_n = \mathbf{n} \cdot \mathbf{T}$ where $\mathbf{n}$ denotes the unit outer normal vector to the surface S.

By standard reasoning, taking into account the balance equations (10), (11), we transform Eq. (12) to the form

$$\frac{\mathrm{d}(\rho_* U)}{\mathrm{d}t} = \boldsymbol{\tau}^T \cdot\cdot \big(\nabla \mathbf{v} + \mathbf{E} \times \tilde{\boldsymbol{\omega}}\big) + \boldsymbol{\mu}^T \cdot\cdot \nabla \tilde{\boldsymbol{\omega}} + \mathbf{T}^T \cdot\cdot \nabla \boldsymbol{\omega} \tag{13}$$

where the double scalar product is defined as $\mathbf{ab} \cdot\cdot \mathbf{cd} = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$. Introducing the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$, which are determined by the formulas

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{E} \times \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \qquad \boldsymbol{\vartheta} = \nabla \boldsymbol{\theta}, \tag{14}$$

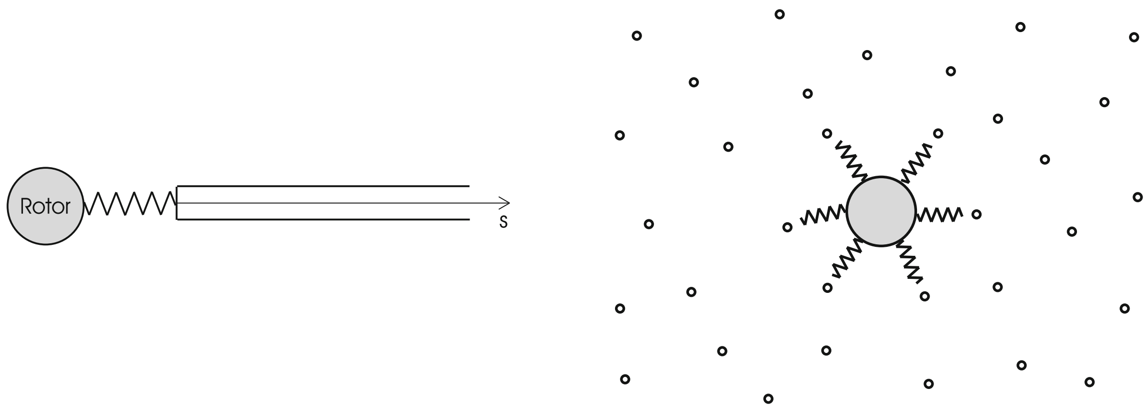

Fig. 3 Two models of interaction between the rotors and the ether: the 1D model on the *left* and the 3D model on the *right*

and taking into account Eq. (9), we rewrite Eq. (13) in the form

$$\frac{\mathrm{d}(\rho_* U)}{\mathrm{d}t} = \boldsymbol{\tau}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} + \boldsymbol{\mu}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} + \mathbf{T}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\vartheta}}{\mathrm{d}t}. \tag{15}$$

Further we consider the special case of the theory of a one-rotor gyrostat continuum. We start with the formulation of two hypotheses.

Hypothesis 1 Vector $\mathbf{L}$ is a sum of the moment $\mathbf{L}_\mathrm{h}$ characterizing external actions of all sorts and the moment of linear viscous damping

$$\mathbf{L}_\mathrm{f} = -\beta J \boldsymbol{\omega} \tag{16}$$

where β is the coefficient of damping. The moment $\mathbf{L}_\mathrm{f}$ models the influence of the ether (the body points positioned in the space between the gyrostats) that causes the dissipation of the rotors energy. The moment $\mathbf{L}_\mathrm{h}$ models the influence of external ponderable bodies that is passed by means of the ether. It can model actions of various physical nature, e.g., heat supply, electromagnetic excitation or some kind of radiation. The main difference between the moment $\mathbf{L}_\mathrm{h}$ and the moment $\mathbf{L}_\mathrm{f}$ is the fact that the moment $\mathbf{L}_\mathrm{h}$ occurs only when there are some ponderable bodies, whereas the moment $\mathbf{L}_\mathrm{f}$ occurs regardless of the presence or absence of other bodies.

The structure of moment (16) is chosen in accordance with the results obtained by solving two model problems [50,51]. The model considered in [50] consists of the semi-infinite inertial elastic rod (a one-dimensional model of the ether) that is connected with the rotor by means of the inertialess spring working in torsion (rotation about the axis of the rod)—see Fig. 3, on the left-hand side. The rotation of the rotor disturbs the elastic rod and causes the torsion waves in it. If the rod had a limited size, the waves would be reflected from the boundary and come back. In this case, the system would be conservative. The dissipation of the rotor energy occurs only due to the infinite length of the rod and the absence of sources at infinity. As shown in [50], after eliminating the variables that characterize the rod motion, the problem is reduced to the set of equations describing the rotor motion. In the set of equations, there is the equation that contains the moment of viscous damping characterizing the energy radiation in the ambient medium. It is proved that the moment of viscous damping is proportional to the angular momentum of the rotor, and the coefficient of damping β depends on the parameters of the rod and the torsional stiffness of the spring connecting the rotor and the rod. In order to have a more appropriate model of the dissipative process, we should consider the interaction of the rotor with a three-dimensional model of the ether occupying the whole space. The model considered in [51] consists of the spherical source (the spherical surface each point of which is the rotor) and the infinite inertial elastic continuum modeling the ether—see Fig. 3, on the right-hand side. All rotors of the spherical source are connected with the continuum by means of the inertialess spring working in torsion (rotation about a radius of the spherical source). The solution of this problem in the case of spherical symmetry can be found in [51]. Comparing the results obtained in the case of the one-dimension model of the ether and the results obtained for the spherically symmetric problem in the case of three-dimension model of the ether, we conclude that although the sets of equations somewhat differ from each other, they have one important similarity. To be exact, both the problems include the equations with the dissipative terms proportional to the angular momentum, and the coefficients of viscous damping have the same dependence on the model parameters. Thus, in view of the results of [50,51], we assume that the moment modeling the influence of the ether is the moment of viscous damping proportional to the angular momentum vector—see Eq. (16). We note that the linear models are studied in [50,51]. A nonlinear statement of the problem modeling the interaction of the rotor with the ether is considered in Sect. 7.4 and Appendix H.

Hypothesis 2 The moment stress tensor **T** characterizing the interactions between the rotors is the spherical part of tensor

$$\mathbf{T} = T\,\mathbf{E}. \tag{17}$$

Assumption (17) is based on the following interpretations. We suppose the interaction of the carrier bodies of gyrostats to be attributed to the mechanical processes. We suppose that the interaction of the rotors of gyrostats models thermal processes, and the interaction of the carrier bodies and the rotors provides the interplay of mechanical and thermal processes. We consider the moment interaction between the rotors to be an analogy of temperature. Since temperature is a scalar, the moment stress tensor **T** must be characterized by one scalar quantity. Hence, it must be the spherical part of the tensor—see Eq. (17).

In view of Eq. (17), the energy balance equation (15) can be reduced to the form

$$\frac{\mathrm{d}(\rho_* U)}{\mathrm{d}t} = \boldsymbol{\tau}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} + \boldsymbol{\mu}^T \cdot\cdot \frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} + T\,\frac{\mathrm{d}\vartheta}{\mathrm{d}t}, \qquad \vartheta = \mathrm{tr}\,\boldsymbol{\vartheta}. \tag{18}$$

If we consider Eq. (18) to be the energy balance equation for the classical medium, then we interpret the last term on the right-hand side of this equation as the thermodynamic one. Since the quantity T has the sense of temperature analogy, the quantity ϑ acquires the meaning of volume density of entropy analogy. The units of measurement of the temperature analogy and the entropy analogy that are introduced within the framework of the suggested model are different from the standard units of measurement of temperature and entropy. Indeed, the unit of measurement of T is N/m, whereas the unit of measurement of temperature is Kelvin; the unit of measurement of ϑ is $1/m$, whereas the unit of measurement of volume density of entropy is $\mathrm{J/(m^3\,K)}$. This obstacle can be overcome by introducing a normalization factor a and changing the variables:

$$T = aT_\mathrm{a}, \qquad \vartheta = \frac{1}{a}\,\vartheta_\mathrm{a}, \qquad \boldsymbol{\theta} = \frac{1}{a}\,\boldsymbol{\theta}_\mathrm{a}, \qquad \boldsymbol{\omega} = \frac{1}{a}\,\boldsymbol{\omega}_\mathrm{a}, \qquad \mathbf{L}_\mathrm{h} = a\mathbf{L}_\mathrm{h}^a, \qquad J = a^2 J_\mathrm{a}. \tag{19}$$

Here T_a is the temperature that can be measured by a thermometer. Its unit of measurement is Kelvin. Correspondingly, ϑ_a is the volume density of entropy. Its unit of measurement is $\mathrm{J/(m^3\,K)}$. As shown in [52], in the case of the linear theory the normalization factor a can be eliminated from all equations of the suggested theory. Therefore, we have no way to determine the numerical value of the normalization factor a. But this is not necessary.

Now we consider the model of the continuum that is based on the hypotheses stated above. In view of the hypotheses, the balance equation of the angular momentum for rotors takes the form

$$\nabla T_\mathrm{a} - \rho_* \beta J_\mathrm{a} \boldsymbol{\omega}_\mathrm{a} + \rho_* \mathbf{L}_\mathrm{h}^a = \rho_* J_\mathrm{a}\,\frac{\mathrm{d}\boldsymbol{\omega}_\mathrm{a}}{\mathrm{d}t}. \tag{20}$$

By taking the divergence of both sides of Eq. (20), we obtain one of the forms of the heat conduction equation, namely

$$\Delta T_\mathrm{a} - \rho_* \beta J_\mathrm{a}\,\frac{\mathrm{d}\vartheta_\mathrm{a}}{\mathrm{d}t} - \rho_* J_\mathrm{a}\,\frac{\mathrm{d}^2\vartheta_\mathrm{a}}{\mathrm{d}t^2} = -\rho_* \nabla \cdot \mathbf{L}_\mathrm{h}^a \tag{21}$$

where the term $\rho_* \nabla \cdot \mathbf{L}_\mathrm{h}^a$ plays the role of a heat supply. We note that the heat conduction equation (21) can be reduced to the conventional form. In order to do this, we have to express ϑ_a in terms of temperature and strain tensors by using the constitutive equation given below.

According to the energy balance equation (18), the internal energy density is a function of the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, and the scalar strain measure ϑ (or the volume density of entropy ϑ_a that is the same thing). Since we construct the linear theory, the internal energy density $\rho_* U$ is assumed to be a quadratic form of the quantities listed above. In this case, the constitutive equations are written as

$$\begin{aligned}
\boldsymbol{\tau}^T &= \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + \mathbf{C}_4\,(\vartheta_\mathrm{a} - \vartheta_\mathrm{a}^*),\\
\boldsymbol{\mu}^T &= \boldsymbol{\mu}_0^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + \mathbf{C}_5\,(\vartheta_\mathrm{a} - \vartheta_\mathrm{a}^*),\\
T_\mathrm{a} &= T_\mathrm{a}^* + \boldsymbol{\varepsilon} \cdot\cdot \mathbf{C}_4 + \boldsymbol{\kappa} \cdot\cdot \mathbf{C}_5 + C_6\,(\vartheta_\mathrm{a} - \vartheta_\mathrm{a}^*).
\end{aligned} \tag{22}$$

Here $\boldsymbol{\tau}_0$ and $\boldsymbol{\mu}_0$ are the initial stresses, T_a^* is the value of absolute temperature at which the thermodynamic parameters are determined, ϑ_a^* is the corresponding value of volume density of entropy, ${}^4\mathbf{C}_1$, ${}^4\mathbf{C}_2$, ${}^4\mathbf{C}_3$ are the fourth-rank stiffness tensors, $\mathbf{C}_4$, $\mathbf{C}_5$ are the second-rank tensors characterizing the interplay of mechanical and thermodynamic processes, and C_6 is the scalar quantity characterizing the specific heat.

In view of the foregoing analogies between the mechanical and thermodynamic quantities, the set of equations (9), (10), (14), (19), (21), (22) can be considered as the mathematical description of a conventional material which possesses elastic and thermodynamic properties.

5 The structure of material tensors of an isotropic medium: classical theory of thermoelasticity

Now we consider some aspect of the determination of the structure of material tensors. First of all, we note that the structure of a material tensor essentially depends on the tensor type. This fact follows from Zhilin's theory of symmetry [15,17,55,56] that is based on the definition of orthogonal transformation given below.

5.1 Zhilin's theory of symmetry

Definition 1 A tensor ${}^k\mathbf{S}'$ is called the orthogonal transformation of a tensor ${}^k\mathbf{S}$ if

$$ {}^k\mathbf{S} = S^{i_1 \ldots i_k}\, \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k} \quad \Rightarrow \quad {}^k\mathbf{S}' = (\det \mathbf{Q})^\alpha\, S^{i_1 \ldots i_k}\, \mathbf{Q}\cdot\mathbf{e}_{i_1} \ldots \mathbf{Q}\cdot\mathbf{e}_{i_k} \tag{23} $$

where $\mathbf{Q}$ is an orthogonal tensor, $\alpha = 0$ for the polar tensor ${}^k\mathbf{S}$, and $\alpha = 1$ for the axial tensor (pseudo-tensor) ${}^k\mathbf{S}$.

The definition of orthogonal transformation (23) differs from the classical one by the multiplier $(\det \mathbf{Q})^\alpha$, which is absent in the classical definition. In the case of the polar tensor ${}^k\mathbf{S}$, this multiplier is equal to unity, and the definition (23) coincides with the classical one. In the case of the axial tensor ${}^k\mathbf{S}$, the multiplier $(\det \mathbf{Q})^\alpha$ is equal to $+1$ for the rotation tensor $\mathbf{Q}$ and -1 for the specular reflection tensor $\mathbf{Q}$. Consequently, in the case of the axial tensor ${}^k\mathbf{S}$ the definition (23) differs from the classical one. The classical theory of symmetry is known to be applicable only for polar tensors. Zhilin's theory of symmetry, which is based on the definition of orthogonal transformation (23), is applicable for both polar and axial tensors. Further both types of tensors are considered, and therefore, Zhilin's theory of symmetry is used.

Definition 2 The set of orthogonal tensors $\mathbf{Q}_\mathrm{s}$ being solutions of the equation

$$ {}^k\mathbf{S}' = {}^k\mathbf{S} \tag{24} $$

where the tensor ${}^k\mathbf{S}'$ is defined by Eq. (23) is called the symmetry group of the tensor ${}^k\mathbf{S}$.

Thus, if the tensor ${}^k\mathbf{S}$ is known, then its symmetry group can be found by solving Eq. (24). The inverse problem is to determine the structure of the tensor ${}^k\mathbf{S}$ on conditions that its symmetry group is known. Only the inverse problem is of practical interest since the Curie–Neumann principle allows us to find symmetry groups of all material tensors characterizing properties of a physical object on conditions that the symmetry group of the physical object is known.

The Curie–Neumann principle The symmetry group of any physical property of a physical object must include the symmetry group of the physical object.

Thus, if the symmetry group of a real physical object is known, then the Curie–Neumann principle along with the theory of symmetry allows us to determine the structure of all material tensors of the physical object.

The model of a continuum discussed above contains both polar and axial material tensors. The fourth-rank tensors ${}^4\mathbf{C}_1$, ${}^4\mathbf{C}_3$, and the second-rank tensor $\mathbf{C}_5$ are polar. The fourth-rank tensor ${}^4\mathbf{C}_2$ and the second-rank tensor $\mathbf{C}_4$ are axial. Further we determine the structure of these tensors for an isotropic chiral medium and an isotropic non-chiral medium.

5.2 An isotropic chiral medium

The symmetry group of an isotropic chiral medium includes only the tensors of rotation by an arbitrary angle about arbitrary axes and does not include any specular reflection tensors. Let us consider the tensors of rotation by an arbitrary angle ψ about three mutually orthogonal axes, the directions of which are determined by the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{Q}_i = (1 - \cos\psi)\,\mathbf{e}_i\mathbf{e}_i + \cos\psi\,\mathbf{E} + \sin\psi\,\mathbf{e}_i \times \mathbf{E}, \qquad i = 1,\ 2,\ 3. \tag{25}$$

Substituting Eq. (25) into Eqs. (23), (24), we determine the structure of all material tensors. Since the symmetry group includes only the rotation tensors (tensors with determinant equal to unity), we obtain the same results for polar and axial tensors. The fourth-rank tensors and the second-rank ones have the form

$${}^4\mathbf{C}_1 = C_1\mathbf{EE} + C_2\sum_{i=1}^{3}\mathbf{e}_i\mathbf{E}\,\mathbf{e}_i + C_3\sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j, \qquad \mathbf{C}_4 = C_4\mathbf{E}. \tag{26}$$

The question is, what sorts of engineering materials can be qualified as chiral media? In our opinion, it depends on what properties of a material and what processes in the material are that we want to study. If we want to study only the mechanical properties and processes, then almost all materials can be qualified as non-chiral media. The only exceptions are materials consisting of a sufficiently large particles that do not have a mirror symmetry, such as materials consisting of large polymer molecules having a helical structure or materials containing the DNA molecules. If we want to model a conventional material taking into account not only its mechanical properties but in addition some other of its physical properties, then a representative volume of the continuum must reflect the properties of the material at the microlevel, i.e., the properties of the material that are conditioned by the state of its atoms. Atoms consist of elementary particles with spin. The presence of spin eliminates the mirror symmetry. That is why, in order to model a conventional material taking into account not only its mechanical properties but in addition some other of its physical properties we should consider this material as a chiral medium. Certainly, the foregoing concerns only the method of modeling that is developed in this paper, i.e., the method based on using the Cosserat continuum with the microstructure.

5.3 An isotropic non-chiral medium

The symmetry group of an isotropic non-chiral medium includes the tensors of rotation by an arbitrary angle about arbitrary axes and the tensors of specular reflection from arbitrary planes. Let us consider three tensors of rotation (25) and three tensors of specular reflection from planes orthogonal to the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively:

$$\mathbf{Q}_i = \mathbf{E} - 2\mathbf{e}_i\mathbf{e}_i, \qquad i = 1,\ 2,\ 3. \tag{27}$$

Substituting Eqs. (25), (27) into Eqs. (23), (24), we obtain the following results. The polar tensors ${}^4\mathbf{C}_1$, ${}^4\mathbf{C}_3$ and $\mathbf{C}_5$ have the form (26). The axial tensors ${}^4\mathbf{C}_2$ and $\mathbf{C}_4$ are equal to zero.

At the same time, it is well known that tensor $\mathbf{C}_4$ characterizing the thermal expansion is not equal to zero. Consequently, if we assume the medium to be isotropic in the classical sense, then we come into conflict with the well-known fact that any material contracts or expands when its temperature is changed. Of course, this is unacceptable. Therefore, further we consider isotropic media which are chiral at least with respect to the microstructure.

5.4 The hyperbolic type thermoelasticity and classical theory of thermoelasticity

It is well known that when describing mechanical processes in three-dimensional media, the moment interactions and the rotation inertia can be neglected. In accordance with this fact, we suppose that

$$\boldsymbol{\mu} = \mathbf{0}, \qquad \mathbf{m} = \mathbf{0}, \qquad I = 0 \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{\tau}^T. \tag{28}$$

Assuming that the medium is chiral with respect to the microstructure and taking into account Eqs. (26), (28), we obtain the following constitutive equations:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + C_1\varepsilon\mathbf{E} + (C_2 + C_3)\boldsymbol{\varepsilon}^s + C_4\,(\vartheta_\mathrm{a} - \vartheta_\mathrm{a}^*)\mathbf{E},$$
$$T_\mathrm{a} = T_\mathrm{a}^* + C_4\varepsilon + C_6\,(\vartheta_\mathrm{a} - \vartheta_\mathrm{a}^*), \qquad \boldsymbol{\varepsilon}^s = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \qquad \varepsilon = \mathrm{tr}\,\boldsymbol{\varepsilon}^s. \tag{29}$$

Let us take the following parameters:

$$C_1 = K_\mathrm{ad} - \frac{2}{3}G, \qquad C_2 + C_3 = G, \qquad C_4 = -\frac{\alpha K_\mathrm{iz} T_\mathrm{a}^*}{\rho_* c_\mathrm{v}}, \qquad C_6 = \frac{T_\mathrm{a}^*}{\rho_* c_\mathrm{v}}, \qquad \beta J_\mathrm{a} = \frac{T_\mathrm{a}^*}{\rho_*\lambda} \tag{30}$$

where K_iz and $K_\mathrm{ad} = K_\mathrm{iz} + \alpha^2 K_\mathrm{iz}^2 T_\mathrm{a}^*/(\rho_* c_\mathrm{v})$ are the isothermal and adiabatic modules of compression, G is the shear modulus, α is the volume coefficient of thermal expansion, c_v is the specific heat at constant volume, and λ is the heat conduction coefficient. It is easy to see that the inverse coefficient of heat conduction is directly proportional to the dynamic coefficient of damping $\rho_*\beta J_\mathrm{a}$, the inverse specific heat is directly proportional to the angular stiffness C_6 characterizing the moment interaction between the rotors, and the volume coefficient of thermal expansion is directly proportional to the stiffness C_4 characterizing the dependence of the stress tensor on the angular strains and the dependence of the moment stress tensor on the linear strains.

In view of Eq. (30), the set of equations (10), (21), (29) can be reduced to the form

$$\nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f}\,\rho_*\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2}, \qquad \boldsymbol{\tau} = \boldsymbol{\tau}_0 + \left(K_\mathrm{iz} - \frac{2}{3}G\right)\varepsilon\mathbf{E} + 2G\,\boldsymbol{\varepsilon}^s - \alpha K_\mathrm{iz}(T_\mathrm{a} - T_\mathrm{a}^*)\,\mathbf{E},$$
$$\Delta T_\mathrm{a} - \frac{\rho_* c_\mathrm{v}}{\lambda}\left(\frac{\mathrm{d}T_\mathrm{a}}{\mathrm{d}t} + \frac{1}{\beta}\frac{\mathrm{d}^2 T_\mathrm{a}}{\mathrm{d}t^2}\right) = \frac{\alpha K_\mathrm{iz} T_\mathrm{a}^*}{\lambda}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \frac{1}{\beta}\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}\right) - \rho_*\nabla\cdot\mathbf{L}_\mathrm{h}^a. \tag{31}$$

The parameter β^{-1} is usually called the heat flow relaxation time scale. If the parameter β^{-1} becomes zero on conditions that the product βJ_a remains finite, then the set of equations (31) is equivalent to the classical statement of the coupled problem of thermoelasticity (see, e.g., [18]). If the parameter β^{-1} is not equal to zero, then Eq. (31) is the statement of the problem of the hyperbolic type thermoelasticity (see, e.g., [57]).

6 The one-rotor gyrostat continuum: basic equations of a nonlinear theory

Constructing a nonlinear model of an isotropic elastic medium that possesses thermodynamic properties, we have to reconsider the thermodynamic analogies (19) and hypotheses (16), (17). Extending the aforesaid hypotheses and thermodynamic analogies on the nonlinear case we are confronted with a number of challenges. This is due to the fact that the problem under discussion can be solved in different ways, and each of the ways can be substantiated by means of appropriate arguments. In fact, choosing one of the possible ways we accept some additional hypotheses. In order to represent the essence of the problem more clear, we start with deriving the general nonlinear equations of the one-rotor gyrostat continuum—see Fig. 2. The general theory that is formulated in this Section possesses both the geometrical nonlinearity (the finite rotations and displacements and the finite deformations) and the physical nonlinearity (nonlinear constitutive equations). The special case of the model, which is intended to describe an elastic medium possessing thermodynamic properties, will be considered in the next Section.

6.1 Kinematics and dynamic structures

In what follows, in deriving the motion equations of the continuum, we apply the spatial description—see, e.g., [60–62]. Let the radius vector $\mathbf{r}$ determine the position of some point of space. We introduce the following notations: $\rho(\mathbf{r}, t)$ is the mass density of the material medium at a given point of space; $\mathbf{v}(\mathbf{r}, t)$ is the velocity field; $\mathbf{u}(\mathbf{r}, t)$ is the displacement field; $\tilde{\mathbf{P}}(\mathbf{r}, t)$, $\tilde{\boldsymbol{\omega}}(\mathbf{r}, t)$ are the rotation tensors and the angular velocity vectors of the carrier bodies; and $\mathbf{P}(\mathbf{r}, t)$, $\boldsymbol{\omega}(\mathbf{r}, t)$ are the rotation tensors and the angular velocity vectors of the rotors. In the spatial description, the kinematic relations have the form

$$\mathbf{v} = \frac{\delta\mathbf{u}}{\delta t}, \qquad \tilde{\boldsymbol{\omega}} = -\frac{1}{2}\left(\frac{\delta\tilde{\mathbf{P}}}{\delta t}\cdot\tilde{\mathbf{P}}^T\right)_\times, \qquad \boldsymbol{\omega} = -\frac{1}{2}\left(\frac{\delta\mathbf{P}}{\delta t}\cdot\mathbf{P}^T\right)_\times. \tag{32}$$

Here the operator $\frac{\delta}{\delta t} = \frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{v} \cdot \nabla$ is the material derivative where the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ is the total derivative. For more details on this point, see [58] where the material and total time derivatives are defined separately and evaluated in the context with local fields as well as during their use in integral formulations, i.e., when applied to balance equations. In particular, it is explained why and how the material and total time derivatives differ and under which circumstances they turn out to be the same.

The volume density of the kinetic energy of the one-rotor gyrostat continuum has the form

$$\rho K = \rho \left(\frac{1}{2}\,\mathbf{v} \cdot \mathbf{v} + \frac{1}{2}\, I\, \tilde{\boldsymbol{\omega}} \cdot \tilde{\boldsymbol{\omega}} + \frac{1}{2}\, J\, \boldsymbol{\omega} \cdot \boldsymbol{\omega} \right). \tag{33}$$

The volume densities of the linear momentum and the angular momentum of the continuum are

$$\rho \mathbf{K}_1 = \rho \mathbf{v}, \qquad \rho \mathbf{K}_2 = \rho (\mathbf{r} \times \mathbf{v} + I\, \tilde{\boldsymbol{\omega}} + J\, \boldsymbol{\omega}) \tag{34}$$

where the angular momentum density is calculated with respect to the origin of the reference frame. The particles of the continuum under consideration possess the internal degrees of freedom. Therefore, in order to describe the motion of this continuum it is not sufficient to formulate the balance equations of linear momentum and angular momentum for the gyrostats in a control volume of the continuum. These equations must be supplemented with the balance equation of the angular momentum for the rotors in a control volume of the continuum. Therefore, in what follows we will need the volume density of the angular momentum of the rotors,

$$\rho \mathbf{K}_2^{(\mathrm{rot})} = \rho (\mathbf{r} \times \mathbf{v} + J\, \boldsymbol{\omega}), \tag{35}$$

which is also calculated with respect to the origin of the reference frame.

The kinetic energy, the linear momentum vector, and two angular momentum vectors (one of them is the angular momentum for the continuum in whole, and the other one is the angular momentum for its component consisting of rotors) constitute the dynamic structure of the continuum.

6.2 The mass conservation law and the dynamics equations

Let V denote some fixed region in the reference frame (control volume) and S denote a closed surface that is the region V boundary. Now we formulate the law of mass conservation for the control volume as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{(V)} \rho(\mathbf{r}, t)\, \mathrm{d}V = - \int_{(S)} \mathbf{n} \cdot \mathbf{v}(\mathbf{r}, t) \rho(\mathbf{r}, t)\, \mathrm{d}S. \tag{36}$$

Using standard line of reasoning, we derive from Eq. (36) the mass conservation law in the local form. By using the material derivative, the local form of mass conservation law can be written as

$$\frac{\delta \rho}{\delta t} + \rho\, \nabla \cdot \mathbf{v} = 0. \tag{37}$$

Now we formulate the balance equation of the linear momentum for the material medium in the control volume V as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{(V)} \rho \mathbf{K}_1 \mathrm{d}V = \int_{(V)} \rho \mathbf{f} \mathrm{d}V + \int_{(S)} \boldsymbol{\tau}_n \mathrm{d}S - \int_{(S)} (\mathbf{n} \cdot \mathbf{v}) \rho \mathbf{K}_1 \mathrm{d}S. \tag{38}$$

Here $\mathbf{f}$ is the mass density of external forces; $\boldsymbol{\tau}_n$ is the force vector modeling the surrounding medium influence on the surface S of the control volume V.

By standard reasoning, we introduce the concept of stress tensor. The stress tensor $\boldsymbol{\tau}$ associated with the force vector $\boldsymbol{\tau}_n$ is defined by the relation $\boldsymbol{\tau}_n = \mathbf{n} \cdot \boldsymbol{\tau}$. Next, also by standard reasoning, we derive the local form of the liner momentum balance equation,

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} = \rho \frac{\delta \mathbf{v}}{\delta t}. \tag{39}$$

By obtaining Eq. (39), we used the expression for the linear momentum density (34) and the mass balance equation (37).

Now we formulate the balance equation of the angular momentum for the material medium in the control volume V as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{(V)}\rho\mathbf{K}_2\mathrm{d}V=\int_{(V)}\rho\big(\mathbf{r}\times\mathbf{f}+\mathbf{m}+\mathbf{L}\big)\mathrm{d}V+\int_{(S)}\big(\mathbf{r}\times\boldsymbol{\tau}_n+\boldsymbol{\mu}_n+\mathbf{T}_n\big)\mathrm{d}S-\int_{(S)}(\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_2\mathrm{d}S. \quad (40)$$

Here $\mathbf{m}$ is the mass density of external moments acting on the carrier bodies of gyrostats; $\mathbf{L}$ is the mass density of external moments acting on the rotors; and $\boldsymbol{\mu}_n$ and $\mathbf{T}_n$ are the moment vectors modeling the influence of the surrounding medium on the surface S of the control volume V. To be exact, vector $\boldsymbol{\mu}_n$ models the action on the carrier bodies of gyrostats, and vector $\mathbf{T}_n$ models the action on the rotors.

In order to explain the difference between vectors $\mathbf{m}$ and $\mathbf{L}$, we use a viscous damping as an example. A moment of viscous damping can depend on the carrier body angular velocity $\tilde{\boldsymbol{\omega}}$ or on the rotor angular velocity $\boldsymbol{\omega}$. In the first case, we deal with the moment $\mathbf{m}$ that characterizes the energy dissipation due to the presence of some material medium (like solids, liquids, and gases). In the second case, we deal with the moment $\mathbf{L}$ that characterizes the energy dissipation due to the presence of the ether (the medium consisting of the body points similar to the rotors).

Since the particles possess the internal rotational degrees of freedom, in order to close the system of equations we must now formulate the balance equation of the angular momentum for the rotors in the control volume V as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{(V)}\rho\mathbf{K}_2^{(\mathrm{rot})}\mathrm{d}V=\int_{(V)}\rho\mathbf{L}\mathrm{d}V+\int_{(S)}\mathbf{T}_n\mathrm{d}S-\int_{(S)}(\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_2^{(\mathrm{rot})}\mathrm{d}S. \quad (41)$$

By obtaining Eq. (41), we assumed that there are no force interactions between the rotors, and external force actions on the rotors are also absent. Thus, in the model under consideration the rotors perceive the moment actions rather than the force ones.

By standard reasoning, we introduce the moment stress tensor $\boldsymbol{\mu}$ associated with the moment vector $\boldsymbol{\mu}_n$ and the moment stress tensor $\mathbf{T}$ associated with the moment vector $\mathbf{T}_n$. They are defined by the relations $\boldsymbol{\mu}_n=\mathbf{n}\cdot\boldsymbol{\mu}$ and $\mathbf{T}_n=\mathbf{n}\cdot\mathbf{T}$, respectively. Also by standard reasoning, we derive from Eqs. (40), (41) two angular momentum balance equations in the local form. After simple transformations by using expressions (34), (35), and balance equations (37), (39), one of the aforesaid angular momentum balance equations is reduced to the form

$$\nabla\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_\times+\rho\mathbf{m}=\rho I\,\frac{\delta\tilde{\boldsymbol{\omega}}}{\delta t}, \quad (42)$$

and the other one takes the form

$$\nabla\cdot\mathbf{T}+\rho\mathbf{L}=\rho J\,\frac{\delta\boldsymbol{\omega}}{\delta t}. \quad (43)$$

Equations (42), (43) describe the rotational motion of the carrier bodies and the rotors, respectively.

6.3 The energy balance equation

Now we formulate the equation of energy balance for the material medium in the control volume V as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{(V)}\rho(K+U)\,\mathrm{d}V=\int_{(V)}\rho(\mathbf{f}\cdot\mathbf{v}+\mathbf{m}\cdot\tilde{\boldsymbol{\omega}}+\mathbf{L}\cdot\boldsymbol{\omega}+Q)\,\mathrm{d}V$$
$$+\int_{(S)}\big(\boldsymbol{\tau}_n\cdot\mathbf{v}+\boldsymbol{\mu}_n\cdot\tilde{\boldsymbol{\omega}}+\mathbf{T}_n\cdot\boldsymbol{\omega}+H_n\big)\mathrm{d}S-\int_{(S)}(\mathbf{n}\cdot\mathbf{v})\,\rho(K+U)\mathrm{d}S. \quad (44)$$

Here U is the internal energy density per unit mass; Q and H_n are the rate of energy supply directly into the volume V and through the surface S, respectively. The rate of energy supply through the surface can be expressed in terms of energy-flux vector $\mathbf{H}$ by the formula

$$H_n=-\mathbf{n}\cdot\mathbf{H}. \quad (45)$$

By standard reasoning, taking into account the mass balance equation (37) and relation (45), we transform the energy balance equation (44) to the local form

$$\rho\frac{\delta K}{\delta t}+\rho\frac{\delta U}{\delta t}=\rho\mathbf{f}\cdot\mathbf{v}+\rho\,\mathbf{m}\cdot\tilde{\boldsymbol{\omega}}+\rho\mathbf{L}\cdot\boldsymbol{\omega}+(\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v}+(\nabla\cdot\boldsymbol{\mu})\cdot\tilde{\boldsymbol{\omega}}$$
$$+(\nabla\cdot\mathbf{T})\cdot\boldsymbol{\omega}+\boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v}+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}-\nabla\cdot\mathbf{H}+\rho Q. \quad (46)$$

Next, using expression (33) for the kinetic energy density and balance equations (39), (42), (43), we transform Eq. (46) to the form

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}^T\cdot\cdot\big(\nabla\mathbf{v}+\mathbf{E}\times\tilde{\boldsymbol{\omega}}\big)+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}-\nabla\cdot\mathbf{H}+\rho Q. \quad (47)$$

If the supply of energy of “non-mechanical nature” is ignored, i.e., the body under consideration is assumed to be isolated, then Eq. (47) takes a more simple form,

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}^T\cdot\cdot\big(\nabla\mathbf{v}+\mathbf{E}\times\tilde{\boldsymbol{\omega}}\big)+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}. \quad (48)$$

In what follows, for simplicity sake, the supply of energy of “non-mechanical nature” is not taken into account.

6.4 The strain measures and the Cauchy–Green relations

The definition of strain measures is based on the energy balance equation. A purely geometrical consideration allows us to give an unambiguous definition of strain characteristics only in a very simple case. In more complicated cases, e.g., in continua with rotational degrees of freedom, by geometrical methods we come up with many tensor quantities that somehow characterize a strain state of continuum. Purely geometrical considerations do not give us grounds to prefer some of the deformation characteristics. Solely the energy balance equation allows us to solve this problem. Following the works by Zhilin [14,15,17,59], we define strain measures as follows.

The tensors on which the stress tensor and the moment stress tensor perform work are called the strain measures.

For convenience and brevity, we introduce the strain measures $\mathbf{g}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\Theta}$ by the formulas

$$\mathbf{g}=\mathbf{E}-\nabla\mathbf{u},\qquad \nabla\tilde{\mathbf{P}}=\boldsymbol{\Phi}\times\tilde{\mathbf{P}},\qquad \nabla\mathbf{P}=\boldsymbol{\Theta}\times\mathbf{P}, \quad (49)$$

and then, we show that these are the quantities that appear in the energy balance equation. It is easy to show (see [14,17]) that the relations between the velocity gradients and (49) have the following form:

$$\nabla\mathbf{v}=-\frac{\delta\mathbf{g}}{\delta t}\cdot\mathbf{g}^{-1},\qquad \nabla\tilde{\boldsymbol{\omega}}=\frac{\delta\boldsymbol{\Phi}}{\delta t}+\boldsymbol{\Phi}\times\tilde{\boldsymbol{\omega}}+(\nabla\mathbf{v})\cdot\boldsymbol{\Phi},\qquad \nabla\boldsymbol{\omega}=\frac{\delta\boldsymbol{\Theta}}{\delta t}+\boldsymbol{\Theta}\times\boldsymbol{\omega}+(\nabla\mathbf{v})\cdot\boldsymbol{\Theta}. \quad (50.1\text{–}3)$$

Let us introduce the energy stress tensor and the energy moment stress tensors

$$\boldsymbol{\tau}_{\rm e}=\mathbf{g}^T\cdot\boldsymbol{\tau}\cdot\tilde{\mathbf{P}},\qquad \boldsymbol{\mu}_{\rm e}=\mathbf{g}^T\cdot\boldsymbol{\mu}\cdot\tilde{\mathbf{P}},\qquad \mathbf{T}_{\rm e}=\mathbf{g}^T\cdot\mathbf{T}\cdot\mathbf{P}, \quad (51.1\text{–}3)$$

and the energy strain measures

$$\mathbf{g}_{\rm e}=\mathbf{g}^{-1}\cdot\tilde{\mathbf{P}},\qquad \boldsymbol{\Phi}_{\rm e}=\mathbf{g}^{-1}\cdot\boldsymbol{\Phi}\cdot\tilde{\mathbf{P}},\qquad \boldsymbol{\Theta}_{\rm e}=\mathbf{g}^{-1}\cdot\boldsymbol{\Theta}\cdot\mathbf{P}. \quad (52)$$

By using Eqs. (49)–(52), the energy balance equation (48) can be reduced to the form

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}_{\rm e}^T\cdot\cdot\frac{\delta\mathbf{g}_{\rm e}}{\delta t}+\boldsymbol{\mu}_{\rm e}^T\cdot\cdot\frac{\delta\boldsymbol{\Phi}_{\rm e}}{\delta t}+\mathbf{T}_{\rm e}^T\cdot\cdot\frac{\delta\boldsymbol{\Theta}_{\rm e}}{\delta t}. \quad (53)$$

From Eq. (53), it is clear the reason why tensors (51) are called the energy stress tensors and why tensors (52) are called the energy strain measures. Thus, it is proved that Eq. (49) really introduces the strain measures.

The energy balance in the form of Eq. (53) allows us to determine the arguments of the function U. If the material is assumed to be elastic, then from Eq. (53) it is seen that the mass density of internal energy is the function of the energy strain measures, i.e.,

$$U = U\big(\mathbf{g}_{\rm e},\ \boldsymbol{\Phi}_{\rm e},\ \boldsymbol{\Theta}_{\rm e}\big). \tag{54}$$

Since in the case of elastic deformations the stress tensor and the moment stress tensors do not depend on the strain rates, by standard reasoning the Cauchy–Green relations[1] can be obtained from Eq. (53). These relations have the form

$$\boldsymbol{\tau}_{\rm e} = \rho\,\frac{\partial U}{\partial \mathbf{g}_{\rm e}}, \qquad \boldsymbol{\mu}_{\rm e} = \rho\,\frac{\partial U}{\partial \boldsymbol{\Phi}_{\rm e}}, \qquad \mathbf{T}_{\rm e} = \rho\,\frac{\partial U}{\partial \boldsymbol{\Theta}_{\rm e}}. \tag{55}$$

In view of Eqs. (51), (52), from Eq. (55) it follows

$$\boldsymbol{\tau} = \rho\,\mathbf{g}^{-T}\cdot\frac{\partial U}{\partial \mathbf{g}_{\rm e}}\cdot\tilde{\mathbf{P}}^{T}, \qquad \boldsymbol{\mu} = \rho\,\mathbf{g}^{-T}\cdot\frac{\partial U}{\partial \boldsymbol{\Phi}_{\rm e}}\cdot\tilde{\mathbf{P}}^{T}, \qquad \mathbf{T} = \rho\,\mathbf{g}^{-T}\cdot\frac{\partial U}{\partial \boldsymbol{\Theta}_{\rm e}}\cdot\mathbf{P}^{T}. \tag{56}$$

In order to elaborate the constitutive equations, it is necessary to specify the function $U\big(\mathbf{g}_{\rm e},\ \boldsymbol{\Phi}_{\rm e},\ \boldsymbol{\Theta}_{\rm e}\big)$. Indeed, the conditions of stability of the material impose certain restrictions upon the choice of function $U\big(\mathbf{g}_{\rm e},\ \boldsymbol{\Phi}_{\rm e},\ \boldsymbol{\Theta}_{\rm e}\big)$.

7 A model of an elastic medium possessing thermodynamic properties

Further, modeling an elastic medium with thermodynamic properties we consider a special case of the one-rotor gyrostat continuum. In respect of the carrier bodies of gyrostats, we follow the assumptions used in Sect. 5.4, when obtaining the equations of the classical theory of thermoelasticity. In respect of the external moment in the angular momentum balance equation for the rotors, we accept an assumption that is similar to Hypothesis 1 in Sect. 4. In respect of the moment stress tensor characterizing the interaction of rotors, we accept an assumption that is a generalization of Hypothesis 2 in Sect. 4.

7.1 The simplifying assumptions

In accordance with the conditions (28), which mean transition to the momentless theory for the carrier bodies of gyrostats, the dynamics equations (39), (42) take the form

$$\nabla\cdot\boldsymbol{\tau} + \rho\mathbf{f} = \rho\,\frac{\delta\mathbf{v}}{\delta t}, \qquad \boldsymbol{\tau}_{\times} = \mathbf{0}. \tag{57.1,2}$$

Hypothesis 1* The external moment $\mathbf{L}$ acting on the rotors of gyrostats is the sum of the moment $\mathbf{L}_{\rm h}$, characterizing external actions of various types, and the moment of viscous damping $\mathbf{L}_{\rm f}$, resulting from the interaction with the ether, i.e.,

$$\mathbf{L} = \mathbf{L}_{\rm h} + \mathbf{L}_{\rm f}. \tag{58}$$

The moment $\mathbf{L}_{\rm h}$ models the influence of external ponderable bodies that results in heat supply, electromagnetic excitation, or some kind of radiation. The main difference between the moment $\mathbf{L}_{\rm h}$ and the moment $\mathbf{L}_{\rm f}$ is the fact that the moment $\mathbf{L}_{\rm h}$ occurs due to the presence of some ponderable bodies, whereas the moment $\mathbf{L}_{\rm f}$ occurs only due to the presence of the ether.

In the nonlinear theory, the moment of viscous damping $\mathbf{L}_{\rm f}$ has a more complicated structure than the moment given by Eq. (16), which is used in the linear theory. But, in order to determine the structure of

[1] This term is not standard. It is accepted only in the works of the St. Petersburg school of mechanics. However, now there is no other term that would be used only in relation to constitutive equations written in the form of partial derivatives of the internal energy with respect to its arguments. All the known terms are used in relation to both the aforesaid equations and constitutive equations written in any other form. However, the formulation of constitutive equations in the form of partial derivatives of the internal energy with respect to its arguments plays an important role in the method of continuum mechanics. That is why this form of constitutive equations deserves a special name, i.e., the name that is not used for other forms of constitutive equations.

moment $\mathbf{L}_\mathrm{f}$ we use the same physical model as in the linear theory. In other words, we assume the rotors to be immersed in the ether, which is a sufficiently rarefied medium occupying the whole infinite space. The ether consists of particles possessing both translational and rotational degrees of freedom. However, interactions between the ether particles are assumed to be the moment interactions, which are related to only the rotational degrees of freedom. To be exact, the interactions of ether particles should be considered as reactions on relative rotations of the particles. We suppose that the rotational motion of the rotors of gyrostats disturbs the ether that results in the appearance of waves in the ether. These waves carry away some part of the rotors energy. Since the ether occupies the whole infinite space, the waves cannot be reflected from the boundaries, and hence, they cannot come back. That is why the part of the rotors energy, which is spent on formation of the waves in the ether, is irretrievably lost. The model of ether and the nature of its interaction with the rotors will be discussed in Sect. 7.4. In the Section, we will also discuss a model problem, which allows us to determine the structure of moment $\mathbf{L}_\mathrm{f}$.

Hypothesis 2* The energy moment stress tensor $\mathbf{T}_\mathrm{e}$, which characterizes the rotor interactions, has the following structure:

$$\mathbf{T}_\mathrm{e} = T_\mathrm{e}\mathbf{E} - \mathbf{M}_\mathrm{e} \times \mathbf{E}, \tag{59}$$

where the scalar quantity T_e characterizes the spherical part of tensor $\mathbf{T}_\mathrm{e}$ and the vector quantity $\mathbf{M}_\mathrm{e}$ characterizes the antisymmetric part of tensor $\mathbf{T}_\mathrm{e}$.

The linear theory of thermoelasticity stated in Sects. 4 and 5 is based on assumption (17), according to which the moment stress tensor $\mathbf{T}$ is the spherical part of the tensor. The linear model based on the representation of the moment stress tensor $\mathbf{T}$ in the form of an antisymmetric tensor is considered in [63]. In the cited work, the mechanical model of electromagnetic processes is suggested, and the moment stress tensor characterizing the interaction of rotors is assumed to be of electromagnetic nature. To be exact, the moment stress tensor $\mathbf{T}$ is represented as $\mathbf{T} = -\mathbf{M} \times \mathbf{E}$, and the vector $\mathbf{M}$ is considered as an analogy of the electric field vector. In the linear theory, the differential equations describing the longitudinal oscillations (associated with the spherical part of the moment stress tensor) and the differential equations describing the transverse oscillation (associated with the antisymmetric part of the moment stress tensor) are independent. Therefore, in the framework of the linear theory, the model of thermal processes and the model of electromagnetic processes can be studied independently. The situation is different in the nonlinear theory. That is why constructing the nonlinear theory it is important to take into account both the spherical part of the moment stress tensor and the antisymmetric one. In addition, we refer to [52] where the hypothesis similar to Eq. (59) is accepted, i.e., the moment stress tensor is assumed to be the sum of the spherical part of the tensor and the antisymmetric tensor, namely $\mathbf{T} = T_\mathrm{e}\mathbf{E} - \mathbf{M} \times \mathbf{E}$. In the cited work, the linear theory of thermoviscoelasticity is constructed, and the vector $\mathbf{M}$ is associated with the internal damping. We note that, in contrast to the linear theory, in the nonlinear theory there are two moment stress tensors. One of them is the true moment stress tensor $\mathbf{T}$, which appears in the angular momentum balance equation (43). The other one is the energy moment stress tensor $\mathbf{T}_\mathrm{e}$, which appears in the energy balance equation (53) as the coefficient of the derivative of the corresponding strain measure. That is why a question arises: Which of the aforesaid moment stress tensors must satisfy the simplifying assumption like Eq. (59)? In the linear theory, the true stress tensors coincide with the energy ones, and the question is solved unambiguously. In the nonlinear theory, hypothesis (59) requires additional arguments. This question will be discussed in more detail in Sect. 8.1 devoted to mechanical analogies of temperature and entropy in the suggested nonlinear theory.

7.2 The angular momentum balance equation for the rotors in view of the simplifying assumptions

In accordance with assumption (59) and Eq. (51.3), the true moment stress tensor $\mathbf{T}$, characterizing the interactions of rotors, takes the form

$$\mathbf{T} = T_\mathrm{e}\, \mathbf{g}^{-T} \cdot \mathbf{P}^T - \mathbf{g}^{-T} \cdot (\mathbf{M}_\mathrm{e} \times \mathbf{P}^T). \tag{60}$$

Now we introduce the notations

$$I_3 = \operatorname{Det} \mathbf{g}, \qquad \mathbf{D} = I_3^{-1}\, \mathbf{g}. \tag{61}$$

We note that tensor $\mathbf{D}$ possesses the property

$$\nabla \cdot \mathbf{D}^{-T} = \mathbf{0}. \tag{62}$$

A proof of identity (62) is found in Appendix A. In view of Eq. (61), the expression (60) takes the form

$$\mathbf{T} = T\,\mathbf{D}^{-T} \cdot \mathbf{P}^T - \mathbf{D}^{-T} \cdot (\mathbf{M} \times \mathbf{P}^T) \tag{63}$$

where the following notations are used:

$$T = I_3^{-1} T_\mathrm{e}, \qquad \mathbf{M} = I_3^{-1} \mathbf{M}_\mathrm{e}. \tag{64}$$

After simple transformations (see Appendix B) taking into account Eqs. (52), (58), (61), (62), (63), the angular momentum balance equation for the rotors (43) can be reduced to the form

$$\Big[\mathbf{g}^{-1} \cdot \nabla T - \mathbf{g}^{-1} \cdot \times \nabla \mathbf{M} - T\big(\boldsymbol{\Theta}_\mathrm{e}\big)_\times + (\mathrm{tr}\,\boldsymbol{\Theta}_\mathrm{e})\,\mathbf{M} - \boldsymbol{\Theta}_\mathrm{e} \cdot \mathbf{M}\Big] \cdot \mathbf{P}^T + \varrho \mathbf{L}_\mathrm{h} + \varrho \mathbf{L}_\mathrm{f} = \varrho J \frac{\delta \boldsymbol{\omega}}{\delta t} \tag{65}$$

where ϱ is the ratio of the density of the material medium to its volume strain, i.e.,

$$\varrho = \frac{\rho}{I_3}, \qquad \frac{\delta \varrho}{\delta t} = 0. \tag{66.1,2}$$

A proof of Eq. (66.2), is found in Appendix C. We note that Eq. (66.2), besides the solution $\varrho = \mathrm{const}$, has also other solutions (for more details see, e.g., [14,17]).

Next we introduce the notations

$$\boldsymbol{\Theta}_\rho = \varrho^{-1} \boldsymbol{\Theta}_\mathrm{e}, \qquad \mathbf{g}_\rho = \varrho \mathbf{g}. \tag{67}$$

In view of the notations (61), (67), Eq. (65) can be reduced to the form

$$\Big[\mathbf{g}_\rho^{-1} \cdot \nabla T - \mathbf{g}_\rho^{-1} \cdot \times \nabla \mathbf{M} - T\big(\boldsymbol{\Theta}_\rho\big)_\times + (\mathrm{tr}\,\boldsymbol{\Theta}_\rho)\,\mathbf{M} - \boldsymbol{\Theta}_\rho \cdot \mathbf{M}\Big] \cdot \mathbf{P}^T + \mathbf{L}_\mathrm{h} + \mathbf{L}_\mathrm{f} = J \frac{\delta \boldsymbol{\omega}}{\delta t}. \tag{68}$$

When the angular momentum balance equation for the rotors is written as Eq. (68), the above-formulated relations between the strain measure $\boldsymbol{\Theta}$ and the rotation tensor $\mathbf{P}$, as well as the strain measure $\boldsymbol{\Theta}$ and the angular velocity vector $\boldsymbol{\omega}$, become not very convenient. Since Eq. (68) contains the tensors $\boldsymbol{\Theta}_\rho$ and $\mathbf{g}_\rho$, it is better to use the relation

$$\nabla \boldsymbol{\omega} = \mathbf{g}_\rho \cdot \frac{\delta \boldsymbol{\Theta}_\rho}{\delta t} \cdot \mathbf{P}^T. \tag{69}$$

Multiplying both sides of Eq. (68) by $\mathbf{P}$ from the right and using the notations

$$\mathbf{L}_\mathrm{h}^* = \mathbf{P}^T \cdot \mathbf{L}_\mathrm{h}, \qquad \mathbf{L}_\mathrm{f}^* = \mathbf{P}^T \cdot \mathbf{L}_\mathrm{f}, \tag{70}$$

we obtain the angular momentum balance equation for the rotors in the form

$$\mathbf{g}_\rho^{-1} \cdot \nabla T - \mathbf{g}_\rho^{-1} \cdot \times \nabla \mathbf{M} - T\big(\boldsymbol{\Theta}_\rho\big)_\times + (\mathrm{tr}\,\boldsymbol{\Theta}_\rho)\,\mathbf{M} - \boldsymbol{\Theta}_\rho \cdot \mathbf{M} + \mathbf{L}_\mathrm{h}^* + \mathbf{L}_\mathrm{f}^* = J \frac{\delta \boldsymbol{\Omega}}{\delta t} \tag{71}$$

where vector $\boldsymbol{\Omega}$ is called the right angular velocity vector. The relation between this vector and the left angular velocity vector $\boldsymbol{\omega}$ is $\boldsymbol{\Omega} = \mathbf{P}^T \cdot \boldsymbol{\omega}$. The relation between vector $\boldsymbol{\Omega}$ and tensor $\boldsymbol{\Theta}_\rho$ has the form

$$\nabla \boldsymbol{\Omega} = \mathbf{g}_\rho \cdot \left[\frac{\delta \boldsymbol{\Theta}_\rho}{\delta t} - \boldsymbol{\Theta}_\rho \times \boldsymbol{\Omega}\right]. \tag{72}$$

As shown in Sect. 7.3, the constitutive equations for the quantities T and $\mathbf{M}$ can be represented in terms of the invariants of tensor $\boldsymbol{\Theta}_\rho$. Therefore, under certain external actions the rotation tensor can be eliminated. In this case, vector $\boldsymbol{\Omega}$ and tensor $\boldsymbol{\Theta}_\rho$ can be considered as the basic variables. This is an advantage of the set of equations (71), (72).

7.3 The Cauchy–Green relations in view of the simplifying assumptions

Taking into account the expression (59) for tensor $\mathbf{T}_e$ and the fact that tensor $\boldsymbol{\mu}$ is equal to zero, we rewrite the energy balance equation (53) as

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}_e^T \cdot\cdot \frac{\delta \mathbf{g}_e}{\delta t} + T_e \, \mathbf{E} \cdot\cdot \frac{\delta \boldsymbol{\Theta}_e}{\delta t} + \big(\mathbf{M}_e \times \mathbf{E}\big) \cdot\cdot \frac{\delta \boldsymbol{\Theta}_e}{\delta t}. \tag{73}$$

Next we reduce the energy balance equation (73) in view of Eq. (57.2), according to which the stress tensor $\boldsymbol{\tau}$ is the symmetric tensor. As a result of the transformation (see Appendix D), we obtain

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}_s \cdot\cdot \frac{\delta \mathbf{g}_s}{\delta t} + T_e \frac{\delta \Theta_e}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \boldsymbol{\Psi}_e}{\delta t}, \tag{74}$$

under the notations

$$\boldsymbol{\tau}_s = \mathbf{g}^T \cdot \boldsymbol{\tau} \cdot \mathbf{g}, \qquad \mathbf{g}_s = \frac{1}{2} \mathbf{g}^{-1} \cdot \mathbf{g}^{-T}, \qquad \Theta_e = \operatorname{tr} \boldsymbol{\Theta}_e, \qquad \boldsymbol{\Psi}_e = \big(\boldsymbol{\Theta}_e\big)_\times. \tag{75.1–4}$$

We consider only the elastic deformations. In this case, in accordance with the energy balance equation (74) the mass density of internal energy is the function of three arguments, namely

$$U = U\big(\mathbf{g}_s, \ \Theta_e, \ \boldsymbol{\Psi}_e\big). \tag{76}$$

Substituting Eq. (76) into Eq. (74), we obtain the Cauchy–Green relations

$$\boldsymbol{\tau}_s = \rho \frac{\partial U}{\partial \mathbf{g}_s}, \qquad T_e = \rho \frac{\partial U}{\partial \Theta_e}, \qquad \mathbf{M}_e = \rho \frac{\partial U}{\partial \boldsymbol{\Psi}_e}. \tag{77}$$

In view of Eq. (75.1) and relations (64), (66), from Eq. (77) we obtain

$$\boldsymbol{\tau} = \rho \, \mathbf{g}^{-T} \cdot \frac{\partial U}{\partial \mathbf{g}_s} \cdot \mathbf{g}^{-1}, \qquad T = \varrho \, \frac{\partial U}{\partial \Theta_e}, \qquad \mathbf{M} = \varrho \, \frac{\partial U}{\partial \boldsymbol{\Psi}_e}. \tag{78}$$

Often it is convenient to split the stress tensor into its spherical part and its deviatoric part, namely

$$\boldsymbol{\tau} = p \, \mathbf{E} + \operatorname{dev} \boldsymbol{\tau}, \qquad \mathbf{E} \cdot\cdot \operatorname{dev} \boldsymbol{\tau} = 0. \tag{79}$$

Here p is the pressure, and $\operatorname{dev} \boldsymbol{\tau}$ is the stress deviator. Let us write the energy balance equation (74) by using the representation of stress tensor (79). After necessary transformations (see Appendix E), we obtain

$$\rho \frac{\delta U}{\delta t} = -\frac{p}{I_3} \frac{\delta I_3}{\delta t} + \Big(\mathbf{G}^T \cdot (\operatorname{dev} \boldsymbol{\tau}) \cdot \mathbf{G}\Big) \cdot\cdot \frac{\delta \mathbf{G}_s}{\delta t} + T_e \frac{\delta \Theta_e}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \boldsymbol{\Psi}_e}{\delta t}, \tag{80}$$

under the notations

$$\mathbf{G} = I_3^{-1/3} \mathbf{g}, \qquad \mathbf{G}_s = \frac{1}{2} \mathbf{G}^{-1} \cdot \mathbf{G}^{-T}. \tag{81}$$

Since deformations are considered to be elastic, in accordance with the energy balance equation (80) we conclude that the mass density of internal energy is the function of four arguments, namely

$$U = U\big(I_3, \ \mathbf{G}_s, \ \Theta_e, \ \boldsymbol{\Psi}_e\big), \tag{82}$$

and then, we obtain the Cauchy–Green relations, two of which (for T_e and for $\mathbf{M}_e$) are given by Eq. (77) and the remainder have the form

$$p = -\rho I_3 \frac{\partial U}{\partial I_3}, \qquad \operatorname{dev} \boldsymbol{\tau} = \rho \, \mathbf{G}^{-T} \cdot \frac{\partial U}{\partial \mathbf{G}_s} \cdot \mathbf{G}^{-1} - \frac{2}{3} \rho \, \mathbf{G}_s \cdot\cdot \frac{\partial U}{\partial \mathbf{G}_s} \mathbf{E}. \tag{83.1,2}$$

Equation (83.1) is obvious. Equation (83.2) is less obvious; however, it can be derived by the standard methods (see Appendix E).

By using Eq. (50), which relates the velocity gradient and the strain measure $\mathbf{g}$, we can rewrite the mass balance equation (37) as

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t}=\frac{\delta\mathbf{g}}{\delta t}\cdot\cdot\,\mathbf{g}^{-1}\quad\Rightarrow\quad\frac{1}{\rho}\frac{\delta\rho}{\delta t}=\frac{1}{I_3}\frac{\delta I_3}{\delta t}. \tag{84}$$

Now we introduce the notations

$$\Theta_\rho=\varrho^{-1}\Theta_{\mathrm{e}},\qquad \boldsymbol{\Psi}_\rho=\varrho^{-1}\boldsymbol{\Psi}_{\mathrm{e}}, \tag{85}$$

where the quantity ϱ is determined by Eq. (66.1). Taking into account Eqs. (64), (66), (85), it is easy to show that

$$T_{\mathrm{e}}\frac{\delta\Theta_{\mathrm{e}}}{\delta t}=\rho T\frac{\delta\Theta_\rho}{\delta t},\qquad \mathbf{M}_{\mathrm{e}}\cdot\frac{\delta\boldsymbol{\Psi}_{\mathrm{e}}}{\delta t}=\rho\mathbf{M}\cdot\frac{\delta\boldsymbol{\Psi}_\rho}{\delta t}. \tag{86}$$

In view of Eqs. (84), (86), the energy balance equation (80) can be rewritten as

$$\rho\frac{\delta U}{\delta t}=-\frac{p}{\rho}\frac{\delta\rho}{\delta t}+\Big(\mathbf{G}^T\cdot(\operatorname{dev}\boldsymbol{\tau})\cdot\mathbf{G}\Big)\cdot\cdot\frac{\delta\mathbf{G}_{\mathrm{s}}}{\delta t}+\rho T\frac{\delta\Theta_\rho}{\delta t}+\rho\mathbf{M}\cdot\frac{\delta\boldsymbol{\Psi}_\rho}{\delta t}. \tag{87}$$

From Eq. (87), it follows that the mass density of internal energy can be considered as the function

$$U=U\big(\rho,\,\mathbf{G}_{\mathrm{s}},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big). \tag{88}$$

Substituting Eq. (88) into Eq. (87), we obtain the Cauchy–Green relations

$$p=-\rho^2\frac{\partial U}{\partial\rho},\qquad T=\frac{\partial U}{\partial\Theta_\rho},\qquad \mathbf{M}=\frac{\partial U}{\partial\boldsymbol{\Psi}_\rho}, \tag{89}$$

and the Gauchy–Green relation for the stress deviator. The latter relation coincides with Eq. (83.2).

We note that the energy balance equation (87), as well as the consequences of this equation, is valid only in the case when there are no sources in the mass balance equation (37), i.e., there is no mass production within the control volume. Otherwise, the relations (84) used in deriving the energy balance equation (87) become more complicated.

7.4 The determination of the structure of the moment of viscous damping acting on the rotors

In order to determine the structure of the moment of viscous damping $\mathbf{L}_{\mathrm{f}}$, we use the physical model which is exactly the same as the model used when constructing the linear theory. To be exact, we suppose that particles of the conventional substance are immersed in the ether. With respect to the ether, we make the assumptions stated in Sect. 4. Now we briefly repeat these assumptions:

(i) The ether particles are much smaller than particles of the conventional substance. With respect to our model, this means that the ether particles are much less than the rotors of gyrostats.
(ii) The ether is the medium having a very low density in comparison with the conventional substance. The ether particles fill the space between elementary particles of the conventional substance.
(iii) The interactions of ether particles with each other are based only on the rotational degrees of freedom and the principle of moment interactions. Thus, the model of the ether is the special case of the Cosserat continuum (see Appendix F).
(iv) The ether is elastic. However, due to its infinite extent the ether can carry away the energy of rotational motion of matter particles located in it. Thereby, the ether creates the moment viscous damping acting on the matter particles.

Particles of the conventional substance are modeled by the gyrostats. With respect to the character of interactions between the gyrostats and the ether, we make the following assumptions:

(i) The rotors of gyrostats interact with the ether directly. The ether can influence the motion of the carrier bodies of gyrostats only indirectly; namely, it can convey its influence via the rotors.

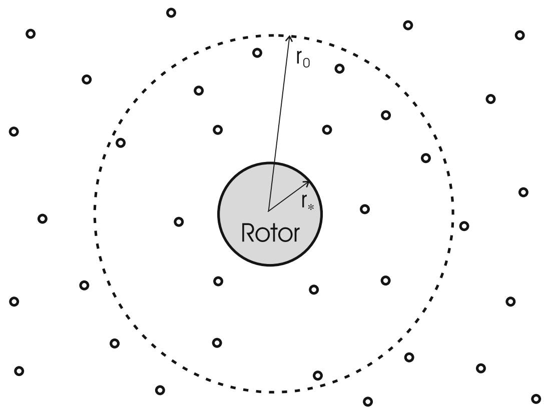

Fig. 4 A model of the interaction between the rotor and the ether

(ii) Rotational motion of the rotors disturbs the ether. The disturbance generated by the given rotor is much smaller at a great distance away from the rotor than in the immediate vicinity of it. Therefore, at all points of space that are located at a distance $r \geq r_0$ from the rotor, the angular strains of the ether can be considered to be small even under finite rotations of the rotor. However, in the layer of ether that occupies the region $r_* < r < r_0$ where r_* is the radius of the rotor, the angular strains of the ether will be finite under finite rotations of the rotor.

(iii) The ratios of parameters of the ether and the material medium satisfy some conditions, under which the layer of ether $r_* < r < r_0$ can be considered to be inertialess. The physical sense of the conditions is that the mass of the layer of ether $r_* < r < r_0$ is small as compared to the mass of the rotor. That is why this part of the ether can be considered as inertialess. A mathematical formulation of the conditions will be discussed below.

Thus, we consider the following simplified model of the interaction between the rotor and the ether—see Fig. 4. The rotor is a spherical body of radius r_*, possessing the mass m and the moment of inertia mJ. It is depicted in the center of Fig. 4. (The carrier body of the gyrostat is not shown in Fig. 4 since it does not interact directly with the ether.) The rotor is surrounded by the layer $r_* < r < r_0$ of an inertialess elastic medium capable of large angular deformations. This layer is, in turn, surrounded by an inertial elastic medium of infinite extent, which can be described by the linear theory. The boundary between two parts of the elastic medium is shown in Fig. 4 by a dotted line. It is important to note that the ether is the same in all areas of space. The only difference is the state of the ether near the rotor and far from the rotor. Therefore, the boundary that divides the ether parts is a fictitious boundary. We have to introduce the fictitious boundary in order to simplify the model problem.

At the initial moment of time, the medium surrounding the rotor is at rest. The rotor has a nonzero initial angular velocity vector, and its initial rotation tensor is not equal to the unit tensor. As a result, the rotor begins to rotate, and its rotation disturbs the ether.

We now turn to the mathematical formulation of the problem. We begin with a description of the ether model. The nonlinear model of the ether, which is presented in Appendix F, is based on the following assumptions: the ether is the Cosserat continuum; the mass production in a volume is impossible; external forces and moments are absent; the stress tensor is equal to zero; and the moment stress tensor has the same structure as the moment stress tensor characterizing the rotor interactions in the material medium.

The ether layer $r_* < r < r_0$ is considered to be inertialess with respect to rotational degrees of freedom. This part of ether is described by the set of equations

$$
\begin{gathered}
\nabla \cdot \hat{\mathbf{T}} = \mathbf{0}, \quad \hat{\mathbf{T}} = \hat{\mathbf{g}}^{-T} \cdot \hat{\mathbf{T}}_{\mathrm{e}} \cdot \hat{\mathbf{P}}^{T}, \quad \hat{\mathbf{T}}_{\mathrm{e}} = \hat{T}_{\mathrm{e}} \mathbf{E} - \hat{\mathbf{M}}_{\mathrm{e}} \times \mathbf{E}, \\
\hat{T}_{\mathrm{e}} = \hat{\rho} \frac{\partial \hat{U}(\hat{\Theta}_{\mathrm{e}}, \hat{\boldsymbol{\Psi}}_{\mathrm{e}})}{\partial \hat{\Theta}_{\mathrm{e}}}, \quad \hat{\mathbf{M}}_{\mathrm{e}} = \hat{\rho} \frac{\partial \hat{U}(\hat{\Theta}_{\mathrm{e}}, \hat{\boldsymbol{\Psi}}_{\mathrm{e}})}{\partial \hat{\boldsymbol{\Psi}}_{\mathrm{e}}}, \quad \hat{\Theta}_{\mathrm{e}} = \operatorname{tr} \hat{\boldsymbol{\Theta}}_{\mathrm{e}}, \quad \hat{\boldsymbol{\Psi}}_{\mathrm{e}} = (\hat{\boldsymbol{\Theta}}_{\mathrm{e}})_{\times}, \\
\hat{\boldsymbol{\Theta}}_{\mathrm{e}} = \hat{\mathbf{g}}^{-1} \cdot \hat{\boldsymbol{\Theta}} \cdot \hat{\mathbf{P}}, \quad \nabla \hat{\mathbf{P}} = \hat{\boldsymbol{\Theta}} \times \hat{\mathbf{P}}, \quad \frac{\hat{\delta} \hat{\rho}}{\delta t} + \hat{\rho} \nabla \cdot \hat{\mathbf{v}} = 0, \quad \frac{\hat{\delta} \hat{\mathbf{g}}}{\delta t} \cdot \hat{\mathbf{g}}^{-1} = -\nabla \hat{\mathbf{v}}, \quad \hat{\rho} \frac{\hat{\delta} \hat{\mathbf{v}}}{\delta t} = \mathbf{0}
\end{gathered}
\qquad (90)
$$

where $\hat{\mathbf{T}}$ is the moment stress tensor in the ether, $\hat{\mathbf{P}}$ is the rotation tensor of ether particles, $\hat{\rho}$ is the ether density, $\hat{\mathbf{g}}$ and $\hat{\boldsymbol{\Theta}}$ are the strain measure, and $\hat{\mathbf{v}}$ is velocity vector of ether particles. The meaning of the other quantities is clear from the equations. The final concretization of the ether model consists in a specification of the mass density of internal energy $\hat{U}(\hat{\Theta}_{\mathrm{e}}, \hat{\boldsymbol{\Psi}}_{\mathrm{e}})$.

Due to the fact that in the considered ether model the velocity field is determined irrespective of the determination of rotational motion characteristics, the solution of the last equation in Eq. (90) allows us to find the strain measure $\hat{\mathbf{g}}$ and the density $\hat{\rho}$. We note that the last three equations in Eq. (90) have the trivial solutions

$$\hat{\mathbf{v}} = \mathrm{const}, \qquad \hat{\rho} = \mathrm{const}, \qquad \hat{\mathbf{g}} = \mathbf{E}, \tag{91.1–3}$$

though other solutions are also possible. Let us assume that the initial conditions for the ether are homogeneous and the rotor motion does not disturb the ether with respect to translational degrees of freedom. In this case, we can assert that the aforesaid trivial solution is realized. In view of Eq. (91), the set of equations (90) is simplified and takes the form

$$\begin{gathered}\nabla \cdot \hat{\mathbf{T}} = \mathbf{0}, \quad \hat{\mathbf{T}} = \hat{\mathbf{T}}_{\mathrm{e}} \cdot \hat{\mathbf{P}}^{T}, \quad \hat{\mathbf{T}}_{\mathrm{e}} = \hat{T}_{\mathrm{e}}\mathbf{E} - \hat{\mathbf{M}}_{\mathrm{e}} \times \mathbf{E}, \quad \hat{\boldsymbol{\Theta}}_{\mathrm{e}} = \hat{\boldsymbol{\Theta}} \cdot \hat{\mathbf{P}}, \quad \nabla\hat{\mathbf{P}} = \hat{\boldsymbol{\Theta}} \times \hat{\mathbf{P}}, \\ \hat{T}_{\mathrm{e}} = \hat{\rho}\frac{\partial \hat{U}(\hat{\Theta}_{\mathrm{e}}, \hat{\boldsymbol{\Psi}}_{\mathrm{e}})}{\partial \hat{\Theta}_{\mathrm{e}}}, \quad \hat{\mathbf{M}}_{\mathrm{e}} = \hat{\rho}\frac{\partial \hat{U}(\hat{\Theta}_{\mathrm{e}}, \hat{\boldsymbol{\Psi}}_{\mathrm{e}})}{\partial \hat{\boldsymbol{\Psi}}_{\mathrm{e}}}, \quad \hat{\Theta}_{\mathrm{e}} = \mathrm{tr}\, \hat{\boldsymbol{\Theta}}_{\mathrm{e}}, \quad \hat{\boldsymbol{\Psi}}_{\mathrm{e}} = (\hat{\boldsymbol{\Theta}}_{\mathrm{e}})_{\times}.\end{gathered} \tag{92}$$

The part of the ether occupying the region $r \geq r_0$ is described by the linearized ether model, which is represented in Appendix G. As shown in Appendix G, the linearized ether model can be reduced to two wave equations

$$\Delta\hat{\vartheta} - \frac{1}{c_{\mathrm{v}}^2}\frac{\mathrm{d}^2\hat{\vartheta}}{\mathrm{d}t^2} = 0, \qquad \Delta\hat{\boldsymbol{\psi}} - \frac{1}{c_{\mathrm{s}}^2}\frac{\mathrm{d}^2\hat{\boldsymbol{\psi}}}{\mathrm{d}t^2} = \mathbf{0}. \tag{93}$$

Here $\hat{\vartheta} = \nabla \cdot \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\psi}} = \nabla \times \hat{\boldsymbol{\theta}}$ have the meaning of angular strains, and $\hat{\boldsymbol{\theta}}$ is the small rotation vector of the ether particles. The wave propagation velocities are expressed in terms of the ether parameters by the formulas

$$c_{\mathrm{v}}^2 = \frac{\hat{k}_{\mathrm{v}}}{\hat{\rho}\hat{J}}, \qquad c_{\mathrm{s}}^2 = \frac{\hat{k}_{\mathrm{s}}}{\hat{\rho}\hat{J}} \tag{94}$$

where $\hat{J}$ is the mass density of inertia moments, $\hat{k}_{\mathrm{v}}$ is the torsional stiffness, and $\hat{k}_{\mathrm{s}}$ is the bending stiffness. The moment stress tensor is determined as

$$\hat{\mathbf{T}} = \hat{T}\mathbf{E} - \hat{\mathbf{M}} \times \mathbf{E}, \qquad \hat{T} = \hat{k}_{\mathrm{v}}\hat{\vartheta}, \qquad \hat{\mathbf{M}} = \hat{k}_{\mathrm{s}}\,\hat{\boldsymbol{\psi}}. \tag{95}$$

Thus, the ether is divided into two parts by the imagined spherical surface $r = r_0$. The considered ether parts are connected on the imagined surface $r = r_0$. There are two conditions of the connection. The first one is the equality of the rotation tensors. The second one is the equality of the moment stress vectors $\mathbf{e}_r \cdot \hat{\mathbf{T}}$ where $\mathbf{e}_r$ is the basis vector of the spherical coordinate system, the origin of which is at the rotor center. For the part of ether occupying the region $r \geq r_0$, the absence of sources at infinity must be assured. For the part of ether occupying the region $r_* < r < r_0$, it is necessary to formulate boundary conditions on the surface $r = r_*$. The latter conditions we discuss in detail.

Let us assume that the ether particles, contacting with the rotor, "adhere" to the rotor and move together with it. For simplicity sake, the rotor center is considered to be fixed. If the ether was modeled by the Cosserat continuum of a general type, the kinematic conditions of contact of the ether particles and the rotor would have the form

$$\hat{\mathbf{v}}\big|_{r=r_*} = \boldsymbol{\omega} \times r_*\mathbf{e}_r, \qquad \hat{\boldsymbol{\omega}}\big|_{r=r_*} = \boldsymbol{\omega} \tag{96.1,2}$$

where $\boldsymbol{\omega}$ and $\hat{\boldsymbol{\omega}}$ are the angular velocities of the rotor and the ether particles, respectively. However, according to the suggested models, the stress tensor in the ether is considered to be equal to zero, and the interactions between the rotors of gyrostats in the material medium are also characterized only by the moment stress tensor. Taking into account the aforesaid features of the models, it is reasonable to assume that the "adhesion" of the ether particles to the rotor occurs only with respect to rotational degrees of freedom. Translational velocities of the ether particles situated on the rotor surface can differ from velocities of the corresponding points of the rotor. Then, the first condition in Eq. (96) can be excluded from consideration. In this case, the use of the particular solution (91) does not contradict anything, and it is quite reasonable. Since at the initial moment of time the ether is at rest, from Eq. (91.1) it follows that $\hat{\mathbf{v}} = \mathbf{0}$. Then from Eq. (96.2), it follows that

$$\hat{\mathbf{P}}\Big|_{r=r_*} = \mathbf{P}(\boldsymbol{\theta}) \tag{97}$$

where $\mathbf{P}(\boldsymbol{\theta})$ is the rotation tensor of the rotor and $\boldsymbol{\theta}$ is the rotation vector of the rotor. We note that the set of equations (92) includes the equation for the rotation tensor, whereas the angular velocity vector is absent in it. Therefore, Eq. (97) should be used as the boundary condition instead of Eq. (96.2). Since vector $\boldsymbol{\theta}$ is an unknown quantity, the boundary condition (97) is not sufficient. It should be supplemented with an integral boundary condition. The angular momentum balance equation for the rotor plays the role of the integral boundary condition. Since we assume that the ether is the only external factor acting on the rotor, the angular momentum balance equation for the rotor is written as

$$mJ\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \int_{\sigma_*} \mathbf{e}_r \cdot \hat{\mathbf{T}}\Big|_{r=r_*} \mathrm{d}\sigma. \tag{98}$$

The formulation of the model problem is completed. Now we turn to a discussion of prospects of its numerical and analytical study. There is no way to solve the problem analytically because of the complexity of the nonlinear set of equations (92). When we try to solve the problem by numerical methods, we are confronted with two obstacles. The first obstacle consists in the fact that the differential equations are formulated in the infinite region of space. Only certain numerical methods allow us to solve such problems. The majority of numerical methods are not applicable for solving differential equations in the infinite region of space. The specificity of the problem in question is such that it is impossible to replace the infinite region by some finite region. Indeed, the energy dissipation takes place because the infinite region is considered and the absence of sources at infinity is assured. If a region is bounded, then all waves will be returning after reflection from the boundary. As a result, there will be no energy dissipation. A constructive approach to overcome the obstacle consists in constructing a semi-analytical solution of the problem. To be exact, it is necessary to construct an analytical solution of the linear problem in the infinite region $r \geq r_0$. As a result, the boundary conditions for the nonlinear problem will be obtained. The nonlinear problem formulated in the finite region $r_* < r < r_0$ can be solved by a numerical method. However, we are confronted with the second obstacle, which consists in the fact that the numerical analysis is impossible without specifying the mass density of the internal energy $\hat{U}\left(\hat{\Theta}_{\mathrm{e}},\ \hat{\boldsymbol{\Psi}}_{\mathrm{e}}\right)$. Any assumptions about the form of function $\hat{U}\left(\hat{\Theta}_{\mathrm{e}},\ \hat{\boldsymbol{\Psi}}_{\mathrm{e}}\right)$ can only be made on the basis of experimental data for specific materials. Consequently, it is important to know how the form of function $\hat{U}\left(\hat{\Theta}_{\mathrm{e}},\ \hat{\boldsymbol{\Psi}}_{\mathrm{e}}\right)$ influences the moment of viscous damping $\mathbf{L}_{\mathrm{f}}$, and in the final analysis, how the form of function $\hat{U}\left(\hat{\Theta}_{\mathrm{e}},\ \hat{\boldsymbol{\Psi}}_{\mathrm{e}}\right)$ influences the heat conduction equation. Thus, in the way of a numerical study we are confronted with two formidable obstacles. Therefore, an analytical study of the problem gains in importance even if we have to take additional simplifying assumptions in order to carry out this study. This is the way that we choose for further advancement.

Let us accept two simplifying assumptions. It is difficult to substantiate these assumptions by means of some asymptotic considerations, but we are compelled to accept these assumptions in order to have the opportunity to construct an analytical solution of the problem.

(i) In the dynamical problem for the part of ether occupying the region $r \geq r_0$, we will disregard the waves propagating along the angular coordinates. These waves are not immediately associated with the energy dissipation. The energy is dissipated due to the waves propagating along the radial coordinate. However, the waves propagating along the radial coordinate interact with the waves propagating along the angular coordinates because of the boundary conditions. This interaction is that what we ignore when we accept the simplifying assumption in question.

(ii) The boundary between the inertial part of the ether and the inertialess part of the ether, i.e., the imagined spherical surface $r = r_0$, will be assumed to move like a rigid body. This statement applies only to rotational degrees of freedom. It means that all ether particles located on the surface $r = r_0$ at the given moment of time are assumed to have the same rotation tensors and the same angular velocities. Due to this simplifying assumption, instead of solving the problem of a quasi-static deformation of the nonlinear ether part, we will introduce an elastic potential (as a function of the vector of the rotor rotation relative to the ether particles located on the imagined spherical surface $r = r_0$), and thereupon, we will calculate the elastic moment corresponding to this elastic potential. This elastic moment is essentially the total moment $\int_{\sigma_*} \mathbf{e}_r \cdot \hat{\mathbf{T}}\Big|_{r=r_*} \mathrm{d}\sigma$ acting on the rotor by the ether. Indeed, in this approach it is impossible to determine the stress–strain state of the ether in the region $r_* < r < r_0$. However, this is not necessary for our research purpose. We need to know only the total moment acting on the rotor.

The solution of the problem formulated above is contained in Appendix H. The main result obtained in Appendix H is the set of equations describing the rotor motion, namely

$$mJ\,\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \hat{\mathbf{T}}_\sigma, \qquad \boldsymbol{\omega} = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}, \qquad \mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\,\mathbf{R}(\boldsymbol{\theta}) + \frac{1 - g(\theta)}{\theta^2}\,\mathbf{R}^2(\boldsymbol{\theta}) \tag{99}$$

where $\mathbf{Z}(\boldsymbol{\theta})$ is the Zhilin tensor, which was first introduced in [64], $\mathbf{R}(\boldsymbol{\theta})$ is the logarithmic rotation tensor, and $g(\theta)$ is the scalar function that depends on the rotation vector magnitude, namely

$$\mathbf{R}(\boldsymbol{\theta}) = \boldsymbol{\theta} \times \mathbf{E}, \qquad g(\theta) = \frac{\theta\,\sin\theta}{2(1 - \cos\theta)}, \qquad \theta = |\boldsymbol{\theta}| \tag{100}$$

The moment $\hat{\mathbf{T}}_\sigma$ is the integrated characteristic that describes the influence of the ambient ether on the rotor. This moment can be expressed in terms of the rotor kinematic characteristics $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ by the formula

$$\hat{\mathbf{T}}_\sigma = -c(\theta^2)\boldsymbol{\theta} - m\boldsymbol{\chi}(\boldsymbol{\theta}) \cdot \left(\boldsymbol{\omega} + p \int_{\mathbf{0}}^{\boldsymbol{\theta}} \mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}}) \cdot \mathrm{d}\tilde{\boldsymbol{\theta}} - q \int_0^t \left[\int_0^t e^{\kappa(\tau - t)} \mathbf{Z}^{-1}\big(\boldsymbol{\theta}(\tau)\big) \cdot \mathrm{d}\boldsymbol{\theta}(\tau) \right] \mathrm{d}t \right) \tag{101}$$

where

$$\boldsymbol{\chi}(\boldsymbol{\theta}) = \chi\left[c(\theta^2)\mathbf{Z}^T(\boldsymbol{\theta}) + k(\theta^2)\boldsymbol{\theta}\boldsymbol{\theta}\right], \qquad c(\theta^2) = -2\,\frac{\mathrm{d}\Pi(\theta^2)}{\mathrm{d}(\theta^2)}, \qquad k(\theta^2) = 4\,\frac{\mathrm{d}^2\Pi(\theta^2)}{\mathrm{d}(\theta^2)^2},$$
$$\chi = \frac{3J}{4\pi r_0^2 \hat{\rho} \hat{J}(c_\mathrm{v} + 2c_\mathrm{s})}, \qquad \kappa = \frac{3c_\mathrm{v}c_\mathrm{s}}{r_0(c_\mathrm{v} + 2c_\mathrm{s})}, \qquad p = \frac{c_\mathrm{v}^2 + 2c_\mathrm{s}^2}{r_0(c_\mathrm{v} + 2c_\mathrm{s})}, \qquad q = \frac{2c_\mathrm{v}c_\mathrm{s}(c_\mathrm{v} - c_\mathrm{s})^2}{r_0^2(c_\mathrm{v} + 2c_\mathrm{s})^2}. \tag{102}$$

Here $\Pi(\theta^2)$ is the potential, which determines the elastic moment acting on the rotor from the part of ether that occupies the region $r_* < r < r_0$. The basic properties of tensors $\mathbf{Z}$ and $\boldsymbol{\chi}$ are set out in Appendix I.

The problem of specification of the potential $\Pi(\theta^2)$ is beyond the scope of this study. We note that even in the simplest case, when $\Pi(\theta^2) = \dfrac{1}{2}\,c\,\theta^2$ where $c = \mathrm{const}$, the expression for the moment $\hat{\mathbf{T}}_\sigma$ is essentially nonlinear.

In deriving formula (101) for the moment $\hat{\mathbf{T}}_\sigma$, we used a number of assumptions, two of which are asymptotic ones. The first assumption is that the part of ether occupying the region $r_* < r < r_0$ is considered to be inertialess. This is acceptable if a representative frequency of the dynamic process is much less than the first eigenfrequency of this layer. The second assumption is that the linear theory can be used for the part of ether occupying the region $r \geq r_0$. This is acceptable if all rotation angles of ether particles are small. The mentioned assumptions impose certain restrictions upon the ratio of parameters of the problem. The corresponding asymptotic analysis is found in Appendix J. These asymptotic estimates allow us to compare the asymptotic order of the terms in the expression for the moment $\hat{\mathbf{T}}_\sigma$ and to draw the following conclusions. The first term on the right-hand side of Eq. (101) is much larger than all other terms. However, this is a conservative moment. This moment has no relation to the rotor energy dissipation, and it need not be taken into account when determining the structure of the viscous damping moment $\mathbf{L}_\mathrm{f}$. The following estimates hold for the terms in parentheses:

$$|\boldsymbol{\omega}| \ll \left| p \int_0^{\boldsymbol{\theta}} \mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}}) \cdot \mathrm{d}\tilde{\boldsymbol{\theta}} \right| \ll \left| q \int_0^t \left[\int_0^t e^{\kappa(\tau - t)} \mathbf{Z}^{-1}\big(\boldsymbol{\theta}(\tau)\big) \cdot \mathrm{d}\boldsymbol{\theta}(\tau) \right] \mathrm{d}t \right|. \tag{103}$$

We note that the second inequality in Eq. (103) is violated when $c_\mathrm{v} \approx c_\mathrm{s}$, because in this case the parameter q tends to zero.

It would seem that the asymptotic analysis convinces us that when determining the structure of the viscous damping moment $\mathbf{L}_f$, first of all, the term containing $\int_0^t [\int_0^t e^{\kappa(\tau-t)} \mathbf{Z}^{-1}(\boldsymbol{\theta}(\tau)) \cdot \mathrm{d}\boldsymbol{\theta}(\tau)]\mathrm{d}t$ should be taken into account. However, this conclusion is contrary to the results that were obtained earlier for the linear approximation. Indeed, in the linear theory, according to Eq. (16) the viscous damping moment has the form $\mathbf{L}_f = -\beta J \boldsymbol{\omega}$. It is this moment that allows us to obtain the hyperbolic type heat conduction equation (as well as the classical heat conduction equation as its special case) within the framework of the suggested model. In order to clarify the situation, we write the expression (101) by using the linear approximation, namely

$$\hat{\mathbf{T}}_\sigma \approx -c\boldsymbol{\theta} - m\chi c\left(\boldsymbol{\omega} + p\boldsymbol{\theta} - q\int_0^t \left[\int_0^t e^{\kappa(\tau-t)}\mathrm{d}\boldsymbol{\theta}(\tau)\right]\mathrm{d}t\right), \qquad c = \text{const.} \tag{104}$$

As can be seen from Eq. (104), all three terms in parentheses are retained in the linearized expression. The term containing $p\boldsymbol{\theta}$ is a conservative moment, and therefore, it need not be taken into account when determining the structure of moment $\mathbf{L}_f$. The term containing the integral is not a conservative moment. Generally speaking, this term must be contained in the expression for $\mathbf{L}_f$. If the integral term is taken into account, then instead of the hyperbolic type heat conduction equation we will obtain a heat conduction equation of more complicated form. It is well known that the classical heat conduction equation possesses sufficiently high accuracy in most cases when a linear approximation is acceptable. Based on this fact, we must conclude that the integral term in the expression (104) is small because of the smallness of parameter q. As noted above, the parameter q is small in the case of $c_v \approx c_s$.

Now we return to the nonlinear expression (101) for the moment $\hat{\mathbf{T}}_\sigma$ and discuss the physical meaning of the non-conservative terms contained therein. The term proportional to vector $\boldsymbol{\omega}$ is the typical dissipative moment both under small rotations and under finite ones. The term containing $\int_{\mathbf{0}}^{\boldsymbol{\theta}} \mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}}) \cdot \mathrm{d}\tilde{\boldsymbol{\theta}}$ is a position moment, which is, in general, not a conservative moment. This moment may cause both the dissipation of the rotor energy into the ether and the pumping of energy from the ether. Let us consider the last term on the right-hand side of Eq. (101). If this term was proportional to $\int_0^t e^{\kappa(\tau-t)} \mathbf{Z}^{-1}(\boldsymbol{\theta}(\tau)) \cdot \mathrm{d}\boldsymbol{\theta}(\tau)$, it would be an analogy of the constitutive equations for the stress tensor in continua with fading memory. Such constitutive equations are typical for viscoelastic materials. Hence, they contain both the conservative part and the dissipative part. It is evident that the additional integral over time cannot transform the moment under discussion into a conservative moment.

In order to define the structure of the viscous damping moment $\mathbf{L}_f$, we will use the results of the model problem analysis. To be exact, we will define the viscous damping moment $\mathbf{L}_f$ by analogy with the viscous damping moment $\hat{\mathbf{T}}_\sigma$. As mentioned above, the non-conservative terms in the expression (101) for the moment $\hat{\mathbf{T}}_\sigma$ are different by nature. Consequently, all these terms can appreciably influence the rotor motion irrespective of their asymptotic orders. That is why, when defining the structure of viscous damping moment $\mathbf{L}_f$, all the non-conservative terms in Eq. (101) should be taken into account.

Now we introduce the mass density of the non-conservative part of the moment (101), namely

$$\hat{\mathbf{L}}_\sigma = -\boldsymbol{\chi}(\boldsymbol{\theta}) \cdot \left(\boldsymbol{\omega} + p\int_{\mathbf{0}}^{\boldsymbol{\theta}} \mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}}) \cdot \mathrm{d}\tilde{\boldsymbol{\theta}} - q\int_0^t \left[\int_0^t e^{\kappa(\tau-t)} \mathbf{Z}^{-1}(\boldsymbol{\theta}(\tau)) \cdot \mathrm{d}\boldsymbol{\theta}(\tau)\right]\mathrm{d}t\right). \tag{105}$$

It is easy to show that the integral equation (105) is equivalent to the differential equation

$$\frac{\mathrm{d}^2(\boldsymbol{\chi}^{-1} \cdot \hat{\mathbf{L}}_\sigma)}{\mathrm{d}t^2} + \kappa\frac{\mathrm{d}(\boldsymbol{\chi}^{-1} \cdot \hat{\mathbf{L}}_\sigma)}{\mathrm{d}t} = -\left(\frac{\mathrm{d}^2\boldsymbol{\omega}}{\mathrm{d}t^2} + \frac{c_v + c_s}{r_0}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \frac{c_v c_s}{r_0^2}\boldsymbol{\omega}\right). \tag{106}$$

Now we define the viscous damping moment $\mathbf{L}_f$ by the differential equation analogous to Eq. (106) for the moment $\hat{\mathbf{L}}_\sigma$. The only difference consists in replacing the total derivatives in Eq. (106) by the corresponding material derivatives, i.e.,

$$\frac{\delta^2(\boldsymbol{\chi}^{-1} \cdot \mathbf{L}_f)}{\delta t^2} + \kappa\frac{\delta(\boldsymbol{\chi}^{-1} \cdot \mathbf{L}_f)}{\delta t} = -\left(\frac{\delta^2\boldsymbol{\omega}}{\delta t^2} + \frac{c_v + c_s}{r_0}\frac{\delta\boldsymbol{\omega}}{\delta t} + \frac{c_v c_s}{r_0^2}\boldsymbol{\omega}\right). \tag{107}$$

In view of the notations (70) and the properties of tensor $\boldsymbol{\chi}$ [see Appendix I, formulas (185)], Eq. (107) can be rewritten as

$$\frac{\delta^2(\boldsymbol{\chi}^{-T} \cdot \mathbf{L}_f^*)}{\delta t^2} + \kappa\frac{\delta(\boldsymbol{\chi}^{-T} \cdot \mathbf{L}_f^*)}{\delta t} = -\left(\frac{\delta^2\boldsymbol{\omega}}{\delta t^2} + \frac{c_v + c_s}{r_0}\frac{\delta\boldsymbol{\omega}}{\delta t} + \frac{c_v c_s}{r_0^2}\boldsymbol{\omega}\right). \tag{108}$$

We note that in the case of $c_\mathrm{v} = c_\mathrm{s}$ Eqs. (107), (108) take the more simple form

$$\frac{\delta(\boldsymbol{\chi}^{-1}\cdot\mathbf{L}_\mathrm{f})}{\delta t} = -\left(\frac{\delta\boldsymbol{\omega}}{\delta t} + \frac{c_\mathrm{v}}{r_0}\,\boldsymbol{\omega}\right), \qquad \frac{\delta(\boldsymbol{\chi}^{-T}\cdot\mathbf{L}_\mathrm{f}^*)}{\delta t} = -\left(\frac{\delta\boldsymbol{\omega}}{\delta t} + \frac{c_\mathrm{v}}{r_0}\,\boldsymbol{\omega}\right). \tag{109.1,2}$$

Thus, based on the model problem analysis we have obtained the following results:

(i) In the general case, the moment of viscous damping $\mathbf{L}_\mathrm{f}$, which occurs in the angular momentum balance equation for the rotors (68), is defined by the differential equation (107). In the special case of $c_\mathrm{v} = c_\mathrm{s}$, this moment is defined by Eq. (109.1).

(ii) In the general case, the moment of viscous damping $\mathbf{L}_\mathrm{f}^*$, which occurs in the angular momentum balance equation for the rotors (71), is defined by Eq. (108). In the special case of $c_\mathrm{v} = c_\mathrm{s}$, the moment $\mathbf{L}_\mathrm{f}^*$ is defined by Eq. (109.2).

7.5 A summary of the basic equations

In this study, we apply the spatial description of continuum kinematics. When the spatial description is used, as a rule, only an actual configuration of the continuum is considered. A velocity vector is the basic kinematic characteristic. Therefore, the relation between the velocity vector and the displacement vector should be considered as the definition of the displacement vector. Often, the displacement vector does not have a definite physical meaning and it is an auxiliary variable that is necessary in order to derive the basic equations. The constitutive equations and the relations between the deformation and kinematic characteristics can be formulated without using the displacement vector. Therefore, when the spatial description is applied, the displacement vector can be eliminated. Somewhat different is the case with the rotation tensor. On the one hand, the rotation tensor is presented in many equations formulated above. On the other hand, in the case of spatial description for rotational degrees of freedom the conception of reference configuration can be introduced by the same way as it is done in the case of material description. Indeed, the rotation tensor $\mathbf{P}(\mathbf{r}, t)$ characterizes the orientation of a particle, located at a given point of space in a given moment of time, relative to the triplet of reference vectors specified at this point of space. Thus, in order to introduce the conception of reference configuration it is sufficient to specify the field of reference vectors $\mathbf{D}_k(\mathbf{r})$ ($k = 1, 2, 3$) and to postulate that at the reference configuration the rotation tensors of all particles are equal to the unit tensor. We note that in the case of spatial description the rotation tensor is introduced by the formal method, and often it has a more vague sense than the angular velocity vector and the angular strain measure. Therefore, it is better to eliminate the rotation tensor and all quantities, related to it by the algebraic formulas, from the set of basic equations. One way to implement this idea is presented in Appendix K. This approach is based on the introduction of the supplementary strain characteristic $\boldsymbol{\vartheta}_\rho = \mathbf{g}_\rho^{-1}\cdot\nabla\boldsymbol{\theta}$.

By using two strain characteristics, $\boldsymbol{\Theta}_\rho$ and $\boldsymbol{\vartheta}_\rho$, the set of basic equations can be written as

$$\nabla\cdot\boldsymbol{\tau} + \rho\mathbf{f} = \rho\frac{\delta\mathbf{v}}{\delta t}, \qquad \boldsymbol{\tau} = p\,\mathbf{E} + \mathrm{dev}\,\boldsymbol{\tau}, \qquad \nabla\mathbf{v} = -\frac{\delta\mathbf{g}}{\delta t}\cdot\mathbf{g}^{-1},$$

$$\mathbf{g}_\rho^{-1}\cdot\nabla T - \mathbf{g}_\rho^{-1}\cdot\times\nabla\mathbf{M} - T\,\boldsymbol{\Psi}_\rho + \Theta_\rho\,\mathbf{M} - \boldsymbol{\Theta}_\rho\cdot\mathbf{M} + \mathbf{L}_\mathrm{h}^* + \mathbf{L}_\mathrm{f}^* = J\,\frac{\delta\boldsymbol{\Omega}}{\delta t},$$

$$\Theta_\rho = \mathrm{tr}\,\boldsymbol{\Theta}_\rho, \qquad \boldsymbol{\Psi}_\rho = \big(\boldsymbol{\Theta}_\rho\big)_\times, \qquad \nabla\boldsymbol{\Omega} = \mathbf{g}_\rho\cdot\left[\frac{\delta\boldsymbol{\Theta}_\rho}{\delta t} - \boldsymbol{\Theta}_\rho\times\boldsymbol{\Omega}\right],$$

$$\mathbf{g}_\rho = \frac{\rho}{I_3}\,\mathbf{g}, \qquad I_3 = \mathrm{Det}\,\mathbf{g}, \qquad \frac{\delta(\rho/I_3)}{\delta t} = 0, \qquad \mathbf{G} = I_3^{-1/3}\mathbf{g}, \qquad \mathbf{G}_\mathrm{s} = \frac{1}{2}\,\mathbf{G}^{-1}\cdot\mathbf{G}^{-T},$$

$$p = -\rho^2\frac{\partial U\big(\rho,\,\mathbf{G}_\mathrm{s},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big)}{\partial\rho}, \qquad T = \frac{\partial U\big(\rho,\,\mathbf{G}_\mathrm{s},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big)}{\partial\Theta_\rho}, \qquad \mathbf{M} = \frac{\partial U\big(\rho,\,\mathbf{G}_\mathrm{s},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big)}{\partial\boldsymbol{\Psi}_\rho},$$

$$\mathrm{dev}\,\boldsymbol{\tau} = \rho\,\mathbf{G}^{-T}\cdot\frac{\partial U\big(\rho,\,\mathbf{G}_\mathrm{s},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big)}{\partial\mathbf{G}_\mathrm{s}}\cdot\mathbf{G}^{-1} - \frac{2}{3}\,\rho\,\mathbf{G}_\mathrm{s}\cdot\cdot\frac{\partial U\big(\rho,\,\mathbf{G}_\mathrm{s},\,\Theta_\rho,\,\boldsymbol{\Psi}_\rho\big)}{\partial\mathbf{G}_\mathrm{s}}\,\mathbf{E},$$

$$\frac{\delta^2(\boldsymbol{\chi}^{-T}\cdot\mathbf{L}_\mathrm{f}^*)}{\delta t^2} + \kappa\,\frac{\delta(\boldsymbol{\chi}^{-T}\cdot\mathbf{L}_\mathrm{f}^*)}{\delta t} = -\left(\frac{\delta^2\boldsymbol{\omega}}{\delta t^2} + \frac{c_\mathrm{v}+c_\mathrm{s}}{r_0}\,\frac{\delta\boldsymbol{\omega}}{\delta t} + \frac{c_\mathrm{v}c_\mathrm{s}}{r_0^2}\,\boldsymbol{\omega}\right),$$

$$\boldsymbol{\omega} = \boldsymbol{\vartheta}_\rho^{-1}\cdot\boldsymbol{\Theta}_\rho\cdot\boldsymbol{\vartheta}_\rho^{T}\cdot\boldsymbol{\Theta}_\rho^{-T}\cdot\boldsymbol{\Omega}, \qquad \frac{\delta\boldsymbol{\vartheta}_\rho}{\delta t} = \mathbf{g}_\rho^{-1}\cdot\nabla\left(\boldsymbol{\Omega}\cdot\boldsymbol{\Theta}_\rho^{-1}\cdot\boldsymbol{\vartheta}_\rho\right),$$

$$\boldsymbol{\chi}^{-T} = \frac{1}{\chi}\left[\frac{1}{c(\theta^2)}\,\boldsymbol{\vartheta}_\rho^{-1}\cdot\boldsymbol{\Theta}_\rho - \frac{k(\theta^2)}{c(\theta^2)\big[c(\theta^2)+\theta^2 k(\theta^2)\big]}\,\boldsymbol{\Theta}_\rho^{-1}\cdot\times\boldsymbol{\vartheta}_\rho\,\boldsymbol{\Theta}_\rho^{-1}\cdot\times\boldsymbol{\vartheta}_\rho\right],$$

$$c(\theta^2) = -2\,\frac{\mathrm{d}\Pi(\theta^2)}{d(\theta^2)},\qquad k(\theta^2) = 4\,\frac{\mathrm{d}^2\Pi(\theta^2)}{d(\theta^2)^2},\qquad \theta^2 = \big(\boldsymbol{\Theta}_\rho^{-1}\cdot\times\boldsymbol{\vartheta}_\rho\big)\cdot\big(\boldsymbol{\Theta}_\rho^{-1}\cdot\times\boldsymbol{\vartheta}_\rho\big). \tag{110}$$

Here $\mathbf{f}$ and $\mathbf{L}_\mathrm{h}^*$ are external actions, which must be specified. The functions $U\big(\rho,\ \mathbf{G}_\mathrm{s},\ \Theta_\rho,\ \boldsymbol{\Psi}_\rho\big)$ and $\Pi(\theta^2)$ must also be specified.

8 A determination of temperature and entropy in the framework of the suggested nonlinear model

In the linear theory, the analogies of temperature and entropy were introduced on the basis of the energy considerations. To be exact, the last term in the energy balance equation (18) was considered as the thermodynamic one. Appropriately, the scalar quantity characterizing the spherical part of the moment stress tensor was interpreted as the analogy of temperature, and the conjugate strain characteristic was interpreted as the analogy of volume density of entropy. In what follows, we extend these mechanical analogies to the nonlinear case. Further, we discuss the physical meaning of the additional terms in the heat conduction equation, and we also suggest a generalization of the ideal gas model taking into account electromagnetic effects.

8.1 An extension of the mechanical analogies of temperature and entropy to the nonlinear case

Now we turn to the energy balance equation (53) in order to introduce mechanical analogies of temperature and entropy in the framework of the nonlinear model. According to the assumption (59), the energy moment stress tensor $\mathbf{T}_\mathrm{e}$, which characterizes the interaction of rotors, is the sum of the spherical part of tensor $T_\mathrm{e}\mathbf{E}$ and the antisymmetric tensor $-\mathbf{M}_\mathrm{e}\times\mathbf{E}$. By substituting Eq. (59) into Eq. (53), we obtained the energy balance equation (80), which contains two terms characterizing the power of moment interaction of the rotors, namely $T_\mathrm{e}\,\dfrac{\delta\Theta_\mathrm{e}}{\delta t}$ and $\mathbf{M}_\mathrm{e}\cdot\dfrac{\delta\boldsymbol{\Psi}_\mathrm{e}}{\delta t}$. Let us assume that Eq. (80) is the energy balance equation for the classical continuum. Then, the term $T_\mathrm{e}\,\dfrac{\delta\Theta_\mathrm{e}}{\delta t}$ can be interpreted as a thermodynamic one.

Now we draw attention to a problem that arises when we turn from the linear theory to the nonlinear one. In the case of the linear theory, in the balance equations of linear momentum, angular momentum, and energy the volume density of mass can be considered as a time-independent quantity that is specified in the reference configuration. Therefore, it does not matter what kind of entropy density, the mass density or the volume density, appears in the energy balance equation under the sign of time derivative. The situation is different in the nonlinear theory. Therefore, when considering the nonlinear model, the interpretation of $T_\mathrm{e}\,\dfrac{\delta\Theta_\mathrm{e}}{\delta t}$ as the thermodynamic term in the energy balance equation does not give us grounds to interpret the quantity T_e as an analogy of temperature. It is necessary to make the change of variables so that the thermodynamic term would take the form conventional for nonlinear continuum mechanics. In other words, the volume density of mass should appear as a coefficient in the thermodynamic term.

The change of variables given by Eqs. (63), (85) allowed us to transform the energy balance equation (80) to the form (87), where the terms $T_\mathrm{e}\,\dfrac{\delta\Theta_\mathrm{e}}{\delta t}$ and $\mathbf{M}_\mathrm{e}\cdot\dfrac{\delta\boldsymbol{\Psi}_\mathrm{e}}{\delta t}$ are replaced by the terms $\rho T\,\dfrac{\delta\Theta_\rho}{\delta t}$ and $\rho\mathbf{M}\cdot\dfrac{\delta\boldsymbol{\Psi}_\rho}{\delta t}$. The term $\rho T\,\dfrac{\delta\Theta_\rho}{\delta t}$ has the same structure as the thermodynamic term in nonlinear continuum mechanics. Consequently, we can interpret the quantity T as a mechanical analogy of temperature and the quantity Θ_ρ as a mechanical analogy of the mass density of entropy. The quantities T and Θ_ρ are related to the absolute temperature T_a and the mass density of entropy Θ_a by

$$T = aT_\mathrm{a},\qquad \Theta_\rho = \frac{1}{a}\,\Theta_\mathrm{a} \tag{111}$$

where a is the normalization factor. As previously shown, in the linear theory the normalization factor a can be eliminated from all equations. Now we cannot answer the question on the possibility to eliminate the normalization factor from the nonlinear equations. Note that the normalization factor a in the suggested theory is analogous to the Boltzmann constant in the kinetic theory.

We note that if assumption (59) is replaced by the similar assumption in relation to the true moment stress tensor **T**, then it will be more difficult to separate a term in the energy balance equation that could be naturally interpreted as a thermodynamic one. This circumstance was the decisive factor for the choice of the underlying assumption, which was necessary in order to extend the previously suggested linear model to the nonlinear case. Certainly, the simplicity of mathematical realization cannot be considered as a serious argument in favor of the chosen way. Especially as various energy stress tensors with the corresponding energy strain measures can be introduced in nonlinear models. All of the energy stress tensors are, in fact, equivalent, and it is difficult to prefer one to another. Therefore, in order to finally decide an issue on the introduction of temperature and entropy in the framework of the suggested nonlinear theory, it is necessary to consider the different ways, as well as to analyze their consequences and to compare these consequences with the known experimental data. A study of this issue is beyond the scope of this paper, and it will be the subject of a separate publication. Here we were only going to identify the problems that arise when trying to extend the thermodynamic analogies to the case of a nonlinear model and to outline the ways of solving the problems.

8.2 The nonlinear heat conduction equation and the meaning of the quantities contained in it

In the linear model, the heat conduction equation (21) was obtained by taking the divergence of both sides of the angular momentum balance equation for the rotors. It is evident that in the nonlinear model the same approach should be used. However, in the nonlinear model there are several different formulations of the angular momentum balance equation for the rotors. They are given in Sect. 7.2. For simplicity sake, we exclude from consideration the possibility of mechanical deformations of the material, i.e., we suppose that $\mathbf{g} = \mathbf{E}$, $\rho = \text{const}$. In this case, there exist two alternatives, namely Eqs. (68) and (71), which differ from each other only by rotation. It is difficult to find physical arguments in favor of one of these equations. For further studies, we choose Eq. (71) because the qualitative results do not depend on the equation choice and Eq. (71) is simpler than Eq. (68). Thus, by taking the divergence of both sides of Eq. (71) and taking into account Eq. (72), we obtain

$$\rho^{-1}\Delta T - (\nabla T)\cdot\boldsymbol{\Psi}_\rho - T\nabla\cdot\boldsymbol{\Psi}_\rho + (\nabla\Theta_\rho)\cdot\mathbf{M} + \Theta_\rho\nabla\cdot\mathbf{M} - \mathbf{M}\cdot(\nabla\cdot\boldsymbol{\Theta}_\rho) - (\nabla\mathbf{M})\cdot\cdot\boldsymbol{\Theta}_\rho^T \\ + \nabla\cdot\mathbf{L}_\mathrm{h}^* + \nabla\cdot\mathbf{L}_\mathrm{f}^* = \rho J\,\frac{\delta}{\delta t}\left(\frac{\delta\Theta_\rho}{\delta t} - \boldsymbol{\Omega}\cdot\boldsymbol{\Psi}_\rho\right). \tag{112}$$

It is easy to see that Eq. (112) depends not only on the mechanical analogies of temperature and entropy but also on the quantities **M** and $\boldsymbol{\Psi}_\rho$. That is why the heat conduction equation (112) cannot be considered as an independent equation. This equation must be considered together with another equation, which also follows from the angular momentum balance equation for the rotors. In order to derive this additional equation, we take the curl operator of both sides of Eq. (71) and take into account Eq. (72). As a result, we obtain

$$\rho^{-1}\big[\Delta\mathbf{M} - \nabla\cdot\nabla\mathbf{M}\big] - (\nabla T)\times\boldsymbol{\Psi}_\rho - T\nabla\times\boldsymbol{\Psi}_\rho + (\nabla\Theta_\rho)\times\mathbf{M} + \Theta_\rho\nabla\times\mathbf{M} \\ - (\nabla\times\boldsymbol{\Theta}_\rho)\cdot\mathbf{M} - (\nabla\mathbf{M})\cdot\times\boldsymbol{\Theta}_\rho^T + \nabla\times\mathbf{L}_\mathrm{h}^* + \nabla\times\mathbf{L}_\mathrm{f}^* = \rho J\,\frac{\delta}{\delta t}\left(\frac{\delta\boldsymbol{\Psi}_\rho}{\delta t} + \Theta_\rho\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\Theta}_\rho\right). \tag{113}$$

Thus, in the nonlinear theory Eqs. (112), (113) are the coupled system of equations. The latter remains true even when the quasi-static processes are considered and the internal energy has a special structure such that the constitutive equations for T and **M** are independent, i.e., $T = T(\Theta_\rho)$ and $\mathbf{M} = \mathbf{M}(\boldsymbol{\Psi}_\rho)$. Therefore, in discussing the heat conduction equation, it is necessary to discuss the physical meaning of vector **M**, which characterizes the antisymmetric part of the moment stress tensor, as well as the physical meaning of the corresponding strain measure $\boldsymbol{\Psi}_\rho$. In order to clarify the physical meaning of vectors **M** and $\boldsymbol{\Psi}_\rho$, we refer to the mechanical model suggested in [63]. This mechanical model is based on the rotational degrees of freedom, and the moment stress tensors are considered to be antisymmetric at that. As shown in [63], the mathematical description of the model is reduced to the equations that are similar to the equations describing the electromagnetic field in a substance. In particular, under certain simplifying assumptions, the analogies of the Lorentz force and Maxwell's equations have been obtained in the framework of the suggested model. Thus, in [63] the antisymmetric parts of moment stress tensors and the corresponding strain measures are considered as analogies of the quantities describing the electromagnetic interactions. Developing the ideas of [63], we attribute to vectors $\mathbf{M}$ and $\boldsymbol{\Psi}_\rho$ in Eqs. (112), (113) the meaning of quantities associated with the electromagnetic interactions between the particles of matter. Now we give some arguments in favor of such an interpretation. We also express our view concerning the need to incorporate some electromagnetic characteristics into the mathematical model describing nonlinear thermal processes.

(i) In many cases, thermal effects exert influence on the processes of chemical reactions, structural transformations, and phase transitions. The chemical reactions, the structural transformations, and the phase transitions are, in fact, the striking changes in interatomic interactions. It is well known that interatomic interactions have electromagnetic character. That is why, when modeling the chemical reactions, the structural transformations, and the phase transitions, it is important to take into account the interdependence between thermal and electromagnetic effects at the microlevel.

(ii) We are convinced that the electromagnetic interactions between particles of matter is one of the factors generating the internal damping. Therefore, the electromagnetic interactions at the microlevel must be taken into account when modeling acoustic processes in material media. The original approach to the description of the internal damping by means of taking into account the spherical and antisymmetric parts of the moment stress tensor has been developed in [52]. The cited paper demonstrates the possibility to determine the model parameters in terms of the known physical characteristics of the material. In particular, the model parameters characterizing the internal damping are determined by using the experimental data on the acoustic wave attenuation factors.

(iii) We believe that it is important to consider the mutual influence of thermal and electromagnetic effects when describing radiation processes. In essence, the model problem considered in Sect. 7.4 demonstrates the description of the radiation process. In view of the suggested physical–mechanical analogies, the "transversal" waves propagating with velocity c_s are considered to be electromagnetic waves, and the "longitudinal" waves propagating with velocity c_v are assumed to be heat waves. The heat transfer by radiation can be represented in different ways. On the one hand, it is generally accepted that only electromagnetic waves propagate, and it are precisely these waves that cause heating of matter when interacting with it. On the other hand, we can assume that waves of different nature, namely heat waves, propagate together with electromagnetic waves. The model problem in Sect. 7.4 is the realization of the latter viewpoint. We note that in order to determine the model problem parameters it is necessary to use experimental data relating to the laws of radiation. However, it is the subject of a separate study that is beyond the scope of this work.

A detailed discussion of electromagnetic processes occurring in matter at the microlevel is also beyond the scope of this study. In [63], the electromagnetic analogies were suggested for the specific cases. In order to extend these analogies to a sufficiently general nonlinear model, we must solve a number of problems similar to those which were solved when extending the thermodynamic analogies, firstly used in the linear theory, to the nonlinear case.

8.3 Generalized models of an ideal gas

Now we show that the known equation of an ideal gas can be obtained in the framework of the suggested model, without excluding the quantities that are assumed to be of an electromagnetic nature.

Let us reduce the energy balance equation (87), eliminating the deviator of stress tensor, using the thermodynamic analogies (111), and taking into account the term that contains vectors $\mathbf{M}$ and $\boldsymbol{\Psi}_\rho$. As a result, we obtain

$$\rho \frac{\delta U}{\delta t} = -\frac{p}{\rho}\frac{\delta \rho}{\delta t} + \rho T_\mathrm{a} \frac{\delta \Theta_\mathrm{a}}{\delta t} + \rho \mathbf{M} \cdot \frac{\delta \boldsymbol{\Psi}_\rho}{\delta t}. \tag{114}$$

As can be seen from Eq. (114), the mass density of internal energy depends on the scalar arguments ρ, Θ_a, and the vector argument $\boldsymbol{\Psi}_\rho$. Since the medium is isotropic, the mass density of internal energy is a function of the form $U = U(\rho,\ \Theta_\mathrm{a},\ \Psi_\rho^2)$, where $\Psi_\rho^2 = \boldsymbol{\Psi}_\rho \cdot \boldsymbol{\Psi}_\rho$. Next, we give a few specific examples.

Example 1 Let us consider the expression for the mass density of internal energy,

$$U = \Upsilon(\Psi_\rho^2) \left(\frac{\rho}{\rho_0}\right)^{(c_\mathrm{p}-c_\mathrm{v})/c_\mathrm{v}} e^{\Theta_\mathrm{a}/c_\mathrm{v}}, \qquad c_\mathrm{p} = \mathrm{const}, \qquad c_\mathrm{v} = \mathrm{const}, \tag{115}$$

where c_p is the specific heat of the gas at constant pressure, c_v is the specific heat of the gas at constant volume, and ρ_0 is the gas density under normal conditions. By using the Cauchy–Green relations (89), we obtain the constitutive equations for pressure and temperature,

$$p = -\Upsilon(\Psi_\rho^2)\,\frac{\rho_0(c_p - c_v)}{c_v}\left(\frac{\rho}{\rho_0}\right)^{c_p/c_v} e^{\Theta_a/c_v}, \qquad T_a = \Upsilon(\Psi_\rho^2)\,\frac{1}{c_v}\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v}, \tag{116}$$

and the constitutive equations for vector $\mathbf{M}$,

$$\mathbf{M} = 2\,\frac{\mathrm{d}\Upsilon(\Psi_\rho^2)}{d(\Psi_\rho^2)}\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v}\,\boldsymbol{\Psi}_\rho. \tag{117}$$

The known equation of the ideal gas and the known relation between the temperature and the internal energy of the ideal gas

$$p = -(c_p - c_v)\rho T_a, \qquad U = c_v T_a \tag{118}$$

follow from Eqs. (115), (116). It is easy to see that the dependence of the internal energy on $\boldsymbol{\Psi}_\rho$ has no effect on the form of Eq. (118). The expression for entropy differs from the known expression for the entropy of an ideal gas. However, the entropy plays a subsidiary role in continuum mechanics. The entropy is usually used for deriving the constitutive equations and the heat conduction equation. As a rule, the entropy is excluded from the final system of equations. In order to appreciate the adequacy of a model, only the behavior of temperature (not the entropy!) is verified by means of a comparison with experimental data. That is why, when the model contains the known equation of the ideal gas and the known relation between the temperature and the internal energy, the expression for the entropy is not very important.

Example 2 Now we modify the expression for the mass density of internal energy (115) as follows:

$$U = \Upsilon\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v} + U_0(\Psi_\rho^2), \qquad c_p = \text{const}, \qquad c_v = \text{const}, \qquad \Upsilon = \text{const}. \tag{119}$$

In this case, the constitutive equations for pressure and temperature take the form

$$p = -\Upsilon\,\frac{\rho_0(c_p - c_v)}{c_v}\left(\frac{\rho}{\rho_0}\right)^{c_p/c_v} e^{\Theta_a/c_v}, \qquad T_a = \Upsilon\,\frac{1}{c_v}\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v}. \tag{120.1,2}$$

The constitutive equation for vector $\mathbf{M}$ is given by the formula

$$\mathbf{M} = 2\,\frac{\mathrm{d}U_0(\Psi_\rho^2)}{d(\Psi_\rho^2)}\,\boldsymbol{\Psi}_\rho. \tag{121}$$

Equation (120.2) coincides with the known relation between the temperature, density, and entropy of the ideal gas. The relations

$$p = -(c_p - c_v)\rho T_a, \qquad U = c_v T_a + U_0(\Psi_\rho^2) \tag{122.1,2}$$

follow from Eqs. (119), (120). It is easy to see that Eq. (122.1) coincides with the equation of the ideal gas. Equation (122.2) demonstrates that the mass density of internal energy linearly depends on temperature, as it should be for an ideal gas.

Example 3 Now we consider the expression for the mass density of internal energy,

$$U = \Upsilon\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v}, \qquad c_p = c_p(\Psi_\rho^2), \qquad c_v = c_v(\Psi_\rho^2), \qquad \Upsilon = \text{const}, \tag{123}$$

wherein the specific heats are the functions of Ψ_ρ^2. The constitutive equations for pressure and temperature are given by the expressions

$$p = -\Upsilon\,\frac{\rho_0(c_p - c_v)}{c_v}\left(\frac{\rho}{\rho_0}\right)^{c_p/c_v} e^{\Theta_a/c_v}, \qquad T_a = \Upsilon\,\frac{1}{c_v}\left(\frac{\rho}{\rho_0}\right)^{(c_p-c_v)/c_v} e^{\Theta_a/c_v}. \tag{124.1,2}$$

The constitutive equation for vector **M** takes the form

$$\mathbf{M} = 2\,\Upsilon \left(\frac{\rho}{\rho_0}\right)^{(c_\mathrm{p}-c_\mathrm{v})/c_\mathrm{v}} e^{\Theta_\mathrm{a}/c_\mathrm{v}} \left[\frac{\ln(\rho/\rho_0)}{c_\mathrm{v}} \frac{\mathrm{d}c_\mathrm{p}(\Psi_\rho^2)}{d(\Psi_\rho^2)} - \frac{\ln(\rho/\rho_0)\,c_\mathrm{p} + \Theta_\mathrm{a}}{c_\mathrm{v}^2} \frac{\mathrm{d}c_\mathrm{v}(\Psi_\rho^2)}{d(\Psi_\rho^2)}\right] \boldsymbol{\Psi}_\rho. \tag{125}$$

From Eqs. (123), (124), it follows that

$$p = -\left(c_\mathrm{p}(\Psi_\rho^2) - c_\mathrm{v}(\Psi_\rho^2)\right)\rho T_\mathrm{a}, \qquad U = c_\mathrm{v}(\Psi_\rho^2) T_\mathrm{a}. \tag{126}$$

Formulas (126) differ from the known relations only by the fact that the specific heats depend on Ψ_ρ^2. The expression for entropy that follows from Eq. (124.2) is also different from the known relation only by the fact that c_p and c_v are functions of Ψ_ρ^2.

We believe that the models of an ideal gas including the quantities $\boldsymbol{\Psi}_\rho$ and **M** can be useful in modeling an ideal plasma.

9 Nonlinear models of heat transfer: state of the art and the peculiarity of the suggested model

Nonlinear thermal processes are actively studied and discussed in the modern literature. Without claiming to be an exhaustive literature review, we indicate the main research areas in the field of nonlinear thermal conductivity and denote a place of the suggested theory among the other models.

Many papers covering only mathematical questions are regularly published for several decades. Various aspects of constructing analytical, semi-analytic and numerical solutions of the nonlinear heat conduction equations are discussed in such papers—see, e.g., [65–69]. Such works usually deal with the simplest nonlinear heat conduction equations. The nonlinearity of these equations consists in the fact that the material constants of the linear equations are replaced by some functions of temperature (more often by polynomials). Among the mathematical works, it is worth mentioning the papers where the authors consider laser heat sources (see, e.g., [70]), as this type of thermal influences is most often found in the modern literature. Another large group of publications consists of applied works, which are devoted to modeling nonlinear thermal processes in technical devices—see, e.g., [71–74]. In such works, the mutual influence of thermal processes and processes of other physical nature (optical, electrical, magnetic) is usually taken into account. Other distinctive features of applied works are the use of numerical methods, the use of parameters of specific technical devices in calculations, and the comparison of modeling results with the experimental data. We note that the models of nonlinear thermal processes in the widest scale range, from geophysical processes (see, e.g., [75]), up to biological processes at the molecular level (see, e.g., [76]), are presented in the modern literature. There are a large number of papers devoted to studying nonlinear effects associated with thermal radiation—see, e.g., [77,78]. There exists a variety of mathematical models used to describe various thermal processes. Some of them are based on classical concepts, and others are based on quantum-mechanical concepts. However, purely empirical relations, which are not based on any models, play an important role in all nonlinear theories.

The main feature of the research presented in this paper is that it provides the description of two fundamentally different processes of heat transfer (heat conduction and thermal radiation) within the framework of one model. Another feature of the present study is the fact that it is based on the mechanical model different from those used in statistical physics and quantum mechanics. Similarity between the suggested model and other modern models consists in considering the mutual influence of thermal processes and processes of different physical nature. In addition, this model (as well as many other modern models) contains a sufficient number of undetermined functions. If we determine these functions by using empirical data, we will be able to model thermal processes that occur in specific materials under specific conditions.

In future, we plan to develop the suggested theory in several directions. The first direction is a comparative analysis of the different methods of introducing temperature and entropy that are based on modifications of assumption (59). The second direction is to define the analogies of electromagnetic quantities using and developing the ideas stated in [63]. The third direction is to specify the form of internal energy and to determine the model parameters based on experimental data on the thermodynamic properties of materials. The fourth direction consists in the further development of the internal damping model by combining the ideas of this paper with the ideas of [52]. The fifth direction is to determine the ether parameters based on a comparative analysis of the heat and electromagnetic waves behavior at the boundary of the ether and some material medium.

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Appendix A: The proof of identity $\nabla \cdot \mathbf{D}^{-T} = \mathbf{0}$

Tensor $\mathbf{D} = I_3^{-1}\mathbf{g}$ possesses the property

$$\nabla \cdot \mathbf{D}^{-T} = \mathbf{0}. \tag{127}$$

In order to prove this fact, we perform the following transformations:

$$\mathbf{D}^{-T} \cdot \mathbf{D}^{T} = \mathbf{E} \quad \Rightarrow \quad \nabla \cdot \big(\mathbf{D}^{-T} \cdot \mathbf{D}^{T}\big) = \mathbf{0} \quad \Rightarrow \quad \big(\nabla \cdot \mathbf{D}^{-T}\big) \cdot \mathbf{D}^{T} + \mathbf{D}^{-1} \cdot\cdot \nabla \mathbf{D}^{T} = \mathbf{0}.$$

Next, we solve the last equation with respect to $\nabla \cdot \mathbf{D}^{-T}$ and carry out a number of identity transformations:

$$\begin{aligned}\nabla \cdot \mathbf{D}^{-T} &= -\big(\mathbf{D}^{-1} \cdot\cdot \nabla \mathbf{D}^{T}\big) \cdot \mathbf{D}^{-T} = \Big[\mathbf{g}^{-1} \cdot\cdot \Big((\nabla I_3)\, \mathbf{g}^{T} - I_3 \nabla \mathbf{g}^{T}\Big)\Big] \cdot \mathbf{g}^{-T} \\ &= \Big[(\nabla I_3) \cdot \mathbf{g}^{-T} \cdot \mathbf{g}^{T} - I_3\, \mathbf{g}^{-1} \cdot\cdot \nabla \mathbf{g}^{T}\Big] \cdot \mathbf{g}^{-T} = \Big[\nabla I_3 - \Big(\frac{\partial I_3}{\partial \mathbf{g}}\Big)^{T} \cdot\cdot \nabla \mathbf{g}^{T}\Big] \cdot \mathbf{g}^{-T}.\end{aligned}$$

The last transformation is done by using the known identity

$$\mathbf{g}^{-1} = \frac{1}{I_3}\Big(\frac{\partial I_3}{\partial \mathbf{g}}\Big)^{T}, \tag{128}$$

which is true for an arbitrary second-rank tensor. Further transformations are based on the specific properties of tensor $\mathbf{g}$:

$$\begin{aligned}\nabla \mathbf{g}^{T} &= \mathbf{r}^{i} \frac{\partial}{\partial q^{i}} \Big(\mathbf{E} - (\nabla \mathbf{u})^{T}\Big) = -\mathbf{r}^{i} \frac{\partial^2 \mathbf{u}}{\partial q^{i} \partial q^{j}}\, \mathbf{r}^{j} = -\frac{\partial (\nabla \mathbf{u})}{\partial q^{j}}\, \mathbf{r}^{j} = \frac{\partial \mathbf{g}}{\partial q^{j}}\, \mathbf{r}^{j} \\ &\Rightarrow \Big(\frac{\partial I_3}{\partial \mathbf{g}}\Big)^{T} \cdot\cdot \nabla \mathbf{g}^{T} = \Big(\frac{\partial I_3}{\partial \mathbf{g}}\Big)^{T} \cdot\cdot \frac{\partial \mathbf{g}}{\partial q^{j}}\, \mathbf{r}^{j} = \frac{\partial I_3}{\partial q^{j}}\, \mathbf{r}^{j} = \nabla I_3 \\ &\Rightarrow \nabla \cdot \mathbf{D}^{-T} = \Big[\nabla I_3 - \Big(\frac{\partial I_3}{\partial \mathbf{g}}\Big)^{T} \cdot\cdot \nabla \mathbf{g}^{T}\Big] \cdot \mathbf{g}^{-T} = \big(\nabla I_3 - \nabla I_3\big) \cdot \mathbf{g}^{-T} \equiv \mathbf{0}.\end{aligned}$$

The proof of identity (127) is complete.

Appendix B: The derivation of the rotors' motion equation in the case of $\mathbf{T} = T\,\mathbf{D}^{-T} \cdot \mathbf{P}^{T} - \mathbf{D}^{-T} \cdot (\mathbf{M} \times \mathbf{P}^{T})$

Let us take the divergence of the moment stress tensor (63) and fulfill some transformations taking into account the last formula in Eq. (61):

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \nabla \cdot \Big(T\mathbf{D}^{-T} \cdot \mathbf{P}^{T} - \mathbf{D}^{-T} \cdot (\mathbf{M} \times \mathbf{P}^{T})\Big) \\ &= \nabla T \cdot \mathbf{D}^{-T} \cdot \mathbf{P}^{T} + T\, \mathbf{D}^{-1} \cdot\cdot \nabla \mathbf{P}^{T} - \mathbf{D}^{-1} \cdot\cdot (\nabla \mathbf{M} \times \mathbf{E}) \cdot \mathbf{P}^{T} + (\mathbf{M} \times \mathbf{D}^{-1}) \cdot\cdot \nabla \mathbf{P}^{T} \\ &= \nabla T \cdot I_3\, \mathbf{g}^{-T} \cdot \mathbf{P}^{T} - \big(T\, I_3\, \mathbf{g}^{-1} + \mathbf{M} \times I_3 \mathbf{g}^{-1}\big) \cdot\cdot \big(\boldsymbol{\Theta} \cdot \mathbf{P} \times \mathbf{P}^{T}\big) - \big(I_3\, \mathbf{g}^{-1} \cdot \times \nabla \mathbf{M}\big) \cdot \mathbf{P}^{T} \\ &= I_3 \Big[\nabla T \cdot \mathbf{g}^{-T} - \big(T\, \mathbf{E} + \mathbf{M} \times \mathbf{E}\big) \cdot\cdot \big(\mathbf{g}^{-1} \cdot \boldsymbol{\Theta} \cdot \mathbf{P} \times \mathbf{E}\big) - \mathbf{g}^{-1} \cdot \times \nabla \mathbf{M}\Big] \cdot \mathbf{P}^{T} \\ &= I_3 \Big[\mathbf{g}^{-1} \cdot \nabla T - \mathbf{g}^{-1} \cdot \times \nabla \mathbf{M} - T\big(\mathbf{g}^{-1} \cdot \boldsymbol{\Theta} \cdot \mathbf{P}\big)_{\times} + \mathrm{tr}(\mathbf{g}^{-1} \cdot \boldsymbol{\Theta} \cdot \mathbf{P})\, \mathbf{M} - (\mathbf{g}^{-1} \cdot \boldsymbol{\Theta} \cdot \mathbf{P}) \cdot \mathbf{M}\Big] \cdot \mathbf{P}^{T}.\end{aligned}$$

By using the notation (52), we write the result of the above transformations in terms of the energy strain measure $\boldsymbol{\Theta}_{\mathrm{e}}$ as

$$\nabla \cdot \mathbf{T} = I_3 \Big[\mathbf{g}^{-1} \cdot \nabla T - \mathbf{g}^{-1} \cdot \times \nabla \mathbf{M} - T\big(\boldsymbol{\Theta}_{\mathrm{e}}\big)_{\times} + (\mathrm{tr}\, \boldsymbol{\Theta}_{\mathrm{e}})\, \mathbf{M} - \boldsymbol{\Theta}_{\mathrm{e}} \cdot \mathbf{M}\Big] \cdot \mathbf{P}^{T}.$$

In view of the last relation, the rotors motion equation (43) takes the form

$$\Big[\mathbf{g}^{-1}\cdot\nabla T-\mathbf{g}^{-1}\cdot\times\nabla\mathbf{M}-T\big(\boldsymbol{\Theta}_{\mathrm{e}}\big)_{\times}+(\operatorname{tr}\boldsymbol{\Theta}_{\mathrm{e}})\,\mathbf{M}-\boldsymbol{\Theta}_{\mathrm{e}}\cdot\mathbf{M}\Big]\cdot\mathbf{P}^{T}+\varrho\mathbf{L}=\varrho J\frac{\delta\boldsymbol{\omega}}{\delta t},\qquad \varrho=\frac{\rho}{I_3}. \quad (129)$$

The derivation of the rotors motion equation is complete.

Appendix C: The proof of identity $\dfrac{\delta\varrho}{\delta t}=0$

Now we prove that the quantity $\varrho=\rho/I_3$ possesses the property

$$\frac{\delta\varrho}{\delta t}=0. \quad (130)$$

In view of identity (128) and Eq. (50.1), which relates the velocity gradient and the strain measure $\mathbf{g}$, we obtain

$$\nabla\mathbf{v}=-\frac{\delta\mathbf{g}}{\delta t}\cdot\mathbf{g}^{-1}\quad\Rightarrow\quad\nabla\cdot\mathbf{v}=-\frac{\delta\mathbf{g}}{\delta t}\cdot\cdot\,\mathbf{g}^{-1}\quad\Rightarrow\quad\nabla\cdot\mathbf{v}=-\frac{1}{I_3}\frac{\delta I_3}{\delta t}.$$

Let us write the mass balance equation (37) taking into account the above identities and carry out a number of simple transformations. As a result, we have

$$\frac{\delta\rho}{\delta t}=\frac{\rho}{I_3}\frac{\delta I_3}{\delta t}\quad\Rightarrow\quad\frac{\delta(\rho/I_3)}{\delta t}=0\quad\Rightarrow\quad\frac{\delta\varrho}{\delta t}=0.$$

The proof of identity (130) is complete.

Appendix D: The transformation of the energy balance equation in the case of $\boldsymbol{\tau}=\boldsymbol{\tau}^{T}$, $\boldsymbol{\mu}=0$, $\mathbf{T}_{\mathrm{e}}=T_{\mathrm{e}}\mathbf{E}-\mathbf{M}_{\mathrm{e}}\times\mathbf{E}$

Let us write the energy balance equation (73) taking into account expressions (51), (52) for the energy stress tensor $\boldsymbol{\tau}_{\mathrm{e}}$ and the energy strain measure $\mathbf{g}_{\mathrm{e}}$:

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}\cdot\cdot\left(\mathbf{g}\cdot\frac{\delta(\mathbf{g}^{-1}\cdot\tilde{\mathbf{P}})}{\delta t}\cdot\tilde{\mathbf{P}}^{T}\right)+T_{\mathrm{e}}\,\mathbf{E}\cdot\cdot\frac{\delta\boldsymbol{\Theta}_{\mathrm{e}}}{\delta t}+\big(\mathbf{M}_{\mathrm{e}}\times\mathbf{E}\big)\cdot\cdot\frac{\delta\boldsymbol{\Theta}_{\mathrm{e}}}{\delta t}.$$

By using the identities

$$T_{\mathrm{e}}\,\mathbf{E}\cdot\cdot\frac{\delta\boldsymbol{\Theta}_{\mathrm{e}}}{\delta t}=T_{\mathrm{e}}\frac{\delta(\operatorname{tr}\boldsymbol{\Theta}_{\mathrm{e}})}{\delta t},\qquad\big(\mathbf{M}_{\mathrm{e}}\times\mathbf{E}\big)\cdot\cdot\frac{\delta\boldsymbol{\Theta}_{\mathrm{e}}}{\delta t}=\mathbf{M}_{\mathrm{e}}\cdot\frac{\delta\big(\boldsymbol{\Theta}_{\mathrm{e}}\big)_{\times}}{\delta t},$$

we reduce the energy balance equation to the form

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}\cdot\cdot\left(\mathbf{g}\cdot\frac{\delta(\mathbf{g}^{-1}\cdot\tilde{\mathbf{P}})}{\delta t}\cdot\tilde{\mathbf{P}}^{T}\right)+T_{\mathrm{e}}\frac{\delta(\operatorname{tr}\boldsymbol{\Theta}_{\mathrm{e}})}{\delta t}+\mathbf{M}_{\mathrm{e}}\cdot\frac{\delta\big(\boldsymbol{\Theta}_{\mathrm{e}}\big)_{\times}}{\delta t}.$$

Since $\boldsymbol{\tau}$ is a symmetric tensor, the energy balance equation depends on only the symmetric part of the expression in parentheses. We take this fact into account as follows:

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}\cdot\cdot\frac{1}{2}\left(\mathbf{g}\cdot\frac{\delta(\mathbf{g}^{-1}\cdot\tilde{\mathbf{P}})}{\delta t}\cdot\tilde{\mathbf{P}}^{T}+\tilde{\mathbf{P}}\cdot\frac{\delta(\tilde{\mathbf{P}}^{T}\cdot\mathbf{g}^{-T})}{\delta t}\cdot\mathbf{g}^{T}\right)+T_{\mathrm{e}}\frac{\delta(\operatorname{tr}\boldsymbol{\Theta}_{\mathrm{e}})}{\delta t}+\mathbf{M}_{\mathrm{e}}\cdot\frac{\delta\big(\boldsymbol{\Theta}_{\mathrm{e}}\big)_{\times}}{\delta t}.$$

By simple transformations, the rotation tensor $\tilde{\mathbf{P}}$ can be eliminated from the expression in parentheses. As a result, the energy balance equation takes a more simple form

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau} \cdot\cdot \frac{1}{2} \left(\mathbf{g} \cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} + \frac{\delta \mathbf{g}^{-T}}{\delta t} \cdot \mathbf{g}^T \right) + T_e \frac{\delta (\mathrm{tr}\, \boldsymbol{\Theta}_e)}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \big(\boldsymbol{\Theta}_e\big)_\times}{\delta t}.$$

Further, in view of the identical transformation

$$\frac{1}{2} \left(\mathbf{g} \cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} + \frac{\delta \mathbf{g}^{-T}}{\delta t} \cdot \mathbf{g}^T \right) = \frac{1}{2} \mathbf{g} \cdot \frac{\delta (\mathbf{g}^{-1} \cdot \mathbf{g}^{-T})}{\delta t} \cdot \mathbf{g}^T,$$

we transform the energy balance equation to the form

$$\rho \frac{\delta U}{\delta t} = \left(\mathbf{g}^T \cdot \boldsymbol{\tau} \cdot \mathbf{g} \right) \cdot\cdot \frac{1}{2} \frac{\delta (\mathbf{g}^{-1} \cdot \mathbf{g}^{-T})}{\delta t} + T_e \frac{\delta (\mathrm{tr}\, \boldsymbol{\Theta}_e)}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \big(\boldsymbol{\Theta}_e\big)_\times}{\delta t}.$$

Introducing the notations

$$\boldsymbol{\tau}_s = \mathbf{g}^T \cdot \boldsymbol{\tau} \cdot \mathbf{g}, \qquad \mathbf{g}_s = \frac{1}{2} \mathbf{g}^{-1} \cdot \mathbf{g}^{-T},$$

we rewrite the energy balance equation as

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}_s \cdot\cdot \frac{\delta \mathbf{g}_s}{\delta t} + T_e \frac{\delta (\mathrm{tr}\, \boldsymbol{\Theta}_e)}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \big(\boldsymbol{\Theta}_e\big)_\times}{\delta t}. \tag{131}$$

The transformation of the energy balance equation is complete.

Appendix E: The derivation of the Cauchy–Green relation for the deviator of the stress tensor

Now we introduce the strain measure

$$\mathbf{G} = I_3^{-1/3} \mathbf{g}, \qquad \mathrm{Det}\, \mathbf{G} = 1,$$

characterizing the change in form. Next, we write the energy balance equation (74) by using expression (79) for the stress tensor. After the necessary transformations of the energy balance equation, we separate the term that characterizes the power of internal interactions related to the change in volume and the term characterizing the power of internal interactions related to the changes in form. As a result, we obtain

$$\rho \frac{\delta U}{\delta t} = -\frac{p}{I_3} \frac{\delta I_3}{\delta t} + \left(\mathbf{G}^T \cdot (\mathrm{dev}\, \boldsymbol{\tau}) \cdot \mathbf{G} \right) \cdot\cdot \frac{1}{2} \frac{\delta (\mathbf{G}^{-1} \cdot \mathbf{G}^{-T})}{\delta t} + T_e \frac{\delta \Theta_e}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \boldsymbol{\Psi}_e}{\delta t}.$$

Making the transition from (74) to the above energy balance equation, we used the identical transformations

$$0 = \frac{\delta (\mathrm{Det}\, \mathbf{G}^{-1})}{\delta t} = \left(\frac{\partial (\mathrm{Det}\, \mathbf{G}^{-1})}{\partial \mathbf{G}^{-1}} \right)^T \cdot\cdot \frac{\delta \mathbf{G}^{-1}}{\delta t} = (\mathrm{Det}\, \mathbf{G}^{-1})\, \mathbf{G} \cdot\cdot \frac{\delta \mathbf{G}^{-1}}{\delta t} = \mathbf{G} \cdot\cdot \frac{\delta \mathbf{G}^{-1}}{\delta t},$$
$$\mathbf{E} \cdot\cdot \left(\mathbf{g} \cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} \right) = \mathrm{tr} \left(\mathbf{g} \cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} \right) = \mathbf{g} \cdot\cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} = -\frac{1}{3 I_3} \frac{\delta I_3}{\delta t} \mathbf{G} \cdot\cdot \mathbf{G}^{-1} + \mathbf{G} \cdot\cdot \frac{\delta \mathbf{G}^{-1}}{\delta t} = -\frac{1}{I_3} \frac{\delta I_3}{\delta t},$$

and the transformations similar to those that were carried out when deriving the energy balance equation in the form of (131). In view of the notation

$$\mathbf{G}_s = \frac{1}{2} \mathbf{G}^{-1} \cdot \mathbf{G}^{-T},$$

the energy balance equation takes the form

$$\rho \frac{\delta U}{\delta t} = -\frac{p}{I_3} \frac{\delta I_3}{\delta t} + \left(\mathbf{G}^T \cdot (\mathrm{dev}\, \boldsymbol{\tau}) \cdot \mathbf{G} \right) \cdot\cdot \frac{\delta \mathbf{G}_s}{\delta t} + T_e \frac{\delta \Theta_e}{\delta t} + \mathbf{M}_e \cdot \frac{\delta \boldsymbol{\Psi}_e}{\delta t}.$$

In the case of elastic strains, in view of Eq. (82) the Cauchy–Green relations for p, T_e, and $\mathbf{M}_e$ can be derived from the last equation in an obvious way. Only the constitutive equation for the deviator of the stress tensor

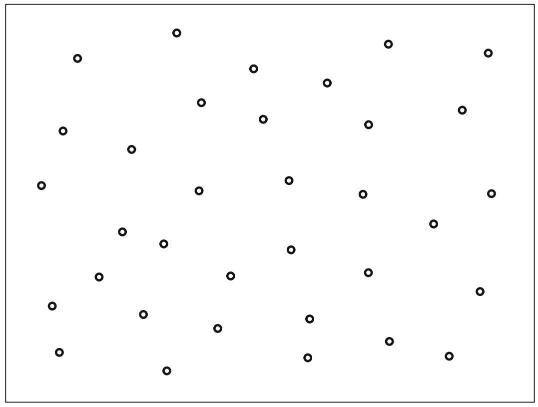

Fig. 5 An elementary volume of the continuum consisting of the body points

deserves a special discussion. Since the determinant of tensor $\mathbf{G}_s$ is equal to the fixed value, the Cauchy–Green relation for $\operatorname{dev}\boldsymbol{\tau}$ is determined up to the spherical part of the tensor, i.e.,

$$\operatorname{dev}\boldsymbol{\tau} = \rho\,\mathbf{G}^{-T}\cdot\frac{\partial U}{\partial \mathbf{G}_s}\cdot\mathbf{G}^{-1} + \lambda\,\mathbf{E}.$$

The coefficient λ is calculated as follows:

$$\operatorname{tr}(\operatorname{dev}\boldsymbol{\tau}) = 0 \quad\Rightarrow\quad \lambda = -\frac{2}{3}\,\rho\,\mathbf{G}_s\cdot\cdot\frac{\partial U}{\partial \mathbf{G}_s}.$$

As a result, the Cauchy–Green relation for $\operatorname{dev}\boldsymbol{\tau}$ takes the form

$$\operatorname{dev}\boldsymbol{\tau} = \rho\,\mathbf{G}^{-T}\cdot\frac{\partial U}{\partial \mathbf{G}_s}\cdot\mathbf{G}^{-1} - \frac{2}{3}\,\rho\,\mathbf{G}_s\cdot\cdot\frac{\partial U}{\partial \mathbf{G}_s}\,\mathbf{E}. \tag{132}$$

The derivation of the Cauchy–Green relation for the deviator of stress tensor is complete.

Appendix F: The nonlinear model of the ether

Now we consider a continuum consisting of body points—see Fig. 5. The body points are similar to the rotors of the gyrostats simulating particles of matter. We introduce the following notations: $\hat{\rho}(\mathbf{r}, t)$ is the mass density of the continuum at a given point of space; $\hat{\mathbf{v}}(\mathbf{r}, t)$ is the velocity field; $\hat{\mathbf{P}}(\mathbf{r}, t)$, $\hat{\boldsymbol{\omega}}(\mathbf{r}, t)$ are the fields of the rotation tensors and the angular velocity vectors of the body points, respectively. The relation between $\hat{\mathbf{P}}(\mathbf{r}, t)$ and $\hat{\boldsymbol{\omega}}(\mathbf{r}, t)$ is

$$\hat{\boldsymbol{\omega}} = -\frac{1}{2}\left(\frac{\hat{\delta}\hat{\mathbf{P}}}{\delta t}\cdot\hat{\mathbf{P}}^T\right)_{\times}, \qquad \frac{\hat{\delta}}{\delta t} = \frac{\mathrm{d}}{\mathrm{d}t} + \hat{\mathbf{v}}\cdot\nabla, \tag{133}$$

where the operator $\dfrac{\hat{\delta}}{\delta t}$ is the material derivative.

The volume densities of the kinetic energy, the linear momentum, and the angular momentum of the continuum are

$$\hat{\rho}\hat{K} = \hat{\rho}\left(\frac{1}{2}\hat{\mathbf{v}}\cdot\hat{\mathbf{v}} + \frac{1}{2}\hat{J}\,\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\omega}}\right), \qquad \hat{\rho}\hat{\mathbf{K}}_1 = \hat{\rho}\hat{\mathbf{v}}, \qquad \hat{\rho}\hat{\mathbf{K}}_2 = \hat{\rho}\Big(\mathbf{r}\times\hat{\mathbf{v}} + \hat{J}\,\hat{\boldsymbol{\omega}}\Big), \tag{134}$$

where constant $\hat{J}$ is the mass density of inertia moments of the body points. The angular momentum density $\hat{\rho}\hat{\mathbf{K}}_2$ is calculated with respect to the origin of the reference frame.

The mass production in a volume is considered to be impossible. There are no external body forces and body moments acting on the ether particles. Interactions between the ether particles that are characterized by forces are also absent. In view of these assumptions, the mass balance equation, the linear momentum balance equation, and the angular momentum balance equation are formulated as

$$\frac{\hat{\delta}\hat{\rho}}{\delta t} + \hat{\rho}\,\nabla\cdot\hat{\mathbf{v}} = 0, \qquad \hat{\rho}\,\frac{\hat{\delta}\hat{\mathbf{v}}}{\delta t} = \mathbf{0}, \qquad \nabla\cdot\hat{\mathbf{T}} = \hat{\rho}\hat{J}\,\frac{\hat{\delta}\hat{\boldsymbol{\omega}}}{\delta t} \tag{135.1–3}$$

where $\hat{\mathbf{T}}$ is the moment stress tensor in the ether. We note that Eq. (135) is not the coupled system of equations. At the beginning, one can find $\hat{\mathbf{v}}(\mathbf{r}, t)$ by solving Eq. (135.2). After that, one can find $\hat{\rho}(\mathbf{r}, t)$ by solving Eq. (135.1). Thus, the problem is reduced to solving Eq. (135.3) where $\hat{\rho}$ and $\hat{\mathbf{v}}$ are considered to be known functions.

Now we introduce the strain measure $\hat{\boldsymbol{\Theta}}$ and give the relation between this strain measure and the angular velocity gradient:

$$\nabla\hat{\mathbf{P}} = \hat{\boldsymbol{\Theta}}\times\hat{\mathbf{P}}, \qquad \nabla\hat{\boldsymbol{\omega}} = \frac{\hat{\delta}\hat{\boldsymbol{\Theta}}}{\delta t} + \hat{\boldsymbol{\Theta}}\times\hat{\boldsymbol{\omega}} + (\nabla\hat{\mathbf{v}})\cdot\hat{\boldsymbol{\Theta}}. \tag{136}$$

Next we introduce the energy moment stress tensor $\hat{\mathbf{T}}_{\rm e}$ and the energy strain measure $\hat{\boldsymbol{\Theta}}_{\rm e}$ in the same way as it is done in Sect. 6.4:

$$\hat{\mathbf{T}}_{\rm e} = \hat{\mathbf{g}}^{T}\cdot\hat{\mathbf{T}}\cdot\hat{\mathbf{P}}, \qquad \hat{\boldsymbol{\Theta}}_{\rm e} = \hat{\mathbf{g}}^{-1}\cdot\hat{\boldsymbol{\Theta}}\cdot\hat{\mathbf{P}}, \qquad \frac{\hat{\delta}\hat{\mathbf{g}}}{\delta t}\cdot\hat{\mathbf{g}}^{-1} = -\nabla\hat{\mathbf{v}}. \tag{137}$$

Further, we assume that an energy supply from on external source is absent and there is not any energy flow of "non-mechanical" nature. In this case, the energy balance equation can be reduced to the form

$$\hat{\rho}\,\frac{\hat{\delta}\hat{U}}{\delta t} = \hat{\mathbf{T}}_{\rm e}^{T}\cdot\cdot\frac{\hat{\delta}\hat{\boldsymbol{\Theta}}_{\rm e}}{\delta t} \tag{138}$$

where $\hat{U}$ is the mass density of internal energy. Assuming the continuum to be elastic, from Eq. (138) we obtain

$$\hat{U} = \hat{U}\big(\hat{\boldsymbol{\Theta}}_{\rm e}\big) \quad\Rightarrow\quad \hat{\mathbf{T}}_{\rm e} = \hat{\rho}\,\frac{\partial\hat{U}}{\partial\hat{\boldsymbol{\Theta}}_{\rm e}} \quad\Rightarrow\quad \hat{\mathbf{T}} = \hat{\rho}\,\hat{\mathbf{g}}^{-T}\cdot\frac{\partial\hat{U}}{\partial\hat{\boldsymbol{\Theta}}_{\rm e}}\cdot\hat{\mathbf{P}}^{T}. \tag{139}$$

In order to specify the model of the ether, we accept an assumption analogous to hypothesis (59), which was accepted in relation to the moment interactions in the material medium, namely

$$\hat{\mathbf{T}}_{\rm e} = \hat{T}_{\rm e}\mathbf{E} - \hat{\mathbf{M}}_{\rm e}\times\mathbf{E}. \tag{140}$$

In view of Eq. (140), the energy balance equation (138) can be rewritten as

$$\hat{\rho}\,\frac{\hat{\delta}\hat{U}}{\delta t} = \hat{T}_{\rm e}\,\frac{\hat{\delta}\hat{\Theta}_{\rm e}}{\delta t} + \hat{\mathbf{M}}_{\rm e}\cdot\frac{\hat{\delta}\hat{\boldsymbol{\Psi}}_{\rm e}}{\delta t}, \qquad \hat{\Theta}_{\rm e} = \operatorname{tr}\hat{\boldsymbol{\Theta}}_{\rm e}, \qquad \hat{\boldsymbol{\Psi}}_{\rm e} = \big(\hat{\boldsymbol{\Theta}}_{\rm e}\big)_{\times}. \tag{141}$$

Taking into account Eq. (141), we obtain

$$\hat{U} = \hat{U}\big(\hat{\Theta}_{\rm e},\,\hat{\boldsymbol{\Psi}}_{\rm e}\big) \quad\Rightarrow\quad \hat{T}_{\rm e} = \hat{\rho}\,\frac{\partial\hat{U}}{\partial\hat{\Theta}_{\rm e}}, \qquad \hat{\mathbf{M}}_{\rm e} = \hat{\rho}\,\frac{\partial\hat{U}}{\partial\hat{\boldsymbol{\Psi}}_{\rm e}}. \tag{142}$$

In order to ultimately elaborate the ether model, we should specify function $\hat{U}\big(\hat{\Theta}_{\rm e},\,\hat{\boldsymbol{\Psi}}_{\rm e}\big)$. This issue is beyond the scope of the present study.

It is easy to see that the suggested model of the ether is based on the same hypotheses as the model of the internal dynamics of those gyrostats that constitute the continuum modeling of some material medium. This coincidence is not accidental. It is the consequence of two ideas that are the foundation of the suggested model. The first idea is that the non-mechanical properties of matter (in particular, the thermodynamic ones) are partly provided by the internal structure of atoms, all the components of which are formed of the ether particles and retain the ether properties. The second idea is that the non-mechanical properties of matter are provided by not only the properties of atoms but also due to the ether that occupies the space between the atoms, interacts with the atoms, and provides the interactions between the atoms.

Appendix G: The linearized model of the ether

In the case of the linear theory, the density of mass is considered to be constant when formulating the balance equations of linear momentum, angular momentum, and energy. The description of rotational motion becomes more simple that leads, in particular, to simplification of the expressions for the strain measures. In the linear theory, the energy stress tensor and the energy moment stress tensor coincide with the true ones, as well as the energy strain measures coincide with the true strain measures. Upon linearizing, we get rid of the material derivatives that leads to simplification of the kinematic relations as well as the dynamic terms in the balance equations. These factors are of a technical nature. They are important in solving problems rather than for stating them. Now we discuss the factor that is important for stating the problems. In the linear theory, the internal energy density is a quadratic form of the strain tensors, whereas in the nonlinear theory the specification of internal energy is a complicated problem, solving which is based on various considerations of the informal nature as well as the results of physical experiments. In this paper, the equations of the ether dynamics are not a subject of investigation. They are used only in order to formulate and to solve the model problem. That is why, for purposes of this study, it is important to specify the constitutive equations and to simplify the differential equations as much as possible.

If for the undisturbed ether the rotation tensors of the body points $\hat{\mathbf{P}}(\mathbf{r}, t)$ are assumed to be equal to the unit tensor, then upon linearizing near this state tensors $\hat{\mathbf{P}}(\mathbf{r}, t)$ take the form

$$\hat{\mathbf{P}}(\mathbf{r}, t) = \mathbf{E} + \hat{\boldsymbol{\theta}}(\mathbf{r}, t) \times \mathbf{E} \tag{143}$$

where $\hat{\boldsymbol{\theta}}(\mathbf{r}, t)$ is the field of the rotation vectors of the body points. In the linear approximation, the expressions for the angular velocity vector and the angular strain tensor are

$$\hat{\boldsymbol{\omega}} = \frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t}, \qquad \hat{\boldsymbol{\vartheta}} = \nabla\hat{\boldsymbol{\theta}}. \tag{144}$$

In view of the above-stated simplifications, the balance equations of linear momentum and angular momentum take the form

$$\frac{\mathrm{d}\hat{\mathbf{v}}}{\mathrm{d}t} = \mathbf{0}, \qquad \nabla \cdot \hat{\mathbf{T}} = \hat{\rho}\hat{J}\,\frac{\mathrm{d}^2\hat{\boldsymbol{\theta}}}{\mathrm{d}t^2}. \tag{145.1,2}$$

The energy balance equation is formulated as

$$\frac{\mathrm{d}(\hat{\rho}\hat{U})}{\mathrm{d}t} = \hat{\mathbf{T}}^T \cdot\cdot \frac{\mathrm{d}\hat{\boldsymbol{\vartheta}}}{\mathrm{d}t}. \tag{146}$$

In view of the fact that in the linear theory assumption (140) takes the form

$$\hat{\mathbf{T}} = \hat{T}\mathbf{E} - \hat{\mathbf{M}} \times \mathbf{E}, \tag{147}$$

the energy balance equation (146) reduces to the form

$$\frac{\mathrm{d}(\hat{\rho}\hat{U})}{\mathrm{d}t} = \hat{T}\,\frac{\mathrm{d}\hat{\vartheta}}{\mathrm{d}t} + \hat{\mathbf{M}} \cdot \frac{\mathrm{d}\hat{\boldsymbol{\psi}}}{\mathrm{d}t}, \qquad \hat{\vartheta} = \nabla \cdot \hat{\boldsymbol{\theta}}, \qquad \hat{\boldsymbol{\psi}} = \nabla \times \hat{\boldsymbol{\theta}}. \tag{148.1–3}$$

By specifying the internal energy density in the simplest form, we obtain the constitutive equations as follows:

$$\hat{\rho}\hat{U} = \frac{1}{2}\hat{k}_{\mathrm{v}}\hat{\vartheta}^2 + \frac{1}{2}\hat{k}_{\mathrm{s}}\,\hat{\boldsymbol{\psi}} \cdot \hat{\boldsymbol{\psi}} \qquad \Rightarrow \qquad \hat{T} = \hat{k}_{\mathrm{v}}\hat{\vartheta}, \qquad \hat{\mathbf{M}} = \hat{k}_{\mathrm{s}}\,\hat{\boldsymbol{\psi}}. \tag{149}$$

Substituting Eqs. (147), (149) into Eq. (145.2), we obtain

$$\hat{k}_{\mathrm{v}}\,\nabla\hat{\vartheta} - \hat{k}_{\mathrm{s}}\,\nabla \times \hat{\boldsymbol{\psi}} = \hat{\rho}\hat{J}\,\frac{\mathrm{d}^2\hat{\boldsymbol{\theta}}}{\mathrm{d}t^2}. \tag{150}$$

It is easy to show that taking into account Eqs. (148.2,3) one can transform Eq. (150) to the two wave equations

$$\Delta\hat{\vartheta} - \frac{1}{c_{\mathrm{v}}^2}\,\frac{\mathrm{d}^2\hat{\vartheta}}{\mathrm{d}t^2} = 0, \qquad c_{\mathrm{v}}^2 = \frac{\hat{k}_{\mathrm{v}}}{\hat{\rho}\hat{J}}, \qquad \Delta\hat{\boldsymbol{\psi}} - \frac{1}{c_{\mathrm{s}}^2}\,\frac{\mathrm{d}^2\hat{\boldsymbol{\psi}}}{\mathrm{d}t^2} = \mathbf{0}, \qquad c_{\mathrm{s}}^2 = \frac{\hat{k}_{\mathrm{s}}}{\hat{\rho}\hat{J}}. \tag{151.1–4}$$

Equations (151.1) and (151.3) describe the propagation of the “longitudinal” and “transverse” waves in the ether, respectively.

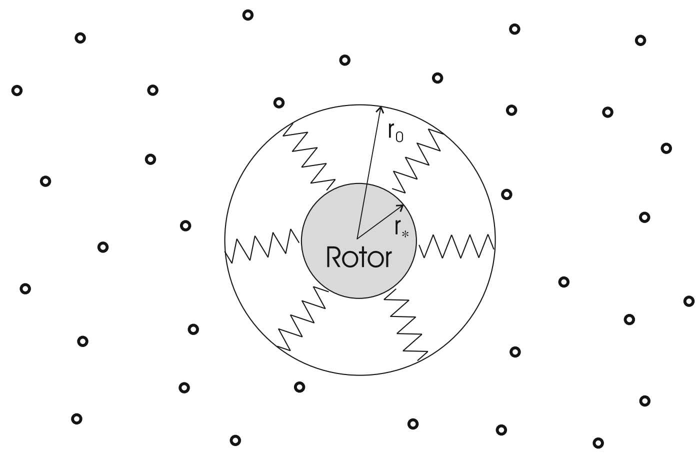

Fig. 6 The simplified model of the interaction between the rotor and the ether

Appendix H: The approximate solution of the model problem

Now we consider the mechanical system that models the interaction of a rotor with the ether. The system consists of the rotor and the ether occupying the whole infinite space—see Fig. 6. The rotor is assumed to be a spherically symmetric rigid body of radius r_* with mass m and inertia moment mJ. The detailed description of the nonlinear model of the ether is contained in Appendix F. The ether layer $r_* < r < r_0$ is considered to be an elastic continuum, capable of large angular deformations and inertialess with respect to the rotational degrees of freedom. The stress strain state of the continuum is described by Eq. (92). The kinematic boundary condition on the contact surface between the rotor and the ether has the form of Eq. (97). The ether located outside the layer $r_* < r < r_0$ is modeled by the inertial elastic continuum that is described by two linear wave equations—see Appendix G. In addition, we suppose that the waves propagating along the angular coordinates can be disregarded. Also, we assume the boundary between the inertial part of ether and the inertialess part of ether to be rotating as a rigid body. At the initial moment of time, the ether is not disturbed.

Since we consider only the waves propagating in the radial direction, the wave equations (93) can be rewritten in the form

$$\frac{\partial^2(r\hat{\vartheta})}{\partial r^2} = \frac{1}{c_{\mathrm{v}}^2}\frac{\partial^2(r\hat{\vartheta})}{\partial t^2}, \qquad \frac{\partial^2(r\hat{\boldsymbol{\psi}})}{\partial r^2} = \frac{1}{c_{\mathrm{s}}^2}\frac{\partial^2(r\hat{\boldsymbol{\psi}})}{\partial t^2}. \tag{152}$$

In view of the absence of sources at infinity, the solutions of Eq. (152) are

$$\hat{\vartheta} = \begin{cases} 0, & r > r_0 + c_{\mathrm{v}}t, \\ \dfrac{1}{r}\, f(r - c_{\mathrm{v}}t), & r_0 \le r \le r_0 + c_{\mathrm{v}}t, \end{cases} \qquad \hat{\boldsymbol{\psi}} = \begin{cases} 0, & r > r_0 + c_{\mathrm{s}}t, \\ \dfrac{1}{r}\, \mathbf{h}(r - c_{\mathrm{s}}t), & r_0 \le r \le r_0 + c_{\mathrm{s}}t \end{cases} \tag{153}$$

where f and $\mathbf{h}$ are arbitrary functions of their arguments. These functions are determined by the boundary conditions on the surface $r = r_0$. For the purposes of this study, the specific forms of functions f and $\mathbf{h}$ are not required.

It is easy to show that expressions (153) satisfy the relations

$$\frac{\partial\hat{\vartheta}}{\partial r} = -\frac{1}{c_{\mathrm{v}}}\frac{\partial\hat{\vartheta}}{\partial t} - \frac{1}{r}\,\hat{\vartheta}, \qquad \frac{\partial\hat{\boldsymbol{\psi}}}{\partial r} = -\frac{1}{c_{\mathrm{s}}}\frac{\partial\hat{\boldsymbol{\psi}}}{\partial t} - \frac{1}{r}\,\hat{\boldsymbol{\psi}}. \tag{154}$$

We note that $\hat{\vartheta}$ and $\hat{\boldsymbol{\psi}}$ depend not only on the arguments r and t. They also can depend on the angular coordinates of the spherical coordinate system. If the wave processes along the angular coordinates are excluded from consideration, it does not mean that the dependence of $\hat{\vartheta}$ and $\hat{\boldsymbol{\psi}}$ on these coordinates is also excluded from consideration. The dependence of $\hat{\vartheta}$ and $\hat{\boldsymbol{\psi}}$ on the angular coordinates as parameters may occur due to the boundary conditions. However, the wave process along the coordinate r is assumed to be sufficiently shortwave in order that the terms containing the derivatives with respect to the angular coordinates can be considered as negligible compared to the terms containing the derivatives with respect to r. In other words, the approximations

$$\nabla\hat{\vartheta} \approx \mathbf{e}_r \frac{\partial\hat{\vartheta}}{\partial r}, \qquad \nabla\times\hat{\boldsymbol{\psi}} \approx \mathbf{e}_r \times \frac{\partial\hat{\boldsymbol{\psi}}}{\partial r} \tag{155}$$

are assumed to be acceptable. In view of expression (95) for the moment stress tensor and Eqs. (154) (155), we obtain

$$\begin{aligned}\nabla\cdot\hat{\mathbf{T}} &\approx k_{\mathrm{v}}\mathbf{e}_r\frac{\partial\hat{\vartheta}}{\partial r} - k_{\mathrm{s}}\mathbf{e}_r\times\frac{\partial\hat{\boldsymbol{\psi}}}{\partial r}\\ &= -\left(\mathbf{C}_w^{-1}\frac{\partial}{\partial t}+\mathbf{E}\frac{1}{r}\right)\cdot\left(k_{\mathrm{v}}\mathbf{e}_r\hat{\vartheta}-k_{\mathrm{s}}\mathbf{e}_r\times\hat{\boldsymbol{\psi}}\right) = -\left(\mathbf{C}_w^{-1}\frac{\partial}{\partial t}+\mathbf{E}\frac{1}{r}\right)\cdot(\mathbf{e}_r\cdot\hat{\mathbf{T}})\end{aligned} \tag{156}$$

where

$$\mathbf{C}_w = c_{\mathrm{v}}\mathbf{e}_r\mathbf{e}_r + c_{\mathrm{s}}(\mathbf{E}-\mathbf{e}_r\mathbf{e}_r). \tag{157}$$

Let us write the angular momentum balance equation for the ether taking into account Eq. (156). As a result, we have

$$-\left(\mathbf{C}_w^{-1}\frac{\partial}{\partial t}+\mathbf{E}\frac{1}{r}\right)\cdot(\mathbf{e}_r\cdot\hat{\mathbf{T}}) = \hat{\rho}\hat{J}\,\frac{\partial^2\hat{\boldsymbol{\theta}}}{\partial t^2}. \tag{158}$$

By solving Eq. (158) with respect to $\mathbf{e}_r\cdot\hat{\mathbf{T}}$ and taking into account the fact that at the initial moment of time the ether is at rest, we obtain

$$\mathbf{e}_r\cdot\hat{\mathbf{T}} = -\hat{\rho}\hat{J}\left(\mathbf{C}_w\cdot\frac{\partial\hat{\boldsymbol{\theta}}}{\partial t} - r^{-1}\mathbf{C}_w^2\cdot\hat{\boldsymbol{\theta}} + r^{-2}\mathbf{C}_w^3\cdot\int_0^t e^{\mathbf{C}_w r^{-1}(\tau-t)}\cdot\hat{\boldsymbol{\theta}}(\tau)\,\mathrm{d}\tau\right). \tag{159}$$

Now we calculate the total moment $\hat{\mathbf{T}}_\sigma \equiv \int_{\sigma_0}\mathbf{e}_r\cdot\hat{\mathbf{T}}\Big|_{r=r_0}\mathrm{d}\sigma$ acting on the spherical surface of radius r_0. We note that vector $\hat{\boldsymbol{\theta}}$ depends not only on t and r but also on the angles of the spherical coordinate system. In accordance with the accepted boundary conditions, the spherical surface $r = r_0$ rotates as a rigid body. This means that $\hat{\boldsymbol{\theta}}_0 = \hat{\boldsymbol{\theta}}\Big|_{r=r_0}$ depends only on time. Because of this, at $r = r_0$ the surface integral of the function $\mathbf{e}_r\cdot\hat{\mathbf{T}}$ given by Eq. (159) can be easily calculated. As a result, after a series of transformations, we obtain the following expression for the moment vector $\hat{\mathbf{T}}_\sigma$:

$$\begin{aligned}\hat{\mathbf{T}}_\sigma = &-\frac{4\pi r_0^2}{3}\left(\frac{k_{\mathrm{v}}}{c_{\mathrm{v}}}+\frac{2k_{\mathrm{s}}}{c_{\mathrm{s}}}\right)\frac{\mathrm{d}\hat{\boldsymbol{\theta}}_0}{\mathrm{d}t} + \frac{4\pi r_0}{3}(k_{\mathrm{v}}+2k_{\mathrm{s}})\,\hat{\boldsymbol{\theta}}_0(t)\\ &-\frac{4\pi}{3}\int_0^t\left[c_{\mathrm{v}}k_{\mathrm{v}}e^{c_{\mathrm{v}}r_0^{-1}(\tau-t)}+2c_{\mathrm{s}}k_{\mathrm{s}}e^{c_{\mathrm{s}}r_0^{-1}(\tau-t)}\right]\hat{\boldsymbol{\theta}}_0(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{160}$$

Next, we consider the part of the ether occupying the region $r_* < r < r_0$. This part of the ether is assumed to be inertialess. Consequently, the total moment acting on this part of the ether must be equal to zero. The inertialess part of the ether undergoes the action of the rotor and the inertial part of the ether. Hence,

$$-\int_{\sigma_*}\mathbf{e}_r\cdot\hat{\mathbf{T}}\Big|_{r=r_*}\mathrm{d}\sigma + \int_{\sigma_0}\mathbf{e}_r\cdot\hat{\mathbf{T}}\Big|_{r=r_0}\mathrm{d}\sigma = \mathbf{0} \quad\Rightarrow\quad \hat{\mathbf{T}}_\sigma = \int_{\sigma_*}\mathbf{e}_r\cdot\hat{\mathbf{T}}\Big|_{r=r_*}\mathrm{d}\sigma. \tag{161}$$

Thus, in fact, the rotor undergoes the action of the moment vector $\hat{\mathbf{T}}_\sigma$ determined by Eq. (160). Hence, the angular momentum balance equation for the rotor can be written as

$$mJ\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \hat{\mathbf{T}}_\sigma \tag{162}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rotor. The system of equations (160), (162) is not closed, because it contains three unknown vectors, namely $\hat{\boldsymbol{\theta}}_0$, $\hat{\mathbf{T}}_\sigma$, and $\boldsymbol{\omega}$. In order to close this system, it is necessary to take into account the elastic properties of the ether occupying the region $r_* < r < r_0$, and also to use the kinematic relation between the angular velocity vector $\boldsymbol{\omega}$ and the corresponding rotation vector $\boldsymbol{\theta}$ (the rotation vector is defined as the product of the rotation angle on the unit vector that is directed along the rotation axis), namely

$$\boldsymbol{\omega} = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}. \tag{163}$$

Here $\mathbf{Z}(\boldsymbol{\theta})$ is the Zhilin tensor [64], which can be expressed in terms of the rotation vector $\boldsymbol{\theta}$ as

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \quad \mathbf{R}(\boldsymbol{\theta}) = \boldsymbol{\theta} \times \mathbf{E}, \quad g(\theta) = \frac{\theta\,\sin\theta}{2(1-\cos\theta)}, \quad \theta = |\boldsymbol{\theta}|. \tag{164}$$

In order to take full account of the elastic properties of the part of the ether occupying the region $r_* < r < r_0$, it is necessary to solve the problem (92). This problem is very complicated. If it can be solved analytically, it is only in the case of the special form of function $\hat{U}\big(\hat{\Theta}_\mathrm{e},\ \hat{\boldsymbol{\Psi}}_\mathrm{e}\big)$. It is unlikely that such $\hat{U}\big(\hat{\Theta}_\mathrm{e},\ \hat{\boldsymbol{\Psi}}_\mathrm{e}\big)$ has some physical meaning. However, for our purposes it is not necessary to know all the functions that characterize the stress–strain state of the ether. We need to know only the total elastic moment $\hat{\mathbf{T}}_\sigma$ acting on the rotor. Taking into account the peculiarity of the boundary conditions, i.e., the fact that the outer boundary of the region $r_* < r < r_0$ rotates as a rigid body and the inner boundary contacting with the rotor also rotates as a rigid body, it can be argued that the elastic moment $\hat{\mathbf{T}}_\sigma$ is a function of rotation vector $\boldsymbol{\gamma}$ characterizing the relative rotation of the inner and outer boundaries. Vector $\boldsymbol{\gamma}$ is defined by the relation

$$\mathbf{P}(\boldsymbol{\gamma}) = \mathbf{P}(\boldsymbol{\theta}) \cdot \mathbf{P}^T(\hat{\boldsymbol{\theta}}_0). \tag{165}$$

Therefore, instead of specifying the mass density of internal energy $\hat{U}\big(\hat{\Theta}_\mathrm{e},\ \hat{\boldsymbol{\Psi}}_\mathrm{e}\big)$ and solving the problem (92), we can directly specify a potential $\Pi(\boldsymbol{\gamma})$ corresponding to the elastic moment that models the influence of the ether occupying the region $r_* < r < r_0$ on the rotor. As shown in [64], the elastic moment corresponding to the potential $\Pi(\boldsymbol{\gamma})$ is calculated by the formula

$$\hat{\mathbf{T}}_\sigma = -\mathbf{Z}^T(\boldsymbol{\gamma}) \cdot \frac{\mathrm{d}\Pi(\boldsymbol{\gamma})}{d\boldsymbol{\gamma}}. \tag{166}$$

In the case of an isotropic continuum (the ether is considered to be isotropic), the elastic potential depends on the square modulus of $\boldsymbol{\gamma}$ rather than vector $\boldsymbol{\gamma}$ itself, i.e., the elastic potential has the form of $\Pi(\boldsymbol{\gamma}) = \Pi(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})$. In this case, the expression for the elastic moment (166) is significantly simplified and takes the form

$$\hat{\mathbf{T}}_\sigma = -2\,\frac{\mathrm{d}\Pi(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})}{d(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})}\,\boldsymbol{\gamma}. \tag{167}$$

Since the rotation vector $\hat{\boldsymbol{\theta}}_0$ is considered to be small, i.e., $\mathbf{P}(\hat{\boldsymbol{\theta}}_0) \approx \mathbf{E} + \hat{\boldsymbol{\theta}}_0 \times \mathbf{E}$, the relative rotation vector $\boldsymbol{\gamma}$ can be represented by the approximate expression

$$\boldsymbol{\gamma} = \boldsymbol{\theta} - \mathbf{Z}^T(\boldsymbol{\theta}) \cdot \hat{\boldsymbol{\theta}}_0. \tag{168}$$

Substituting Eq. (168) into Eq. (167), after a series of asymptotic transformations retaining the terms linear with respect to $\hat{\boldsymbol{\theta}}_0$, we obtain

$$\hat{\mathbf{T}}_\sigma = -c(\theta^2)\boldsymbol{\theta} + \Big[c(\theta^2)\mathbf{Z}^T(\boldsymbol{\theta}) + k(\theta^2)\boldsymbol{\theta}\boldsymbol{\theta}\Big] \cdot \hat{\boldsymbol{\theta}}_0 \tag{169}$$

where the scalar functions $c(\theta^2)$ and $k(\theta^2)$ have the form

$$c(\theta^2) = -2\,\left.\frac{\mathrm{d}\Pi(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})}{d(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})}\right|_{\boldsymbol{\gamma}=\boldsymbol{\theta}}, \qquad k(\theta^2) = 4\,\left.\frac{\mathrm{d}^2\Pi(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})}{d(\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})^2}\right|_{\boldsymbol{\gamma}=\boldsymbol{\theta}}. \tag{170}$$

Equations (160), (162), (163), (169) are a closed system of equations for vectors $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}_0$, and $\hat{\mathbf{T}}_\sigma$. Next we eliminate vector $\hat{\boldsymbol{\theta}}_0$ from this system. In order to do this, first we substitute Eq. (160) into Eq. (162), and after simple transformations, we obtain the differential equation relating $\hat{\boldsymbol{\theta}}_0$ and $\boldsymbol{\omega}$:

$$\frac{\mathrm{d}^2\hat{\boldsymbol{\theta}}_0}{\mathrm{d}t^2} + \kappa\,\frac{\mathrm{d}\hat{\boldsymbol{\theta}}_0}{\mathrm{d}t} = -m\chi\left(\frac{\mathrm{d}^2\boldsymbol{\omega}}{\mathrm{d}t^2} + \frac{c_\mathrm{v}+c_\mathrm{s}}{r_0}\,\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \frac{c_\mathrm{v}c_\mathrm{s}}{r_0^2}\,\boldsymbol{\omega}\right) \tag{171}$$

where the parameters κ and χ are determined by the formulas

$$\kappa = \frac{3c_\mathrm{v}c_\mathrm{s}}{r_0(c_\mathrm{v}+2c_\mathrm{s})}, \qquad \chi = \frac{3J}{4\pi r_0^2\hat{\rho}\hat{J}(c_\mathrm{v}+2c_\mathrm{s})}. \tag{172}$$

In view of Eq. (163), the solution of Eq. (171) with respect to $\hat{\boldsymbol{\theta}}_0$ is

$$\hat{\boldsymbol{\theta}}_0 = -m\chi\left(\boldsymbol{\omega} + p\int_0^{\boldsymbol{\theta}} \mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}})\cdot\mathrm{d}\tilde{\boldsymbol{\theta}} - q\int_0^t\left[\int_0^t e^{\kappa(\tau-t)}\mathbf{Z}^{-1}\big(\boldsymbol{\theta}(\tau)\big)\cdot\mathrm{d}\boldsymbol{\theta}(\tau)\right]\mathrm{d}t\right) \tag{173}$$

where

$$p = \frac{c_\mathrm{v}^2+2c_\mathrm{s}^2}{r_0(c_\mathrm{v}+2c_\mathrm{s})}, \qquad q = \frac{2c_\mathrm{v}c_\mathrm{s}(c_\mathrm{v}-c_\mathrm{s})^2}{r_0^2(c_\mathrm{v}+2c_\mathrm{s})^2}. \tag{174}$$

Substituting Eq. (173) into Eq. (169), we obtain the final expression for the moment acting on the rotor, namely

$$\begin{aligned}\hat{\mathbf{T}}_\sigma &= -c(\theta^2)\boldsymbol{\theta} - m\chi\left[c(\theta^2)\mathbf{Z}^T(\boldsymbol{\theta}) + k(\theta^2)\boldsymbol{\theta}\boldsymbol{\theta}\right]\cdot\boldsymbol{\omega}\\ &\quad -m\chi\left[c(\theta^2)\mathbf{Z}^T(\boldsymbol{\theta}) + k(\theta^2)\boldsymbol{\theta}\boldsymbol{\theta}\right]\cdot\left(p\int_0^{\boldsymbol{\theta}}\mathbf{Z}^{-1}(\tilde{\boldsymbol{\theta}})\cdot\mathrm{d}\tilde{\boldsymbol{\theta}} - q\int_0^t\left[\int_0^t e^{\kappa(\tau-t)}\mathbf{Z}^{-1}\big(\boldsymbol{\theta}(\tau)\big)\cdot\mathrm{d}\boldsymbol{\theta}(\tau)\right]\mathrm{d}t\right).\end{aligned} \tag{175}$$

Equations (162), (163), (175) are a closed system of equations for vectors $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, and $\hat{\mathbf{T}}_\sigma$. The influence of the ambient ether on the rotor is taken into account integrally in this system of equations.

Appendix I: The properties of tensor $\mathbf{Z}(\boldsymbol{\theta})$ and tensor $\boldsymbol{\chi}(\boldsymbol{\theta})$

In [64], it is shown that the Zhilin tensor

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\,\mathbf{R} + \frac{1-g(\theta)}{\theta^2}\,\mathbf{R}^2, \qquad g(\theta) = \frac{\theta\,\sin\theta}{2(1-\cos\theta)}, \qquad \theta = |\boldsymbol{\theta}| \tag{176}$$

possesses the following properties:

$$\mathbf{Z}^T(\boldsymbol{\theta}) = \mathbf{P}(\boldsymbol{\theta})\cdot\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{Z}(\boldsymbol{\theta})\cdot\mathbf{P}(\boldsymbol{\theta}), \qquad \mathbf{Z}(\boldsymbol{\theta})\cdot\boldsymbol{\theta} = \boldsymbol{\theta}, \qquad \boldsymbol{\theta}\cdot\mathbf{Z}(\boldsymbol{\theta}) = \boldsymbol{\theta}. \tag{177}$$

In [64], it is also shown that the determinant of tensor $\mathbf{Z}(\boldsymbol{\theta})$ is not zero (except the singular points of $\theta = 2\pi k$ where k is an integer), and the inverse tensor has the form

$$\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \mathbf{E} + \frac{1-\cos\theta}{\theta^2}\,\mathbf{R} + \frac{\theta-\sin\theta}{\theta^3}\,\mathbf{R}^2. \tag{178}$$

If the rotation about a fixed axis is ignored, without loss of generality, we can assume that $0 \le \theta < \pi$. The inverse tensor exists for any θ within this range. The inverse tensor possesses properties similar to (177), namely

$$\mathbf{Z}^{-T}(\boldsymbol{\theta}) = \mathbf{P}^T(\boldsymbol{\theta})\cdot\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \mathbf{Z}^{-1}(\boldsymbol{\theta})\cdot\mathbf{P}^T(\boldsymbol{\theta}), \qquad \mathbf{Z}^{-1}(\boldsymbol{\theta})\cdot\boldsymbol{\theta} = \boldsymbol{\theta}, \qquad \boldsymbol{\theta}\cdot\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \boldsymbol{\theta}. \tag{179}$$

Taking into account the formula $\mathbf{R} = \boldsymbol{\theta}\times\mathbf{E}$, it is easy to show that from Eq. (176) it follows that

$$\boldsymbol{\theta} = \mathbf{Z}_\times(\boldsymbol{\theta}), \qquad \theta = |\mathbf{Z}_\times(\boldsymbol{\theta})|\,. \tag{180}$$

We note that Eq. (177) allows us to express the rotation tensor in terms of the Zhilin tensor, namely

$$\mathbf{P}(\boldsymbol{\theta}) = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \mathbf{Z}^{T}(\boldsymbol{\theta}). \tag{181}$$

Let us now discuss the properties of tensor $\boldsymbol{\chi}(\boldsymbol{\theta})$ defined by the formula

$$\boldsymbol{\chi}(\boldsymbol{\theta}) = \chi \left[c(\theta^2)\mathbf{Z}^{T}(\boldsymbol{\theta}) + k(\theta^2)\boldsymbol{\theta}\boldsymbol{\theta} \right]. \tag{182}$$

The determinant of tensor $\boldsymbol{\chi}(\boldsymbol{\theta})$ is related to the determinant of tensor $\mathbf{Z}(\boldsymbol{\theta})$ by

$$\mathrm{Det}\, \boldsymbol{\chi}(\boldsymbol{\theta}) = \chi^3 \left[c(\theta^2) + \theta^2 k(\theta^2) \right] c^2(\theta^2)\ \mathrm{Det}\, \mathbf{Z}(\boldsymbol{\theta}). \tag{183}$$

If $\mathrm{Det}\, \boldsymbol{\chi}(\boldsymbol{\theta}) \neq 0$, there exists the inverse tensor, which has the form

$$\boldsymbol{\chi}^{-1}(\boldsymbol{\theta}) = \frac{1}{\chi} \left[\frac{1}{c(\theta^2)} \mathbf{Z}^{-T}(\boldsymbol{\theta}) - \frac{k(\theta^2)}{c(\theta^2)\left[c(\theta^2) + \theta^2 k(\theta^2)\right]} \boldsymbol{\theta}\boldsymbol{\theta} \right]. \tag{184}$$

Taking into account Eqs. (177), (179), it is easy to show that tensors $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$ possess the following properties:

$$\boldsymbol{\chi}^{T}(\boldsymbol{\theta}) = \mathbf{P}^{T}(\boldsymbol{\theta}) \cdot \boldsymbol{\chi}(\boldsymbol{\theta}) = \boldsymbol{\chi}(\boldsymbol{\theta}) \cdot \mathbf{P}^{T}(\boldsymbol{\theta}), \qquad \boldsymbol{\chi}^{-T}(\boldsymbol{\theta}) = \mathbf{P}(\boldsymbol{\theta}) \cdot \boldsymbol{\chi}^{-1}(\boldsymbol{\theta}) = \boldsymbol{\chi}^{-1}(\boldsymbol{\theta}) \cdot \mathbf{P}(\boldsymbol{\theta}). \tag{185}$$

Appendix J: The asymptotic analysis of the model problem solution

When deriving formula (101) for moment $\hat{\mathbf{T}}_{\sigma}$, we used a number of assumptions (see Appendix H), two of which are of an asymptotic nature. They impose certain restrictions on the ratio of the problem parameters. Now we carry out the appropriate asymptotic analysis. We suppose that the dynamic process with characteristic frequency ν is considered, i.e., $\boldsymbol{\omega} \sim \nu \boldsymbol{\theta}$.

When stating the problem, we supposed that the part of ether occupying the region $r_* < r < r_0$ can be considered to be inertialess. Such a simplification is acceptable, if the characteristic frequency of the dynamic process ν is small in comparison with the first eigenfrequency of this layer of the ether. Now we carry out non-rigorous asymptotic considerations allowing us to estimate the order of the first eigenfrequencies. To describe free oscillations of the ether layer $r_* < r < r_0$, we will use the linear equations (152). In addition, we assume the following conditions at the boundaries of the region:

$$\hat{\vartheta}\Big|_{r=r_*} = 0, \qquad \hat{\vartheta}\Big|_{r=r_0} = 0, \qquad \hat{\boldsymbol{\psi}}\Big|_{r=r_*} = \mathbf{0}, \qquad \hat{\boldsymbol{\psi}}\Big|_{r=r_0} = \mathbf{0}. \tag{186}$$

The solutions of Eq. (152) satisfying the boundary conditions (186) are

$$\hat{\vartheta}_n = \frac{A_n}{r} \sin\!\left(\frac{\pi n}{r_0 - r_*}(r - r_*)\right) \sin\!\left(\frac{\pi n c_{\mathrm{v}}}{r_0 - r_*}\, t\right), \quad \hat{\boldsymbol{\psi}}_n = \frac{\mathbf{B}_n}{r} \sin\!\left(\frac{\pi n}{r_0 - r_*}(r - r_*)\right) \sin\!\left(\frac{\pi n c_{\mathrm{s}}}{r_0 - r_*}\, t\right). \tag{187}$$

Certainly, ignoring the nonlinear effects and using some boundary conditions in place of the other ones, we make a significant error in the determination of eigenfrequencies, but unlikely this may affect the order of their magnitudes. Thus, we can conclude that the frequency ν of the dynamic process should be much smaller than the first eigenfrequencies that are determined by the solution (187). Consequently, the problem parameters must satisfy the inequalities

$$r_0 - r_* \ll \frac{\pi c_{\mathrm{v}}}{\nu}, \qquad r_0 - r_* \ll \frac{\pi c_{\mathrm{s}}}{\nu}. \tag{188}$$

When stating the problem, we supposed that the part of the ether occupying the region $r \geq r_0$ can be described by the linear theory. This assumption means that the rotation vector $\hat{\boldsymbol{\theta}}_0$ that is given by Eq. (173) has to be small.

At first, we assume that $c_{\mathrm{v}} \sim c_{\mathrm{s}} \sim |c_{\mathrm{v}} - c_{\mathrm{s}}|$. Let us consider the different variants:

(i) All the terms in Eq. (173) have the same order. This is possible if $r_0 \sim \dfrac{c_v}{\nu}$ and $r_0 \sim \dfrac{c_s}{\nu}$. These conditions do not contradict Eq. (188) if $\dfrac{r_0 - r_*}{\pi r_0} \ll 1$. The smallness of the rotation vector $\hat{\boldsymbol{\theta}}_0$ is ensured by the condition $\nu m \chi \ll 1$. In the present case, this condition takes the form $\dfrac{mJ}{\hat{m}\hat{J}} \ll 1$ where $\hat{m} = \dfrac{4\pi r_0^3 \hat{\rho}}{3}$.

(ii) Let $r_0 \gg \dfrac{c_v}{\nu}$ and $r_0 \gg \dfrac{c_s}{\nu}$. In this case, the third term in Eq. (173) is much smaller than the second one, and the second term is much smaller than the first one. Contradiction with Eq. (188) does not occur if $\dfrac{r_0 - r_*}{\pi r_0} \ll 1$ while $r_0 - r_* \ll \dfrac{\pi c_v}{\nu} \ll \pi r_0$ and $r_0 - r_* \ll \dfrac{\pi c_s}{\nu} \ll \pi r_0$. The smallness of the rotation vector $\hat{\boldsymbol{\theta}}_0$ is ensured by the condition $\nu m \chi \ll 1$, as in the previous case. However, in this case the latter condition can be reduced to a slightly different form, namely $\dfrac{mJ}{\hat{m}\hat{J}} \ll \dfrac{c_v + 2c_s}{\nu r_0} \ll 1$.

(iii) Let $r_0 \ll \dfrac{c_v}{\nu}$ and $r_0 \ll \dfrac{c_s}{\nu}$. In this case, the first term in Eq. (173) is much smaller than the second one, and the second term is much smaller than the third one. Conditions (188) are satisfied automatically. The smallness of the rotation vector $\hat{\boldsymbol{\theta}}_0$ is ensured by the condition $\dfrac{m\chi q}{\nu} \ll 1$. In the present case, this condition takes the form $\dfrac{mJ}{\hat{m}\hat{J}} \ll \dfrac{\nu r_0 (c_v + 2c_s)^3}{2c_v c_s (c_v - c_s)^2} \ll 1$.

Thus, for any ratio of the parameters, the condition $\dfrac{mJ}{\hat{m}\hat{J}} \ll 1$ is a necessary smallness condition for the rotation vector $\hat{\boldsymbol{\theta}}_0$. Since the parameters $\hat{m}$, $\hat{J}$ are the inertial characteristics of the ether and the parameters m, J are the inertial characteristics of the rotor, it is clear that the above condition does not contradict common sense only if r_0 is sufficiently large. This means that $r_0 \gg r_*$. Consequently, only the last of the three options is realistic. Since we consider the case of $c_v \sim c_s \sim |c_v - c_s|$, the main result can be rewritten as

$$\frac{mJ}{\hat{m}\hat{J}} \ll \frac{\nu r_0}{c_v} \sim \frac{\nu r_0}{c_s} \ll 1. \tag{189}$$

Now we assume that $c_v \gg c_s$ or $c_s \gg c_v$. After analyzing the options similar to those described above, we conclude that

$$c_v \gg c_s: \quad \frac{mJ}{\hat{m}\hat{J}} \ll \frac{\nu r_0}{c_s} \ll 1; \qquad c_s \gg c_v: \quad \frac{mJ}{\hat{m}\hat{J}} \ll \frac{\nu r_0}{c_v} \ll 1. \tag{190}$$

In this case, as well as in the case of $c_v \sim c_s \sim |c_v - c_s|$, the first term in Eq. (173) is much smaller than the second one, and the second term is much smaller than the third one.

Next we assume that $c_v \approx c_s$. After analyzing the options similar to those considered above, we obtain

$$\frac{mJ}{\hat{m}\hat{J}} \ll 1, \qquad \frac{\nu r_0}{c_v} \approx \frac{\nu r_0}{c_s} \ll 1. \tag{191}$$

In this case, the second term in Eq. (173) is much larger than the first and third terms. The smallness of the third term is ensured by the smallness of $(c_v - c_s)^2$ compared to $c_v c_s$.

We note that the radius r_0, which appears in the above inequalities, is not a physical parameter, because it is related to the imagined spherical surface in the ether.

Appendix K: The representation of tensors $\mathbf{Z}$, $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$ in terms of the strain characteristics

According to Eqs. (180), (182), (184), tensors $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$ are related to the tensor $\mathbf{Z}$ by the formulas

$$\begin{aligned} \boldsymbol{\chi} &= \chi \left[c(\theta^2)\, \mathbf{Z}^T + k(\theta^2)\, \mathbf{Z}_\times \mathbf{Z}_\times \right], \qquad \theta = |\mathbf{Z}_\times|, \\ \boldsymbol{\chi}^{-1} &= \frac{1}{\chi} \left[\frac{1}{c(\theta^2)} \mathbf{Z}^{-T} - \frac{k(\theta^2)}{c(\theta^2)\left[c(\theta^2) + \theta^2 k(\theta^2)\right]} \mathbf{Z}_\times \mathbf{Z}_\times \right]. \end{aligned} \tag{192}$$

Therefore, the problem of representation of tensors $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{-1}$ in terms of the strain characteristics is reduced to the representation of tensor $\mathbf{Z}$ in terms of these strain characteristics.

It is known that the relation between the rotation and the angular velocity vector can be written in the following equivalent forms [64]:

$$\frac{\delta \mathbf{P}}{\delta t} = \boldsymbol{\omega} \times \mathbf{P}, \qquad \frac{\delta \mathbf{P}}{\delta t} = \mathbf{P} \times \boldsymbol{\Omega} \qquad \Leftrightarrow \qquad \frac{\delta \boldsymbol{\theta}}{\delta t} = \boldsymbol{\omega} \cdot \mathbf{Z}^T, \qquad \frac{\delta \boldsymbol{\theta}}{\delta t} = \boldsymbol{\Omega} \cdot \mathbf{Z}. \tag{193}$$

Similarly, the relation between the rotation and the angular strain measure $\boldsymbol{\Theta}$ can be written in two equivalent forms

$$\nabla \mathbf{P} = \boldsymbol{\Theta} \times \mathbf{P} \qquad \Leftrightarrow \qquad \nabla \boldsymbol{\theta} = \boldsymbol{\Theta} \cdot \mathbf{Z}^T. \tag{194}$$

Since, when tensor $\boldsymbol{\Theta}$ is a non-degenerate tensor, the second relation in Eq. (194) allows us to express tensor $\mathbf{Z}^T$ in terms of the strain characteristics of the continuum: $\mathbf{Z}^T = \boldsymbol{\Theta}^{-1} \cdot \nabla \boldsymbol{\theta}$. In view of Eqs. (52), (67), (177), the last formula can be rewritten in the form

$$\mathbf{Z} = \boldsymbol{\Theta}_\rho^{-1} \cdot \mathbf{g}_\rho^{-1} \cdot \nabla \boldsymbol{\theta}. \tag{195}$$

Taking the gradient operator of both sides of Eq. (193) and performing the following transformations:

$$\nabla \frac{\delta \boldsymbol{\theta}}{\delta t} = \frac{\delta (\nabla \boldsymbol{\theta})}{\delta t} + (\nabla \mathbf{v}) \cdot \nabla \boldsymbol{\theta} = \frac{\delta (\nabla \boldsymbol{\theta})}{\delta t} + \mathbf{g} \cdot \frac{\delta \mathbf{g}^{-1}}{\delta t} \cdot \nabla \boldsymbol{\theta} = \mathbf{g} \cdot \frac{\delta \left(\mathbf{g}^{-1} \cdot \nabla \boldsymbol{\theta}\right)}{\delta t} = \mathbf{g}_\rho \cdot \frac{\delta \left(\mathbf{g}_\rho^{-1} \cdot \nabla \boldsymbol{\theta}\right)}{\delta t},$$

in view of Eq. (195), we obtain

$$\mathbf{g}_\rho \cdot \frac{\delta \left(\mathbf{g}_\rho^{-1} \cdot \nabla \boldsymbol{\theta}\right)}{\delta t} = \nabla \left(\boldsymbol{\Omega} \cdot \boldsymbol{\Theta}_\rho^{-1} \cdot \mathbf{g}_\rho^{-1} \cdot \nabla \boldsymbol{\theta}\right). \tag{196}$$

Introducing the notation $\boldsymbol{\vartheta}_\rho = \mathbf{g}_\rho^{-1} \cdot \nabla \boldsymbol{\theta}$, we rewrite Eqs. (195), (196) as

$$\mathbf{Z} = \boldsymbol{\Theta}_\rho^{-1} \cdot \boldsymbol{\vartheta}_\rho, \qquad \frac{\delta \boldsymbol{\vartheta}_\rho}{\delta t} = \mathbf{g}_\rho^{-1} \cdot \nabla \left(\boldsymbol{\Omega} \cdot \boldsymbol{\Theta}_\rho^{-1} \cdot \boldsymbol{\vartheta}_\rho\right). \tag{197}$$

Taking into account identity (177), the problem (197) can be reformulated via the left angular velocity vector $\boldsymbol{\omega} = \mathbf{P} \cdot \boldsymbol{\Omega}$:

$$\mathbf{Z} = \boldsymbol{\Theta}_\rho^{-1} \cdot \boldsymbol{\vartheta}_\rho, \qquad \frac{\delta \boldsymbol{\vartheta}_\rho}{\delta t} = \mathbf{g}_\rho^{-1} \cdot \nabla \left(\boldsymbol{\Theta}_\rho^{-1} \cdot \boldsymbol{\vartheta}_\rho \cdot \boldsymbol{\omega}\right). \tag{198}$$

From the above expression for tensor $\mathbf{Z}$ and the identity (177), it follows that

$$\mathbf{P} = \boldsymbol{\vartheta}_\rho^{-1} \cdot \boldsymbol{\Theta}_\rho \cdot \boldsymbol{\vartheta}_\rho^T \cdot \boldsymbol{\Theta}_\rho^{-T}, \qquad \boldsymbol{\theta} = \boldsymbol{\Theta}_\rho^{-1} \cdot \times \boldsymbol{\vartheta}_\rho, \qquad \boldsymbol{\omega} = \boldsymbol{\vartheta}_\rho^{-1} \cdot \boldsymbol{\Theta}_\rho \cdot \boldsymbol{\vartheta}_\rho^T \cdot \boldsymbol{\Theta}_\rho^{-T} \cdot \boldsymbol{\Omega}. \tag{199}$$

Thus, tensors $\mathbf{Z}$ and $\mathbf{P}$ as well as vector $\boldsymbol{\theta}$ are expressed in terms of the strain characteristics $\boldsymbol{\Theta}_\rho$ and $\boldsymbol{\vartheta}_\rho$. Tensor $\boldsymbol{\vartheta}_\rho$ is an auxiliary quantity that is determined by means of $\mathbf{g}_\rho$ and $\boldsymbol{\Theta}_\rho$ as the solution of the differential equation.

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