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Introducing damage mechanics templates for the systematic and consistent formulation of holistic material damage models

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Abstract An attempt is presented to provide for a global and holistic approach to the study of continuum damage mechanics. For this purpose, the authors classify material damage models into three categories. Furthermore, the concept of ready-made model templates for damage mechanics is proposed. In this regard, the authors provide details for fourteen basic damage mechanics templates. Each template comes with a schematic diagram along with a set of equations that govern the respective material damage template. The user can then use the chosen template and fill in the details to obtain the constitutive equations. The obtained material damage model is guaranteed to be systematic and consistent provided that proper use is made of the chosen template. Furthermore, three additional sections are added to provide details on how to generate advanced, special, and more complex damage mechanics templates that go beyond the rule of mixtures and for specific types of materials. The details of 14 complete and comprehensive damage mechanics templates for both metals and composite materials are presented in this work. Finally, using one chosen basic template, it is shown how to generate five additional and more complex templates.

List of symbols
Material configurations

C	Damaged material configuration
$\overline{C}$	Effective fictitious undamaged material configuration
C^m	Damaged material configuration of the matrix
$\overline{C}^m$	Effective fictitious damaged material configuration of the matrix
C^f	Damaged material configuration of the fibers
$\overline{C}^f$	Effective fictitious damaged material configuration of the fibers
C^c	Damaged material configuration when cracks are present only
$\overline{C}^c$	Effective fictitious damaged material configuration when cracks are present and voids are removed
C^v	Damaged material configuration when voids are present only
$\overline{C}^v$	Effective fictitious damaged material configuration when voids are present and cracks are removed
C^{mc}	Damaged material configuration of the matrix when cracks are present only

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$\overline{C}^{mc}$ Effective fictitious damaged material configuration of the matrix when cracks are present and voids are removed
C^{mv} Damaged material configuration of the matrix when voids are present only
$\overline{C}^{mv}$ Effective fictitious damaged material configuration of the matrix when voids are present and cracks are removed
C^{fc} Damaged material configuration of the fibers when cracks are present only
$\overline{C}^{fc}$ Effective fictitious damaged material configuration of the fibers when cracks are present and voids are removed
C^{fv} Damaged material configuration of the fibers when voids are present only
$\overline{C}^{fv}$ Effective fictitious damaged material configuration of the fibers when voids are present and cracks are removed
$\overline{C}^{d}$ Effective fictitious configuration with respect to de-bonding, i.e., interfacial damage

Fourth-rank damage effect tensors

M Fourth-rank damage effect tensor
M^m Fourth-rank matrix damage effect tensor
M^f Fourth-rank fiber damage effect tensor
M^c Fourth-rank damage effect tensor when cracks are present only
M^v Fourth-rank damage effect tensor when voids are present only
M^{mc} Fourth-rank matrix damage effect tensor of the matrix when cracks are present only
M^{mv} Fourth-rank matrix damage effect tensor of the matrix when voids are present only
M^{fc} Fourth-rank matrix damage effect tensor of the fibers when cracks are present only
M^{fv} Fourth-rank matrix damage effect tensor of the fibers when voids are present only
M^d De-bonding (i.e., interfacial damage) fourth-rank damage effect tensor

Strain concentration factors/tensors

A^m Matrix strain concentration factor/tensor
A^f Fiber strain concentration factor/tensor
$\overline{A}^m$ Effective matrix strain concentration factor/tensor
$\overline{A}^f$ Effective fiber strain concentration factor/tensor
A^c Strain concentration factor/tensor when cracks are present only
A^v Strain concentration factor/tensor when voids are present only
$\overline{A}^c$ Effective strain concentration factor/tensor when cracks are present only and voids are removed
$\overline{A}^v$ Effective strain concentration factor/tensor when voids are present only and cracks are removed

Stress concentration factors/tensors

B^m Matrix stress concentration factor/tensor
B^f Fiber stress concentration factor/tensor
$\overline{B}^m$ Effective matrix stress concentration factor/tensor
$\overline{B}^f$ Effective fiber stress concentration factor/tensor
B^c Stress concentration factor/tensor when cracks are present only
B^v Stress concentration factor/tensor when voids are present only
$\overline{B}^c$ Effective stress concentration factor/tensor when cracks are present only and voids are removed
$\overline{B}^v$ Effective stress concentration factor/tensor when voids are present only and cracks are removed

Other symbols

ϕ Scalar cross-sectional damage variable
ϕ^c Scalar cross-sectional damage variable due to cracks
ϕ^v Scalar cross-sectional damage variable due to voids
A Cross-sectional area in the deformed/damaged configuration
$\overline{A}$ Cross-sectional area in the effective/undamaged configuration

σ	Cauchy stress
σ^m	Matrix Cauchy stress
σ^f	Fiber Cauchy stress
$\overline{\sigma}$	Effective Cauchy stress
$\overline{\sigma}^m$	Effective matrix Cauchy stress
$\overline{\sigma}^f$	Effective fiber Cauchy stress
E	Elastic modulus in the deformed/damaged configuration
E^m	Matrix elastic modulus in the deformed/damaged configuration
E^f	Fiber elastic modulus in the deformed/damaged configuration
$\overline{E}$	Effective elastic modulus (in the fictitious/undamaged configuration)
$\overline{E}^m$	Matrix effective elastic modulus (in the fictitious/undamaged configuration)
$\overline{E}^f$	Fiber effective elastic modulus (in the fictitious/undamaged configuration)
G	Shear modulus in the deformed/damaged configuration
G^m	Matrix shear modulus in the deformed/damaged configuration
G^f	Fiber shear modulus in the deformed/damaged configuration
υ	Poisson's ratio in the deformed/damaged configuration
υ^m	Matrix Poisson's ratio in the deformed/damaged configuration
υ^f	Fiber Poisson's ratio in the deformed/damaged configuration
S	Eshelby tensor
c^m	Matrix volume fraction
c^f	Fiber volume fraction
$\overline{c}^m$	Effective matrix volume fraction
$\overline{c}^f$	Effective fiber volume fraction
c^c	Crack volume fraction
c^v	Void volume fraction
$\overline{c}^c$	Effective crack volume fraction
$\overline{c}^v$	Effective void volume fraction
V^m	Matrix volume fraction
V^f	Fiber volume fraction
η	Stress-portioning factor
η^m	Stress-portioning factor
η^f	Shear-portioning factor

Plasticity related

s	Deviatoric stress tensor
f	Yield function
α	Backstress tensor
ε^e	Elastic strain tensor
ε^p	Plastic strain tensor
c	Scalar parameter related to kinematic hardening
λ	Scalar parameter related to flow rule
I_2	Second-rank identity tensor
I_4	Fourth-rank identity tensor
δ	Kronecker delta

1 Introduction

The subject of continuum damage mechanics was pioneered by [23] and more fully developed by [31,36,55,59,65,67,68]; and [25,29,30].

In 1958, Kachanov [23] pioneered the concept of effective stress and introduced the topic of continuum damage mechanics. This development was followed by [53] and by others later [29,30,33,65,67–69]. In the framework of continuum damage mechanics, a scalar damage variable φ is introduced that has values in the range $0 \leq \varphi \leq 1$. Thus, the value of the damage variable is zero when the virgin material is undamaged, while the value approaches 1 upon complete rupture. However, practically the damage value cannot exceed 0.3–0.4 without violating the concept of a continuum.

Research on damage mechanics accelerated rapidly in the past few years [16,20,24,25,34,36,46,47,54,55,57–59]. In addition, there have been some notable research efforts in the corresponding subject of healing mechanics [15,74–76]. Some of the recent research efforts are seen to be in the form of combined damage/healing models for different types of materials. Another approach to damage characterization is to use an entropy generation rate as a damage metric rather than the damage potential surface [6,7]. In addition, damage mechanics was extended to solids subjected to electrical loading and thermomigration [5,42,79]. Finally, [70] introduced new concepts related to damage, healing, damageability, and integrity of materials. In addition, [71] used a damage mechanics model to investigate elastic damage in graphene.

Damage in composite materials has been studied extensively in the literature [1–3,9–12,37–41,60]. Micromechanical characterization of damage in these types of materials has been only recently attempted. Voyiadjis and Kattan [61] proposed a *local approach* in which several damage variables were introduced: one for each constituent. These constituent damage variables were called local damage variables. This approach was labeled the local approach in the sense that local damage variables were used in the analysis. In fact, it is local only at the constituent level. Kattan and Voyiadjis [26–28] and Voyiadjis and Kattan [60] proposed an *overall approach* to damage in elasto-plastic fiber-reinforced composites in which one single damage tensor is used to model damage in the composite system. The damage tensor reflected all types of different damage mechanisms including fiber cracking, matrix microcrack and microvoid growth and coalescence, and de-bonding at the fiber-matrix interface. Both overall and local approaches to damage were applied to the problem of damage modeling in an elasto-plastic composite system consisting of elastic fibers embedded in an elasto-plastic matrix. Voyiadjis and Kattan [64] showed that the two approaches of overall and local damage characterization in composite materials are basically equivalent.

The objective of this work is to use the overall and local approaches to damage characterization in composite materials to build what are called damage mechanics templates. These are templates for damage models to be used to characterize damage in materials. The following are the advantages of using the presented damage mechanics templates:

1. The analyst can use these templates to model and characterize the damage process in composite materials without having to worry about the mathematical and mechanical consistency of the damage model.
2. Using the prescribed templates ensures a systematic material damage model. In this sense, the various steps of damage modeling are outlined and the key equations pinned down for each specific step.
3. The various steps used in each damage mechanics template are logical, are mathematically and mechanically sound, and follow the arrows in an illustrative schematic diagram that accompanies each template.
4. The overall and local damage mechanics templates for composite materials are applied to metals in which the two constituents of matrix and fibers are replaced with mathematical constituents made of up of cracks and voids.
5. More complex damage mechanics templates are generated for composite materials by considering four constituents: cracks in matrix, cracks in fibers, voids in matrix, and voids in fibers. In this regard, two types of templates can be generated: symmetrical templates and unsymmetrical templates.

The design of the various damage mechanics templates in this work is based on the observation that the overall and local approaches to damage give similar results when applied to fiber-reinforced metal matrix composites [64]. The fibers are assumed to be continuous and perfectly aligned. In addition, a perfect bond is assumed to exist at the matrix–fiber interface. Both the matrix and fibers are assumed to be linearly elastic and follow the three-dimensional generalization of Hooke's Law. A consistent mathematical formulation is used to prove the equivalence of the two approaches in this case. The elastic stiffness matrix derived from both approaches is shown to be identical in both cases.

Direct tensor notation is used throughout this work. The matrix notation $[A]\{\sigma\}$ and $[A][B]$ is used to denote the tensor operations $A_{ijkl}\sigma_{kl}$ and $A_{ijmn}B_{mnkl}$, respectively, for the fourth-rank $[A]$, $[B]$ and the second-rank tensor $\{\sigma\}$. The emphasis is on composite materials, specifically fiber-reinforced metal matrix composites, as 12 templates of the basic 14 templates are designed for these types of composite materials. Only two basic templates are designed for metals.

In Sect. 2, the authors propose a classification of materials according to the nature of their damage mechanisms. Three types of material damages are classified. In addition, the concept of damage mechanics templates is introduced. In this regard, two tables are presented: One table explains the naming methodology of the new damage mechanics templates, while the other table lists the main features of each template.

Section 3 presents two simple templates for the characterization of damage mechanisms in metal matrix composites (type of damage is unspecified and will have to be included into the template by the user). The templates are shown to be symmetrical, simple to use, and each one consists of two steps for its implementation.

Section 4 presents two similar simple templates for the characterization of damage mechanisms in metals. The metal templates are also shown to be symmetrical, simple to use, and each one consists of two steps for its implementation. Two types of damages are considered in the metal templates: damage due to distributions of cracks and damage due to distributions of voids.

Section 5 deals with more complex templates for metal matrix composites that account for damages due to the distributions of cracks and distributions of voids. Thus, instead of dealing with two constituents (as in the templates in the previous two Sections), each template in this Section is built based on four different constituents. Therefore, it is seen that a combination of six different but symmetrical templates is generated in this case.

In Sect. 6, the authors present four more complex templates for metal matrix composites that are also based on the same four different constituents but are unsymmetrical.

Sections 7, 8, and 9 provide guidelines on how to generate advanced, special, and more complex templates. Section 7 deals with generating damage mechanics templates that go beyond the use of the rule of mixtures and move into homogenization methods. Section 8 shows how to generate damage mechanics templates for other types of materials using elasto-plastic and hyper-elastic models. Finally, in Sect. 9 it is shown how to extend the existing templates of Sects. 3, 4, 5, and 6 to account for more types of damage mechanisms like de-bonding of composites. In this case, the damaged composite system will include more than two or three constituents, and the resulting templates will be far more complex. In this regard, one basic template is chosen, and it is shown how five different and more complex templates are generated using the basic template.

As can be seen from the list of nomenclature and the various schematic diagrams of Sects. 3, 4, 5 and 6, a total of 19 different configurations of the material are considered. However, a maximum number of 11 configurations are used simultaneously in some of the more complicated templates. In addition, there are 10 different fourth-rank damage effect tensors that are introduced, but not all of them are used at the same time or in the same model. Finally, there are eight different stress concentration factors/tensors and eight different strain concentration factors/tensors. Again, not all of them are used at the same time or in the same model.

2 Classification of material damage models

In this Section, an attempt is made to classify material damage models into three categories. The classification system is based on the number of damage mechanisms involved in each specific damage model. For this purpose, the following three categories are proposed.

Material Damage Models of the First Kind (Type MDM1): these are damage models that use one single damage mechanism to characterize the damage process in a material. Examples in this category include matrix cracks [51], transverse matrix cracks [44], interfacial cracks [45], fiber de-bonding [21], and de-bonding [52].

Material Damage Models of the Second Kind (Type MDM2): these are damage models that use two damage mechanisms along with their interactions to characterize the damage process in a material. It is noted that the interactions of the two damage mechanisms may be somewhat simple and easy to characterize. Examples in this category include matrix cracking and interfacial de-bonding [43], de-bonding and matrix cracking [14], and fiber breaks and matrix cracking [19].

Material Damage Models of the Third Kind (Type MDM3): these are damage models that use three or more damage mechanisms to characterize the damage process in a material. It is noted that the interactions of the three or more damage mechanisms may prove to be difficult to characterize. Examples in this category include matrix cracks, fiber breaks, and de-bonding [56] and multi-damage mechanisms in unidirectional ceramic matrix composites [13].

The remaining Sections of this work present various proposed damage mechanics templates. Each template comes with a schematic diagram along with a Table listing the key equations of the corresponding template. The detailed mathematical derivation is presented only for the first two templates. For all the other templates, only the key equations are shown. The interested reader should be able to verify the key equations for the selected template by following the arrows on the schematic diagram of each template and thus produce a detailed mathematical derivation of the governing equations for the corresponding template.

Table 1 Naming a damage mechanics template

Template name	First part L, O, D, or a combination of them	Second part M for metal or C for composite	Third part C for cracks and V for voids	Fourth part M for matrix and F for fibers
ODCMF	OD	C	–	MF
LDCMF	LD	C	–	MF
ODMCV	OD	M	CV	–
LDMCV	LD	M	CV	–
OOCMFCV	OO	C	CV	MF
OLCMFCV	OL	C	CV	MF
LOCMFCV	LO	C	CV	MF
LOCCVMF	LO	C	CV	MF
OOCCVMF	OO	C	CV	MF
OLCCVMF	OL	C	CV	MF
OLLOCMFCV	OLLO	C	CV	MF
LOOLCMFCV	LOOL	C	CV	MF
OLLOCCVMF	OLLO	C	CV	MF
LOOLCCVMF	LOOL	C	CV	MF

Table 2 Main features of damage mechanics templates

Template name	Material type	Damage type	Number of steps	Number of sub-steps	Symmetry
ODCMF LDCMF	Composite	Matrix (unspecified), Fiber (unspecified)	2	5	Symmetrical
ODMCV LDMCV	Metal	Cracks, voids	2	5	Symmetrical
OOCMFCV OLCMFCV LOCMFCV LOCCVMF OOCCVMF OLCCVMF	Composite	Matrix cracks Matrix voids Fiber cracks Fiber voids	3	7	Symmetrical
OLLOCMFCV LOOLCMFCV OLLOCCVMF LOOLCCVMF	Composite	Matrix cracks Matrix voids Fiber cracks Fiber voids	3	12	Unsymmetrical

Table 1 shows the procedure for naming a damage mechanics template, while Table 2 shows the main features of each one of the damage mechanics templates presented in this work.

One new feature that becomes apparent immediately by observing the numerous schematic diagrams is that some damage mechanics templates are symmetrical, while others are unsymmetrical. Thus, the following new terminology is introduced here: A *symmetrical damage model* is a damage mechanics model in which the geometry of its schematic diagram is symmetrical. On the other hand, an *unsymmetrical damage model* is a damage mechanics model in which the geometry of its schematic diagram is unsymmetrical. The templates appearing in Sects. 4 and 5 are all symmetrical, while the templates appearing in Sect. 6 are all unsymmetrical. It is not clear whether using a symmetrical damage model offers any noticeable advantages over using unsymmetrical damage models. However, it is clear that using a symmetrical damage mechanic template is much simpler than using an unsymmetrical damage mechanics template.

The symmetry of the damage models introduced in the previous paragraph is a mathematical symmetry. It is different than the symmetry obtained through the process of isotropic damage, which is a material damage symmetry. The mathematical symmetry introduced here in the damage mechanics templates is based on how the mathematical equations of damage are applied. If the equations are applied equally to both cracks and voids, then the resulting damage model is symmetrical, and this feature is clearly evident in the schematic diagram of the corresponding template. However, if the mathematical equations are not applied equally to both cracks and voids, then the resulting damage model is unsymmetrical, and this feature is also clearly evident in the schematic diagram of the corresponding template. In this regard, an isotropic damage model may be symmetrical or unsymmetrical. Also, an anisotropic damage model may also be symmetrical or unsymmetrical. The mathematical symmetry has no relation to the material damage symmetry.

3 Damage mechanics templates for composite materials: I

For simplicity, the composite system is assumed to consist of a matrix reinforced with continuous fibers. Both the matrix and fibers are linearly elastic with different matrix constants. Let $\overline{C}$ denote the configuration of the undamaged composite system, and let $\overline{C}^m$ and $\overline{C}^f$ denote the configurations of the undamaged matrix and fibers, respectively. A superscript "m" is used to denote a matrix-related quantity, while a superscript "f" is used to denote a fiber-related quantity. Since the composite system assumes a perfect bond at the matrix–fiber interface, it is clear that $\overline{C}^m \cap \overline{C}^f = \phi$ and $\overline{C}^m \cup \overline{C}^f = \overline{C}$. In the overall approach, the problem reduces to transforming the undamaged configuration $\overline{C}$ into the damaged configuration C. In contrast, two intermediate damage configurations C^m and C^f are considered in the local approach for the matrix and fibers, respectively. In the latter approach, the problem is reduced to transforming each of the undamaged constituent configurations $\overline{C}^m$ and $\overline{C}^f$ into the damaged constituent configurations C^m and C^f, respectively.

In each of the elastic fiber-reinforced composites, the following linear elastic relation is used for each constituent in its respective undamaged configuration $\overline{C}^k$:

$$\left\{\overline{\sigma}^k\right\} = \left[\overline{E}^k\right]\left\{\overline{\varepsilon}^k\right\}, \quad k = m, f \tag{1}$$

where $\left\{\overline{\sigma}^k\right\}$ is a vector that represents the effective constituent stress tensor, $\left\{\overline{\varepsilon}^k\right\}$ is a vector that represents the effective constituent strain tensor (the total strain tensor is the same as the elastic strain tensor in this case), and $[\overline{E}^k]$ is a square matrix that represents the effective fourth-rank elasticity tensor. For the case of an isotropic constituent, $[\overline{E}^k]$ is given by the formula:

$$\left[\overline{E}^k\right] = \overline{\lambda}^k\,[I_2] \otimes [I_2] + 2\overline{\mu}^k\,[I_4] \tag{2}$$

where $\overline{\lambda}^k$ and $\overline{\mu}^k$ are the effective constituent Lame's constants, while $[I_2]$ and $[I_4]$ are matrices that represent the second-rank and fourth-rank identity tensors, respectively.

Within the framework of the micromechanical analysis of composite materials, the effective constituent stress tensor $\left[\overline{\sigma}^k\right]$ is related to the effective composite stress tensor $\{\overline{\sigma}\}$ by:

$$\left\{\overline{\sigma}^k\right\} = \left[\overline{B}^k\right]\{\overline{\sigma}\} \tag{3}$$

where the square matrix $[\overline{B}^k]$ represents the effective fourth-rank constituent stress concentration tensor. The components of this tensor can be determined by several models such as the Voigt model and the Mori-Tanaka model [35,50,78]. The interested reader is referred to the work of [17] and Voyiadjis and Kattan [61–63]. A summary of the basic equations in the determination of the stress and strain concentration tensors is presented in the "Appendix." In addition, the effective constituent strain tensor is represented by vector $\left\{\overline{\varepsilon}^k\right\}$ and is determined by using the following equation:

$$\left\{\overline{\varepsilon}^k\right\} = \left[\overline{A}^k\right]\{\overline{\varepsilon}\} \tag{4}$$

where $\{\overline{\varepsilon}\}$ is a vector that represents the effective composite strain tensor (the total effective strain tensor is the same as the elastic effective strain tensor in this case), and $[\overline{A}^k]$ is a square matrix that represents the effective fourth-rank strain concentration tensor. Evaluating the components of the strain concentration tensor follows similar steps to those of evaluating the components of the stress concentration tensor, and the reader is referred to the same references cited above.

Next, the first two damage mechanics templates are formulated based on the overall and local approaches to damage characterization in elastic composite materials as outlined above [64]. The two damage mechanics templates introduced in this Section have the following properties:

1. Each template is described by *two* stages of damage characterization. These two stages are very clear from Figs. 1 and 2.
2. Each template is *symmetrical*. The template symmetry is clearly evident in the schematic diagrams of Figs. 1 and 2.

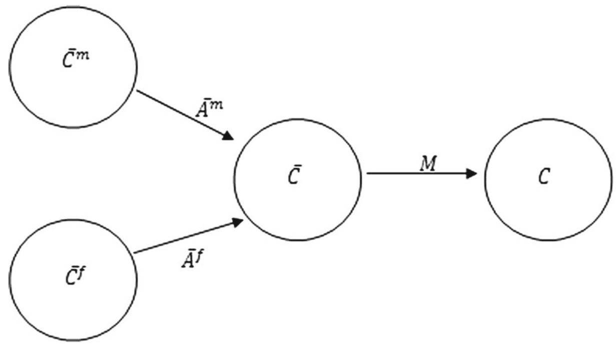

Fig. 1 Template ODCMF

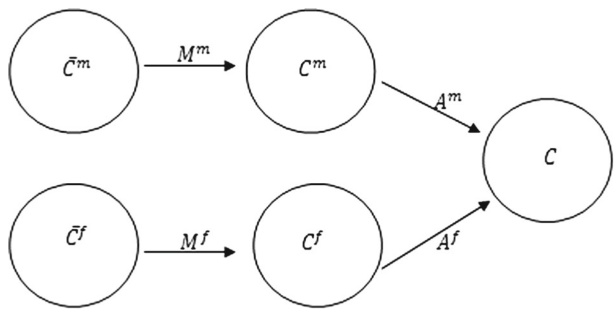

Fig. 2 Template LDCMF

Table 3 Key equations of template ODCMF

Step	Equation	Description
Step 1-a	$\{\bar{\sigma}\} = \bar{c}^m \{\bar{\sigma}^m\} + \bar{c}^f \{\bar{\sigma}^f\}$	Law of mixtures applied to stress
Step 1-b	$[\bar{E}] = \bar{c}^m [\bar{E}^m][\bar{A}^m] + \bar{c}^f [\bar{E}^f][\bar{A}^f]$	Law of mixtures applied to elasticity tensor
Step 2-a	$\{\bar{\sigma}\} = [M]\{\sigma\}$	Damage transformation of stress
Step 2-b	$\{\bar{\varepsilon}\} = [M]^{-T}\{\varepsilon\}$	Damage transformation of strain
Step 2-c	$[E] = [M]^{-1}\left(\bar{c}^m [\bar{E}^m][\bar{A}^m] + \bar{c}^f [\bar{E}^f][\bar{A}^f]\right)[M]^{-T}$	Elasticity tensor

Table 4 Key equations of template LDCMF

Step	Equation	Description
Step 1-a	$\{\bar{\sigma}^k\} = [M^k]\{\sigma^k\}, \quad k = m, f$	Damage transformation of stress
Step 1-b	$\{\bar{\varepsilon}^k\} = [M^k]^{-T}\{\varepsilon^k\}, \quad k = m, f$	Damage transformation of strain
Step 1-c	$[E^k] = [M^k]^{-1}[\bar{E}^k][M^k]^{-T}, \quad k = m, f$	Damage transformation of elasticity tensor
Step 2-a	$\{\sigma\} = c^m \{\sigma^m\} + c^f \{\sigma^f\}$	Law of mixtures applied to stress
Step 2-b	$[E] = c^m [E^m][A^m] + c^f [E^f][A^f]$	Elasticity tensor

3. The two templates in this Section are the two *simplest* templates possible. This is again evident from Figs. 1 and 2. These templates apply to linear elastic materials. Check Sect. 8 on how to generate special templates for other types of materials.
4. The detailed mathematical formulation is presented for both templates in Sects. 3.1 and 3.2. In addition, the *key equations* for the two templates for composite materials of this Section are listed in Tables 3 and 4.

3.1 Template ODCMF (overall damage in composites with matrix and fibers)

In this template, damage is incorporated in the composite system as a whole using one single damage tensor called the *overall damage tensor*. The two steps needed for this template are shown schematically in Fig. 1 for a two-phase composite consisting of matrix and fibers. In the first step, the elastic equations are formulated in an undamaged composite system. This is performed here using the *law of mixtures* as follows (by following the two inclined arrows on the left part of Fig. 1):

$$\{\overline{\sigma}\} = \overline{c}^m \left\{\overline{\sigma}^m\right\} + \overline{c}^f \left\{\overline{\sigma}^f\right\} \tag{5}$$

where $\overline{c}^m$ and $\overline{c}^f$ are the effective matrix and fibers fractions, respectively, with $\overline{c}^m + \overline{c}^f = 1$. Check Sect. 7 for details how to replace the rule of mixtures with more elaborate schemes and generate advanced damage mechanics templates.

In the effective composite configuration $\overline{C}$, the following linear elastic relation holds:

$$\{\overline{\sigma}\} = \left[\overline{E}\right] \{\overline{\varepsilon}\} \tag{6}$$

where $\left[\overline{E}\right]$ is a square matrix that represents the fourth-rank effective constant elasticity tensor. Substituting Eqs. (1), (4), and (6) into equation (5) and simplifying, one obtains the following expression for $\left[\overline{E}\right]$:

$$\left[\overline{E}\right] = \overline{c}^m \left[\overline{E}^m\right] \left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^f\right] \left[\overline{A}^f\right]. \tag{7}$$

In the second step of the formulation, damage is introduced by the fourth-rank damage effect tensor, represented by the square matrix $[M]$, as follows (by following the horizontal arrow on the right part of the schematic diagram of Fig. 1):

$$\{\overline{\sigma}\} = [M] \{\sigma\} \tag{8}$$

where $\{\sigma\}$ is a vector that represents the composite stress tensor. The damage effect tensor $[M]$ has been investigated extensively in the literature. See, for example, the recent work of Voyiadjis and Kattan [59] and Voyiadjis et al. [77]. Equation (8) represents the damage transformation equation for the stress tensor. In order to derive a similar relation for the strain tensor, one needs to use the *hypothesis of elastic energy equivalence* [55]. In this hypothesis, the elastic strain energy of the damaged system is equal to the elastic strain energy of the effective system. Applying this hypothesis to the composite system by equating the two elastic strain energies, one obtains:

$$\frac{1}{2} \{\varepsilon\}^T \{\sigma\} = \frac{1}{2} \{\overline{\varepsilon}\}^T \{\overline{\sigma}\} \tag{9}$$

where $\{\varepsilon\}$ is a vector that represents the composite strain tensor (the total elastic strain tensor is the same as the elastic strain tensor in this case). Substituting Eq. (8) into Eq. (9) and simplifying, one obtains:

$$\{\overline{\varepsilon}\} = [M]^{-T} \{\varepsilon\} \tag{10}$$

where the superscript $-T$ indicates the inverse transpose of a matrix. It should be noted that other expressions for the damage transformation relation of strain of Eq. (10) may be obtained using other equivalence hypotheses of damage mechanics, but the one used here will suffice.

In order to derive the final elastic relation in the damaged composite system, one substitutes Eqs. (8) and (10) into equation (6) to obtain:

$$\{\sigma\} = [E] \{\varepsilon\} \tag{11}$$

where the fourth-rank damaged elasticity tensor is given by the following square matrix representation:

$$[E] = [M]^{-1} \left[\overline{E}\right] [M]^{-T}. \tag{12.1}$$

Substituting for $\left[\overline{E}\right]$ from Eq. (7) into Eq. (12.1), one obtains:

$$[E] = [M]^{-1} \left(\overline{c}^m \left[\overline{E}^m\right] \left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^f\right] \left[\overline{A}^f\right]\right) [M]^{-T}. \tag{12.2}$$

The expression given by Eq. (12.2) above represents the fourth-rank elasticity tensor in the damaged composite system according to the overall approach. The key equations of this template are listed in Table 3. As shown in the Table, this template consists of five sub-steps (Step 1 consists of two sub-steps, while Step 2 consists of three sub-steps).

3.2 Template LDCMF (local damage in composites with matrix and fibers)

In the local approach, damage is introduced in the first step of the formulation using two independent damage tensors for the matrix and fibers. However, more damage tensors may be introduced to account for other types of damages such as de-bonding and de-lamination (check Sect. 9 for more details about this issue). The two steps involved in this approach are shown schematically in Fig. 2. One first introduces the fourth-rank matrix and fiber damage effect tensors, represented by the square matrices $\left[M^m\right]$ and $\left[M^f\right]$, respectively, as follows (by following the two horizontal arrows on the left part of Fig. 2):

$$\left\{\overline{\sigma}^k\right\} = \left[M^k\right]\left\{\sigma^k\right\}, k = m, f. \tag{13}$$

The above equation can be interpreted in a similar way as Eq. (8), except that it applies at the constituent level. It also represents the damage transformation equation for each constituent stress tensor. In order to derive a similar transformation equation for each constituent strain tensor, one applies the *hypothesis of elastic energy equivalence* [55] to each constituent separately as follows:

$$\frac{1}{2}\left\{\varepsilon^k\right\}^T\left\{\sigma^k\right\} = \frac{1}{2}\left\{\overline{\varepsilon}^k\right\}^T\left\{\overline{\sigma}^k\right\}, \quad k = m, f. \tag{14}$$

On using Eq. (14), one assumes that there are no micromechanical or constituent elastic interactions between the matrix and fibers. This assumption is not valid in general. From micromechanical consideration, there should be interactions between the elastic strain energies in the matrix and fibers. However, such interactions are beyond the scope of this work as the resulting equations will be complicated and the sought equivalence relations will consequently be unattainable. It should be clear to the reader that Eq. (14) is the single most important assumption that is needed to prove the equivalence between the overall and local (constituent) approaches. In fact, the proposed proof (see Sect. 3.3) hinges entirely on the assumption given by Eq. (14). Therefore, the sought equivalence between the overall and local (constituent) approaches is a very special case when Eq. (14) is valid.

Substituting Eq. (13) into Eq. (14) and simplifying, one obtains the required transformation equation for the constituent strain tensor as follows:

$$\left\{\overline{\varepsilon}^k\right\} = \left[M^k\right]^{-T}\left\{\varepsilon^k\right\}, \quad k = m, f. \tag{15}$$

Substituting Eqs. (13) and (15) into Eq. (1) and simplifying, one obtains:

$$\left\{\sigma^k\right\} = \left[E^k\right]\left\{\varepsilon^k\right\}, \quad k = m, f \tag{16}$$

where the fourth-rank constituent elasticity tensor $\left[E^k\right]$ is given by the following square matrix representation:

$$\left[E^k\right] = \left[M^k\right]^{-1}\left[\overline{E}^k\right]\left[M^k\right]^{-T}, \quad k = m, f. \tag{17}$$

Equation (16) represents the elasticity equation for the damaged constituents. The second step of the formulation involves transforming Eq. (17) into the whole composite system using the law of mixtures as follows (by following the two inclined arrows on the right part of the schematic diagram of Fig. 2):

$$\{\sigma\} = c^m\left\{\sigma^m\right\} + c^f\left\{\sigma^f\right\} \tag{18}$$

where c^m and c^f are the matrix and fiber volume fractions, respectively, in the damaged composite system, with $c^m + c^f = 1$. Check Sect. 7 for details how to replace the rule of mixtures with more elaborate schemes and generate advanced damage mechanics templates.

Before proceeding with Eq. (18), one needs to derive a strain constituent relation similar to Eq. (4). Substituting Eqs. (10) and (15) into Eq. (4) and simplifying, one obtains:

$$\left\{\varepsilon^k\right\} = \left[A^k\right]\{\varepsilon\}, \quad k = m, f \tag{19}$$

where the constituent strain concentration tensor $\left[A^k\right]$ in the damaged state is given by the following matrix representation:

$$\left[A^k\right] = \left[M^k\right]^T \left[\overline{A}^k\right] [M]^{-T}, \quad k = m, f. \tag{20}$$

The expression given by Eq. (20) above represents the damage transformation equation for the constituent strain concentration tensor.

Finally, one substitutes Eqs. (11), (16), and (19) into equation (18) and simplifies to obtain:

$$[E] = c^m \left[E^m\right]\left[A^m\right] + c^f \left[E^f\right]\left[A^f\right]. \tag{21}$$

The expression given by Eq. (21) above represents the fourth-rank elasticity tensor in the damaged composite system according to the local approach. The key equations of this template are listed in Table 4. As shown in the Table, this template consists of five sub-steps (Step 1 consists of three sub-steps, while Step 2 consists of two sub-steps).

3.3 Equivalence of the two templates ODCMF and LDCMF

In this Subsection, it is shown that both templates ODCMF and LDCMF are equivalent, i.e., the overall and local approaches to damage in composite materials are equivalent, by making use of the assumptions made in their derivations. The equivalence proof is performed by showing that both the elasticity tensors given in Eqs. (12.2) and (21) are the same. In fact, it is shown that Eq. (21) reduces to Eq. (12.2) after making the appropriate substitutions.

First, one needs to find a damage transformation equation for the volume fractions. This is performed by substituting Eqs. (9) and (13) into Eq. (5), simplifying, and comparing the result with Eq. (18). One therefore obtains:

$$c^k [I_4] = \overline{c}^k [M]^{-1} \left[M^k\right], \quad k = m, f. \tag{22}$$

Substituting Eqs. (17) and (20) into Eq. (21) and simplifying, one obtains:

$$[E] = \left(c^m \left[M^m\right]^{-1} \left[\overline{E}^m\right]\left[\overline{A}^m\right] + c^F \left[M^f\right]^{-1} \left[\overline{E}^f\right]\left[\overline{A}^f\right]\right) [M]^{-T}. \tag{23}$$

Finally, one substitutes Eq. (22) into Eq. (23) and simplifies to obtain:

$$[E] = [M]^{-1} \left(\overline{c}^m \left[\overline{E}^m\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^f\right]\left[\overline{A}^f\right]\right) [M]^{-T}. \tag{24}$$

It is clear that the expression in Eq. (24) above is exactly the same as that of Eq. (12.2). Therefore, both the overall and local approaches yield the same elasticity tensor in the damaged composite system, and thus templates ODCMF and LDCMF are equivalent.

4 Damage mechanics templates for metals

For simplicity, the damaged metal is assumed to consist of damage in the form of distributions of cracks and voids. Two configurations of the damaged system are considered as two different constituents: one damaged constituent consisting of damage in the form of distributions of cracks only, and another damaged constituent consisting of damage in the form of distributions of voids. The damaged material is assumed to be linearly elastic. Let $\overline{C}$ denote the configuration of the undamaged metal, and let $\overline{C}^c$ and $\overline{C}^v$ denote the configurations of the undamaged constituents, respectively: one with cracks only (i.e., voids are removed) and the other one with voids only (i.e., cracks are removed). A superscript "c" is used to denote a crack-related quantity, while a superscript "v" is used to denote a void-related quantity. For including other types of defects in addition to cracks and voids, check Sect. 9 for more details. For applying these templates to other types of materials, check Sect. 8 for more details. The two templates for damage in metals described in this Section utilize the rule of mixtures. Check Sect. 7 to generate advanced templates that go beyond the rule of mixtures and utilize more elaborate schemes of homogenization.

In the overall approach, the problem reduces to transforming the undamaged configuration $\overline{C}$ into the damaged configuration C. In contrast, two intermediate damage configurations C^c and C^v are considered in the local approach for the cracks and voids, respectively. In the latter approach, the problem is reduced to transforming each of the undamaged constituent configurations $\overline{C}^c$ and $\overline{C}^v$ into the damaged constituent configurations C^c and C^v, respectively. It is noted that the same mathematical formulation utilized in Sect. 3 for composites applies here and will not be repeated. The total strain tensor is the same as the elastic strain tensor in this case.

One key difference between the two metal templates in this Section and the two composites templates in Sect. 3 is the use of the stress and strain concentration factors/tensors. While use was made of matrix stress concentration factors/tensors and fiber stress concentration factors/tensors in Sect. 3, use is made of crack stress concentration factors/tensors $\left[B^c\right]$ and void stress concentration factors/tensors $[B^v]$ in this Section. Similarly, use is made of crack strain concentration factors/tensors $\left[A^c\right]$ and void strain concentration factors/tensors $[A^v]$ in this Section. Evaluation of these newly introduced concentration tensors for metals follows the same procedure outlined in the "Appendix" for the concentration tensors for composite materials.

The two damage mechanics templates introduced in this Section have the following properties:

1. Each template is described by *two* stages of damage characterization. These two stages are very clear from Figs. 3 and 4
2. Each template is *symmetrical*. The template symmetry is clearly evident in schematic diagrams in Figs. 3 and 4.
3. The two templates in this Section are the two *simplest* templates possible. This is again evident from Figs. 3 and 4. These templates apply to linear elastic materials. Check Sect. 8 on how to generate special templates for other types of materials.
4. The detailed mathematical formulation is *not* presented for both templates here, but is similar to the derivation for composites of Sects. 3.1 and 3.2. Only the *key equations* for the two templates for metals are listed in Tables 5 and 6.

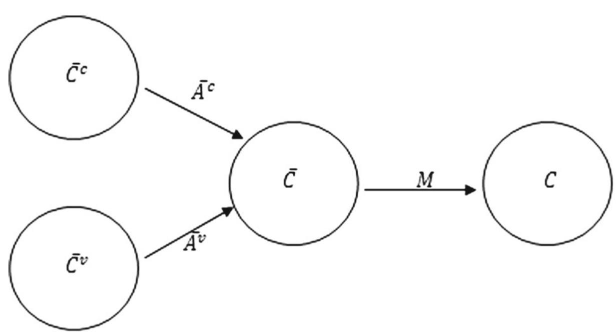

Fig. 3 Template ODMCV

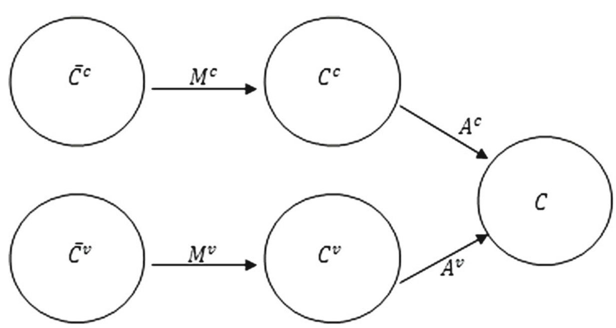

Fig. 4 Template LDMCV

Table 5 Key equations of template ODMCV

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}\} = \overline{c}^c \{\overline{\sigma}^c\} + \overline{c}^v \{\overline{\sigma}^v\}$	Law of mixtures applied to stress
Step 1-b	$\left[\overline{E}\right] = \overline{c}^c \left[\overline{E}^c\right]\left[\overline{A}^c\right] + \overline{c}^v \left[\overline{E}^v\right]\left[\overline{A}^v\right]$	Law of mixtures applied to elasticity tensor
Step 2-a	$\{\overline{\sigma}\} = [M]\{\sigma\}$	Damage transformation of stress
Step 2-b	$\{\overline{\varepsilon}\} = [M]^{-T}\{\varepsilon\}$	Damage transformation of strain
Step 2-c	$[E] = [M]^{-1}\left(\overline{c}^c \left[\overline{E}^c\right]\left[\overline{A}^c\right] + \overline{c}^v \left[\overline{E}^v\right]\left[\overline{A}^v\right]\right)[M]^{-T}$	Elasticity tensor

Table 6 Key equations of template LDMCV

Step	Equation	Description
Step 1-a	$\left\{\overline{\sigma}^k\right\} = \left[M^k\right]\left\{\sigma^k\right\}, \quad k = c, v$	Damage transformation of stress
Step 1-b	$\left\{\overline{\varepsilon}^k\right\} = \left[M^k\right]^{-T}\left\{\varepsilon^k\right\}, \quad k = c, v$	Damage transformation of strain
Step 1-c	$\left[E^k\right] = \left[M^k\right]^{-1}\left[\overline{E}^k\right]\left[M^k\right]^{-T}, \quad k = c, v$	Damage transformation of elasticity tensor
Step 2-a	$\{\sigma\} = c^c \{\sigma^c\} + c^v \{\sigma^v\}$	Law of mixtures applied to stress
Step 2-b	$[E] = c^c \left[E^c\right]\left[A^c\right] + c^v [E^v][A^v]$	Elasticity tensor

4.1 Template ODMCV (overall damage in metals of cracks and voids)

The first template to describe damage in metals is named ODMCV. This template accounts for two types of damage, namely cracks and voids. It is assumed that the material is linearly elastic when using this template. Furthermore, the use of this template implies the assumption of the rule of mixtures. In order to generate more elaborate templates that go beyond the rule of mixtures, check Sect. 7; for more elaborate templates for other types of materials, check Sect. 8; and for more elaborate templates that include other types of defects, check Sect. 9.

The schematic diagram for this template is shown in Fig. 3, while the key equations are listed in Table 5. It is noted that the schematic diagram and key equations for this template ODMCV for damaged metals are very similar to the schematic diagram and key equations for the template ODCMF for damaged composites of Sect. 3.1. The interested reader in the detailed mathematical derivation is advised to follow the derivation of Sect. 3.1.

4.2 Template LDMCV (local damage in metals of cracks and voids)

The second template to describe damage in metals is named LDMCV. This template accounts for two types of damage, namely cracks and voids. It is assumed that the material is linearly elastic when using this template. Furthermore, the use of this template implies the assumption of the rule of mixtures. In order to generate more elaborate templates that go beyond the rule of mixtures, check Sect. 7; for more elaborate templates for other types of materials, check Sect. 8; and for more elaborate templates that include other types of defects, check Sect. 9.

The schematic diagram for this template is shown in Fig. 4, while the key equations are listed in Table 6. It is noted that the schematic diagram and key equations for this template LDMCV for damaged metals are very similar to the schematic diagram and key equations for the template LDCMF for damaged composites of Sect. 3.2. The interested reader in the detailed mathematical derivation is advised to follow the derivation of Sect. 3.2.

5 Damage mechanics templates for composite materials: II

For simplicity, the damaged composite system is assumed to consist of damage in the form of distributions of cracks and voids in both the matrix and fibers. Four configurations of the damaged system are considered as four different constituents: the damaged matrix consisting of damage in the form of distributions of cracks only (i.e., voids are removed)—this is the configuration C^{mc}, the damaged matrix consisting of damage in the form of distributions of voids only (i.e., cracks are removed)—this is the configuration C^{mv}, the damaged

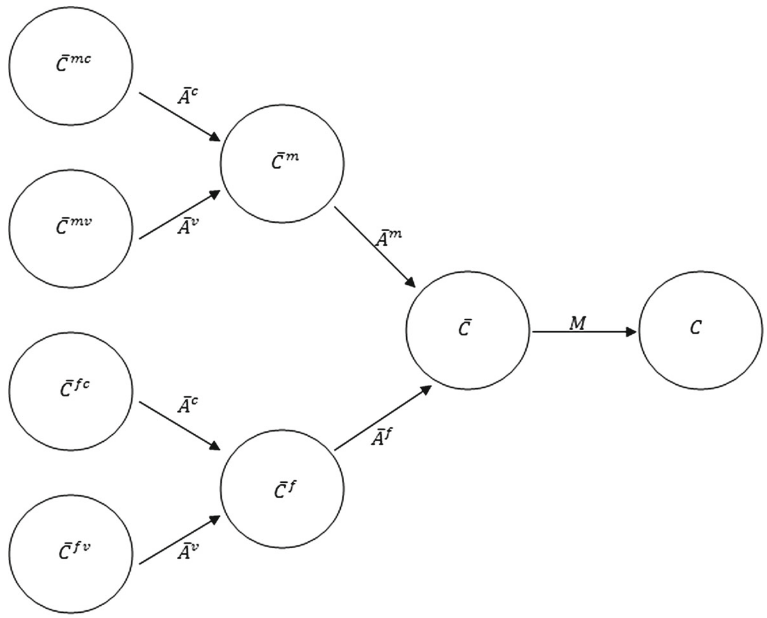

Fig. 5 Template OOCMFCV

fibers consisting of damage in the form of distributions of cracks only (i.e., voids are removed)—this is the configuration C^{fc}, and the damaged fibers consisting of damage in the form of voids only (i.e., cracks are removed)—this is the configuration C^{fv}. The damaged material is assumed to be linearly elastic. Let $\overline{C}$ denote the configuration of the undamaged composite system (consisting of matrix and fibers), and let $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$ denote the configurations of the effective undamaged matrix with cracks only, matrix with voids only, fibers with cracks only, and fibers with voids only, respectively. A superscript "mc" is used to denote a crack-related matrix quantity, "mv" is used to denote a void-related matrix quantity, "fc" is used to denote a crack-related fiber quantity, and "fv" is used to denote a void-related fiber quantity. For including other types of defects in addition to cracks and voids, check Sect. 9 for more details. For applying these templates to other types of materials, check Sect. 8 for more details. The six templates for damage in composites described in this Section utilize the rule of mixtures. Check Sect. 7 to generate advanced templates that go beyond the rule of mixtures and utilize more elaborate schemes of homogenization.

Using the four configurations outlined above (both deformed/damaged and effective/undamaged), one finds that there are six possible symmetrical combinations of the overall and local damage approaches in a metal matrix composite system. Thus, six different damage mechanics templates are generated in this Section. These six templates are presented in Sects. 5.1, 5.2, 5.3, 5.4, 5.5, and 5.6. All six templates are symmetrical as can be seen from their respective schematic diagrams. Each damage mechanics template shown in this Section is distinct from the others by the way and order of implementing the rule of mixtures, the damage transformation equations, etc. The distinguishing features of each template are clearly evident from its schematic diagram as well as its Table that lists the key equations. The total strain tensor is the same as the elastic strain tensor in this case.

Writing the governing equations for each template should be smooth and direct if one follows the arrows logically as shown in each schematic diagram. This process is illustrated in detail in Sect. 3.1 and is not repeated here. The authors decide to present the key equations only for each template in tabular form. To present the full mathematical formulation for each template would increase the size of this work unnecessarily. The six damage mechanics templates introduced in this Section have the following properties:

1. Each template is described by *three* stages of damage characterization. The three stages are very clear in the figure (i.e., schematic diagram) corresponding to each template.
2. Each template consists of *seven* sub-steps of consistent mathematical formulations. The seven sub-steps are listed in the table corresponding to each template.
3. Each template is *symmetrical*. The template symmetry is clearly evident in schematic diagrams of Figs. 5, 6, 7, 8, 9, and 10.

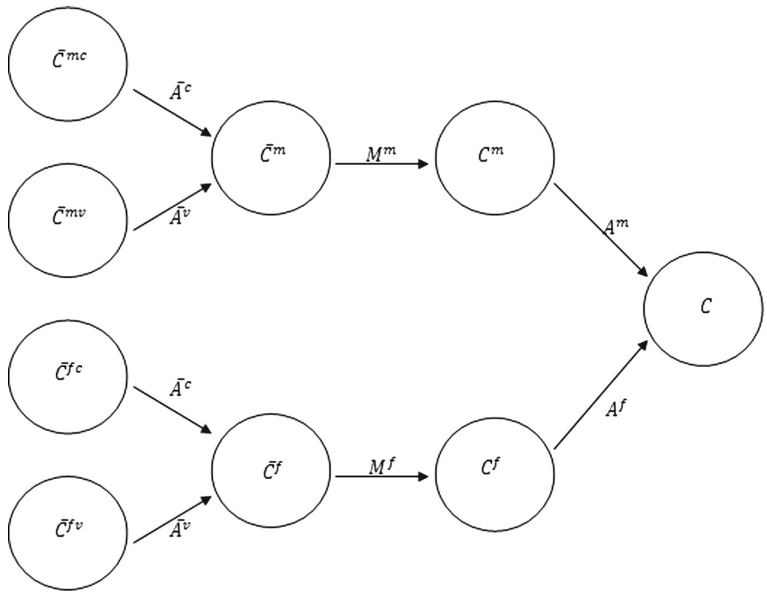

Fig. 6 Template OLCMFCV

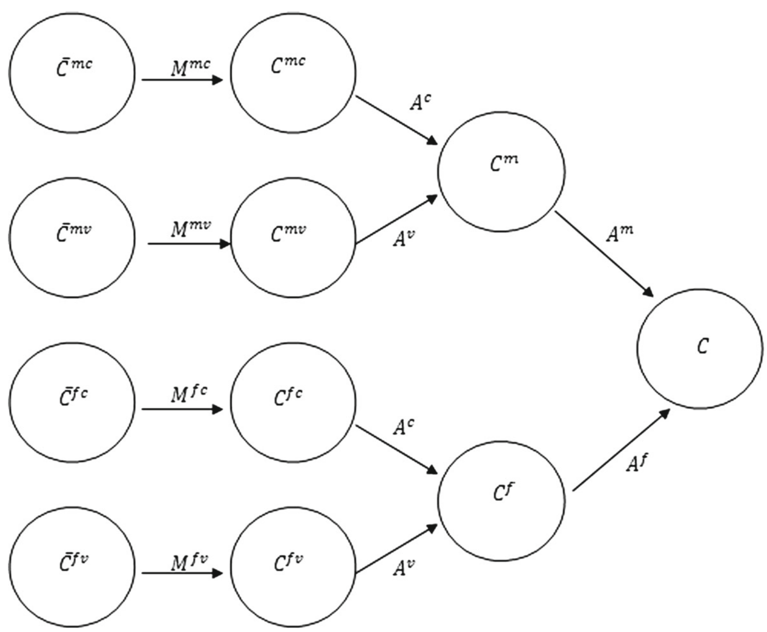

Fig. 7 Template LOCMFCV

5.1 Template OOCMFCV (overall damage)

This is the first damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 5, while Table 7 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical.

The damage mechanics template OOCMFCV consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$ and proceeds to the two intermediate effective configurations $\overline{C}^{m}$ and $\overline{C}^{f}$, where the law of mixtures is applied twice—for the matrix and fibers. The second

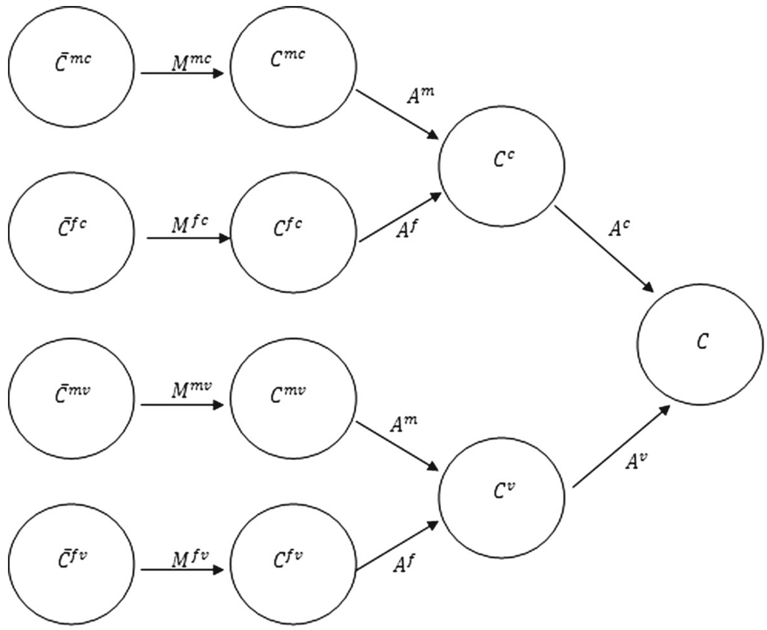

Fig. 8 Template LOCCVMF

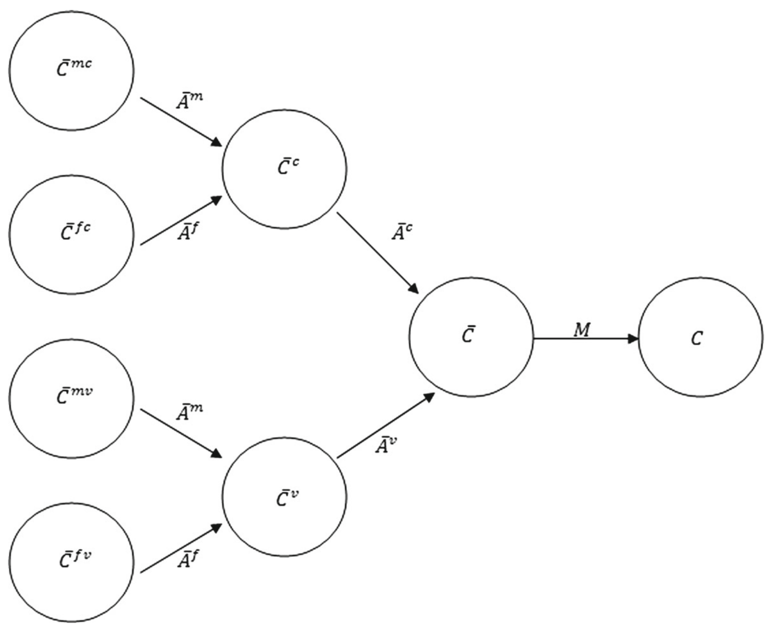

Fig. 9 Template OOCCVMF

step starts with the two intermediate effective configurations $\overline{C}^m$ and $\overline{C}^f$ and proceeds to one single intermediate effective configuration $\overline{C}$, where again the law of mixtures is used for the third time. Finally, the third step proceeds from the intermediate effective configuration $\overline{C}$ to the final deformed/damaged configuration C, where the damage transformation equations for stress and strains are used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 5, and the key equations are listed in Table 7.

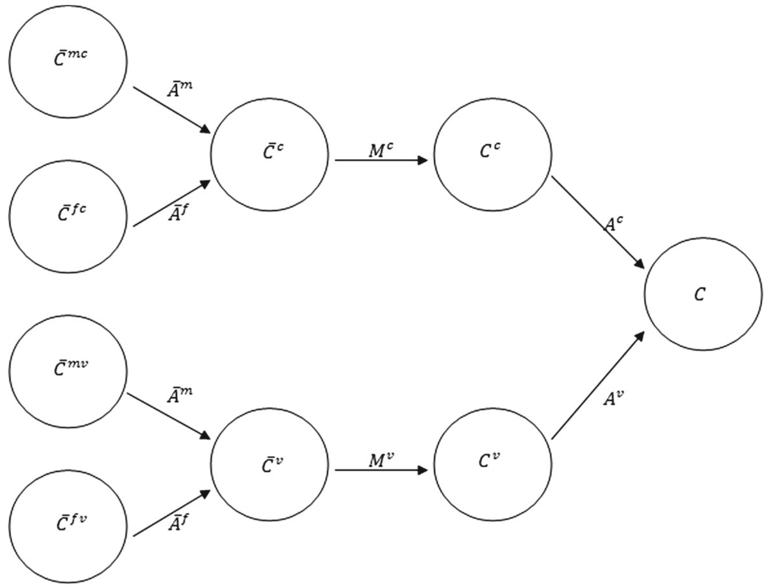

Fig. 10 Template OLCCVMF

Table 7 Key equations of template OOCMFCV

Step	Equation	Description
Step 1-a	$\left\{\overline{\sigma}^{m}\right\} = \overline{c}^{c}\left\{\overline{\sigma}^{mc}\right\} + \overline{c}^{v}\left\{\overline{\sigma}^{mv}\right\}$ $\left\{\overline{\sigma}^{f}\right\} = \overline{c}^{c}\left\{\overline{\sigma}^{fc}\right\} + \overline{c}^{v}\left\{\overline{\sigma}^{fv}\right\}$	Law of mixtures (applied twice to constituent stress)
Step 1-b	$\left[\overline{E}^{m}\right] = \overline{c}^{c}\left[\overline{E}^{mc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{mv}\right]\left[\overline{A}^{v}\right]$ $\left[\overline{E}^{f}\right] = \overline{c}^{c}\left[\overline{E}^{fc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{fv}\right]\left[\overline{A}^{v}\right]$	Law of mixtures (applied twice to constituent elasticity tensor)
Step 2-a	$\{\overline{\sigma}\} = \overline{c}^{m}\left\{\overline{\sigma}^{m}\right\} + \overline{c}^{f}\left\{\overline{\sigma}^{f}\right\}$	Law of mixtures applied to stress
Step 2-b	$\left[\overline{E}\right] = \overline{c}^{m}\overline{c}^{c}\left[\overline{E}^{mc}\right]\left[\overline{A}^{c}\right]\left[\overline{A}^{m}\right] + \overline{c}^{m}\overline{c}^{v}\left[\overline{E}^{mv}\right]\left[\overline{A}^{v}\right]\left[\overline{A}^{m}\right]$ $+ \overline{c}^{f}\overline{c}^{c}\left[\overline{E}^{fc}\right]\left[\overline{A}^{c}\right]\left[\overline{A}^{f}\right] + \overline{c}^{f}\overline{c}^{v}\left[\overline{E}^{fv}\right]\left[\overline{A}^{v}\right]\left[\overline{A}^{f}\right]$	Law of mixtures applied to elasticity tensor
Step 3-a	$\{\overline{\sigma}\} = [M]\{\sigma\}$	Damage transformation of stress
Step 3-b	$\{\overline{\varepsilon}\} = [M]^{-T}\{\varepsilon\}$	Damage transformation of strain
Step 3-c	$[E] = [M]^{-1}\begin{pmatrix} \overline{c}^{m}\overline{c}^{c}\left[\overline{E}^{mc}\right]\left[\overline{A}^{c}\right]\left[\overline{A}^{m}\right] \\ +\overline{c}^{m}\overline{c}^{v}\left[\overline{E}^{mv}\right]\left[\overline{A}^{v}\right]\left[\overline{A}^{m}\right] \\ +\overline{c}^{f}\overline{c}^{c}\left[\overline{E}^{fc}\right]\left[\overline{A}^{c}\right]\left[\overline{A}^{f}\right] \\ +\overline{c}^{f}\overline{c}^{v}\left[\overline{E}^{fv}\right]\left[\overline{A}^{v}\right]\left[\overline{A}^{f}\right] \end{pmatrix}[M]^{-T}$	Elasticity tensor

5.2 Template OLCMFCV (local damage)

This is the second damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 6, while Table 8 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical.

The damage mechanics template OLCMFCV consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to the two intermediate effective

Table 8 Key equations of template OLCMFCV

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^m\} = \overline{c}^c \{\overline{\sigma}^{mc}\} + \overline{c}^v \{\overline{\sigma}^{mv}\}$	
	$\{\overline{\sigma}^f\} = \overline{c}^c \{\overline{\sigma}^{fc}\} + \overline{c}^v \{\overline{\sigma}^{fv}\}$	Law of mixtures (applied twice to constituent stress)
Step 1-b	$[\overline{E}^m] = \overline{c}^c [\overline{E}^{mc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{mv}][\overline{A}^v]$	
	$[\overline{E}^f] = \overline{c}^c [\overline{E}^{fc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{fv}][\overline{A}^v]$	Law of mixtures (applied twice to constituent elasticity tensor)
Step 2-a	$\{\overline{\sigma}^m\} = [M^m]\{\sigma^m\}$	
	$\{\overline{\sigma}^f\} = [M^f]\{\sigma^f\}$	Damage transformation of stress (applied twice)
Step 2-b	$\{\overline{\varepsilon}^m\} = [M^m]^{-T}\{\varepsilon^m\}$	
	$\{\overline{\varepsilon}^f\} = [M^f]^{-T}\{\varepsilon^f\}$	Damage transformation of strain (applied twice)
Step 2-c	$[E^m] = [M^m]^{-1}\left(\overline{c}^c [\overline{E}^{mc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{mv}][\overline{A}^v]\right)[M^m]^{-T}$	
	$[E^f] = [M^f]^{-1}\left(\overline{c}^c [\overline{E}^{fc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{fv}][\overline{A}^v]\right)[M^f]^{-T}$	Damage transformation of elasticity (applied twice)
Step 3-a	$\{\sigma\} = c^m\{\sigma^m\} + c^f\{\sigma^f\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^m [M^m]^{-1}\left(\overline{c}^c [\overline{E}^{mc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{mv}][\overline{A}^v]\right)[M^m]^{-T}[A^m]$	
	$+ c^f [M^f]^{-1}\left(\overline{c}^c [\overline{E}^{fc}][\overline{A}^c] + \overline{c}^v [\overline{E}^{fv}][\overline{A}^v]\right)[M^f]^{-T}[A^f]$	Elasticity tensor

configurations $\overline{C}^m$ and $\overline{C}^f$, where the law of mixtures is applied twice—for the matrix and fibers. The second step starts with the two intermediate effective configurations $\overline{C}^m$ and $\overline{C}^f$ and proceeds to the two intermediate damaged configurations C^m and C^f, where the damage transformation equations for both the stress and strain are applied twice—for the matrix and fibers. In this regard, expressions are obtained for the fourth-rank matrix and fiber elasticity tensors. Finally, the third step proceeds form the two intermediate damaged configurations C^m and C^f to the final deformed/damaged configuration C, where the law of mixtures is used for the third time to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 6, and the key equation are listed in Table 8.

5.3 Template LOCMFCV (overall damage)

This is the third damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 7, while Table 9 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical.

The damage mechanics template LOCMFCV consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to the four intermediate damaged configurations C^{mc}, C^{mv}, C^{fc}, and C^{fv}, where damage transformation equations for stress and strain are applied—each four times. The second step starts with the four intermediate damaged configurations C^{mc}, C^{mv}, C^{fc}, and C^{fv}, and proceeds to the two intermediate damaged configurations C^m and C^f, where the law of mixtures is applied twice—for the matrix and fibers. In this regard, expressions are obtained for the fourth-rank matrix and fiber elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^m and C^f to the final deformed/damaged configuration C, where the law of mixtures is used for the third time to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 7, and the key equations are listed in Table 9.

Table 9 Key equations of template LOCMFCV

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^{mc}\} = [M^{mc}]\{\sigma^{mc}\}$ $\{\overline{\sigma}^{mv}\} = [M^{mv}]\{\sigma^{mv}\}$ $\{\overline{\sigma}^{fc}\} = [M^{fc}]\{\sigma^{fc}\}$ $\{\overline{\sigma}^{fv}\} = [M^{fv}]\{\sigma^{fv}\}$	Damage transformation equations for stresses (applied four times)
Step 1-b	$\{\overline{\varepsilon}^{mc}\} = [M^{mc}]^{-T}\{\varepsilon^{mc}\}$ $\{\overline{\varepsilon}^{mv}\} = [M^{mv}]^{-T}\{\varepsilon^{mv}\}$ $\{\overline{\varepsilon}^{fc}\} = [M^{fc}]^{-T}\{\varepsilon^{fc}\}$ $\{\overline{\varepsilon}^{fv}\} = [M^{fv}]^{-T}\{\varepsilon^{fv}\}$	Damage transformation equations for strains (applied four times)
Step 1-c	$[E^{mc}] = [M^{mc}]^{-1}[\overline{E}^{mc}][M^{mc}]^{-T}$ $[E^{mv}] = [M^{mv}]^{-1}[\overline{E}^{mv}][M^{mv}]^{-T}$ $[E^{fc}] = [M^{fc}]^{-1}[\overline{E}^{fc}][M^{fc}]^{-T}$ $[E^{fv}] = [M^{fv}]^{-1}[\overline{E}^{fv}][M^{fv}]^{-T}$	Damage transformation equations for elasticity tensors (applied four times)
Step 2-a	$\{\sigma^{m}\} = c^{c}\{\sigma^{mc}\} + c^{v}\{\sigma^{mv}\}$ $\{\sigma^{f}\} = c^{c}\{\sigma^{fc}\} + c^{v}\{\sigma^{fv}\}$	Law of mixtures (applied twice to stresses)
Step 2-b	$[E^{m}] = c^{c}[E^{mc}][A^{c}] + c^{v}[E^{mv}][A^{v}]$ $[E^{f}] = c^{c}[E^{fc}][A^{c}] + c^{v}[E^{fv}][A^{v}]$	Law of mixtures (applied twice to elasticity tensors)
Step 3-a	$\{\sigma\} = c^{m}\{\sigma^{m}\} + c^{f}\{\sigma^{f}\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^{m}c^{c}[E^{mc}][A^{m}][A^{c}] + c^{m}c^{v}[E^{mv}][A^{m}][A^{v}]$ $+ c^{f}c^{c}[E^{fc}][A^{f}][A^{c}] + c^{f}c^{v}[E^{fv}][A^{f}][A^{v}]$	Elasticity tensor

5.4 Template LOCCVMF (local damage)

This is the fourth damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 8, while Table 10 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical. This template is similar geometrically to the template LOCMFCV of Sect. 5.3. They differ only in the order of the starting the four effective configurations.

The damage mechanics template LOCMFCV consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{fc}$, $\overline{C}^{mv}$, and $\overline{C}^{fv}$, and proceeds to the four intermediate damaged configurations C^{mc}, C^{fc}, C^{mv}, and C^{fv}, where damage transformation equations for stress and strain are applied—each four times. The second step starts with the four intermediate damaged configurations C^{mc}, C^{fc}, C^{mv}, and C^{fv}, and proceeds to the two intermediate damaged configurations C^{c} and C^{v}, where the law of mixtures is applied twice—for the cracks and voids. In this regard, expressions are obtained for the fourth-rank crack and void elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^{c} and C^{v}, to the final deformed/damaged configuration C, where the law of mixtures is used for the third time to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 8, and the key equations are listed in Table 10.

Table 10 Key equations of template LOCCVMF

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^{mc}\} = \left[M^{mc}\right]\{\sigma^{mc}\}$ $\left\{\overline{\sigma}^{fc}\right\} = \left[M^{fc}\right]\left\{\sigma^{fc}\right\}$ $\{\overline{\sigma}^{mv}\} = \left[M^{mv}\right]\{\sigma^{mv}\}$ $\left\{\overline{\sigma}^{fv}\right\} = \left[M^{fv}\right]\left\{\sigma^{fv}\right\}$	Damage transformation equations for stresses (applied four times)
Step 1-b	$\{\overline{\varepsilon}^{mc}\} = \left[M^{mc}\right]^{-T}\{\varepsilon^{mc}\}$ $\left\{\overline{\varepsilon}^{fc}\right\} = \left[M^{fc}\right]^{-T}\left\{\varepsilon^{fc}\right\}$ $\{\overline{\varepsilon}^{mv}\} = \left[M^{mv}\right]^{-T}\{\varepsilon^{mv}\}$ $\left\{\overline{\varepsilon}^{fv}\right\} = \left[M^{fv}\right]^{-T}\left\{\varepsilon^{fv}\right\}$	Damage transformation equations for strains (applied four times)
Step 1-c	$\left[E^{mc}\right] = \left[M^{mc}\right]^{-1}\left[\overline{E}^{mc}\right]\left[M^{mc}\right]^{-T}$ $\left[E^{fc}\right] = \left[M^{fc}\right]^{-1}\left[\overline{E}^{fc}\right]\left[M^{fc}\right]^{-T}$ $\left[E^{mv}\right] = \left[M^{mv}\right]^{-1}\left[\overline{E}^{mv}\right]\left[M^{mv}\right]^{-T}$ $\left[E^{fv}\right] = \left[M^{fv}\right]^{-1}\left[\overline{E}^{fv}\right]\left[M^{fv}\right]^{-T}$	Damage transformation equations for elasticity tensors (applied four times)
Step 2-a	$\{\sigma^{c}\} = c^{m}\{\sigma^{mc}\} + c^{f}\left\{\sigma^{fc}\right\}$ $\{\sigma^{v}\} = c^{m}\{\sigma^{mv}\} + c^{f}\left\{\sigma^{fv}\right\}$	Law of mixtures (applied twice to stresses)
Step 2-b	$\left[E^{c}\right] = c^{m}\left[E^{mc}\right]\left[A^{m}\right] + c^{f}\left[E^{fc}\right]\left[A^{f}\right]$ $\left[E^{v}\right] = c^{m}\left[E^{mv}\right]\left[A^{m}\right] + c^{f}\left[E^{fv}\right]\left[A^{f}\right]$	Law of mixtures (applied twice to elasticity tensors)
Step 3-a	$\{\sigma\} = c^{c}\{\sigma^{c}\} + c^{v}\{\sigma^{v}\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^{c}\left[M^{c}\right]^{-1}\left(\overline{c}^{m}\left[\overline{E}^{mc}\right]\left[\overline{A}^{m}\right] + \overline{c}^{f}\left[\overline{E}^{fc}\right]\left[\overline{A}^{f}\right]\right)\left[M^{c}\right]^{-T}\left[A^{c}\right]$ $+c^{v}\left[M^{v}\right]^{-1}\left(\overline{c}^{m}\left[\overline{E}^{mv}\right]\left[\overline{A}^{m}\right] + \overline{c}^{f}\left[\overline{E}^{fv}\right]\left[\overline{A}^{f}\right]\right)\left[M^{v}\right]^{-T}\left[A^{v}\right]$	Elasticity tensor

5.5 Template OOCCVMF (overall damage)

This is the fifth damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 9, while Table 11 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical. This template is similar geometrically to the template OOCMFCV of Sect. 5.1. They differ only in the order of the starting four effective configurations.

The damage mechanics template OOCCVMF consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to the two intermediate effective configurations $\overline{C}^{c}$ and $\overline{C}^{v}$, where the law of mixture is applied twice—for the cracks and voids. The second step starts with the two intermediate effective configurations $\overline{C}^{c}$ and $\overline{C}^{v}$ and proceeds to one single intermediate effective configuration $\overline{C}$, where again the law of mixtures is used for the third time. Finally, the third step proceeds from the intermediate effective configuration $\overline{C}$ to the final deformed/damaged configuration C, where the damage transformation equations for stress and strains are used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 9, and the key equations are listed in Table 11.

5.6 Template OLCCVMF (local damage)

This is the sixth damage mechanics template that consists of three steps. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic

Table 11 Key equations of template OOCCVMF

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^c\} = \overline{c}^m \{\overline{\sigma}^{mc}\} + \overline{c}^f \{\overline{\sigma}^{fc}\}$ $\{\overline{\sigma}^v\} = \overline{c}^m \{\overline{\sigma}^{mv}\} + \overline{c}^f \{\overline{\sigma}^{fv}\}$	Law of mixtures (applied twice to stresses)
Step 1-b	$\left[\overline{E}^c\right] = \overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]$ $\left[\overline{E}^v\right] = \overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]$	Law of mixtures (applied twice to elasticity tensors)
Step 2-a	$\{\overline{\sigma}\} = \overline{c}^c \{\overline{\sigma}^c\} + \overline{c}^v \{\overline{\sigma}^v\}$	Law of mixtures applied to stress
Step 2-b	$\left[\overline{E}\right] = \overline{c}^c\overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right]\left[\overline{A}^c\right] + \overline{c}^c\overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]\left[\overline{A}^c\right]$ $+\overline{c}^v\overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right]\left[\overline{A}^v\right] + \overline{c}^v\overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]\left[\overline{A}^v\right]$	Law of mixtures applied to elasticity tensor
Step 3-a	$\{\overline{\sigma}\} = [M]\{\sigma\}$	Damage transformation of stress
Step 3-b	$\{\overline{\varepsilon}\} = [M]^{-T}\{\varepsilon\}$	Damage transformation of strain
Step 3-c	$[E] = [M]^{-1}\begin{pmatrix} \overline{c}^c\overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right]\left[\overline{A}^c\right] \\ +\overline{c}^c\overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]\left[\overline{A}^c\right] \\ +\overline{c}^v\overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right]\left[\overline{A}^v\right] \\ +\overline{c}^v\overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]\left[\overline{A}^v\right] \end{pmatrix}[M]^{-T}$	Elasticity tensor

diagram of this template is shown in Fig. 10, while Table 12 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is symmetrical. This template is similar geometrically to the template OLCMFCV of Sect. 5.2. They differ only in the order of the starting four effective configurations.

The damage mechanics template OLCCVMF consists of three distinct steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to the two intermediate effective configurations $\overline{C}^c$ and $\overline{C}^v$, where the law of mixtures is applied twice—for the cracks and voids. The second step starts with the two intermediate effective configurations $\overline{C}^c$ and $\overline{C}^v$ and proceeds to the two intermediate damaged configurations C^c and C^v, where the damage transformation equations for both the stress and strain are applied twice—for the cracks and voids. In this regard, expressions are obtained for the fourth-rank matrix and fiber elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^c and C^v, to the final deformed/damaged configuration C, where the law of mixtures is used for the third time to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 10 and the key equations listed in Table 12.

Looking at the six damage mechanics templates presented in this Section, one concludes that the two templates of Sects. 5.1 and 5.5 are *homogeneous* in the sense that they use the overall approach exclusively, while the four templates of Sects. 5.2, 5.3, 5.4, and 5.6 are *mixed* (or *inhomogeneous*) in the sense that they use a mix of overall and local aspects. In this regard, it is very clear from the final expression of the damaged elasticity tensor, as shown in the last row of the table of each respective template, that the expressions obtained for the two homogeneous templates are totally different in form from those obtained for the four inhomogeneous (or mixed) templates. As shown also in these expressions, the four mixed templates provide for a more diverse representation of damage mechanisms that include more than one approach of damage characterization.

6 Damage mechanics templates for composite materials: III

For simplicity, the damaged composite system is assumed to consist of damage in the form of distributions of cracks and voids in both the matrix and fibers. The notation used in Sect. 5 is utilized here and is not repeated—check the first paragraph of Sect. 5 for details. Using the four configurations outlined (both deformed/damaged and effective/undamaged), one finds that there are four possible unsymmetrical combinations of the overall and local damage approaches in a metal matrix composite system. Thus, four different damage mechanics

Table 12 Key equations of template OLCCVMF

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^c\} = \overline{c}^m \{\overline{\sigma}^{mc}\} + \overline{c}^f \{\overline{\sigma}^{fc}\}$ $\{\overline{\sigma}^v\} = \overline{c}^m \{\overline{\sigma}^{mv}\} + \overline{c}^f \{\overline{\sigma}^{fv}\}$	Law of mixtures (applied twice to stresses)
Step 1-b	$\left[\overline{E}^c\right] = \overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]$ $\left[\overline{E}^v\right] = \overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]$	Law of mixtures (applied twice to elasticity tensors)
Step 2-a	$\{\overline{\sigma}^c\} = \left[M^c\right]\{\sigma^c\}$ $\{\overline{\sigma}^v\} = \left[M^v\right]\{\sigma^v\}$	Damage transformation of stress (applied twice)
Step 2-b	$\{\overline{\varepsilon}^c\} = \left[M^c\right]^{-T}\{\varepsilon^c\}$ $\{\overline{\varepsilon}^v\} = \left[M^v\right]^{-T}\{\varepsilon^v\}$	Damage transformation of strain (applied twice)
Step 2-c	$\left[E^c\right] = \left[M^c\right]^{-1}\left(\overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]\right)\left[M^c\right]^{-T}$ $\left[E^v\right] = \left[M^v\right]^{-1}\left(\overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]\right)\left[M^v\right]^{-T}$	Damage transformation of elasticity (applied twice)
Step 3-a	$\{\sigma\} = c^c\{\sigma^c\} + c^v\{\sigma^v\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^c\left[M^c\right]^{-1}\left(\overline{c}^m \left[\overline{E}^{mc}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fc}\right]\left[\overline{A}^f\right]\right)\left[M^c\right]^{-T}\left[A^c\right]$ $+c^v\left[M^v\right]^{-1}\left(\overline{c}^m \left[\overline{E}^{mv}\right]\left[\overline{A}^m\right] + \overline{c}^f \left[\overline{E}^{fv}\right]\left[\overline{A}^f\right]\right)\left[M^v\right]^{-T}\left[A^v\right]$	Elasticity tensor

templates are generated in this Section. These four templates are presented in Sects. 6.1 , 6.2, 6.3, and 6.4. All four templates are unsymmetrical as can be seen from their respective schematic diagrams. The four templates apply to linear elastic composite materials. The total strain tensor is the same as the elastic strain tensor in this case. For other types of materials, for using homogenization methods, or for including the effects of additional defects, the reader is referred to Sects. 7, 8, and 9.

Writing the governing equations for each template should be smooth and direct if one follows the arrows logically as shown in each schematic diagram. This process is illustrated in detail in Sect. 3.1 and is not repeated here. It is emphasized that the four templates presented in this Section are more complex than the six templates of Sect. 5, mainly because the present four templates are "mixed" in the sense that overall and local steps are used interchangeably as shown in the schematic diagrams. Unlike the templates of Sect. 5, it is especially noted that these templates are unsymmetrical.

The four damage mechanics templates introduced in this Section have the following properties:

1. Each template is described by *three* stages of damage characterization. The three stages are very clear in the Figure corresponding to each template.
2. Each template consists of *twelve* sub-steps of consistent mathematical formulation. The twelve sub-steps are listed in the table corresponding to each template.
3. Each template is *unsymmetrical*. The template asymmetry is clearly evident in schematic diagrams of Figs. 11, 12, 13, and 14.

6.1 Template OLLOCMFCV (overall damage in composites with matrix and fibers of cracks and voids)

This is the first damage mechanics template that consists of three steps and is unsymmetrical. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 11, while Table 13 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is unsymmetrical in the sense that local damage tensors are introduced partially and locally (only for the fibers) in the first step, then followed by the local damage tensor for the matrix which is introduced asymmetrically in the second step. Thus, unlike the six templates of Sect. 5, clearly there is no symmetry in the way the damage tensors are introduced for the matrix and fibers.

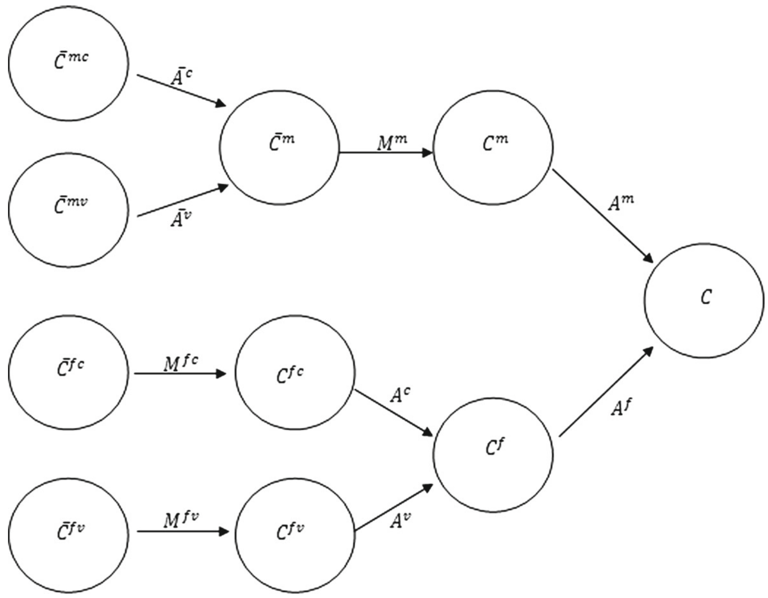

Fig. 11 Template OLLOCMFCV

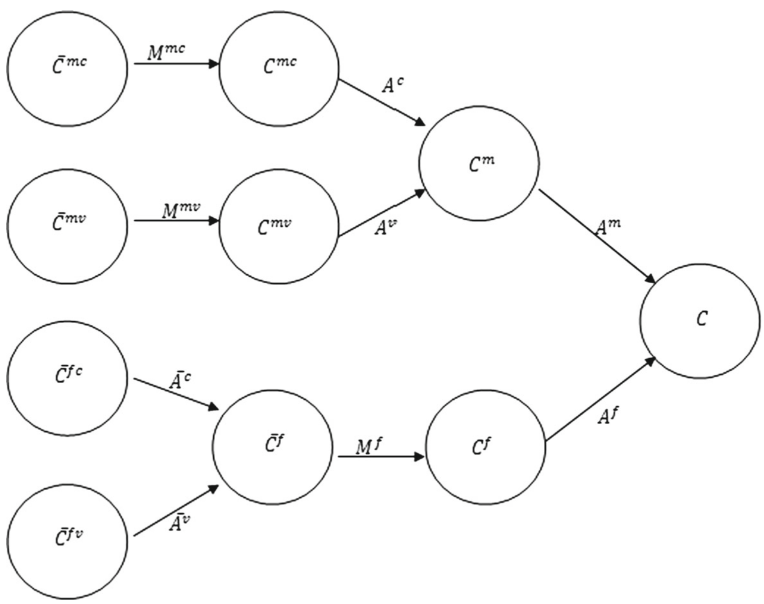

Fig. 12 Template LOOLCMFCV

The damage mechanics template OLLOCMFCV consists of three distinct steps comprising a total of twelve sub-steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to three intermediate configurations: the effective matrix configuration $\overline{C}^{m}$, and the two damaged fiber configurations C^{fc}, and C^{fv}. The second step starts with the three intermediate configurations $\overline{C}^{m}$, C^{fc}, and C^{fv}, and proceeds to the two intermediate damaged configurations C^{m} and C^{f}. In this regard, expressions are obtained for the fourth-rank matrix and fiber elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^{m} and C^{f} to the final deformed/damaged configuration C, where the law of mixtures is used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 11, and the key equations are listed in Table 13. It is noted that both the first and second steps consists each of five sub-steps, while the third step consists of two sub-steps.

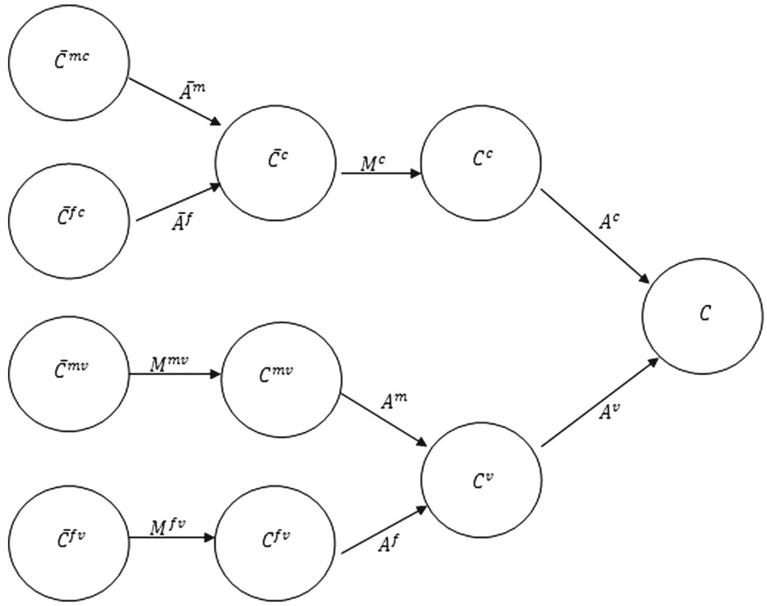

Fig. 13 Template OLLOCCVMF

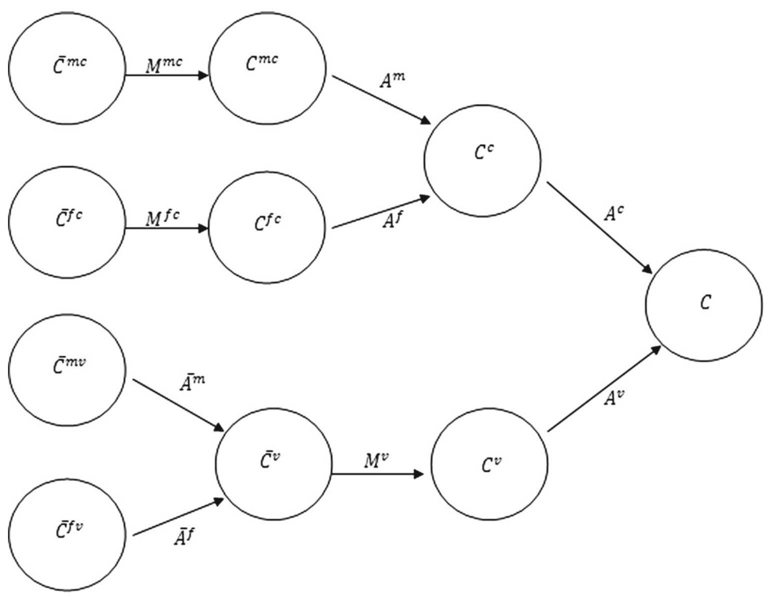

Fig. 14 Template LOOLCCVMF

6.2 Template LOOLCMFCV (local damage in composites with matrix and fibers of cracks and voids)

This is the second damage mechanics template that consists of three steps and is unsymmetrical. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 12, while Table 14 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is unsymmetrical in the sense that local damage tensors are introduced partially and locally (only for the matrix) in the first step, then followed by the local damage tensor for the fibers which is introduced asymmetrically in the second step. Thus, unlike the six templates of Sect. 5, clearly there is no symmetry in the way the damage tensors are introduced for the matrix and fibers.

Table 13 Key equations of template OLLOCMFCV

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^{m}\} = \overline{c}^{c}\{\overline{\sigma}^{mc}\} + \overline{c}^{v}\{\overline{\sigma}^{mv}\}$	Law of mixtures for matrix applied to stress
Step 1-b	$\{\overline{\sigma}^{fc}\} = [M^{fc}]\{\sigma^{fc}\}$ $\{\overline{\sigma}^{fv}\} = [M^{fv}]\{\sigma^{fv}\}$	Damage transformation for fiber stresses applied twice
Step 1-c	$\{\overline{\varepsilon}^{fc}\} = [M^{fc}]^{-T}\{\varepsilon^{fc}\}$ $\{\overline{\varepsilon}^{fv}\} = [M^{fv}]^{-T}\{\varepsilon^{fv}\}$	Damage transformation for fiber strains applied twice
Step 1-d	$\left[\overline{E}^{m}\right] = \overline{c}^{c}\left[\overline{E}^{mc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{mv}\right]\left[\overline{A}^{v}\right]$	Law of mixtures for matrix applied to elasticity tensor
Step 1-e	$[E^{fc}] = [M^{fc}]^{-1}\left[\overline{E}^{fc}\right][M^{fc}]^{-T}$ $[E^{fv}] = [M^{fv}]^{-1}\left[\overline{E}^{fv}\right][M^{fv}]^{-T}$	Damage transformation for fiber elasticity applied twice
Step 2-a	$\{\overline{\sigma}^{m}\} = [M^{m}]\{\sigma^{m}\}$	Damage transformation of matrix stress
Step 2-b	$\{\overline{\varepsilon}^{m}\} = [M^{m}]^{-T}\{\varepsilon^{m}\}$	Damage transformation for matrix strain
Step 2-c	$\{\sigma^{f}\} = c^{c}\{\sigma^{fc}\} + c^{v}\{\sigma^{fv}\}$	Law of mixtures applied to stress
Step 2-d	$[E^{m}] = [M^{m}]^{-1}\left[\overline{E}^{m}\right][M^{m}]^{-T}$	Damage transformation for matrix elasticity
Step 2-e	$[E^{f}] = c^{c}[E^{fc}][A^{c}] + c^{v}[E^{fv}][A^{v}]$	Law of mixtures applied to fiber elasticity tensor
Step 3-a	$\{\sigma\} = c^{m}\{\sigma^{m}\} + c^{f}\{\sigma^{f}\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^{m}[E^{m}] + c^{f}[E^{f}]$ $= c^{m}[M^{m}]^{-1}\left[\overline{E}^{m}\right][M^{m}]^{-T} + c^{f}\left(c^{c}[E^{fc}][A^{c}] + c^{v}[E^{fv}][A^{v}]\right)$ $= c^{m}[M^{m}]^{-1}\left(\overline{c}^{c}\left[\overline{E}^{mc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{mv}\right]\left[\overline{A}^{v}\right]\right)[M^{m}]^{-T}$ $+c^{f}\left(c^{c}[M^{fc}]^{-1}\left[\overline{E}^{fc}\right][M^{fc}]^{-T}[A^{c}] + c^{v}[M^{fv}]^{-1}\left[\overline{E}^{fv}\right][M^{fv}]^{-T}[A^{v}]\right)$	Elasticity tensor

The damage mechanics template LOOLCMFCV consists of three distinct steps comprising a total of twelve sub-steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{mv}$, $\overline{C}^{fc}$, and $\overline{C}^{fv}$, and proceeds to three intermediate configurations: the two damaged matrix configurations C^{mc} and C^{mv}, and the effective fiber configuration $\overline{C}^{f}$. The second step starts with the three intermediate configurations C^{mc}, C^{mv}, and $\overline{C}^{f}$, and proceeds to the two intermediate damaged configurations C^{m} and C^{f}. In this regard, expressions are obtained for the fourth-rank matrix and fiber elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^{m} and C^{f} to the final deformed/damaged configuration C, where the law of mixtures is used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 12, and the key equations are listed in Table 14. It is noted that both the first and second steps consist each of five sub-steps, while the third step consists of two sub-steps.

Table 14 Key equations of template OLLOCMFCV

Step	Equation	Description
Step 1-a	$\left\{\overline{\sigma}^{f}\right\} = \overline{c}^{c}\left\{\overline{\sigma}^{fc}\right\} + \overline{c}^{v}\left\{\overline{\sigma}^{fv}\right\}$	Law of mixtures for fibers applied to stress
Step 1-b	$\{\overline{\sigma}^{mc}\} = \left[M^{mc}\right]\{\sigma^{mc}\}$ $\{\overline{\sigma}^{mv}\} = \left[M^{mv}\right]\{\sigma^{mv}\}$	Damage transformation for matrix stresses applied twice
Step 1-c	$\{\overline{\varepsilon}^{mc}\} = \left[M^{mc}\right]^{-T}\{\varepsilon^{mc}\}$ $\{\overline{\varepsilon}^{mv}\} = \left[M^{mv}\right]^{-T}\{\varepsilon^{mv}\}$	Damage transformation for matrix strains applied twice
Step 1-d	$\left[\overline{E}^{f}\right] = \overline{c}^{c}\left[\overline{E}^{fc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{fv}\right]\left[\overline{A}^{v}\right]$	Law of mixtures for fibers applied to elasticity tensor
Step 1-e	$\left[E^{mc}\right] = \left[M^{mc}\right]^{-1}\left[\overline{E}^{mc}\right]\left[M^{mc}\right]^{-T}$ $\left[E^{mv}\right] = \left[M^{mv}\right]^{-1}\left[\overline{E}^{mv}\right]\left[M^{mv}\right]^{-T}$	Damage transformation for matrix elasticity applied twice
Step 2-a	$\left\{\overline{\sigma}^{f}\right\} = \left[M^{f}\right]\left\{\sigma^{f}\right\}$	Damage transformation of fiber stress
Step 2-b	$\left\{\overline{\varepsilon}^{f}\right\} = \left[M^{f}\right]^{-T}\left\{\varepsilon^{f}\right\}$	Damage transformation for fiber strain
Step 2-c	$\{\sigma^{m}\} = c^{c}\{\sigma^{mc}\} + c^{v}\{\sigma^{mv}\}$	Law of mixtures applied to stress
Step 2-d	$\left[E^{f}\right] = \left[M^{f}\right]^{-1}\left[\overline{E}^{f}\right]\left[M^{f}\right]^{-T}$	Damage transformation for fiber elasticity
Step 2-e	$\left[E^{m}\right] = c^{c}\left[E^{mc}\right]\left[A^{c}\right] + c^{v}\left[E^{mv}\right]\left[A^{v}\right]$	Law of mixtures applied to matrix elasticity tensor
Step 3-a	$\{\sigma\} = c^{m}\{\sigma^{m}\} + c^{f}\left\{\sigma^{f}\right\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^{m}\left[E^{m}\right] + c^{f}\left[E^{f}\right]$ $= c^{m}\left(c^{c}\left[E^{mc}\right]\left[A^{c}\right] + c^{v}\left[E^{mv}\right]\left[A^{v}\right]\right) + c^{f}\left[M^{f}\right]^{-1}\left[\overline{E}^{f}\right]\left[M^{f}\right]^{-T}$ $= c^{m}\left(c^{c}\left[M^{mc}\right]^{-1}\left[\overline{E}^{mc}\right]\left[M^{mc}\right]^{-T}\left[A^{c}\right] + c^{v}\left[M^{mv}\right]^{-1}\left[\overline{E}^{mv}\right]\left[M^{mv}\right]^{-T}\left[A^{v}\right]\right)$ $+ c^{f}\left[M^{f}\right]^{-1}\left(\overline{c}^{c}\left[\overline{E}^{fc}\right]\left[\overline{A}^{c}\right] + \overline{c}^{v}\left[\overline{E}^{fv}\right]\left[\overline{A}^{v}\right]\right)\left[M^{f}\right]^{-T}$	Elasticity tensor

6.3 Template OLLOCCVMF (overall damage in composites with matrix and fibers of cracks and voids)

This is the third damage mechanics template that consists of three steps and is unsymmetrical. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 13, while Table 15 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by following the arrows in the schematic diagram. It is clear from the Figure that this template is unsymmetrical in the sense that local damage tensors are introduced partially and locally (only for the voids in both matrix and fibers) in the first step, then followed by the local damage tensor for the cracks (in both matrix and fibers) which is introduced asymmetrically in the second step. Thus, unlike the six templates of Sect. 5, clearly there is no symmetry in the way the damage tensors are introduced for the cracks and voids.

The damage mechanics template OLLOCCVMF consists of three distinct steps comprising a total of twelve sub-steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{fc}$, $\overline{C}^{mv}$, and $\overline{C}^{fv}$, and proceeds to three intermediate configurations: the effective "crack" configuration $\overline{C}^{c}$, and the two damaged "void" configurations C^{mv} and C^{fv}. The second step starts with the three intermediate configurations $\overline{C}^{c}$, C^{mv}, and

Table 15 Key equations of template OLLOCCVMF

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^c\} = \overline{c}^m \{\overline{\sigma}^{mc}\} + \overline{c}^f \{\overline{\sigma}^{fc}\}$	Law of mixtures for matrix applied to stress
Step 1-b	$\{\overline{\sigma}^{mv}\} = [M^{mv}] \{\sigma^{mv}\}$ $\{\overline{\sigma}^{fv}\} = [M^{fv}] \{\sigma^{fv}\}$	Damage transformation for "void" stresses applied twice
Step 1-c	$\{\overline{\varepsilon}^{mv}\} = [M^{mv}]^{-T} \{\varepsilon^{mv}\}$ $\{\overline{\varepsilon}^{fv}\} = [M^{fv}]^{-T} \{\varepsilon^{fv}\}$	Damage transformation for "void" strains applied twice
Step 1-d	$[\overline{E}^c] = \overline{c}^m [\overline{E}^{mc}] [\overline{A}^m] + \overline{c}^f [\overline{E}^{fc}] [\overline{A}^f]$	Law of mixtures for cracks applied to elasticity tensor
Step 1-e	$[E^{mv}] = [M^{mv}]^{-1} [\overline{E}^{mv}] [M^{mv}]^{-T}$ $[E^{fv}] = [M^{fv}]^{-1} [\overline{E}^{fv}] [M^{fv}]^{-T}$	Damage transformation for "void" elasticity applied twice
Step 2-a	$\{\overline{\sigma}^c\} = [M^c] \{\sigma^c\}$	Damage transformation of "crack" stress
Step 2-b	$\{\overline{\varepsilon}^c\} = [M^c]^{-T} \{\varepsilon^c\}$	Damage transformation for "crack" strain
Step 2-c	$\{\sigma^v\} = c^m \{\sigma^{mv}\} + c^f \{\sigma^{fv}\}$	Law of mixtures applied to stress
Step 2-d	$[E^c] = [M^c]^{-1} [\overline{E}^c] [M^c]^{-T}$	Damage transformation for "crack" elasticity tensor
Step 2-e	$[E^v] = c^m [E^{mv}] [A^m] + c^f [E^{fv}] [A^f]$	Law of mixtures applied to "void" elasticity tensor
Step 3-a	$\{\sigma\} = c^c \{\sigma^c\} + c^v \{\sigma^v\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^c [E^c] + c^v [E^v]$ $= c^c [M^c]^{-1} [\overline{E}^c] [M^c]^{-T} + c^v \left(c^m [E^{mv}] [A^m] + c^f [E^{fv}] [A^f]\right)$ $= c^c [M^c]^{-1} \left(\overline{c}^m [\overline{E}^{mc}] [\overline{A}^m] + \overline{c}^f [\overline{E}^{fc}] [\overline{A}^f]\right) [M^c]^{-T}$ $+ c^v \left(c^m [M^{mv}]^{-1} [\overline{E}^{mv}] [M^{mv}]^{-T} [A^m] + c^f [M^{fv}]^{-1} [\overline{E}^{fv}] [M^{fv}]^{-T} [A^f]\right)$	Elasticity tensor

C^{fv}, and proceeds to the two intermediate damaged configurations C^c and C^v. In this regard, expressions are obtained for the fourth-rank "crack" and "void" elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^c and C^v to the final deformed/damaged configuration C, where the law of mixtures is used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 13, and the key equations are listed in Table 15. It should be noted that both the first and second steps consist each of five sub-steps, while the third step consists of two sub-steps.

6.4 Template LOOLCCVMF (local damage in composites with matrix and fibers of cracks and voids)

This is the fourth damage mechanics template that consists of three steps and is unsymmetrical. The template accounts for four types of damages: cracks in the matrix, voids in the matrix, cracks in the fibers, and voids in the fibers. The schematic diagram of this template is shown in Fig. 14, while Table 16 lists the key equations of this template without a detailed formulation. The interested reader can derive the key equations in the Table by

Table 16 Key equations of template LOOLCCVMF

Step	Equation	Description
Step 1-a	$\{\overline{\sigma}^{v}\} = \overline{c}^{m}\{\overline{\sigma}^{mv}\} + \overline{c}^{f}\{\overline{\sigma}^{fv}\}$	Law of mixtures for voids applied to stress
Step 1-b	$\{\overline{\sigma}^{mc}\} = [M^{mc}]\{\sigma^{mc}\}$ $\{\overline{\sigma}^{fc}\} = [M^{fc}]\{\sigma^{fc}\}$	Damage transformation for "crack" stresses applied twice
Step 1-c	$\{\overline{\varepsilon}^{mc}\} = [M^{mc}]^{-T}\{\varepsilon^{mc}\}$ $\{\overline{\varepsilon}^{fc}\} = [M^{fc}]^{-T}\{\varepsilon^{fc}\}$	Damage transformation for "crack" strains applied twice
Step 1-d	$[\overline{E}^{v}] = \overline{c}^{m}[\overline{E}^{mv}][\overline{A}^{m}] + \overline{c}^{f}[\overline{E}^{fv}][\overline{A}^{f}]$	Law of mixtures for voids applied to "void" elasticity tensor
Step 1-e	$[E^{mc}] = [M^{mc}]^{-1}[\overline{E}^{mc}][M^{mc}]^{-T}$ $[E^{fc}] = [M^{fc}]^{-1}[\overline{E}^{fc}][M^{fc}]^{-T}$	Damage transformation for "crack" elasticity tensor
Step 2-a	$\{\overline{\sigma}^{v}\} = [M^{v}]\{\sigma^{v}\}$	Damage transformation of "void" stress
Step 2-b	$\{\overline{\varepsilon}^{v}\} = [M^{v}]^{-T}\{\varepsilon^{v}\}$	Damage transformation for "void" strain
Step 2-c	$\{\sigma^{c}\} = c^{m}\{\sigma^{mc}\} + c^{f}\{\sigma^{fc}\}$	Law of mixtures applied to stress
Step 2-d	$[E^{v}] = [M^{v}]^{-1}[\overline{E}^{v}][M^{v}]^{-T}$	Damage transformation for "void" elasticity tensor
Step 2-e	$[E^{c}] = c^{m}[E^{mc}][A^{m}] + c^{f}[E^{fc}][A^{f}]$	Law of mixtures applied to "crack" elasticity tensor
Step 3-a	$\{\sigma\} = c^{c}\{\sigma^{c}\} + c^{v}\{\sigma^{v}\}$	Law of mixtures applied to stress
Step 3-b	$[E] = c^{c}[E^{c}] + c^{v}[E^{v}]$ $= c^{c}(c^{m}[E^{mc}][A^{m}] + c^{f}[E^{fc}][A^{f}]) + c^{v}[M^{v}]^{-1}[\overline{E}^{v}][M^{v}]^{-T}$ $= c^{c}\left(c^{m}[M^{mc}]^{-1}[\overline{E}^{mc}][M^{mc}]^{-T}[A^{m}] + c^{f}[M^{fc}]^{-1}[\overline{E}^{fc}][M^{fc}]^{-T}[A^{f}]\right)$ $+ c^{v}[M^{v}]^{-1}\left(\overline{c}^{m}[\overline{E}^{mv}][\overline{A}^{m}] + \overline{c}^{f}[\overline{E}^{fv}][\overline{A}^{f}]\right)[M^{v}]^{-T}$	Elasticity tensor

following the arrows in the schematic diagram. It is clear from the Figure that this template is unsymmetrical in the sense that local damage tensors are introduced partially and locally (only for the cracks in both matrix and fibers) in the first step, then followed by the local damage tensor for the voids (in both matrix and fibers) which is introduced asymmetrically in the second step. Thus, unlike the six templates of Sect. 5, clearly there is no symmetry in the way the damage tensors are introduced for the cracks and voids.

The damage mechanics template LOOLCCVMF consists of three distinct steps comprising a total of twelve sub-steps. The first step starts with the four effective configurations $\overline{C}^{mc}$, $\overline{C}^{fc}$, $\overline{C}^{mv}$, and $\overline{C}^{fv}$, and proceeds to three intermediate configurations: the two damaged "crack" configurations C^{mc} and C^{fc}, and the effective "void" configuration $\overline{C}^{v}$. The second step starts with the three intermediate configurations C^{mc}, C^{fc}, and $\overline{C}^{v}$, and proceeds to the two intermediate damaged configurations C^{c} and C^{v}. In this regard, expressions are obtained for the fourth-rank "crack" and "void" elasticity tensors. Finally, the third step proceeds from the two intermediate damaged configurations C^{c} and C^{v} to the final deformed/damaged configuration C, where the law of mixtures is used to derive the final expression for the fourth-rank elasticity tensor. The details of these three steps are illustrated schematically in Fig. 14, and the key equations are listed in Table 16. It is noted that both the first and second steps consist each of five sub-steps, while the third step consists of two sub-steps.

Looking at the four damage mechanics templates presented in this Section, one concludes that all four templates of Sects. 6.1, 6.2, 6.3, and 6.4 are *inhomogeneous* (or mixed) in the sense that they use a mix of overall and local aspects. In this regard, it is very clear from the final expression of the damaged elasticity tensor, as shown in the last row of the Table of each respective template, that the expressions obtained for these four unsymmetrical and inhomogeneous templates are totally different in form from those obtained for the six symmetrical templates of Sect. 5. As shown also in these expressions, the four mixed templates provide for a more diverse representation of damage mechanisms that include more than one approach of damage characterization. However, due to the unsymmetrical nature of these four templates, it is anticipated that using the templates in this Section will prove more complicated than using the symmetrical templates of Sect. 5.

7 Generating advanced damage mechanics templates

In the damage mechanics templates introduced in the previous Section, use was made of the *law of mixtures*, as shown in Eqs. (5) and (18), as evident eventually in Eqs. (12.2) and (21). The templates presented can be extended to generalized advanced damage mechanics templates that go beyond the *law of mixtures*. In this regard, Eqs. (5) and (18) should be replaced with more potent expressions using a modified law of mixtures or advanced homogenization schemes. Possible procedures for this task are shown below using micromechanics. The total strain tensor is the same as the elastic strain tensor in this case.

The purpose of this Section is to predict the material constants (also called the elastic constants) by studying the micromechanics of the problem (basically in fiber-reinforced composite materials). This is achieved by studying how the matrix and fibers interact. Computing the stresses within the matrix, within the fibers, and at the interface of the matrix and fibers is very important for understanding some of the underlying failure mechanisms. Considering the fibers and surrounding matrix, the following assumptions are utilized [22,66]: (i) Both the matrix and fibers are linearly elastic, (ii) the fibers are infinitely long, and (iii) the fibers are spaced periodically in square-packed or hexagonal-packed arrays.

There are three different approaches that can be used to determine the elastic constants for the composite material based on micromechanics. The three approaches are [22,66]: (i) using numerical methods such as the finite element method, (ii) using models based on the theory of elasticity, and (iii) using rule-of-mixtures models based on a strength-of-materials approach.

Consider a unit cell in either a square-packed array or a hexagonal-packed array—see [22] and [66] for more details. The ratio of the cross-sectional area of the fiber to the total cross-sectional area of the unit cell is called the *fiber volume fraction* and is denoted by V^f (denoted by c^f previously). The fiber volume fraction satisfies the relation $0 < V^f < 1$ and is usually 0.5 or smaller. Similarly, the *matrix volume fraction* V^m (denoted by c^m previously) is the ratio of the cross-sectional area of the matrix to the total cross-sectional area of the unit cell. Note that V^m also satisfies $0 < V^m < 1$. The following relation can be shown to exist between V^f and V^m:

$$V^f + V^m = 1. \tag{25}$$

In the above, the notation is used such that a superscript m indicates a matrix quantity, while a superscript f indicates a fiber quantity. In addition, the matrix material is assumed to be isotropic, so that $E_1^m = E_2^m = E^m$, and $\upsilon_{12}^m = \upsilon^m$. However, the fiber material is assumed to be only transversely isotropic, so that $E_3^f = E_2^f$, $\upsilon_{13}^f = \upsilon_{12}^f$, and $\upsilon_{23}^f = \upsilon_{32}^f = \upsilon^f$. The subscripts “1”, “2”, and “3” indicate the principal directions in a composite lamina under plane stress, where “1” denotes the longitudinal direction.

Using the strength-of-materials approach and the simple rule of mixtures, one can write the following relations for the elastic constants of the composite material [22,66]. For Young’s modulus in the 1-direction (also called the longitudinal stiffness), one can write the following relation:

$$E_1 = E_1^f V^f + E^m V^m \tag{26}$$

where E_1^f is Young’s modulus of the fiber in the 1-direction, while E^m is Young’s modulus of the matrix. For Poisson’s ratio υ_{12}, the following relation can be written:

$$\upsilon_{12} = \upsilon_{12}^f V^f + \upsilon^m V^m \tag{27}$$

where υ_{12}^{f} and υ^{m} are Poisson's ratios for the fiber and matrix, respectively. For Young's modulus in the 2-direction (also called the transverse stiffness), the following relation can be written:

$$\frac{1}{E_2} = \frac{V^f}{E_2^f} + \frac{V^m}{E^m} \tag{28}$$

where E_2^f is Young's modulus of the fiber in the 2-direction, while E^m is Young's modulus of the matrix. For the shear modulus G_{12}, the following relation can be written:

$$\frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{V^m}{G^m} \tag{29}$$

where G_{12}^f and G^m are the shear moduli of the fiber and matrix, respectively.

It should be noted that Eqs. (26–29) are all based on applying the *rule of mixtures*. This rule is also the basis of Eqs. (5) and (18) which were used in the damage mechanics templates of the previous Sections and applied to the stresses. Next, it is shown how Eqs. (26–29) can be modified to allow the use of more potent expressions that can be used to replace Eqs. (5) and (18), in order to generate advanced damage mechanics templates.

When the simple rule-of-mixtures models used above give accurate results for E_1 and υ_{12}, the results obtained for E_2 and G_{12} do not agree well with finite element analysis and elasticity theory results. Therefore, one needs to modify the simple rule-of-mixtures models shown above. For E_2, one can write the following modified rule-of-mixtures formula [22,66]:

$$\frac{1}{E_2} = \frac{\frac{V^f}{E_2^f} + \frac{\eta V^m}{E^m}}{V^f + \eta V^m} \tag{30}$$

where η is called the *stress-partitioning factor* (related to the stress σ_2). This factor satisfies the relation $0 < \eta < 1$ and is usually taken 0.4 and 0.6. Another alternative rule-of-mixtures formula for E_2 is [22,66]:

$$\frac{1}{E_2} = \frac{\eta^f V^f}{E_2^f} + \frac{\eta^m V^{,m}}{E^m} \tag{31}$$

where the factors η^f and η^m are given by [22,66]:

$$\eta^f = \frac{E_1^f V^f + \left[\left(1 - \upsilon_{12}^f \upsilon_{21}^f\right) E^m + \upsilon^m \upsilon_{21}^f E_1^f\right] V^m}{E_1^f V^f + E^m V^m}, \tag{32}$$

$$\eta^m = \frac{\left[\left[1 - (\upsilon^m)^2\right] E_1^f - \left(1 - \upsilon^m \upsilon_{12}^f\right) E^m\right] V^f + E^m V^m}{E_1^f V^f + E^m V^m}. \tag{33}$$

The above alternative for E_2 gives accurate results and is used whenever the modified rule-of-mixtures model of Eq. (30) cannot be applied, i.e., when the factor η is unknown.

The modified rule-of-mixtures formula for G_{12} is given by the following expression [22,66]:

$$\frac{1}{G_{12}} = \frac{\frac{V^f}{G_{12}^f} + \frac{\eta' V^m}{G^m}}{V^f + \eta' V^m} \tag{34}$$

where η' is called the *shear-partitioning factor*. Note that η' satisfies the relation $0 < \eta' < 1$, but using $\eta' = 0.6$ gives results that correlate with the elasticity solution. Finally, the elasticity solution gives the following formula for G_{12} [22,66]:

$$G_{12} = G^m \left[\frac{\left(G^m + G_{12}^f\right) - V^f \left(G^m - G_{12}^f\right)}{\left(G^m + G_{12}^f\right) + V^f \left(G^m - G_{12}^f\right)}\right]. \tag{35}$$

In order to generate advanced damage mechanics templates, use is made of the templates of the previous sections while replacing the rule-of-mixtures formulas (5) and (18) with the modified rule-of-mixtures formulas of Eqs. (26) and (34), or with the elasticity solution of Eq. (35). Other modifications to the rule of mixtures formula are also possible, but the above examples are sufficient for this work. More elaborate schemes can be included by utilizing what are called homogenization methods [4,49].

8 Generating specific damage mechanics templates

Linear elastic material behavior is assumed in all the damage mechanics templates introduced in the previous Sections. It is shown in this Section how to extend the previous templates to generate specific damage mechanics templates for specific types of material behavior other than linearly elastic materials. This is performed by replacing the constitutive equations for linear elasticity, i.e., Eqs. (1), (6), and (11) with the needed constitutive laws. For example, materials such as nonlinear elastic, hypo-elastic, hyper-elastic, elasto-plastic, visco-plastic, visco-elastic materials can be used. In this Section, the example of rate-independent elasto-plastic material behavior is illustrated.

For fiber-reinforced metal matrix composites where the matrix is assumed to follow a rate-independent elasto-plastic behavior while the fiber is assumed to follow a linear elastic behavior, the simple linear elastic relations of Eqs. (1), (6), and (11) no longer apply to the matrix. In this case, use is made of a yield function f for the matrix that is generally written in the following form [27,61,64]:

$$f\left(\sigma_{ij}, \alpha_{ij}\right) = \sqrt{\frac{3}{2}\left[s_{ij} - \alpha_{ij}\right]\left(s_{ij} - \alpha_{ij}\right)} - Y \equiv 0 \tag{36}$$

where the indicial notation of tensor representation is used, σ_{ij} are the components of the stress tensor, α_{ij} are the components of the backstress tensor (i.e., shift tensor), s_{ij} are the components of the deviatoric stress tensor, and Y is the yield stress which is assumed to be a constant in this case (no isotropic hardening). The presence of the backstress tensor indicates that kinematic hardening is accounted for in this example. All the stresses in Eq. (36) should have a superscript “m” indicating that they are matrix stresses. The deviatoric stress tensor is given by the following formula:

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{37}$$

where δ_{ij} are the components of the identity tensor, known as Kronecker delta. In this example of rate-independent elasto-plasticity, the strain tensor is assumed to be additively decomposed into elastic and plastic parts provided that the strains are small. This assumption is given by the following equation:

$$\varepsilon_{ij} = \varepsilon^{e}_{ij} + \varepsilon^{p}_{ij} \tag{38}$$

where ε_{ij} are the components of the strain tensor, ε^{e}_{ij} are the components of the elastic strain tensor, and ε^{p}_{ij} are the components of the plastic strain tensor. Sometimes, the additive decomposition of strain of Eq. (38) is written in terms of strain rates (i.e., strain increments) instead of strains. In this case, Eq. (38) becomes:

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij} \tag{39}$$

where $d\varepsilon_{ij}$ represents the components of the increment of the strain tensor. The elastic strain increment tensor is evaluated from the elastic constitutive relation of Eqs. (1), (6), and (11). The plastic strain increment tensor is usually evaluated by assuming some form of flow rule. In its simplest form, the flow rule is written in the following form which applies to the strains in the matrix material only:

$$d\varepsilon^{p}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{40}$$

where λ is a scalar parameter that can be determined by utilizing what is called the consistency condition. In this example, it is assumed that the material undergoes kinematic hardening but no isotropic hardening. The characterization of kinematic hardening is performed by the use of the backstress tensor that appears in the expression of the yield function of Eq. (36). The backstress tensor (i.e., shift tensor) is usually evaluated using an evolution equation written in terms of its increment as follows:

$$d\alpha_{ij} = \frac{2}{3} c\, d\varepsilon_{ij}^{p} \tag{41}$$

where c is a scalar material constant. Other forms of the backstress evolution equations are available in the literature [48], but the simple form of Eq. (41) suffices to illustrate this example.

The equations presented in this Section, i.e., Eqs. (36–41), suffice to illustrate the use of a specific material model, and the interested reader is referred to the work of [27,48,61,64] for more details. It should be clear that the example of rate-independent plasticity as presented here for the matrix material is far more complicated than the equations of linear elasticity appearing in the previous Sections. In this regard, all the quantities appearing in Eqs. (36–41) should have a superscript "m" indicating that they apply to the matrix.

To modify a template from the set of basic templates of Sects. 5 and 6, one considers, for example, template OOCMFCV in Fig. 5. The governing equations for this template are shown in Table 7. The linear elasticity modulus is shown in Steps 1-b, 2-b, and 3-c. These three steps should be modified to include the constitutive relations of the new specific material model to be used. Other templates can be modified appropriately to generate specific material model templates provided the constitutive equations for the new material model are available.

9 Generating more complex damage mechanics templates

In the basic templates presented in Sects. 5 and 6, only two damage mechanisms are employed: cracks and voids. It is shown here how to generate more complex damage mechanics templates considering three or more damage mechanisms such as cracks, voids, de-bonding. For this purpose, one of the fourteen templates introduced in Sects. 5 and 6 is chosen as an example.

As an example, the template OOCMFCV in Fig. 5 is utilized. It is shown how this template can be used to generate five additional templates that are more complex and that account for more than two mechanisms of damage. The five generated templates based on template OOCMFCV are illustrated below. Each template consists of three steps except one which consists of four steps. The five different methods of generating more advanced templates are listed below.

1. In addition to the matrix and fibers, a third constituent representing the interface is introduced and is denoted by the letter "i". The corresponding generated template is shown in Fig. 15. In this Figure, it is clear that the template represents a general state of damage in the interface with both cracks and voids. In this manner, the new template represents one way of characterizing interfacial damage, in addition to both matrix and fiber damages. This template consists of three steps as shown in Fig. 15. The governing equations for this complex template can be derived from the equations of Table 7 (not shown here) by noting that Steps 1-a and 1-b should be revised to include the third constituent. The template generated using this method is symmetrical. For more details about including de-bonding and interfacial damage in damaged fibrous composites, the reader is referred to the work of [72,73].
2. Another way to represent interfacial damage is by introducing a new damage variable that is specific to the de-bonding process that occurs across the interface. The new fourth-rank de-bonding damage effect tensor is denoted by M^d and is applied in the last step of the damage characterization process. The generated template utilizing this method is shown in Fig. 16. In this way, there is no need to introduce a separate third constituent that accounts for interfacial damage. This template consists of four steps as shown in Fig. 16. The governing equations for this complex template can be obtained from the equations in Table 7 (not shown here) noting that a fourth step should be added to account for the damage transformation equations for the new fourth-rank de-bonding damage effect tensor M^d. The template generated using this method is symmetrical. For more details about the new de-bonding damage effect tensor M^d, the reader is referred to the work of [72,73].
3. A new template can be generated from template OOCMFCV by introducing a third damage mechanism in each constituent. In addition to cracks and voids, a third defect can be accounted for and is denoted by the letter "r". In this way, the matrix has three types of damage mechanisms, namely cracks, voids, and the third defect. Similarly, the fibers have also three corresponding damage mechanisms. The new generated template is shown in Fig. 17. It is noted that the new template is symmetrical. This template consists of three steps as shown in the Figure. The governing equations for this complex template can be obtained

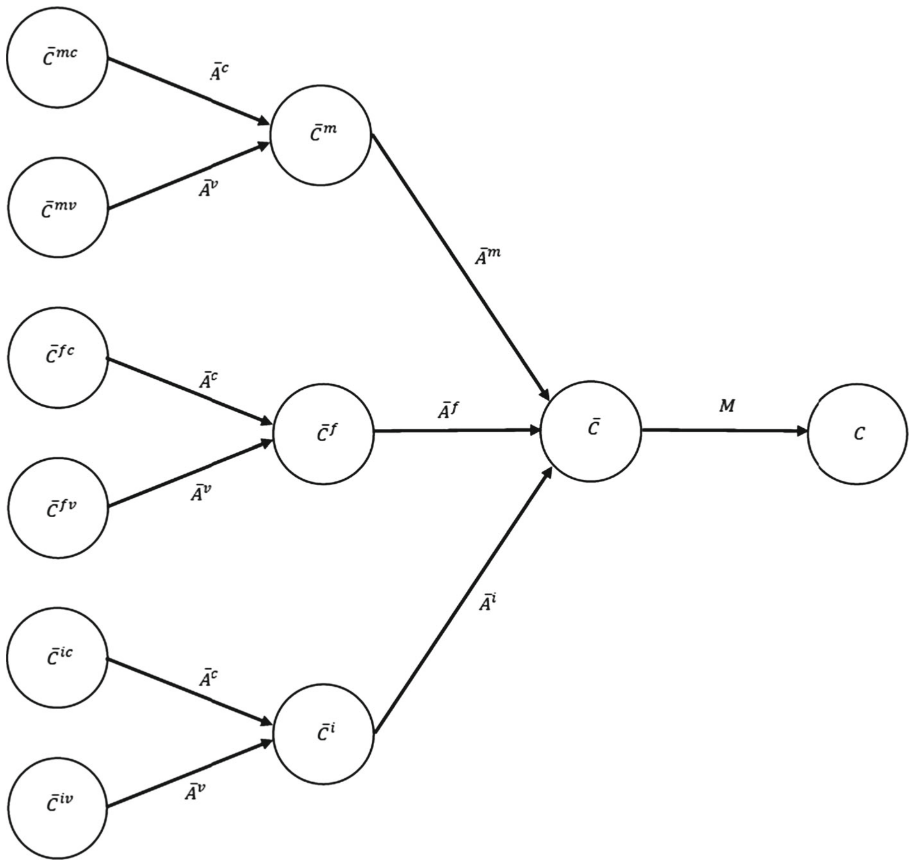

Fig. 15 Schematic diagram for complex template of item # 1

from the equations in Table 7 (not shown here) noting that Steps 1-a and 1-b should be revised to include a third term in each equation to account for the third damage mechanism.

4. The same method used in item # 3 above can be used again to generate a new template with a third damage mechanism denoted by the letter "r". However, this time it is assumed that the third damage mechanism occurs in the matrix only. Thus, the matrix undergoes three different types of damage mechanisms, while the fibers undergo only two types of damage mechanisms. The generated template is shown in Fig. 18. It is noted that the new template is unsymmetrical. This template consists of three steps as shown in Fig. 18. The governing equations for this complex template can be obtained from the equation in Table 7 (not shown here) noting that Steps 1-a and 1-b should be revised by adding a third term for the third damage mechanism, but only for those equations related to the matrix.
5. The same method used in item # 3 above can be used again to generate a new template with a third damage mechanism denoted by the letter "r". However, this time it is assumed that the third damage mechanism occurs in the fibers only. Thus, the matrix undergoes two different types of damage mechanisms, while the fibers undergo three types of damage mechanisms. The generated template is shown in Fig. 19. It is noted that the new template is unsymmetrical. This template consists of three steps as shown in Fig. 19. The governing equations for this complex template can be obtained from the equations in Table 7 (not shown here) noting that Steps 1-a and 1-b should be revised by adding a third term for each equation to account for the third damage mechanism, but only for the fiber constituent.

Finally, one can use any one of the other thirteen templates of Sects. 5 and 6 to generate more advanced templates using one of the five methods outlined in this Section for template OOCMFCV. In this regard, it is estimated that at least fifty such advanced templates similar to those in Figs. 15, 16, 17, 18, and 19 can be generated.

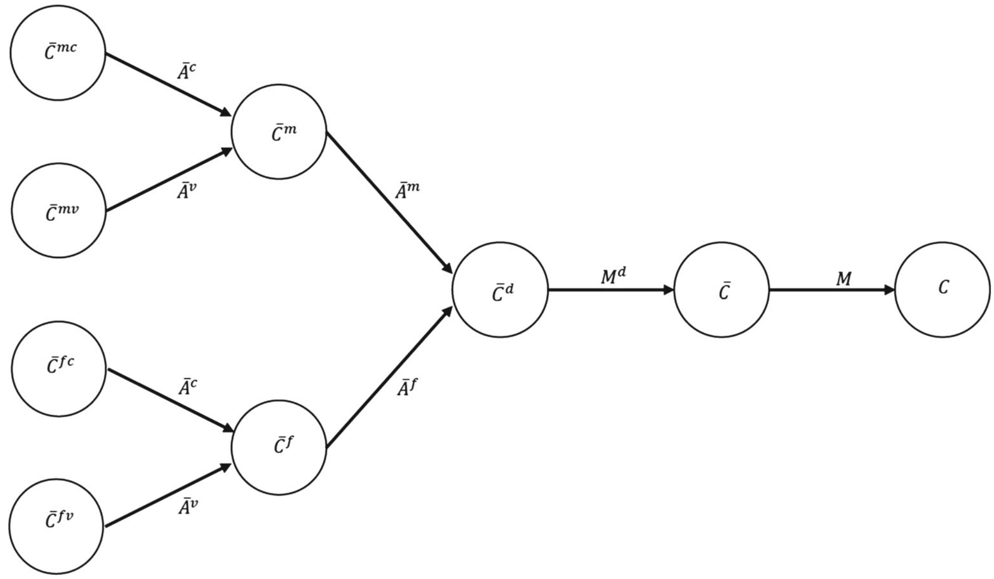

Fig. 16 Schematic diagram for complex template of item # 2

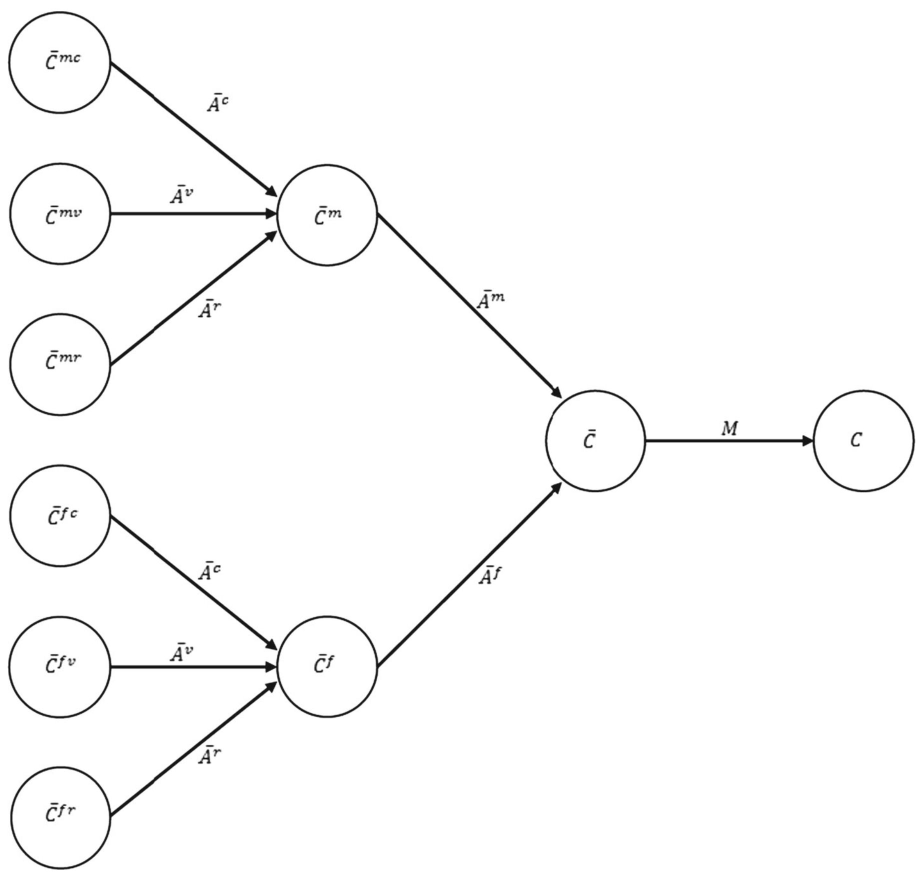

Fig. 17 Schematic diagram for complex template of item # 3

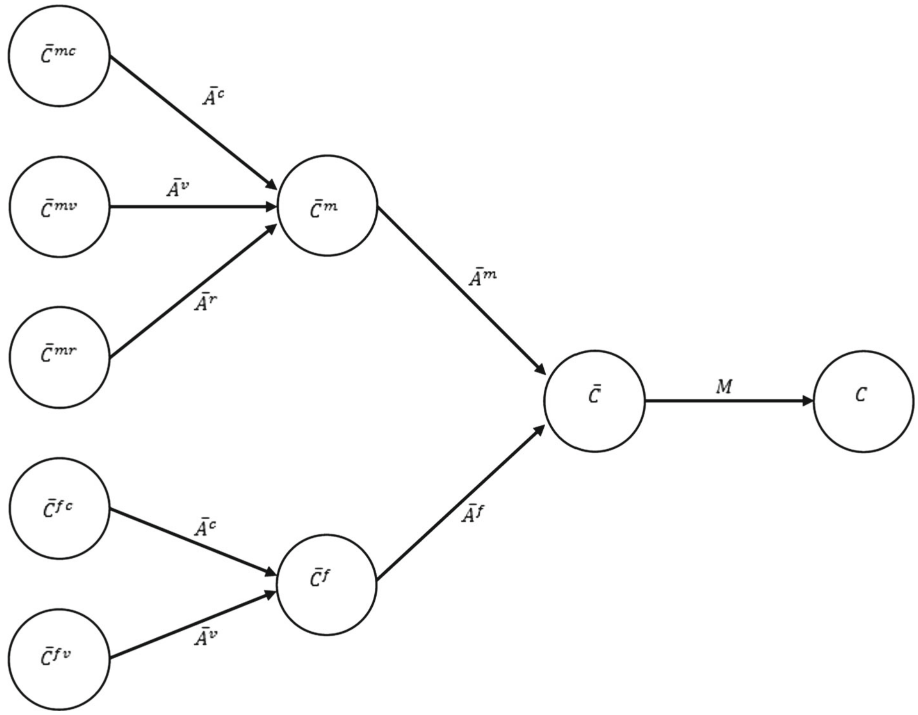

Fig. 18 Schematic diagram for complex template of item # 4

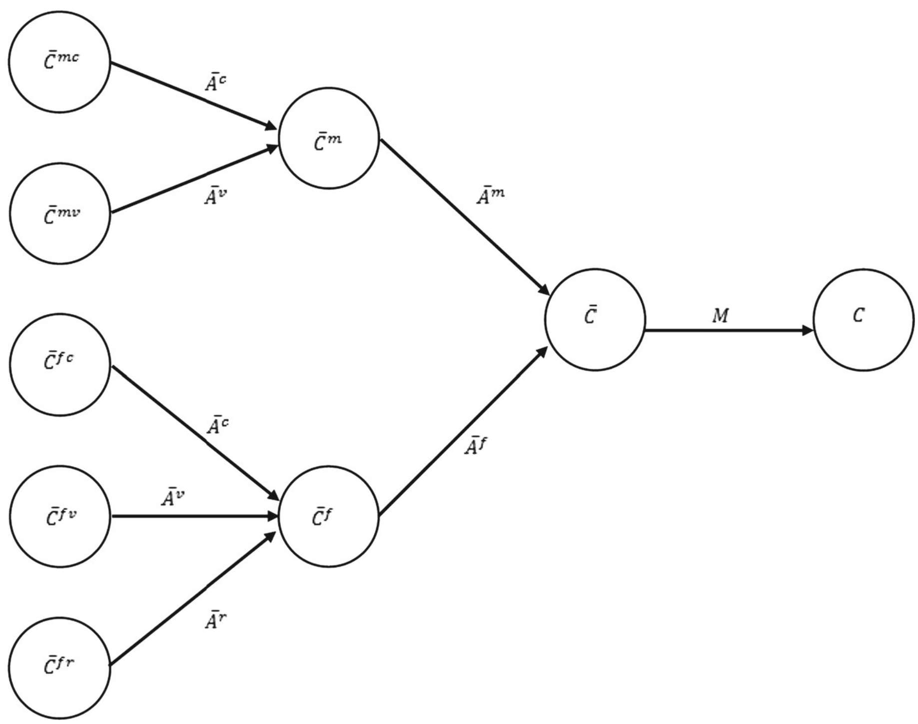

Fig. 19 Schematic diagram for complex template of item # 5

10 Conclusions and discussion

The new concept of a damage mechanics template to be used to formulate material damage models is introduced. A total of 14 different basic templates are introduced. Two of the basic templates are specifically built for metals, while the other twelve templates are built for fiber-reinforced metal matrix composite materials. All the provided templates assume a linearly elastic material. The use of these damage mechanics templates guarantees a material damage model that is systematic and consistent both mathematically and mechanically. The provided templates are designed to be used as a tool for producing holistic material damage models. Each template comes equipped with a schematic diagram along with a Table listing the key equations. The interested reader can verify the key equations and expand on them by following the respective arrows in the schematic diagram. Furthermore, Sections are added at the end of this work to show the reader how to generate additional templates: advanced templates that go beyond the rule of mixtures, special templates for other types of materials, and more complex templates that account for more than two different damage mechanisms.

The following is a summary of the main features of the present work;

1. A classification is proposed for material damage models. Different material damage models are classified according to the number of different damage mechanism present in the model.
2. Damage mechanics templates are introduced for fiber-reinforced composite materials. No specific damage mechanism is considered in these templates. In this case, two such templates are illustrated. These templates are used to build upon them more complex templates in the subsequent Sections.
3. Damage mechanics templates are introduced for metals. Certain assumptions are made in this case. Two such templates are illustrated. Each template comes with a schematic diagram and a Table showing the governing equations.
4. Six basic damage mechanics templates are introduced for composite materials where both crack damage and void damage are considered. The damage is assumed to exist in both the matrix and fibers. All six templates introduced are symmetrical, i.e., the same damage mechanisms are assumed to exist in both matrix and fibers.
5. Four additional damage mechanics templates are introduced for composite materials where both crack damage and fiber damage are considered. However, in this case, all four templates are unsymmetrical in the sense that different damage mechanisms are assumed to exist in the matrix and fibers.
6. It is shown how to generate advanced damage mechanics templates that go beyond the simple rule of mixtures. In this case, the rule of mixtures is modified to generate new templates.
7. It is shown how to generate specific damage mechanics templates that are based on material models other than linear elasticity. In the example presented, rate-independent elasto-plasticity is used and applied to the matrix material.
8. It is shown how to generate more complex damage mechanics template by including more damage mechanisms. In this case, a basic template is chosen and is used as an example to generate five additional and more complex templates.

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Appendix

In this Appendix, the expressions for the elastic stress and strain concentration factors/tensors are given based on the *Mori-Tanaka method*. In this method, the equivalence principle of *Eshelby* is used to find the stress in the fibers. According to this principle, the stress field in the fiber material (inhomogeneity) is the same as the stress field of an equivalent inclusion that has the same material properties as the matrix. The reader is referred to [18,32,35,50,78], and [8] for more details. The final expressions for the elastic stress and strain concentration factors/tensors, $[\overline{B}^f]$ and $[\overline{A}^f]$, respectively, for the fibers, are given by:

$$\left[\overline{A}^f\right] = \left([I_4] + \overline{c}^m [S] \left[\overline{E}^m\right]^{-1} \left(\left[\overline{E}^f\right] - \left[\overline{E}^m\right]\right)\right)^{-1}, \tag{42}$$

$$\left[\overline{B}^f\right] = \left([I_4] + \overline{c}^m \left[\overline{E}^m\right] ([I_4] - [S]) \left(\left[\overline{E}^f\right] - \left[\overline{E}^m\right]\right)\right)^{-1} \tag{43}$$

where $[S]$ is a square matrix that represents the fourth-rank Eshelby's tensor for elasticity (given explicitly below). Using the expressions in Eqs. (42) and (43) results in the following constraint equations:

$$\bar{c}^m \left[\overline{A}^m\right] + \bar{c}^f \left[\overline{A}^f\right] = [I_4], \tag{44}$$

$$\bar{c}^m \left[\overline{B}^m\right] + \bar{c}^f \left[\overline{B}^f\right] = [I_4]. \tag{45}$$

One can now easily obtain the elastic stress and strain concentration factors/tensors $[\overline{B}^m]$ and $[\overline{A}^m]$, respectively, for the matrix, by using the expressions in the constraint Eqs. (44) and (45) above. In Eqs. (42) and (43), the nonzero components of the fourth-rank Eshelby's tensor $[S]$ for elasticity for cylindrical fibers with a circular cross section are given by [77]:

$$S_{1111} = \frac{5 - \upsilon^m}{8\left(1 - \upsilon^m\right)}, \tag{46}$$

$$S_{2222} = S_{1111}, \tag{47}$$

$$S_{1122} = \frac{4\upsilon^m - 1}{8\left(1 - \upsilon^m\right)}, \tag{48}$$

$$S_{2211} = S_{1122}, \tag{49}$$

$$S_{2323} = \frac{1}{4}, \tag{50}$$

$$S_{2233} = \frac{\upsilon^m}{2\left(1 - \upsilon^m\right)}, \tag{51}$$

$$S_{1133} = S_{2233}, \tag{52}$$

$$S_{1212} = \frac{3 - 4\upsilon^m}{8\left(1 - \upsilon^m\right)}, \tag{53}$$

$$S_{3131} = \frac{1}{4} \tag{54}$$

where υ^m is the matrix Poisson's ratio.

The elastic stress and strain concentration factors/tensors, $\left[\overline{B}^c\right]$ and $\left[\overline{A}^c\right]$, respectively, for the effective configuration with cracks only (i.e., voids are removed) have not appeared before in the literature. The same observation is true also for the elastic stress and strain concentration factors/tensors, $\left[\overline{B}^v\right]$ and $\left[\overline{A}^v\right]$, respectively, for the effective configuration with voids only (i.e., cracks are removed). However, one can safely utilize the above expressions of Eqs. (42–54) and apply them to these new types of stress and strain concentration factors. This issue needs further investigation and is beyond the scope of this work. For now, one can write the following constraint relations for these two new concentration factors/tensors (similar to the ones in Eqs. (44) and (45):

$$\bar{c}^c \left[\overline{A}^c\right] + \bar{c}^v \left[\overline{A}^v\right] = [I_4], \tag{55}$$

$$\bar{c}^c \left[\overline{B}^c\right] + \bar{c}^v \left[\overline{B}^v\right] = [I_4]. \tag{56}$$

It is clear from Eqs. (55) and (56) above that once two of the four concentration factors/tensors are known, then the other two can be immediately obtained.

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