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von Mises hypothesis revised

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Abstract The most popular isotropic yield conditions, verified for many ductile metals, were proposed by *Tresca* in 1864 (maximum shearing stresses theory) and *von Mises* in 1913. The *von Mises* yield theory (von Mises in Mathematisch-physikalische Klasse 582–592, 1913), also known as maximum distortion energy criterion, finds considerable experimental support, especially for very ductile materials and plane stress (Banabic et al. in Int. J. Mater. Form. 3:165–189, 2010). For this reason, and for its simplicity, it is common in design. During 100 years, this theory has been developed and improved systematically by *Hosford, Christensen, Tsai-Hill,* etc. The modified *von Mises* hypothesis combines the theories of maximum strain energy and maximum distortion energy, and it involves the Poisson ratio. It ensures a smooth transition from the *von Mises* to the *Beltrami* criterion. The results obtained by this new yield hypothesis are compared with those obtained both by the classic *von Mises* criterion and by experiments on different metallic materials. A quite good concordance is observed between these results.

1 Introduction

Equations for predicting the conditions at which plastic yielding begins when a material is subjected to a compound state of stress are very important in the field of plasticity. In order to obtain these equations, usually it is expected that yielding under combined stress can be related to some particular combination of principal stresses. Unfortunately, there is no yet a theoretical way to correlate yielding for triaxial or biaxial state of stress with yielding for uniaxial tension test. In these circumstances, the yielding criteria are essentially empirical relationships, but which must be consistent with experiments, "the chief of which is that pure hydrostatic pressure does not cause yielding in continuous solid" [3]. The special state of stress, uniform in all directions, is created by equi-triaxial compression or tension, when the principal stresses are identical in magnitude and sign, and Mohr's circle is a point. Experiments have shown that materials can be hydrostatically compressed well beyond their ultimate strength in uniaxial compression without failure, and very large amounts of strain energy can be stored in material. The equi-triaxial tension or compression reduces only the volume of specimens, without changing its shape. The uniform stress in all directions causes no distortions and thus no shear stress. It is much more difficult to create equi-triaxial tension than compression, and therefore, only few reliable experiments for this state of stress are reported.

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Many metallic materials have shown a transition from ductile to brittle type of failure, which depends on the loading conditions. The main factors that influence this transition are [4] the following: state of stress, loading speed and temperature.

Generally, the complex states of stress, high loading speed and low temperatures favour the brittle failure of the materials. According to Christensen [14], the failure in equi-triaxial tension is always brittle.

The yield theories (*Tresca*, *von Mises*, etc.) can be used for ductile materials, but not for brittle materials. For brittle materials, other failure theories are used (*Rankine*, etc.). For example, for the transition domain ductile–brittle, it is impossible to know with exact precision when to replace the *von Mises* criterion with the *Rankine* criterion.

Unfortunately, some of the most popular yield theories cannot be used for equi-triaxial tension or compression. For this particular state of stress, the theories build on the assumption that yielding depends only on deviatoric stress and the volumetric component of stress has no role in material yielding, leading to the conclusion: no shear stress = no distortion = no failure.

As a result, some adaptations of classical theories or new theories would be necessary in order to be used for equi-triaxial tension or compression.

Unfortunately, there is no universal yield theory. The success of one or other theory depends to a large degree on the material with which it is associated.

2 Energy-based theories

Historically, several energy-based theories have been formulated in order to explain the yield under static loading.

In 1885, Italian scientist *F. Beltrami* proposed the yield theory of maximum strain energy (total strain energy of deformation per volume unit). Because it was not well confirmed by experiments, today this theory is not used any more in plasticity and engineering and it has only a historical importance. However, it formed the foundation for new energetic theories and for some particulars states of stress and for some materials; *Beltrami* theory can give more accurate results regarding the experimental data compared to the *von Mises* theory [5].

It is known that the yielding mechanism is due to relative sliding of atoms within their network. The sliding is produced by shear stress, which causes distortion of the shape. Because the plastic deformation takes place practically without change in volume, *M. T. Huber* in 1904 (paper written in Polish) [7,8] proposed another energetic theory, which considers only a part of the total energy, respective distortion energy. His work was first cited by *Hencky*, 20 years after [9], and *H. Hencky* in 1924 [43]. However, the first scientist to have formulated this hypothesis was *J. Maxwell* who in a letter addressed to *W. Thomson* states: “I have strong reasons for believing that when [the strain energy of distortion] reaches a certain limit then the element will begin to give way” [10]. The classical *Maxwell–Huber–Hencky–von Mises* theory does not depend on any material constant.

According to energetic theories, yielding occurs when the strain energy per unit volume in a state of combined stress, or a part of them (distortion energy, for example), reaches the same energy for yield as in uniaxial tension. According to *Robert Norton* [31], “it appears that distortion is the culprit in tensile failure too.”

It is known that the total strain energy per unit volume U_1 can be split into two parts: the distortion energy $U_{1\mathrm{D}}$ and the volumetric component $U_{1\mathrm{V}}$:

$$U_1 = U_{1D} + U_{1V}, \tag{1}$$

where

$$U_1 = \frac{I_1^2 - 2(1+\mu) I_2}{2E}; \quad U_{1D} = \frac{(1+\mu)\left(I_1^2 - 3I_2\right)}{3E}; \quad U_{1V} = \frac{(1-2\mu) I_1^2}{6E}, \tag{2}$$

and I_1, I_2 are stress invariants and μ is the *Poisson* ratio.

Mathematically, these yield theories are expressed as following:

- Total strain energy

$$I_1^2 - 2(1+\mu) I_2 \leq \sigma_{yp}^2 \tag{3}$$

- Distortion energy (*von Mises*)

$$I_1^2 - 3I_2 \leq \sigma_{yp}^2 \tag{4}$$

- Energy due to change in volume (volumetric component)

$$I_1 \leq \sigma_{yp}, \tag{5}$$

where σ_{yp} is the uniaxial yield point stress of the material.
The maximum distortion energy criterion, also known as *von Mises* yield theory (or criterion), finds considerable experimental support, especially for very ductile materials and plane stress. For this reason, and for its simplicity, it is common in design, although it has some important disadvantages:

- It does not give good results for all isotropic and ductile materials. *Hershey* in 1954 and *Hosford* in 1972 presented a yield criterion for isotropic polycrystalline materials [2,6,11], which is a generalization of the *von Mises* criterion:

$$\frac{1}{2}\left[|\sigma_1 - \sigma_2|^m + |\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m\right] = \sigma_{yp}^m, \tag{6}$$

where m is a material-dependent exponent.
A series of yield criteria can be given when m goes from 1 to infinity. This yield surface lies inside the *von Mises* yield criterion and outside the *Tresca* yield criterion. When $m = 2$ and when $m = 4$, the *Hershey–Hosford* criterion reduces to the *von Mises* yield criterion. When $m = 1$ and as $m \to \infty$, it becomes the *Tresca* yield criterion;
- It does not give good results for non-isotropic materials. *Hill* proposed in 1948 a widely used yield theory for metals, as an extension of the *von Mises* criterion. The *Tsai-Hill* failure criterion is an extension of the *von Mises* yield criterion to orthotropic materials.

The generalized *Hill* criterion has the form [6]

$$\begin{aligned} F\,|\sigma_2 - \sigma_3|^m + G\,|\sigma_3 - \sigma_1|^m + H\,|\sigma_1 - \sigma_2|^m + L\,|2\sigma_1 - \sigma_2 - \sigma_3|^m + \\ M\,|2\sigma_2 - \sigma_1 - \sigma_3|^m + N\,|2\sigma_3 - \sigma_1 - \sigma_2|^m = \sigma_{yp}^m, \end{aligned} \tag{7}$$

where:

- σ_{i} are the principal stresses, aligned with the directions of anisotropy;
- F, G, H, L, M and N are parameters characterizing the anisotropy;
- $m \geq 1$ to ensure convexity of the yield surface.

For the isotropic materials, $F=G=H$ and $L=M=N$ and it becomes a three parameter criterion.
For an isotropic material, the *Tsai-Hill* failure criterion is identical to the *von Mises* criterion [11,15];

- The *Drucker–Prager* criterion [16] is an extension of the *von Mises* criterion for pressure-dependent materials [6]. It is often used for modelling of rock and soil materials. This yield function is rarely used for metal plasticity [17];
- Apparently, the *von Mises* criterion does not depend on the material, because it does not include some material characteristics. This theory is consistent with experiments for very ductile polycrystalline metals (*von Mises* materials). Though experimental curves for torsion-tensile shaft show small differences even among very ductile materials (mild steel, aluminium and copper), it looks like the *von Mises* criterion fits better with data obtained for aluminium and copper than those for mild steel [18,19], etc.

Christensen proposed a criterion that can be applied to brittle materials and for ductile materials as well [12–14]. The mathematical modelling of this theory is:

$$(\sigma_c - \sigma_t)\,(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \sigma_t\sigma_c, \tag{8}$$

where σ_{t} and σ_{c} are the tensile and compressive strengths of the isotropic material, respectively. For ductile materials, $\sigma_{\mathrm{t}} = \sigma_{\mathrm{c}}$ and the *Christensen* criterion is essentially the *von Mises* criterion.

About the number of material parameters, according to *Richard M. Christensen*: "All existing failure forms with three or more parameters appear to be of empirical origin."

The *von Mises* criterion is often used to estimate the yield of very ductile materials, with the same yield strength in tension and in compression. Because of its simplicity, the *von Mises* yield condition is used sometimes by engineers as failure condition in cases where no macroscopic flow is involved.

The hypothesis of maximum energy due to change in volume does not agree well with most experiments and is not used any more.

Table 1 Metals classification according to ratio of shear yield stress to tensile yield stress [6]

τ_{yp}/σ_{yp}	Yield criterion agreed
0.31–0.41	No yield criterion agreed
0.48–0.53	Single shear theory (*Tresca*)
0.54–0.62	Tri-shear theory (*von Mises*)
0.67–0.71	Twin-shear theory (*Yu* et al.)

Kishkin and *Ratner* divided the metals in four categories (see Table 1), according the ratio of shear yield stress to tensile yield stress [6,20].

The twin-shear yield criterion was introduced and developed successively by Yu and Yu et al. [21,22]. It assumes that yielding begins when the sum of the two larger shear stresses reaches a critical magnitude. The shear stresses related to principal normal stresses are

$$\tau_{ij} = \frac{\sigma_i - \sigma_j}{2}; \quad \sigma_{ij} = \frac{\sigma_i + \sigma_j}{2}; \quad i, j = 1, 2, 3. \tag{9}$$

It is noted that only two shear stresses τ_{ij} are independent, because:

$$\tau_{13} = \tau_{12} + \tau_{23}. \tag{10}$$

The twin-shear yield criterion, proposed by *Yu* [6] in 1962, is

$$\begin{cases} \tau_{13} + \tau_{12} = \sigma_1 - \frac{\sigma_2+\sigma_3}{2} \le \sigma_{yp}; \quad \sigma_2 \le \frac{\sigma_1+\sigma_3}{2}, \\ \tau_{13} + \tau_{23} = \frac{\sigma_1+\sigma_2}{2} - \sigma_3 \le \sigma_{yp}; \quad \sigma_2 \ge \frac{\sigma_1+\sigma_3}{2}. \end{cases} \tag{11}$$

The mathematical expression of *von Mises*'s criterion can be written as

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 (\sigma_3 - \sigma_1)^2 = 2\sigma_{yp}^2 \tag{12}$$

or

$$3J_2 = \sigma_{yp}^2, \tag{13}$$

where J_2 is the second invariant of the deviatoric stress tensor.

For a biaxial state of stress ($\sigma_3 = 0$), Eq. (12) becomes

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_{yp}^2. \tag{14}$$

3 Modified *von Mises* hypothesis

The goal of this paper is to modify the *von Mises* hypothesis, presented above, so the criteria built on it (revised *von Mises*) be used for both yielding and brittle failure.

In order to do this, a correction factor k_{w} is proposed for the classical *von Mises* criterion, such that Eq. (4) becomes

$$I_1^2 - (2 + k_w)\, I_2 \le \sigma_{yp}^2, \tag{15}$$

where k_w is calculated as ratio of energies:

$$k_w = \frac{U_{1D} + 2\mu U_{1V}}{U_1} \tag{16}$$

or

$$k_w = \frac{2}{3}\left[1 + \frac{2\mu\,(1-\mu)\, I_1^2 - (1+\mu)\, I_2}{I_1^2 - 2\,(1+\mu)\, I_2}\right]. \tag{17}$$

It is observed that

$$k_w \in [2\mu, 1]\,. \tag{18}$$

In engineering practice, we can distinguish completely ductile fracture (gold, for example); ductile fracture (mild steel); brittle fracture (steel with high percentage of C), etc. Very ductile fracture occurs only due to sliding of atoms and involves distortion energy. Completely brittle fracture occurs only due to separation of atomic planes and involves volumetric component of energy. The ductile fracture involves both sliding of atoms (in some zones of the cross section) and separation of atomic planes (in the rest of the cross section), and consequently, both components of energy. The hypothesis proposed in this paper suggests just that: to consider both components of energy, in different percentages (ranging between 0 and 100%) in function of the state of stress. Coefficient k_w itself allows a combined yield and strength hypothesis. As shown below, when it has extreme values, only distortion energy (*von Mises* hypothesis) and, respectively, only volumetric component of energy are considered. When it has intermediate values, the two components of energy are considered by different percentages. This would correspond to transition from ductile to brittle fracture.

For isotropic materials, classical elasticity predicts μ to be between -1 and 0.5 [23]. For most usual materials, $\mu \in (0, 0.5)$, where the 0.5 value corresponds to incompressible materials (ideal fluids). Anyway, μ is just about 0.5 for rubber and zero for cork. However, some special composite materials (cellular materials, or unidirectional reinforced plies with some particular stacking sequences) can exhibit negative or greater than 0.5 *Poisson* ratios [24]. Only materials with *Poisson* ratio between 0.2 and 0.5 are considered in this paper.

When there is no change in volume (there is only distortion), it results that $k_w = 1$, because

$$U_{1V} = 0 \quad \text{and} \quad U_{1D} = U_1 \tag{19}$$

or

$$\mu = \frac{1}{2}, \tag{20}$$

and Eq. (15) becomes identical to Eq. (4).

When there is only change in volume, we have

$$U_{1D} = 0 \text{ and } U_{1V} = U_1 \tag{21}$$

and $k_w = 2\mu$. In this case, Eq. (15) becomes identical with Eq.(3).

For other values of k_w, results between those obtained by Eqs. (4) and (3) are expected.

Other forms of Eq. (15) are the following:

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - k_w\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + (2 + k_w)\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \le \sigma_{yp}^2, \tag{22}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - k_w\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \le \sigma_{yp}^2. \tag{23}$$

The corresponding expressions of k_w are:

$$k_w = \frac{2}{3}\left[1 + \frac{2\mu(1-\mu)\left(\sigma_x + \sigma_y + \sigma_z\right)^2 - (1+\mu)\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right)}{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + 2(1+\mu)\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}\right], \tag{24}$$

$$k_w = \frac{2}{3}\left[1 + \frac{2\mu(1-\mu)(\sigma_1 + \sigma_2 + \sigma_3)^2 - (1+\mu)(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}\right]. \tag{25}$$

For the plane state of stress, Eqs. (22) and (23) become, respectively,

$$\sigma_x^2 + \sigma_y^2 - k_w\sigma_x\sigma_y + (2 + k_w)\tau_{xy}^2 \le \sigma_{yp}^2, \tag{26}$$

$$\sigma_1^2 + \sigma_2^2 - k_w\sigma_1\sigma_2 \le \sigma_{yp}^2. \tag{27}$$

For torsion-tensile, Eq. (26) becomes

$$\sigma_x^2 + (2 + k_w)\tau_{xy}^2 \le \sigma_{yp}^2 \tag{28}$$

with

$$k_w = \frac{2}{3}\left[1 + \frac{2\mu(1-\mu)\sigma_x^2 + (1+\mu)\tau_{xy}^2}{\sigma_x^2 + 2(1+\mu)\tau_{xy}^2}\right]. \tag{29}$$

Equation (28) can be used for shaft design, for example. For $k_w = 1$, the classic *von Mises* criterion is obtained. Nevertheless, some empirical equations, used in machine design, have a coefficient of shear stress less than 3 [25].

Using Eq.(25), coefficient k_w for the plane state of stress can be written. In Fig. 1, coefficient k_w versus *Poisson* ratio μ, for different values of main stresses ratio $s = \sigma_1/\sigma_2$, is presented. It is observed that k_w coefficient decreases with μ. $k_w = 1$ for $\mu = 0.5$, when it does not depend on s, and for $s = -1$, when it does not depend on μ.

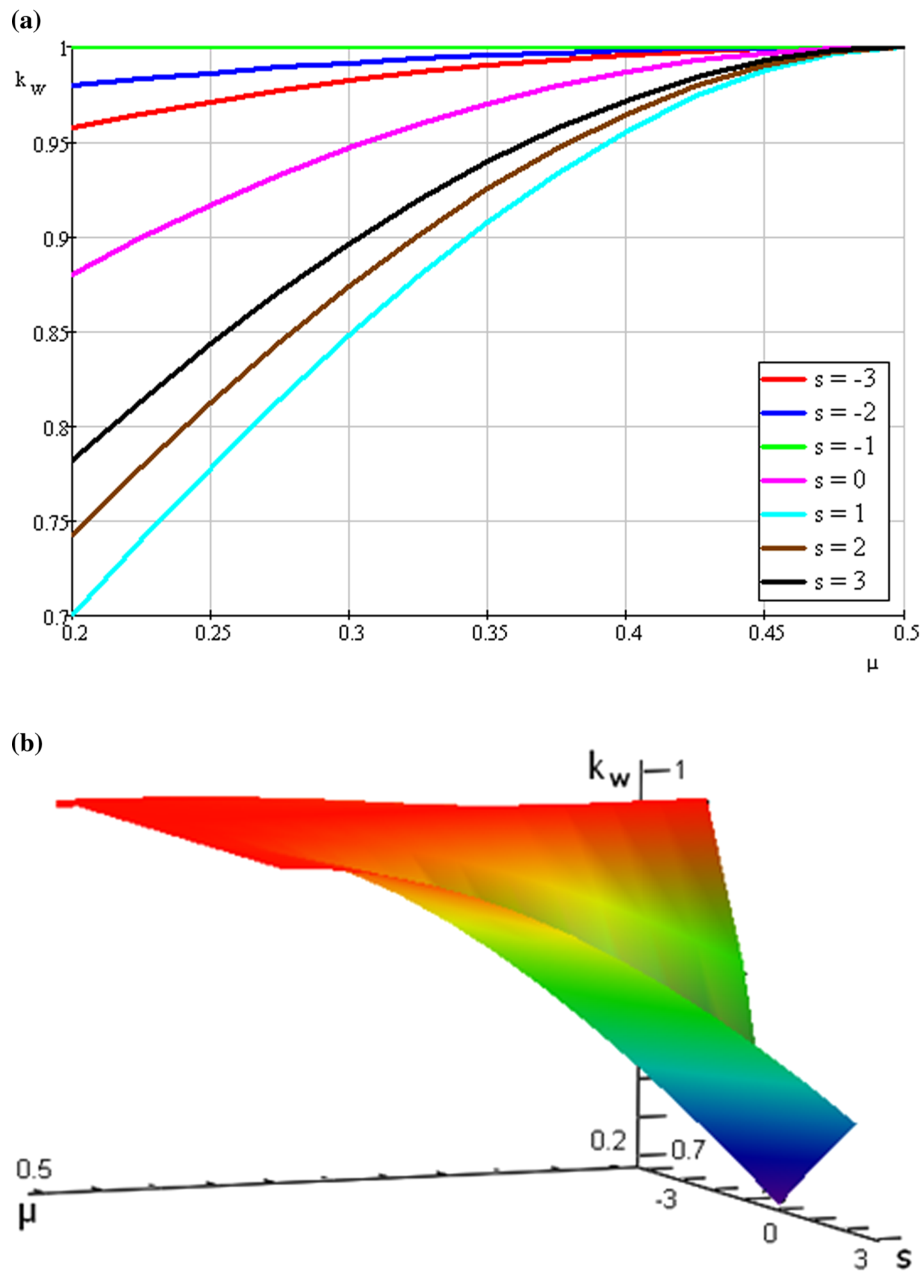

Fig. 1 Coefficient k_w representation: **a** versus Poisson's ratio μ, for different values of main stress ratio $s = \sigma_1/\sigma_2$ and **b** versus μ and s

3.1 Particular cases

3.1.1 Pure shear

For pure shear state of stress $\sigma_1 = \tau_{yp}$, $\sigma_2 = -\tau_{yp}$, and from Eq. (27) follows

$$\tau_{yp} = \frac{\sigma_{yp}}{\sqrt{2 + k_w}}. \tag{30}$$

In Table 2, the yielding shear stress value predicted by Eq. (30) for different values of k_w is presented. For $k_w = 1$, there are only distortion energy ($U_{1v} = 0$) and the same value of τ_{yp} is predicted by both the classic and the revised *von Mises* hypothesis.

Table 2 Yielding shear stress τ_{yp} predicted by revised and classic *von Mises* hypothesis, Eq. (30)

Revised *von Mises* hypothesis		Classic *von Mises* hypothesis
k_w	τ_{yp}	τ_{yp}
1	$\tau_{yp} = \sigma_{yp}/\sqrt{3} \approx 0.5773\sigma_{yp}$	$\tau_{yp} = \sigma_{yp}/\sqrt{3}$
0.8	$\tau_{yp} = \sigma_{yp}/\sqrt{2.8} \approx 0.5976\sigma_{yp}$	
0.6	$\tau_{yp} = \sigma_{yp}/\sqrt{2.6} \approx 0.6202\sigma_{yp}$	

Table 3 Points of yield curves for torsion-tension shaft ($x = \sigma_x/\sigma_{yp}$; $y = \tau_{xy}/\sigma_{yp}$, y_M = classic *von Mises*)

x	0	0.1	0.2	0.3	0.4	
y_M	0.57735	0.57445	0.56568	0.55075	0.52915	
$y_{0.29}$	0.57735	0.57452	0.56594	0.55131	0.53009	
$y_{0.32}$	0.57735	0.57450	0.56587	0.55115	0.52982	
x	0.5	0.6	0.7	0.8	0.9	1
y_M	0.5	0.46188	0.41231	0.34641	0.25166	0
$y_{0.29}$	0.50136	0.46367	0.41444	0.34871	0.25373	0
$y_{0.32}$	0.50098	0.46317	0.41386	0.34808	0.25317	0

A comparison between data presented in Tables 1 and 2 shows that the both the classic and revised *von Mises* hypothesis aim at the same field: some materials with $\tau_{yp}/\sigma_{yp} \in$ [0.54-0.62]. Nevertheless, the value of shear yield stress predicted by the classic *von Mises* criterion is constant (the same for all materials), while the one that is predicted by the revised hypothesis depends on the *Poisson* ratio.

3.1.2 Equi-triaxial tension/compression

In this case $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, $k_w = 2\mu$ and Eq. (23) becomes

$$\sigma = \frac{\sigma_{yp}}{\sqrt{3(1-2\mu)}}. \tag{31}$$

From Eq. (31), $\sigma = \sigma_{yp}$ is obtained for $\mu = 1/3$. For $\mu = 0.3$, the value predicted by the revised *von Mises* hypothesis is $\sigma \approx 0.913\sigma_{yp}$. This value is 8.7 % lower than the one predicted by the maximum principal stress failure theory (*Lamé G.* and *Rankine J.M.*) used for brittle materials, $\sigma = \sigma_{yp}$, respectively. The revised hypothesis fails only for $\mu = 0.5$, when it is identical with the classic *von Mises* hypothesis.

3.1.3 Torsion and tension

By removing of the factor k_w from Eqs. (28) and (29), the following equation is obtained:

$$18(1+\mu)z^2 + 2\left[\left(7+5\mu-2\mu^2\right)x^2 - 3(1+\mu)\right]z + 3x^2\left(x^2-1\right) = 0, \tag{32}$$

where

$$z = y^2; \quad y = \frac{\tau_{xy}}{\sigma_{yp}}; \quad x = \frac{\sigma_x}{\sigma_{yp}}. \tag{33}$$

Only one solution of Eq. (32) has a physical significance:

$$z = \frac{3(1+\mu) - \left(7+5\mu-2\mu^2\right)x^2 + \sqrt{\left[\left(7+5\mu-2\mu^2\right)x^2 - 3(1+\mu)\right]^2 - 54(1+\mu)x^2\left(x^2-1\right)}}{18(1+\mu)}. \tag{34}$$

Using the solution (34), data from Table 3 was calculated for mild steel ($\mu = 0.29$) and aluminium ($\mu = 0.32$) and the yield curves have been drawn in Fig. 2.

Experiments show that the *von Mises* curve is positioned between the yield curves of aluminium (below) and those of mild steel (above) [18,19]. According to these experiments and Fig. 2b, it is observed that the *von Mises* yield curve fits better with aluminium, and the revised *von Mises* curves with mild steel.

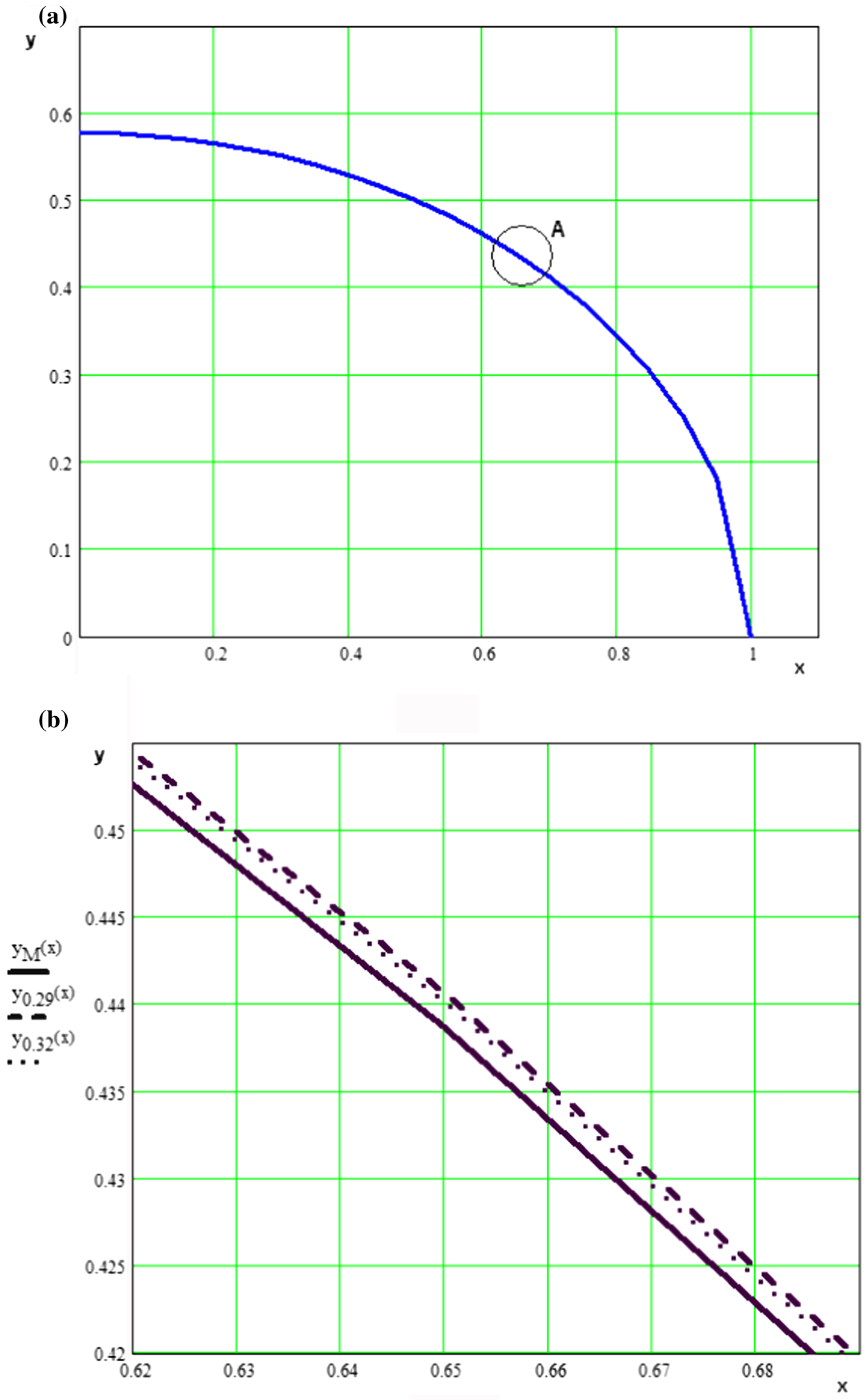

Fig. 2 Yield curves for torsion-tension shaft: **a** *von Mises*(y_M), revised *von Mises* for mild steel ($y_{0.29}$) and aluminium ($y_{0.32}$) curves, superposed at a small scale; **b** the detail A, presented at a bigger scale. *Legend*: $x = \sigma_x / \sigma_{yp}$; $y = \tau_{xy}/\sigma_{yp}$

3.2 Some mathematical aspects

For the plane state of stress, the *von Mises* criterion can be rewritten as

$$\left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 - \frac{\sigma_1\sigma_2}{\sigma_{yp}^2} = 1. \tag{35}$$

This is an oblique ellipse, oriented at 45°, with semi-axis $\sqrt{2}\sigma_{yp}$ and $\sqrt{2/3}\sigma_{yp}$, respectively.

Then again, Eq. (27) can be rewritten as

$$\left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 - k_w\frac{\sigma_1\sigma_2}{\sigma_{yp}^2} = 1. \tag{36}$$

This is also an ellipse oriented at 45°, but having the different semi-axis

$$\sqrt{\frac{2}{2-k_w}}\cdot\sigma_{yp};\quad \sqrt{\frac{2}{2+k_w}}\cdot\sigma_{yp}. \tag{37}$$

When $k_w = 1$, the two ellipses are superposed. For $k_w < 1$, the classic *von Mises* ellipse has a bigger major semi-axis and a smaller minor semi-axis than revised one.

In Fig. 3, it can be observed that the big diameter of ellipse decreases and the small diameter increases with decrease in k_w coefficient. The revised ellipse is more accurate with experimental points obtained by *Dowling* [26] for Ni–Cr–Mo steel than the classic *von Mises* ellipse.

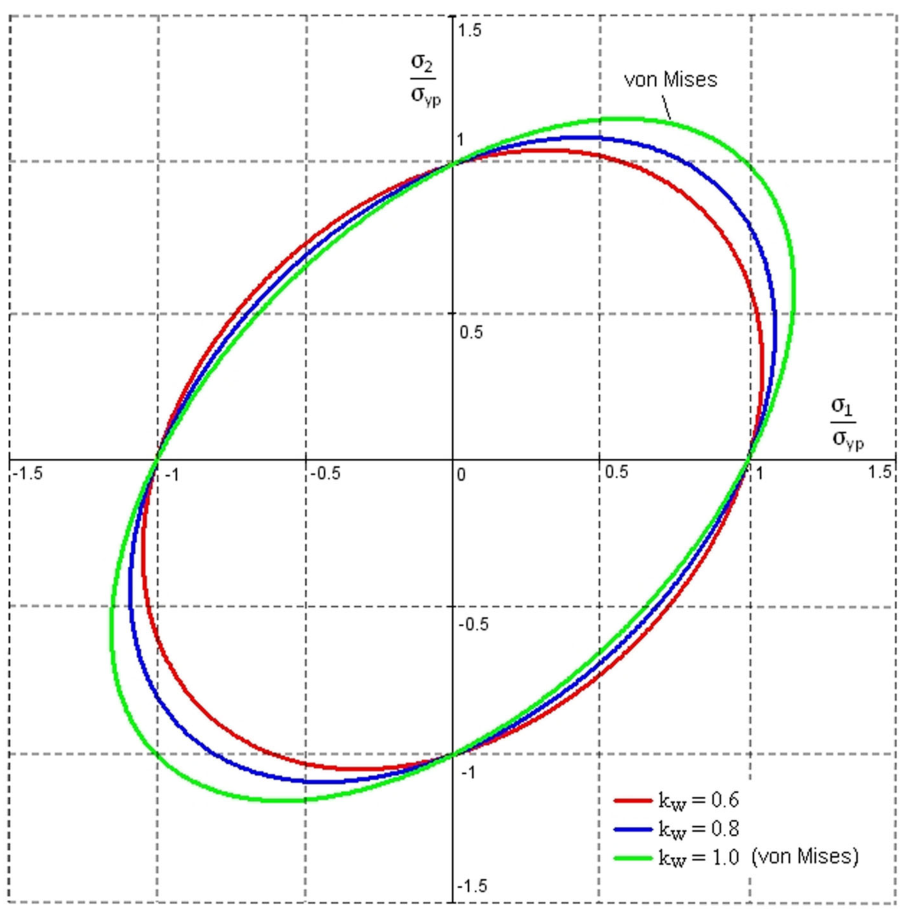

Fig. 3 Ellipses obtained for different values of k_w coefficient (for $k_w = 1$, both classic and revised *von Mises* ellipses are superposed)

4 Testing the hypothesis with experiments

In order to have a comparison between the two theories (classic and revised *von Mises*), on the one hand, experimental results [27] obtained for some metallic and nonmetallic materials are presented. A *Poisson*'s ratio $\mu = 0.33$ has been considered for aluminium alloy data obtained by the revised *von Mises* hypothesis. The two hypotheses give quite similar results for materials and states of stress presented in Tables 4 and 5. The k_w coefficient was calculated with Eq. (25).

It is known, for example, that the *von Mises* hypothesis fits better with the experimental results for aluminium and copper than for mild steel [18]. To get a better fit with the experimental results, it is proposed to amend the coefficient k_w as follows:

$$k_w = \frac{U_{1D} + 2m\mu U_{1V}}{U_1}, \tag{38}$$

where m is a coefficient which depends on the material.

For biaxial state of stress, it becomes

$$k_w = \frac{2}{3}\left[1 + \frac{\mu\,(1+m-2m\mu)\,(\sigma_1+\sigma_2)^2 - (1+\mu)\,\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2}\right]. \tag{39}$$

For

$$m = \frac{1}{2\mu}, \tag{40}$$

it follows that $k_w = 1$ and the classic *von Mises* criterion is obtained.

On the other hand, for aluminium wrought alloy AA6016, Hill'90 [28] and Yld2000-2D [29] yield criteria are in good agreement with experimental values [30].

The classical *von Mises* criterion is widely used in design. It is especially incorporated in finite element analysis software. A major deficiency in this case is the fact that the classical hypothesis cannot be used for equi-triaxial tension or near equi-triaxial tension. For these state of stress in ductile materials, the classic *von Mises* yielding theory cannot be applied, as well as the *Tresca, Hosford* or *Christensen* theories. It assumed that there are not only two points (with main stress ratio 1/1/1 or −1/−1/−1), but two regions around these points, where the classic *von Mises* hypothesis, as well as other theories build on this foundation, cannot be applied. On the other hand, the revised *von Mises* hypothesis can be used for ductile isotropic materials under equi-triaxial tension. For equi-triaxial state of stress, the revised hypothesis fails only at $\mu = 0.5$, when it is identical with the classic *von Mises* hypothesis. Unfortunately, there are very few reliable tests with materials subjected to equi-triaxial tension, presented in an adequate form, able to be used for verification of the revised

Table 4 Prediction of failure stress and relative error for different metallic materials, for triaxial tests

No.	Material	μ	σ_{yp}	σ_1	σ_2	σ_3	k_w	Predicted stress [MPa]		Relative error [%]		References
								σ_{vM}	$\sigma_{vM-revised}$	e_{vM}	$e_{vM-revised}$	
1	Grey cast iron	0.21	253	176	38	−100	0.97	239.02	238.10	−5.52	−5.89	[42]
		0.21	253	136	−32	−200	0.99	290.98	290.35	15.01	14.76	
		0.21	253	82	−109	−300	0.89	330.82	330.68	30.76	30.70	
		0.21	253	130	130	−100	0.94	230.00	228.80	−9.09	−9.56	
		0.21	253	97	97	−200	1.00	297.00	297.00	17.39	17.39	
		0.21	253	106	−100	−306	0.92	356.80	355.33	41.03	40.45	
		0.21	253	176	38	−100	0.97	239.02	238.10	−5.52	−5.89	
		0.21	253	136	−32	−200	0.99	290.98	290.35	15.01	14.76	
		0.21	253	82	−109	−300	0.89	330.82	330.68	30.76	30.70	
		0.21	253	10	−195	−400	0.77	355.07	377.55	40.34	49.23	
		0.21	253	106	−100	−306	0.92	356.80	355.33	41.03	40.45	
		0.21	253	40	−200	−440	0.81	415.69	429.95	64.31	69.94	
2	Magnesium alloy	0.28	181	126	−100	−100	0.99	226.00	225.73	24.86	24.71	[42]
		0.28	181	59	−200	−200	0.90	259.00	262.18	43.09	44.85	
		0.28	181	120	10	−133	1.00	219.72	219.72	21.39	21.39	
		0.28	181	56	−72	−238	0.94	255.32	255.29	41.06	41.04	
		0.28	181	95	95	−129	0.99	224.00	223.81	23.76	23.65	
		0.28	181	20	20	−233	0.96	253.00	252.29	39.78	39.39	
		0.28	181	−48	−48	−338	0.88	290.00	297.24	60.22	64.22	

Table 5 Prediction of yield stress and relative error for different materials, for biaxial tests

No.	Material	μ	σ_{yp}	σ_1	σ_3	k_w	Predicted stress [MPa]		Relative error [%]		References
							σ_{vM}	$\sigma_{vM-revised}$	e_{vM}	$e_{vM-revised}$	
Aluminium alloys											
1	Al6061_T6	0.35	290	213.00	−112.00	1.00	285.95	285.78	−1.40	−1.45	[27]
		0.35	290	317.00	112.00	0.94	278.44	282.40	−3.99	−2.62	
		0.35	290	172.00	−168.00	1.00	294.46	294.46	1.54	1.54	
		0.35	290	109.00	−212.00	1.00	282.72	282.55	−2.51	−2.57	
		0.35	290	−141.00	−347.00	0.93	302.26	307.67	4.23	6.09	
2	Al 6016-T4	0.34	385.4	398.85	−11.21	0.97	404.57	404.39	4.97	4.93	[34]
		0.34	385.4	407.57	109.41	0.94	365.37	369.12	−5.20	−4.23	
		0.34	385.4	423.02	235.04	0.91	367.10	378.93	−4.75	−1.68	
		0.34	385.4	391.45	327.29	0.90	363.64	381.15	−5.65	−1.10	
		0.34	385.4	347.93	347.53	0.90	347.73	365.27	−9.77	−5.22	
		0.34	385.4	240.90	428.27	0.91	371.86	384.03	−3.51	−0.35	
		0.34	385.4	116.89	431.98	0.94	387.01	391.03	0.42	1.46	
		0.34	385.4	−10.55	408.89	0.97	414.26	414.10	7.49	7.45	
3	AA6016	0.3	125	125.32	0.65	0.95	125.00	125.02	0.00	0.01	[35]
		0.3	125	119.04	121.44	0.85	120.26	129.09	−3.79	3.27	
		0.3	125	−0.25	118.17	0.95	118.30	118.29	−5.36	−5.37	
4	Al 2.5% wt Mg	0.33	350	363.37	105.73	0.93	323.72	327.96	−7.51	−6.30	[37]
		0.33	350	326.91	328.13	0.88	327.52	345.84	−6.42	−1.19	
		0.33	350	216.32	390.10	0.90	338.51	350.62	−3.28	0.18	
		0.33	350	103.30	390.10	0.93	350.08	354.02	0.02	1.15	
		0.33	350	−3.65	353.65	0.96	355.48	355.42	1.57	1.55	
		0.33	350	327.72	329.67	0.88	328.70	347.09	−6.09	−0.83	[38]
		0.33	350	210.91	390.67	0.90	338.69	350.37	−3.23	0.11	
		0.33	350	103.18	391.97	0.93	351.91	355.83	0.55	1.67	
		0.33	350	−5.84	349.13	0.96	352.09	351.98	0.60	0.57	
5	Ni3Al−based alloy IC10	0.26	825	866.99	109.09	0.90	817.92	823.82	−0.86	−0.14	[41]
		0.26	825	855.50	522.49	0.81	746.92	801.42	−9.46	−2.86	
		0.26	825	752.15	717.70	0.79	735.53	808.06	−10.84	−2.05	
		0.26	825	212.44	775.12	0.87	693.74	709.43	−15.91	−14.01	
Magnesium alloy											
6	Mg alloy	0.28	181	174	87	0.85	150.69	158.03	−16.75	−12.69	[42]
		0.28	181	150	150	0.82	150.00	162.89	−17.13	−10.01	
Steel											
7	EN14301	0.29	322	231.00	−132.00	0.99	318.24	317.93	−1.17	−1.26	[32]
		0.29	322	242.00	−142.00	0.99	336.29	335.99	4.44	4.34	
		0.29	322	370.00	88.00	0.90	334.79	339.53	3.97	5.44	
		0.29	322	123.00	−233.00	0.99	313.17	312.79	−2.74	−2.86	
8	EN14462	0.25	665	502.00	−351.00	1.00	742.57	742.08	11.66	11.59	
		0.25	665	681.00	322.00	0.82	590.05	623.15	−11.27	−6.29	
		0.25	665	−316.00	−644.00	0.81	557.75	590.72	−16.13	−11.17	
9	TRIP	0.31	198	130.00	−128.00	1.00	223.44	223.44	12.85	12.85	[36]
		0.31	198	152.00	35.00	0.92	137.87	139.41	−30.37	−29.59	
		0.31	198	292.00	71.00	0.92	263.77	266.97	33.22	34.83	
		0.31	198	76.00	256.00	0.91	227.72	231.51	15.01	16.93	
Mettallic glass											
10	Mettallic glass* $Zr_{52.5}Cu_{17.9}Ni_{14.6}$ $Al_{10}Ti_5MG_{20}$	0.35	2231	2000.00	640.00	0.94	1769.07	1790.47	−20.71	−19.75	[33]
		0.35	2231	1752.30	932.20	0.92	1518.57	1559.60	−31.93	−30.09	
		0.35	2231	1706.20	1188.20	0.91	1515.13	1571.97	−32.09	−29.54	
		0.35	2231	1472.30	1330.40	0.91	1406.73	1469.26	−36.95	−34.14	
		0.35	2231	1280.60	1458.40	0.91	1378.13	1438.80	−38.23	−35.51	
		0.35	2231	958.50	1572.40	0.92	1372.60	1417.04	−38.48	−36.48	
		0.35	2231	613.40	1700.40	0.94	1491.51	1513.48	−33.15	−32.16	
		0.35	2231	306.70	1687.80	0.95	1557.27	1564.94	−30.20	−29.85	
PVC											
11	PVC	0.38	55	58.46	2.28	0.98	57.36	57.38	4.28	4.33	[39]
		0.38	55	50.30	49.72	0.94	50.01	51.54	−9.07	−6.29	
		0.38	55	−2.10	55.63	0.98	56.72	56.70	3.12	3.09	
		0.38	55	29.36	−49.35	1.00	68.90	68.88	25.27	25.23	[40]
		0.38	55	44.08	−27.62	1.00	62.64	62.62	13.88	13.86	
		0.38	55	47.10	52.43	0.94	49.98	51.47	−9.13	−6.41	

von Mises hypothesis. Nevertheless, for 6061-T6 aluminium, the value predicted by the revised the *von Mises* hypothesis is only 1% lower than that predicted by the maximum principal stress failure theory for equi-triaxial tension (*Rankine*).

Although the *Rankine* and reviewed *von Mises* theories give close results for equi-triaxial tension, the second one presents several advantages:

(1) Unfortunately, we do not know a clear demarcation of areas of triaxial stresses that one of the theories presented above can shape in a better way the experimental determinations. Consequently, near the hydrostatic axis we cannot surely say when the *von Mises* criterion must be replaced by the *Rankine* criterion;
(2) In finite element analysis it is preferable to work with a single function instead of two (classical *von Mises* or *Rankine*, respectively).

5 Conclusions

The revised *von Mises* yield hypothesis, presented in this paper, combines the theories of maximum distortion energy (*von Mises*) and maximum strain energy (*Beltrami*), ensuring a smooth transition from the *von Mises* to the *Beltrami* criterion. It can be used for ductile isotropic materials and different states of stress, including those generated by equi-triaxial tension or compression, when some of the most popular yield theories such as *Tresca*, *von Mises*, etc. cannot be used. Because the equi-triaxial tension is by no means accompanied by macroscopic yielding, in this case the proposed "yielding condition" is in fact a "failure condition."

All theories build on the assumption that yielding depends only on deviatoric stress and volumetric component of stress has no role in material yielding, inevitably cannot be applied in the case of equi-triaxial tension or compression.

In order to address the situation where it is possible to have (near) equi-triaxial stress states, it was necessary to rewrite the classical *von Mises* hypothesis. However, like other classic criteria, the mathematical model presented in this article cannot distinguish between hydrostatic tension and hydrostatic pressure (compression) and this is a notable disadvantage.

Unfortunately, there is very few reliable experimental data for materials subjected to equi-triaxial tension, adequately disclosed, and therefore, the revised *von Mises* hypothesis could not be verified for this particular state of stress, within this paper. Nevertheless, the value predicted by revised yield hypothesis is lower than that predicted by the maximum principal stress failure criterion for equi-triaxial tension. In order to validate the new yield hypothesis, some theoretical and experimental results have been presented. For available data, both theories (classic and revised *von Mises*) give quite similar results. Nevertheless, for most of them the revised *von Mises* criterion gives smaller errors.

To get a better fit with the experimental results, a modified coefficient k_{w} has been proposed. It includes a coefficient which depends on the material.

The classic *von Mises* criterion predicts the same yield loci (ellipses) for all isotropic and ductile materials. The revised *von Mises* hypothesis contains the *Poisson* ratio and, for this reason, the ellipses for aluminium and mild steel, for example, are not exactly the same, as it is confirmed experimentally. However, for a given Poisson ratio, k_w coefficient should be determined, and therefore, it needs some extra calculus when evaluating the failure condition.

The classic *von Mises* criterion is used for finite elements analysis by a significant number of software programs. Unfortunately, near hydrostatic axis we cannot surely say when the *von Mises* criterion must be replaced by the *Rankine* criterion. On the other hand, in finite element analysis it is preferable to work with a single function instead of two (classical *von Mises* or *Rankine*, respectively).

The criterion based on the revised *von Mises* hypothesis ensures a smooth transition from the *von Mises* to the *Beltrami* criterion.

Sometimes, finite element analysis of a structure (for example a metallic shell) may indicate (even within a single point) a *von Mises* stress whose value is zero, although in fact at that point there could be a dangerous (near) equi-triaxial state of stress. Even if such cases would be rare, they pose a major risk which cannot be ignored.

The new model presented in this paper proposes an extension of the classical yield hypothesis. It can be used for isotropic materials subjected to equi-triaxial tension. The key strengths that it possesses include the possibility of its use for equi-triaxial or near equi-triaxial state of stress and can be used for some metallic materials. The revised *von Mises* hypothesis can be also used in order to extend the *Hosford* or *Christensen* theories.

Hence, every yielding hypothesis needs a very careful examination and must be agreed with many experiments, made for different materials. In fact, every yielding criterion must be associated with some materials and achieving this involves a huge amount of work.

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