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ORIGINAL PAPER

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Frictionless contact of a rigid punch indenting an elastic layer having piezoelectric properties

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Abstract This article is concerned with the study of frictionless contact between a rigid punch and an elastic layer having piezoelectric properties. The rigid punch is assumed to be axially symmetric and is supposed to be pressing the elastic layer through an applied load on it. The layer is resting on a rigid base and is assumed to be sufficiently thick in comparison with the amount of indentation by the rigid punch. The relationship between the applied load and the contact area is obtained by solving the mathematically formulated problem through the use of Hankel transform of different orders. Variations of stresses and electric displacements on the surface of the layer and the piezoelectric effects on the load contact area relationship as well as normal stress have been numerically evaluated and shown graphically.

1 Introduction

In the study of various properties of solid materials, a class of materials has attracted the attention of the scientists, known as piezoelectric materials. The piezoelectric effect was discovered in 1880 by Jacques and Pierre Curie. It consists of the apparition of electric charges on the surfaces of some crystals after their deformation. The reverse effect was also outlined in 1881: where there occurs generation of stress and strain in crystals under the action of electric field on the boundary. These materials turn out to be very useful with very specific and unusual properties. In fact, these materials have the ability to produce electrical energy through the use of mechanical loadings. Piezoelectric materials, particularly piezoelectric ceramics, have been widely used for applications such as sensors, filters, ultrasonic generators, actuators, laser, supersonics, microwave, navigation and biology. Piezoelectric composite materials are also in use in hydrophone application and transducers for medical imaging. Considering the huge applicability of these materials in various fields, solid mechanics problems are being studied using elastic solids with piezoelectric properties.

The determination of the state of stress in the media when two solids are in contact has been the subject of study in literature for many years and the problems are usually called as contact problems. Contact problems are well known in solid mechanics and have been investigated thoroughly since their initial appearance in literature through the investigation by Hertz in 1882. A systematic analysis may be found in the book by

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Galin [1]. The importance of contact problems lies in the fact that whenever a solid is subjected to mechanical loading, it is done through contact between these bodies. Depending upon the characteristics of the materials of the two bodies, deformation occurs in one or both the bodies and the area of contact between the two bodies, called the contact area, may or may not change. There is some kind of relationship between this area and the loading process. This relationship is important from engineering point of view. In particular, the contact problem of electro-elasticity is very interesting from the point of view of application. Due to the application of load, the area of contact may increase, decrease or even remain stationary. Accordingly, contact problems are classified as advancing, receding or stationary. There are quite a good number of contact problems investigated in which the frictional forces at the contact surface have been ignored. Among several works done, we may mention a few: Shvets et al. [2], Chaudhuri and Ray [3,4], Comez et al. [5], Barik et al. [6,7], El-Borgi et al. [8] and Fabrikant [9]. Various types of contact problems are discussed in books and journals by Johnson [10], Gladwell [11], Hills et al. [12], Raous et al. [13], Rogowski [14], Giannakopoulos and Suresh [15], Giannakopoulos and Parmaklis [16], Chen et al. [17], Fan et al. [18], Rogowski and Kalinski [19], Sofonea and Essoufi [20], Karapetian et al. [21], Kirilyuk and Levehuk [22], Li and Wang [23], Ding et al. [24], etc. Owing to their applications in a great variety of structural systems, such as foundations, pavements in roads and runways, automotive disk brake systems and many other technological applications, considerable progress has been made with the analysis of contact problems in the theory of elasticity.

It has been noted that piezoelectric materials, although behaves like usual solids, have some additional properties in respect of generating electric field in the medium with the application of load. This additional property is expected to produce some kind of information due to electrical effects in addition to those produced by mechanical loadings. Such expectations demand reinvestigation of solid mechanics problems in piezoelectric media. In literature, several studies of contact problems in piezoelectric medium have been reported. Among these we may mention the works of Chen [25,26], Ramirez and Heylinger [27] and Zhou and Lee [28] assuming that there is no frictional forces acting at the contact surface. Frictionless contact problem for a piezoelectric layered half plane has also been investigated by Wang et al. [29]. Taking into consideration of sliding frictional force, the problems were investigated by Hao [30], Makagon et al. [31] and in relatively recently by Zhou and Lee [32] and Ma et al. [33]. Among the most recently published works on contact problems in piezoelectric media we may mention the works of Su et al. [34–36]. The investigations are on fretting contact problems in piezoelectric half space, taking into account of frictional force. Papers have been nicely presented and are very informative.

The present investigation aims to find the solution of an axially symmetric frictionless contact between a piezoelectro-elastic layer and a rigid cylindrical, spherical or conical indenter which is loaded by a concentrated force P. Using the operator theory, we derive a general solution that is expressed in terms of the three potentials. These functions satisfy differential equations of the second order and are quasi-harmonic in nature. Making use of these fundamental solutions, the punch problem in the aforesaid three cases, is investigated. The solution of the problem has been reduced to the solution of two Fredholm-type integral equations of second kind which require numerical treatment. The numerical results are discussed and presented graphically to show the influence of piezoelectricity of the layer on various subjects of interest. The novelty of this discussion lies in the nature of the problem and the way to find its solution. The problem considered here is a contact problem in an elastic layer having piezoelectric properties. Usually handling of such problems with two boundaries turns out to be more difficult compared with similar problems in a half space. The Method of solution here demands application of the Hankel transform of different orders followed by a special method as applied in [37]. In contrast with the closed-form solution available for the half space problem in piezoelectric medium as discussed in [37], no closed-form solution can be possible for our present problem. The solution available here is in the form of two integral equations, which are to be solved by numerical methods. For comparison's sake we have computed the variations of the applied load P with contact radius for different values of h in ascending order. It is observed that as the thickness of the layer increases, the value of the applied load P is approaching the corresponding value for a half space as considered in [37].

2 Formulation of the problem

We consider an elastic layer of thickness H lying on a rigid base. The material of the layer is elastically transversely isotropic, having piezoelectric properties. An axisymmetric rigid punch is placed on the free surface of the layer and is pressed toward the layer by an applied concentrated force P (Fig. 1). The axis of symmetry of the punch is along the normal to the free surface of the layer. It is assumed that the thickness of

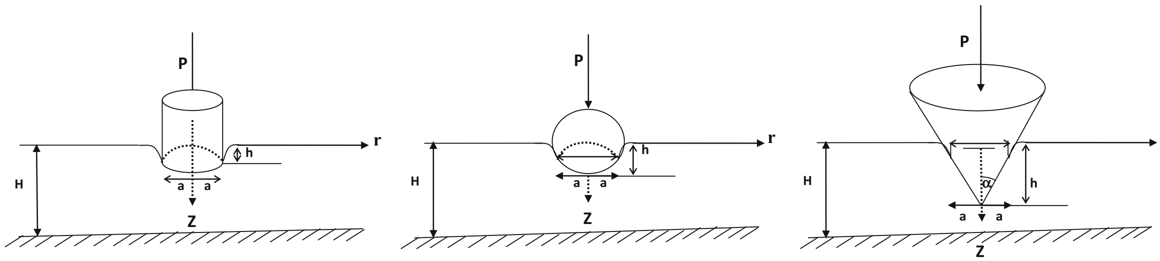


Fig. 1 **a** Geometry of the problem for flat-ended cylindrical punch. **b** Geometry of the problem for spherical punch. **c** Geometry of the problem for conical punch

the layer is much larger than the indented depth h caused by the punch. We shall use a cylindrical coordinate system (r, θ, z) with origin at the undeformed free surface of the layer and z-axis along the inward normal to the free surface. We shall also make the following additional assumptions:

(a) The force of gravity is ignored,
(b) The axis of symmetry of the transversely isotropic material is along the z-axis,
(c) The axis of polarization of the indented piezoelectric material coincides with the z-axis,
(d) Strains and displacements are small so as to apply linear theory.

Because of the axisymmetric structure of the indenter, the displacement vector $\overrightarrow{u}$ will have its cross-radial component $u_\theta = 0$, i.e., $\overrightarrow{u} = (u_r, 0, u_z)$ and all the physical quantities are independent of θ. The solution of the frictionless contact problem demands a relation between the applied load and the contact area. The mathematical formulation of the problem consists of

(i) *Equilibrium equations*

$$\begin{aligned}
&C_{11}\pounds_1 u_r + C_{44}D^2 u_r + (C_{13}+C_{44})D\frac{\partial u_z}{\partial r} + (e_{31}+e_{15})D\frac{\partial \phi}{\partial r} = 0,\\
&C_{44}\pounds_0 u_z + C_{33}D^2 u_z + (C_{13}+C_{44})D\frac{\partial [r u_r]}{r\partial r} + e_{15}\pounds_0\phi + e_{33}D^2\phi = 0,\\
&(e_{31}+e_{15})D\frac{\partial [r u_r]}{r\partial r} + e_{15}\pounds_0 u_z + e_{33}D^2 u_z - \epsilon_{11}\pounds_0\phi - \epsilon_{33}D^2\phi = 0,
\end{aligned} \tag{1}$$

where $\pounds_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k}{r^2}, k = 0, 1, D = \frac{\partial}{\partial z}$,
and

(ii) *The boundary conditions*

$$u_z(r, 0) = \delta(r), \quad 0 \le r \le a, \tag{2}$$
$$u_z(r, H) = 0, \quad r \ge 0, \tag{3}$$
$$\sigma_{rz}(r, 0) = 0, \quad r \ge 0, \tag{4}$$
$$\sigma_{zz}(r, 0) = 0, \quad r > a, \tag{5}$$
$$\sigma_{rz}(r, H) = 0, \quad r \ge 0, \tag{6}$$
$$D_z(r, 0) = 0, \quad r > a, \tag{7}$$
$$\frac{\partial \phi}{\partial r}(r, 0) = 0, \quad 0 < r < a, \tag{8}$$
$$\phi(r, H) = \phi_0, \quad r \ge 0. \tag{9}$$

The parameters C_{ij} appearing in (1) are the elastic coefficients, whereas e_{kl} and ϵ_{kl} are the piezoelectric and dielectric constants, respectively, of the material. In addition to the boundary conditions, the displacement components and the potential function ϕ should satisfy the regularity condition $u_r, u_z, \phi \to 0$ as $\sqrt{r^2+z^2} \to$

∞. At the surface of contact of the material with the indenter $0 \leq r \leq a$, the boundary condition will depend on the shape of the indenter. If h is the indented depth of the solid into the material, then
(a) for a cylindrical indenter the condition will be

$$u_z(r, 0) = \delta(r) = h; \tag{10a}$$

(b) for a conical indenter having α as the semi-vertical angle, the condition is

$$u_z(r, 0) = \delta(r) = h - (a - r)\cot\alpha; \tag{10b}$$

(c) for a spherical indenter having radius R, the condition is

$$u_z(r, 0) = \delta(r) = h - \frac{r^2}{2R}. \tag{10c}$$

3 Method of solution

Solution of the partial differential equations (1) requires the Hankel transform technique of different orders. We shall outline the method adopted by Dyka and Rogowski [37] for the first part of the discussion and thereafter apply those results in our considered problem.

$$\begin{aligned} \widehat{u_r}(\xi, z) &= H_1[u_r(r, z); r \to \xi] = \int_0^\infty u_r(r, z) r J_1(r\xi) \mathrm{d}r, \\ \{\widehat{u_z}(\xi, z), \widehat{\phi}(\xi, z)\} &= H_0[u_z(r, z), \phi(r, z); r \to \xi] = \int_0^\infty \{u_z(r, z), \phi(r, z)\} r J_0(r\xi) \mathrm{d}r \end{aligned} \tag{11}$$

are applied, where J_0 and J_1 are the Bessel functions of the first kind and of order one or zero, respectively, and ξ is the transform parameter. We use the following properties of Hankel transforms:

$$\begin{aligned} &H_\nu[\pounds_\nu f(r, z); r \to \xi] = -\xi^2 \widehat{f_\nu}(\xi, z), \\ &H_1\left[\frac{\partial f(r, z)}{\partial r}; r \to \xi\right] = -\xi \widehat{f_0}(\xi, z), \\ &H_0\left[\frac{\partial [r f(r, z)]}{r \partial r}; r \to \xi\right] = \xi \widehat{f_1}(\xi, z). \end{aligned} \tag{12}$$

Applying the Hankel transformations (11) to Eq. (1) we get three coupled ordinary differential equations, which may be written in the form

$$\mathbf{A} \begin{bmatrix} \widehat{u}_r \\ \widehat{u}_z \\ \widehat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{13}$$

where $\mathbf{A}$ is the following operator matrix

$$\mathbf{A} = \begin{bmatrix} -C_{11}\xi^2 + C_{44}D^2 & -\xi(C_{13} + C_{44})D & -\xi(e_{31} + e_{15})D \\ \xi(C_{13} + C_{44})D & -C_{44}\xi^2 + C_{33}D^2 & -e_{15}\xi^2 + e_{33}D^2 \\ \xi(e_{31} + e_{15})D & -e_{15}\xi^2 + e_{33}D^2 & \epsilon_{11}\xi^2 - \epsilon_{33}D^2 \end{bmatrix}. \tag{14}$$

We have

$$|\mathbf{A}| = -a_0(D^2 - \lambda_1^2\xi^2)(D^2 - \lambda_2^2\xi^2)(D^2 - \lambda_3^2\xi^2), \tag{15}$$

where λ_i^2 $(i = 1, 2, 3)$ are the roots of the following cubic algebraic equation:

$$a_0\lambda^6 + b_0\lambda^4 + c_0\lambda^2 + d_0 = 0 \tag{16}$$

with the coefficients defined by

$$\begin{aligned}
a_0 &= C_{44}\left(C_{33}\epsilon_{33} + e_{33}^2\right),\\
b_0 &= (e_{31}+e_{15})[2C_{13}e_{33} - C_{33}(e_{31}+e_{15})] + 2C_{44}e_{33}e_{31} - C_{11}e_{33}^2,\\
&\quad -\epsilon_{11}C_{33}C_{44} - \epsilon_{33}c^2,\\
c_0 &= 2e_{15}[C_{11}e_{33} - C_{13}(e_{31}+e_{15})] + C_{44}e_{31}^2 + \epsilon_{33}C_{11}C_{44} + \epsilon_{11}c^2,\\
d_0 &= -C_{11}(C_{44}\epsilon_{11} + e_{15}^2),\\
c^2 &= C_{11}C_{33} - C_{13}(C_{13} + 2C_{44}).
\end{aligned} \tag{17}$$

Using the operator theory, we obtain the general solution to Eq. (1) as

$$\begin{aligned}
\widehat{u}_r(\xi, z) &= A_{i1}\widehat{F}(\xi, z),\\
\widehat{u}_z(\xi, z) &= A_{i2}\widehat{F}(\xi, z),\\
\widehat{\phi}(\xi, z) &= A_{i3}\widehat{F}(\xi, z),
\end{aligned} \tag{18}$$

where A_{ij} are the algebraic cominors of the matrix operator and $\widehat{F}(\xi, z)$ is the zero-order Hankel transform of the general solution $F(r, z)$, satisfying the equations

$$\begin{aligned}
&|\mathbf{A}|\widehat{F}(\xi, z) = 0,\\
&\left(D^2 + \lambda_1^2\Delta\right)\left(D^2 + \lambda_2^2\Delta\right)\left(D^2 + \lambda_3^2\Delta\right)F(r, z) = 0.
\end{aligned} \tag{19}$$

Here, $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ and $D^2 = \frac{\partial^2}{\partial z^2}$.

Taking $i = 3$ and writing down the expression for A_{3j}, we obtain

$$\begin{aligned}
\widehat{u}_r(\xi, z) &= \left(a_1 D^2 + b_1\xi^2\right)\xi D\widehat{F}(\xi, z),\\
\widehat{u}_z(\xi, z) &= -\left(a_2 D^4 + b_2\xi^2 D^2 + c_2\xi^4\right)\widehat{F}(\xi, z),\\
\widehat{\phi}(\xi, z) &= \left(a_3 D^4 + b_3\xi^2 D^2 + c_3\xi^4\right)\widehat{F}(\xi, z),
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&a_1 = C_{33}(e_{31}+e_{15}) - (C_{13}+C_{44})e_{33}, && b_1 = C_{13}e_{15} - C_{44}e_{31},\\
&a_2 = C_{44}e_{33}, && b_2 = (C_{13}+C_{44})e_{31} + C_{13}e_{15} - C_{11}e_{33},\\
&c_2 = C_{11}e_{15}, && a_3 = C_{44}C_{33},\\
&b_3 = C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33}, && c_3 = C_{11}C_{44}.
\end{aligned} \tag{21}$$

Using the inverse Hankel transforms the Eq. (20), the original solution for the displacements and electric potential is obtained as:

$$\begin{aligned}
u_r(r, z) &= -\left(a_1 D^2 - b_1\Delta\right)\frac{\partial^2}{\partial r\partial z}F(r, z),\\
u_z(r, z) &= -\left(a_2 D^4 - b_2\Delta D^2 + c_2\Delta^2\right)F(r, z),\\
\phi(r, z) &= \left(a_3 D^4 - b_3\Delta D^2 + c_3\Delta^2\right)F(r, z).
\end{aligned} \tag{22}$$

Using the generalized Almansi's theorem [38], the function $F(r, z)$, which satisfies Eq. $(19)_2$, can be expressed in terms of three quasi-harmonic functions

$$F = \begin{cases} F_1 + F_2 + F_3 & \text{for distinct } \lambda_i,\\ F_1 + F_2 + zF_3 & \text{for } \lambda_1 \neq \lambda_2 = \lambda_3,\\ F_1 + zF_2 + z^2F_3 & \text{for } \lambda_1 = \lambda_2 = \lambda_3, \end{cases} \tag{23}$$

where $F_i(r, z)$ satisfies, respectively

$$\left(\Delta + \frac{1}{\lambda_i^2 D^2}\right)F_i(r, z) = 0, \quad i = 1, 2, 3. \tag{24}$$

As we shall see later, at the roots of Eq. (16) are all distinct in our considered problem, so we shall consider only the first solution in Eq. (23).

Using

$$\Delta F_i = -\frac{1}{\lambda_i^2} D^2 F_i$$

and summing in Eq. (22), we obtain

$$\begin{aligned} u_r(r,z) &= -\sum_{i=1}^{3} \alpha_{i1} \frac{\partial^4 F_i}{\partial r \partial z^3}, \\ u_z(r,z) &= -\sum_{i=1}^{3} \alpha_{i2} \frac{\partial^4 F_i}{\partial z^4}, \\ \phi(r,z) &= \sum_{i=1}^{3} \alpha_{i3} \frac{\partial^4 F_i}{\partial z^4}. \end{aligned} \tag{25}$$

The coefficients α_{ij} are

$$\alpha_{ij} = a_j + \frac{b_j}{\lambda_i^2} + \frac{c_j}{\lambda_i^4},$$

where a_j, b_j and c_j are defined by Eq. (21) and $c_1 = 0$. It is now assumed that

$$\alpha_{i2} \frac{\partial^3}{\partial z^3} F_i(r,z) = -\frac{1}{\lambda_i} \varphi_i(r,z),$$

then Eq. (25) can be further simplified to

$$\begin{aligned} u_r(r,z) &= \sum_{i=1}^{3} a_{i1} \lambda_i \frac{\partial \varphi_i}{\partial r}, \\ u_z(r,z) &= \sum_{i=1}^{3} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial z}, \\ \phi(r,z) &= -\sum_{i=1}^{3} \frac{a_{i3}}{\lambda_i} \frac{\partial \varphi_i}{\partial z}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} a_{i1} &= \frac{\alpha_{i1}}{\alpha_{i2}} \frac{1}{\lambda_i^2} = \frac{a_1 \lambda_i^2 + b_1}{a_2 \lambda_i^4 + b_2 \lambda_i^2 + c_2^2}, \\ a_{i3} &= \frac{\alpha_{i3}}{\alpha_{i2}} = \frac{a_3 \lambda_i^4 + b_3 \lambda_i^2 + c_3}{a_2 \lambda_i^4 + b_2 \lambda_i^2 + c_2} = \frac{C_{13} + C_{44}}{e_{31} + e_{15}} - \frac{C_{11} - C_{44} \lambda_i^2}{e_{31} + e_{15}} a_{i1}, \end{aligned} \tag{27}$$

and for the quasi-harmonic function $\varphi_i(r,z)$

$$\left(\Delta + \tfrac{1}{\lambda_i^2} \tfrac{\partial^2}{\partial z^2} \right) \varphi_i(r,z) = 0. \tag{28}$$

The relations between stress, displacement and electric potential for a transversely isotropic piezoelectric medium (the so-called Duhamel–Neumann relation), in the case of axial symmetry, are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{zr} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & C_{13} & 0 & e_{31} \\ C_{12} & C_{11} & 0 & 0 & C_{13} & 0 & e_{31} \\ C_{13} & C_{13} & 0 & 0 & C_{33} & 0 & e_{33} \\ 0 & 0 & C_{44} & C_{44} & 0 & e_{15} & 0 \end{bmatrix} \begin{bmatrix} u_{r,r} \\ \frac{u_r}{r} \\ u_{r,z} \\ u_{z,r} \\ u_{z,z} \\ \phi_r \\ \phi_z \end{bmatrix}. \tag{29}$$

Substituting Eq. (26) into Eq. (29), we obtain

$$\sigma_{rr}(r,z) = -\sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial z^2} - (C_{11} - C_{12}) \frac{u_r}{r}, \quad \sigma_{zz}(r,z) = \sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i^3} \frac{\partial^2 \varphi_i}{\partial z^2},$$
$$\sigma_{\theta\theta}(r,z) = -\sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i^2} \frac{\partial^2 \varphi_i}{\partial z^2} - (C_{11} - C_{12}) \frac{\partial u_r}{\partial r}, \quad \sigma_{zr}(r,z) = \sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial r \partial z}, \tag{30}$$

where

$$a_{i4} = \frac{e_{31} C_{44} \lambda_i^2 + e_{15} C_{11}}{e_{31} + e_{15}} a_{i1} + \frac{C_{44} e_{31} - C_{13} e_{15}}{e_{31} + e_{15}}, \tag{31}$$

The components of the electric field vector E_r and E_z are obtained from the relations

$$E_r = -\frac{\partial \phi}{\partial r} = \sum_{i=1}^{3} \frac{a_{i3}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial r \partial z},$$
$$E_z = -\frac{\partial \phi}{\partial z} = \sum_{i=1}^{3} \frac{a_{i3}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial z^2}. \tag{32}$$

The electric displacements are defined by the equations

$$\begin{bmatrix} D_r \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & e_{15} & e_{15} & 0 & \epsilon_{11} & 0 \\ e_{31} & e_{31} & 0 & 0 & e_{33} & 0 & \epsilon_{33} \end{bmatrix} \begin{bmatrix} u_{r,r} \\ \frac{u_r}{r} \\ u_{r,z} \\ u_{z,r} \\ u_{z,z} \\ E_r \\ E_z \end{bmatrix}. \tag{33}$$

In terms of φ_i,

$$D_r = \sum_{i=1}^{3} a_{i5} \lambda_i \frac{\partial^2 \varphi_i}{\partial r \partial z},$$
$$D_z = \sum_{i=1}^{3} \frac{a_{i5}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial z^2}, \tag{34}$$

where

$$a_{i5} = \frac{e_{33} \epsilon_{11} - e_{15} \epsilon_{33}}{\epsilon_{11} - \epsilon_{33} \lambda_i^2} - \frac{e_{31} \epsilon_{11} - e_{15} \epsilon_{33} \lambda_i^2}{\epsilon_{11} - \epsilon_{33} \lambda_i^2} a_{i1}. \tag{35}$$

It can be easily verified that Gauss' law [39]

$$\frac{\partial D_r}{\partial r} + \frac{D_r}{r} + \frac{\partial D_z}{\partial z} = 0 \tag{36}$$

and the equilibrium equations for stresses [40]

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0,$$
$$\frac{\partial \sigma_{zr}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{zr}}{r}=0 \tag{37}$$

are satisfied.

In the vacuum, the constitutive equations (33) and the governing equations (36) become

$$D_r=\epsilon_0 E_r, \quad D_z=\epsilon_0 E_z,$$
$$\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{\partial^2 \phi}{\partial z^2}=0, \tag{38}$$

where ϵ_0 is the electric permittivity of the vacuum.

For axially symmetric problems, the general solution of the differential equation (28) may be written as

$$\varphi_i(r,z)=\int_0^\infty \left[A_i(\xi)e^{-\lambda_i \xi z}+B_i(\xi)e^{\lambda_i \xi z}\right] J_0(r\xi)\mathrm{d}\xi, \tag{39}$$

where $A_i(\xi)$, $B_i(\xi)(i=1,2,3)$ are arbitrary functions of the transform parameter ξ, which are to be determined from the boundary conditions Eqs. (2–9), and λ_i are the roots of Eq. (16).

Assuming $\phi_0=0$ and using Eqs. (39), (26) and (30) in the boundary conditions (2–9), we obtain

$$\sum_{i=1}^{3} a_{i4}\left[A_i(\xi)-B_i(\xi)\right]=0, \quad r\geq 0, \tag{40}$$

$$\sum_{i=1}^{3} a_{i4}\left[A_i(\xi)e^{-\lambda_i \xi H}-B_i(\xi)e^{\lambda_i \xi H}\right]=0, \quad r\geq 0, \tag{41}$$

$$\sum_{i=1}^{3}\left[A_i(\xi)e^{-\lambda_i \xi H}-B_i(\xi)e^{\lambda_i \xi H}\right]=0, \quad r\geq 0, \tag{42}$$

$$\sum_{i=1}^{3} a_{i3}\left[A_i(\xi)e^{-\lambda_i \xi H}-B_i(\xi)e^{\lambda_i \xi H}\right]=0, \quad r\geq 0, \tag{43}$$

$$\sum_{i=1}^{3}\int_0^\infty \left[-A_i(\xi)+B_i(\xi)\right]\xi J_0(r\xi)\mathrm{d}\xi=\delta(r), \quad 0\leq r\leq a, \tag{44}$$

$$\sum_{i=1}^{3}\int_0^\infty \frac{a_{i4}}{\lambda_i}\left[A_i(\xi)+B_i(\xi)\right]\xi^2 J_0(r\xi)\mathrm{d}\xi=0, \quad r>a, \tag{45}$$

$$\sum_{i=1}^{3}\int_0^\infty a_{i3}\left[-A_i(\xi)+B_i(\xi)\right]\xi^2 J_1(r\xi)\mathrm{d}\xi=0, \quad 0\leq r\leq a, \tag{46}$$

$$\sum_{i=1}^{3}\lambda_i a_{i5}\int_0^\infty \left[A_i(\xi)+B_i(\xi)\right]\xi^2 J_0(r\xi)\mathrm{d}\xi=0, \quad r>a. \tag{47}$$

Equations (40–43) yield

$$B_i(\xi) = A_i(\xi)e^{-2\lambda_i \xi H} \quad (i = 1, 2, 3) \tag{48}$$

and

$$A_1(\xi) = \chi_1(\xi)A_2(\xi) + \chi_2(\xi)A_3(\xi), \tag{49}$$

where

$$\chi_1(\xi) = -\frac{a_{24}}{a_{14}} \frac{1 - e^{-2\lambda_2 \xi H}}{1 - e^{-2\lambda_1 \xi H}},$$
$$\chi_2(\xi) = -\frac{a_{34}}{a_{14}} \frac{1 - e^{-2\lambda_3 \xi H}}{1 - e^{-2\lambda_1 \xi H}}. \tag{50}$$

Using Eq. (48) in Eqs. (44) and (45) we get, respectively,

$$\int_0^\infty \left[\left\{-(1 - e^{-2\lambda_1 \xi H})\chi_1(\xi) - (1 - e^{-2\lambda_2 \xi H})\right\} A_2(\xi) + \{-(1 - e^{-2\lambda_1 \xi H})\chi_2(\xi) - (1 - e^{-2\lambda_3 \xi H})\}A_3(\xi)\right] \xi J_0(r\xi)\mathrm{d}\xi = \delta(r), \quad 0 \le r \le a, \tag{51}$$

$$\int_0^\infty \{\chi_3(\xi)A_2(\xi) + \chi_4(\xi)A_3(\xi)\}\xi^2 J_0(r\xi)\mathrm{d}\xi = 0, \quad r > a, \tag{52}$$

where

$$\chi_3(\xi) = \frac{a_{14}}{\lambda_1}(1 + e^{-2\lambda_1 \xi H})\chi_1(\xi) + \frac{a_{24}}{\lambda_2}(1 + e^{-2\lambda_2 \xi H}),$$
$$\chi_4(\xi) = \frac{a_{14}}{\lambda_1}(1 + e^{-2\lambda_1 \xi H})\chi_2(\xi) + \frac{a_{34}}{\lambda_3}(1 + e^{-2\lambda_3 \xi H}). \tag{53}$$

Now we assume that

$$\{\chi_3(\xi)A_2(\xi) + \chi_4(\xi)A_3(\xi)\}\,\xi = \sqrt{\frac{2}{\pi}} \int_0^a \phi_1(x)\cos(\xi x)\mathrm{d}x. \tag{54}$$

Then Eq. (52) is automatically satisfied. Substituting Eq. (48) into (46) and (47) leads to

$$\int_0^\infty \left[\left\{-(1 - e^{-2\lambda_1 \xi H})\chi_1(\xi) - (1 - e^{-2\lambda_2 \xi H})\right\} A_2(\xi) + \left\{-(1 - e^{-2\lambda_1 \xi H})\chi_2(\xi) - (1 - e^{-2\lambda_3 \xi H})\right\} A_3(\xi)\right] \xi^2 J_1(r\xi)\mathrm{d}\xi = 0, \quad 0 \le r \le a \tag{55}$$

and

$$\int_0^\infty \{\chi_5(\xi)A_2(\xi) + \chi_6(\xi)A_3(\xi)\}\,\xi^2 J_0(r\xi)\mathrm{d}\xi = 0, \quad r > a, \tag{56}$$

where

$$\chi_5(\xi) = \lambda_1 a_{15}(1 + e^{-2\lambda_1 \xi H})\chi_1(\xi) + \lambda_2 a_{25}(1 + e^{-2\lambda_2 \xi H}),$$
$$\chi_6(\xi) = \lambda_1 a_{15}(1 + e^{-2\lambda_1 \xi H})\chi_2(\xi) + \lambda_3 a_{35}(1 + e^{-2\lambda_3 \xi H}). \tag{57}$$

Now we assume that

$$\{\chi_5(\xi)A_2(\xi)+\chi_6(\xi)A_3(\xi)\}\xi=\sqrt{\frac{2}{\pi}}\int_0^a \phi_2(x)\cos(\xi x)\mathrm{d}x. \tag{58}$$

Then Eq. (56) is automatically satisfied. Solving Eqs. (54) and (58) we get

$$A_2(\xi)=\beta_1(\xi)\sqrt{\frac{2}{\pi}}\int_0^a \phi_1(x)\cos(\xi x)\mathrm{d}x+\beta_2(\xi)\sqrt{\frac{2}{\pi}}\int_0^a \phi_2(x)\cos(\xi x)\mathrm{d}x \tag{59}$$

and

$$A_3(\xi)=\beta_3(\xi)\sqrt{\frac{2}{\pi}}\int_0^a \phi_1(x)\cos(\xi x)\mathrm{d}x+\beta_4(\xi)\sqrt{\frac{2}{\pi}}\int_0^a \phi_2(x)\cos(\xi x)\mathrm{d}x, \tag{60}$$

where

$$\begin{aligned}\beta_1(\xi)&=\frac{\chi_6(\xi)}{\{\chi_3(\xi)\chi_6(\xi)-\chi_4(\xi)\chi_5(\xi)\}\xi},\quad \beta_2(\xi)=-\frac{\chi_4(\xi)}{\{\chi_3(\xi)\chi_6(\xi)-\chi_4(\xi)\chi_5(\xi)\}\xi},\\ \beta_3(\xi)&=\frac{\chi_5(\xi)}{\{\chi_4(\xi)\chi_5(\xi)-\chi_3(\xi)\chi_6(\xi)\}\xi},\quad \beta_4(\xi)=-\frac{\chi_3(\xi)}{\{\chi_4(\xi)\chi_5(\xi)-\chi_3(\xi)\chi_6(\xi)\}\xi}.\end{aligned} \tag{61}$$

From Eqs. (51) and (59)–(61) we get

$$\int_0^a \phi_1(x)k_{11}(r,x)\mathrm{d}x+\int_0^a \phi_2(x)k_{12}(r,x)\mathrm{d}x=\delta(r),\quad 0\le r\le a, \tag{62}$$

where

$$k_{1i}(r,x)=\int_0^\infty G_i(\xi)\xi J_0(\xi r)\cos(\xi x)\mathrm{d}\xi,\quad (i=1,2), \tag{63}$$

$$\begin{aligned}G_1(\xi)=\sqrt{\frac{2}{\pi}}\Big[&\beta_1(\xi)\Big\{-(1-e^{-2\lambda_1\xi H})\chi_1(\xi)-(1-e^{-2\lambda_2\xi H})\Big\}\\ &+\beta_3(\xi)\Big\{-(1-e^{-2\lambda_1\xi H})\chi_2(\xi)-(1-e^{-2\lambda_3\xi H})\Big\}\Big],\end{aligned} \tag{64}$$

$$\begin{aligned}G_2(\xi)=\sqrt{\frac{2}{\pi}}\Big[&\beta_2(\xi)\Big\{-(1-e^{-2\lambda_1\xi H})\chi_1(\xi)-(1-e^{-2\lambda_2\xi H})\Big\}\\ &+\beta_4(\xi)\Big\{-(1-e^{-2\lambda_1\xi H})\chi_2(\xi)-(1-e^{-2\lambda_3\xi H})\Big\}\Big].\end{aligned} \tag{65}$$

Again from Eq. (55) we get

$$\int_0^a \phi_1(x)k_{21}(r,x)\mathrm{d}x+\int_0^a \phi_2(x)k_{22}(r,x)\mathrm{d}x=0,\quad 0\le r\le a, \tag{66}$$

where

$$k_{2i}(r,x)=\int_0^\infty G_i(\xi)\xi^2 J_1(\xi r)\cos(\xi x)\mathrm{d}\xi=-\frac{\mathrm{d}}{\mathrm{d}r}k_{1i}(r,x),\quad (i=1,2). \tag{67}$$

Now Eq. (62) can be written as

$$\int_0^r \frac{\mathrm{d}x}{\sqrt{x^2-r^2}}\left[\phi_1(x)+\int_0^a \phi_1(u)L_1(u,x)\mathrm{d}u\right]=f_1(r), \tag{68}$$

which is an Abel-type integral equation, where

$$f_1(r)=\delta(r)-\int_0^a \phi_2(u)k_{12}(r,u)\mathrm{d}u. \tag{69}$$

After some work, we get the integral equation in ϕ_1 as

$$\phi_1(x)+\int_0^a \phi_1(u)L_1(u,x)\mathrm{d}u+\int_0^a \phi_2(u)L_2(u,x)\mathrm{d}u=\frac{2}{\pi}f(x),\quad 0\le x\le a, \tag{70}$$

where

$$\begin{aligned} f(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{r\delta(r)}{\sqrt{x^2-r^2}}\mathrm{d}r &= h,\quad \text{for cylindrical indenter,}\\ &= h+\left(\frac{\pi}{2}x-a\right)\cot\alpha,\quad \text{for conical indenter,}\\ &= h-\frac{2x^2}{R},\quad \text{for spherical indenter,}\end{aligned}$$

$$L_1(u,x)=\frac{2}{\pi}\int_0^\infty \Omega_1(\xi)\cos(\xi u)\cos(\xi x)\mathrm{d}\xi, \tag{71}$$

$$L_2(u,x)=\frac{2}{\pi}\int_0^\infty G_2(\xi)\cos(\xi u)\cos(\xi x)\xi\mathrm{d}\xi, \tag{72}$$

$$\Omega_1(\xi)=G_1(\xi)\xi-1. \tag{73}$$

Again Eq. (66) can be written as

$$\int_0^r \frac{\mathrm{d}x}{\sqrt{x^2-r^2}}\left[\phi_2(x)+\int_0^a \phi_2(u)L_3(u,x)\mathrm{d}u\right]=f_2(r), \tag{74}$$

which is an Abel-type integral equation, where

$$f_2(r)=\int_0^a \{\phi_1(u)k_{11}(0,u)-\phi_1(u)k_{11}(r,u)+\phi_2(u)k_{12}(0,u)\}\,\mathrm{d}u. \tag{75}$$

We get the integral equation in ϕ_2 as

$$\begin{aligned}\phi_2(x)&+\int_0^a \phi_2(u)\left\{L_3(u,x)-\frac{2}{\pi}k_{12}(0,u)\right\}\mathrm{d}u\\ &+\int_0^a \phi_1(u)\left\{L_4(u,x)-\frac{2}{\pi}k_{11}(0,u)\right\}\mathrm{d}u=0,\quad 0\le x\le a,\end{aligned} \tag{76}$$

where

$$L_3(u,x)=\frac{2}{\pi}\int_0^\infty \Omega_2(\xi)\cos(\xi u)\cos(\xi x)\mathrm{d}\xi, \tag{77}$$

$$L_4(u,x)=\frac{2}{\pi}\int_0^\infty G_1(\xi)J_0(\xi u)\cos(\xi x)\xi\mathrm{d}\xi, \tag{78}$$

$$\Omega_2(\xi)=G_2(\xi)\xi-1. \tag{79}$$

Before proceeding further, it will be convenient to introduce non-dimensional variables u', x' and r' by rescaling by the length scale a:

$$u' = \frac{u}{a}, \quad x' = \frac{x}{a}, \quad r' = \frac{r}{a}. \tag{80}$$

For notational convenience, we shall use only dimensionless variables and shall ignore the dashes on the transformed non-dimensional variables, and then the integral equations (70) and (76) become

$$\phi_1(x) + \int_0^1 \phi_1(u)L_1(u,x)\mathrm{d}u + \int_0^1 \phi_2(u)L_2(u,x)\mathrm{d}u = \frac{2}{\pi}f(x), \quad 0 \le x \le 1, \tag{81}$$

$$\phi_2(x) + \int_0^1 \phi_2(u)\left\{L_3(u,x) - \frac{2}{\pi}k_{12}(0,u)\right\}\mathrm{d}u + \int_0^1 \phi_1(u)\left\{L_4(u,x) - \frac{2}{\pi}k_{11}(0,u)\right\}\mathrm{d}u = 0, \quad 0 \le x \le 1. \tag{82}$$

These equations determine the functions ϕ_1 and ϕ_2.

Now the equilibrium condition demands

$$P + \int_0^{a^*} 2\pi r \mathrm{d}r \sigma_{zz}(r,0) = 0$$
$$\Rightarrow P + 2\pi \int_0^\infty M(\omega)\left[\int_0^{a^*} J_0(\omega r) r \mathrm{d}r\right]\mathrm{d}\omega = 0, \tag{83}$$

where

$$M(\omega) = \sqrt{\frac{2}{\pi}}\omega^2 \left[\{\beta_1(\omega)\chi_3(\omega) + \beta_3(\omega)\chi_4(\omega)\}\int_0^1 \phi_1(x)\cos(\omega x)\mathrm{d}x \right.$$
$$\left. + \{\beta_2(\omega)\chi_3(\omega) + \beta_4(\omega)\chi_4(\omega)\}\int_0^1 \phi_2(x)\cos(\omega x)\mathrm{d}x\right] \tag{84}$$

and

$$a^* = \frac{a}{h}, \quad \omega = \xi h.$$

Equation (83) is the relationship between the applied load P and the radius of the contact area.

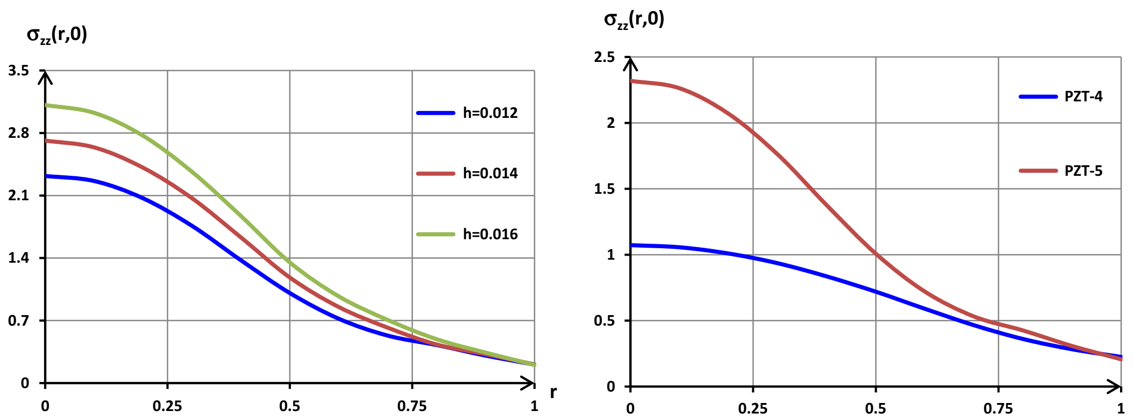
4 Numerical results and discussion

The present study aims at investigating a frictionless contact problem in a finite piezo-electric layer. The main objective of the present discussion is to study the effects of indentation on normal stress and electric displacement. Stresses and displacements have been computed numerically through Eqs. (81) and (82) and are shown graphically. In our numerical computation the piezoelectric materials considered are PZT-4 and PZT-5 with the following nonzero constitutive coefficients [41]:

	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	e_{15}	e_{31}	e_{33}	ϵ_{11}	ϵ_{33}
PZT-4	13.90	7.78	7.43	11.30	2.56	13.44	−6.98	13.84	60.00	54.70
PZT-5	12.60	5.50	5.30	11.70	3.53	17.00	−6.50	23.30	151.00	130.00

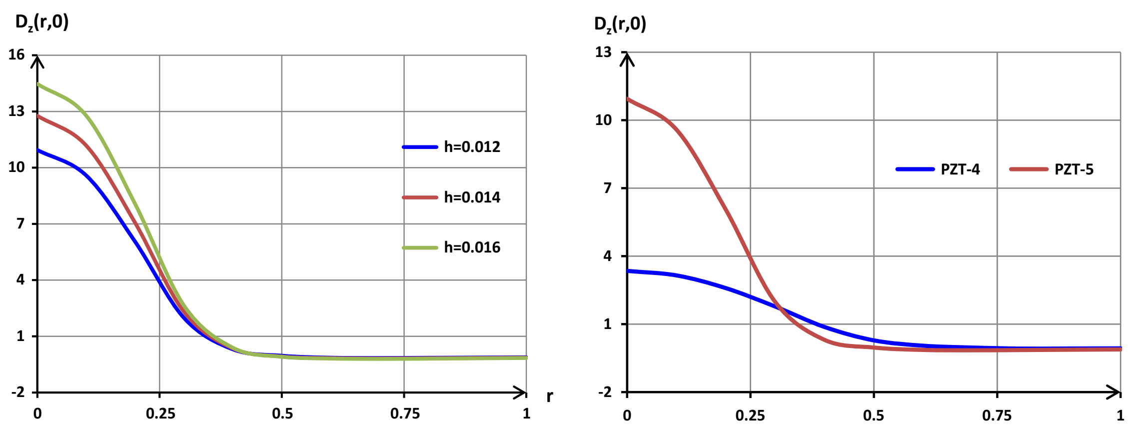
The actual C-values are the values in the table multiplied by 10^{10} in $\mathrm{N/m^2}$ and the e-values in $\mathrm{C/m^2}$, and the ϵ-values are the values in the table multiplied by 10^{10} in C/V m.

Our numerical study will cover three different types of rigid indenter, namely cylindrical shaped, spherical shaped and conical shaped. The numerical results based on the values in the table above are displayed in Figs. 2 and 3 for a cylindrical indenter, in Figs. 4 and 5 for a spherical indenter and in Figs. 6 and 7 for



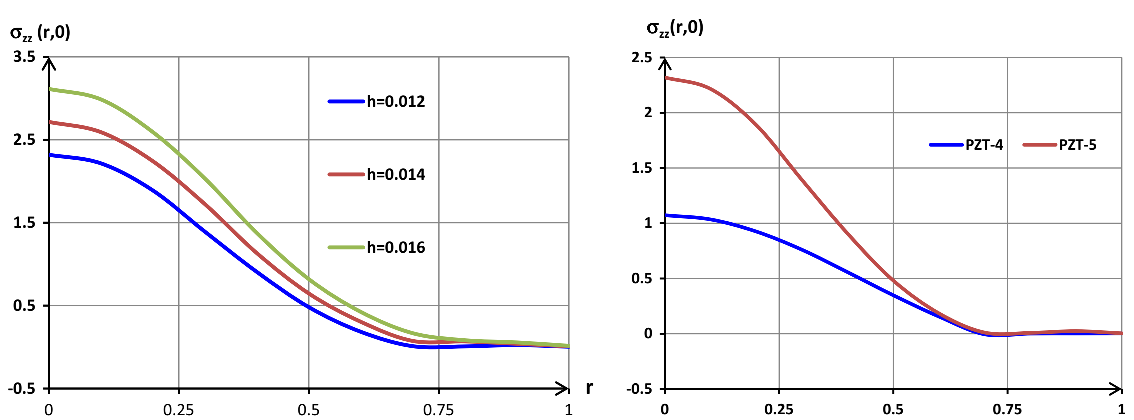
(a) (b)

Fig. 2 **a** Effect of indentation h on $\sigma_{zz}(r, 0)$ for flat-ended cylindrical punch. **b** Variation of $\sigma_{zz}(r, 0)$ for various piezoelectric ceramics with fixed indentation of the flat-ended cylindrical punch ($h = 0.012$)



(a) (b)

Fig. 3 **a** Effect of indentation h on $D_z(r, 0)$ for the flat-ended cylindrical punch. **b** Variation of $D_z(r, 0)$ for various piezoelectric ceramics with fixed indentation of the flat-ended cylindrical punch ($h = 0.012$)



(a) (b)

Fig. 4 **a** Effect of indentation h on $\sigma_{zz}(r, 0)$ for the spherical punch ($R = 20$). **b** Variation of $\sigma_{zz}(r, 0)$ for various piezoelectric ceramics with fixed indentation of the spherical punch ($h = 0.012$, $R = 20$)

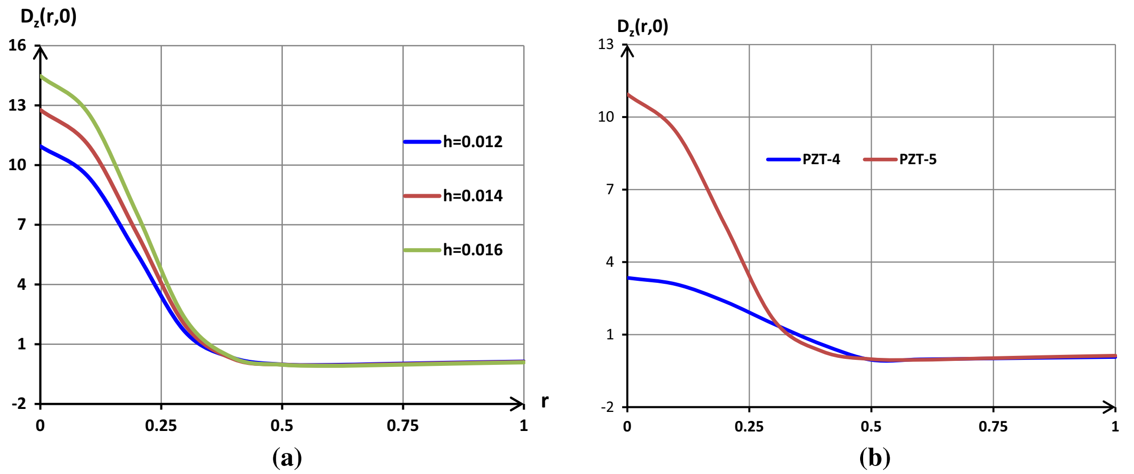


Fig. 5 **a** Effect of indentation h on $D_z(r, 0)$ for the spherical punch ($R = 20$). **b** Variation of $D_z(r, 0)$ for various piezoelectric ceramics with fixed indentation h of the spherical punch ($h = 0.012$, $R = 20$)

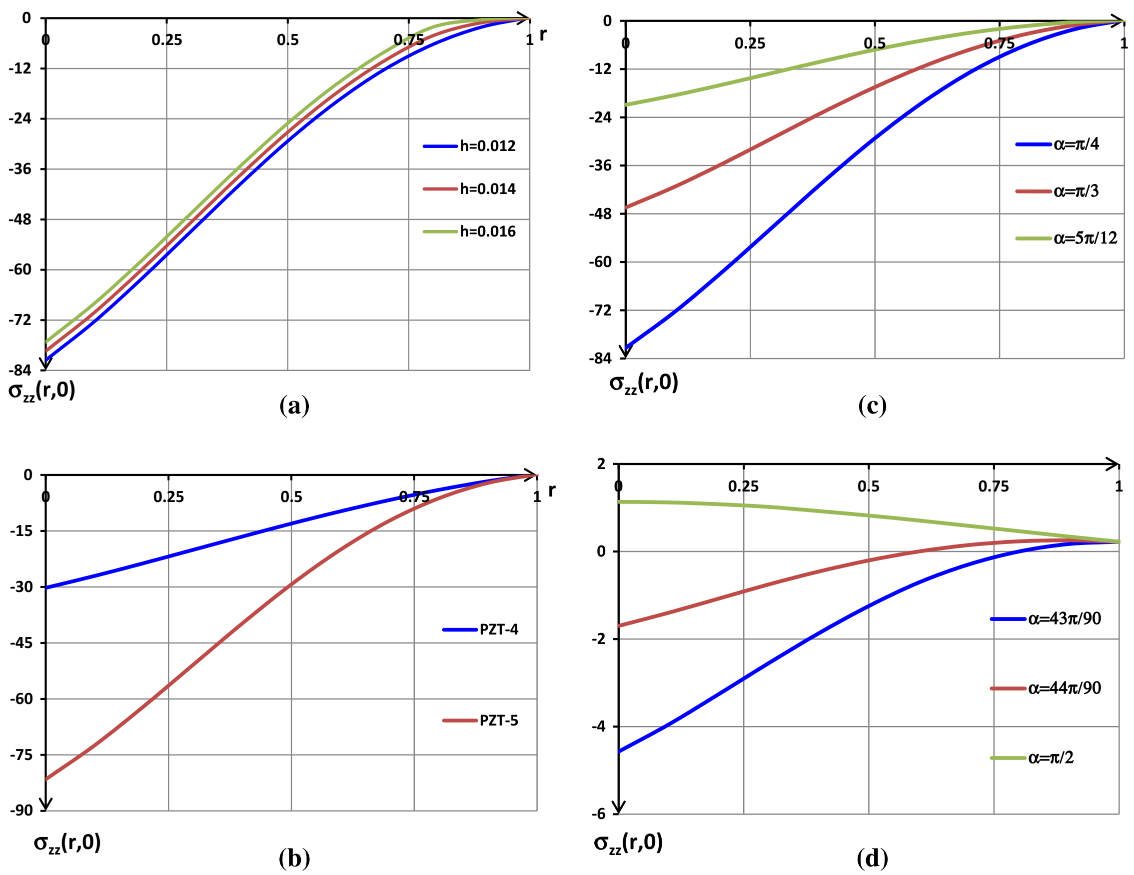


Fig. 6 **a** Effect of indentation h on $\sigma_{zz}(r, 0)$ for the conical punch ($\alpha = \pi/4$). **b** Variation of $\sigma_{zz}(r, 0)$ for various piezoelectric ceramics with fixed indentation of the conical punch ($h = 0.012$, $\alpha = \pi/4$). **c** Variation of $\sigma_{zz}(r, 0)$ for different α of the conical punch ($h = .014$). **d** Variation of $\sigma_{zz}(r, 0)$ for semi-vertical angle $\alpha \to \pi/2$ of the conical punch ($h = 0.014$)

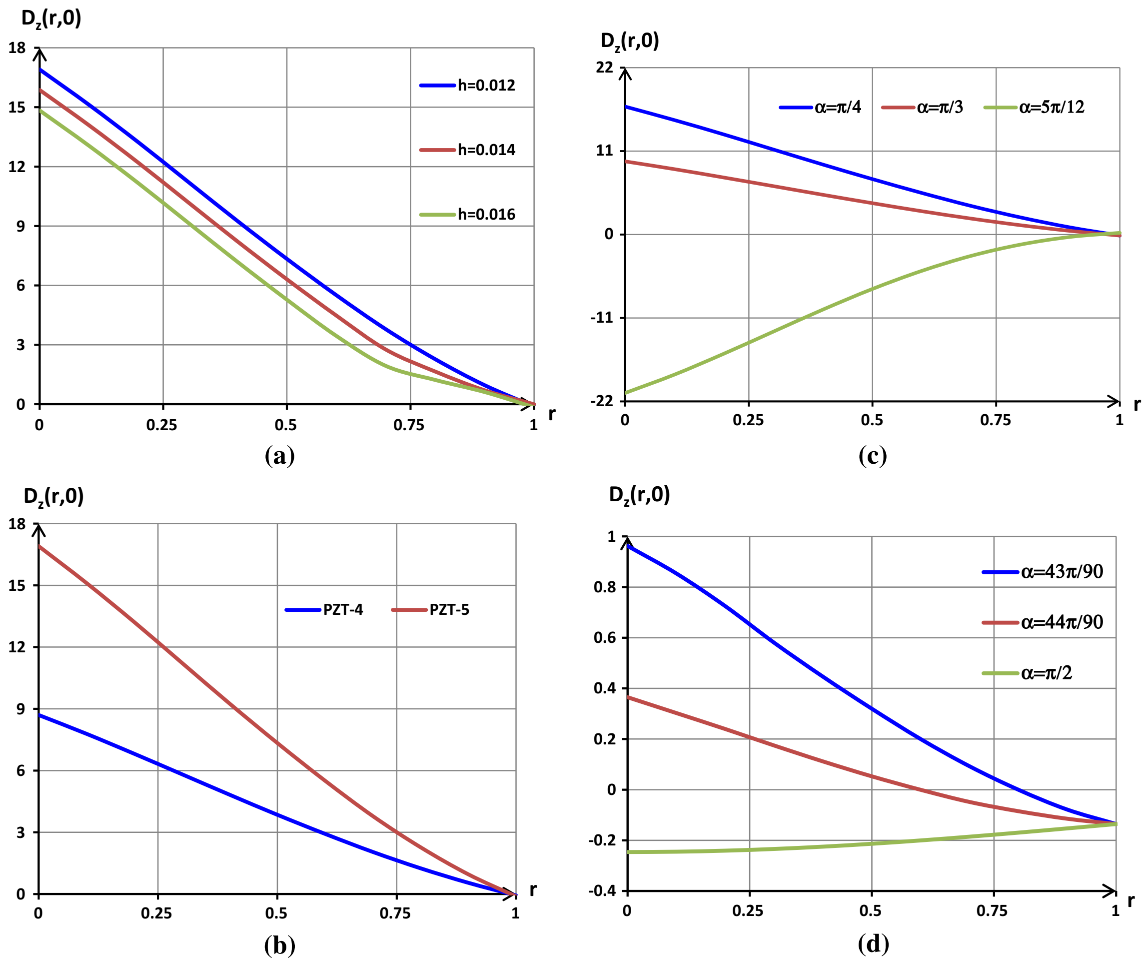


Fig. 7 **a** Effect of indentation h on $D_z(r, 0)$ for the conical punch ($\alpha = \pi/4$). **b** Variation of $D_z(r, 0)$ for fixed indentation h of the conical punch ($h = 0.014, \alpha = \pi/4$). **c** Effect of semi-vertical angle α on $D_z(r, 0)$ of the conical punch ($h = 0.014$). **d** Variation of $D_z(r, 0)$ for semi-vertical angle $\alpha \to \pi/2$ of the conical punch ($h = 0.014$)

a conical indenter. Figures 2a, 4a and 6a represent the effects of indentation h on $\sigma_{zz}(r, 0)$ for flat-ended cylindrical punch (case 1), spherical punch (case 2) and conical punch (case 3), respectively. The normal stress $\sigma_{zz}(r, 0)$ is seen to be increasing with h in cases 1 and 2, and the increase is very significant near the center of the punch, while the effects are almost negligible near the end of the punch. In case 3 the numerical values of $\sigma_{zz}(r, 0)$ are seen to be decreasing with increasing h, but the decrease is not as significant as in cases 1 and 2. Figures 2b, 4b and 6b show the behavior of $\sigma_{zz}(r, 0)$ for the two types of piezoelectric materials considered here for a fixed value of h. The results indicate that the normal stress at the boundary surface $z = 0$ is numerically greater for PZT-5 compared to PZT-4. The variations of electrical surface displacement $D_z(r, 0)$ with r are displayed for different h in Fig. 3a and for different piezoelectric materials with fixed value of h in case 1, while the corresponding results for the other cases are shown in Figs. 5a, b and 7a–c, respectively. Here it is found that in cases 1 and 2 the distribution of electric displacement increases with the increase in indentation and the displacement has its maximum at $r = 0$; then, it gradually decreases as r increases up to approximately $r = 0.5$, after which the displacement becomes constant. The variations of $D_z(r, 0)$ for various piezoelectric ceramics in these two cases are almost similar. In the case of conical punch (case 3) some kinds of dissimilarities from the other two punches are observed. The behavior of the electric displacement is quite different from that for the cylindrical and spherical punches. From Fig. 7 it is clear that for $\alpha = \frac{\pi}{4}$ the electric displacement gradually diminishes with r and becomes zero at $r = 1$, a result which is totally different from the other two cases. Also, the displacement decreases with an increase in h. Figures 7c, d depict the behavior of the electric displacement with changing r, when the semi-vertical angle varies. Figure 8 shows that the applied load P for a particular contact radius is greater for a transversely

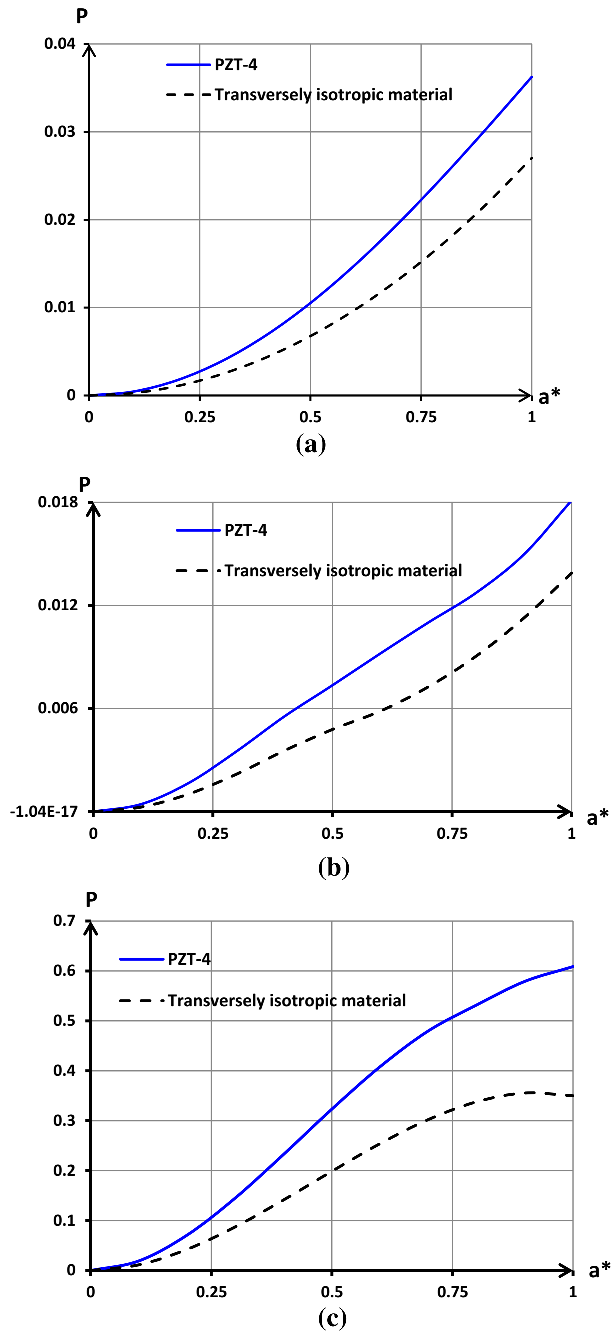


Fig. 8 **a** Variation of applied load P with contact radius for the flat-ended cylindrical punch. **b** Variation of allied load with contact radius for the spherical punch ($R = 20$). **c** Variation of applied load P with contact radius for the conical punch ($\alpha = \pi/3$)

isotropic medium with piezoelectric property for all kinds of considered punches than the corresponding results for a transversely isotropic medium with no piezoelectric property. Figure 9 indicates that the normal stress is greater in a transversely isotropic medium with piezoelectric property. Figures 8 and 9 agree with the result obtained in [7] for a transversely isotropic layer. Finally, as a rough check on our results we have computed the values of the applied load P for different values of h and displayed them in Fig. 10. It is observed that as h increases, the value of P approaches the value of P for a half space as predicted by Dyka [37].

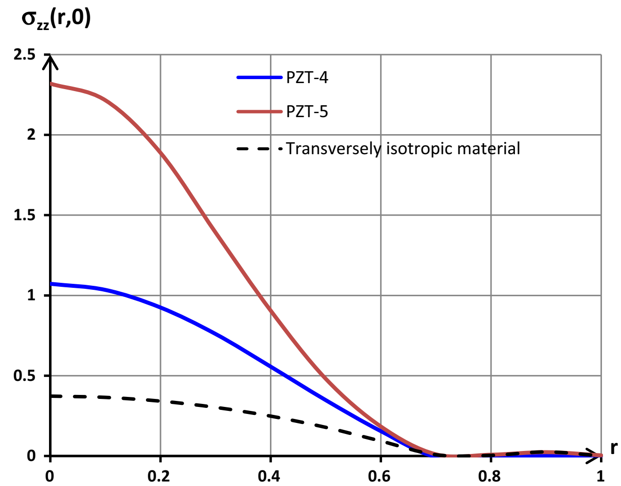


Fig. 9 Piezoelectric effect on $\sigma_{zz}(r, 0)$ for the spherical punch ($h = 0.012$, $R = 20$)

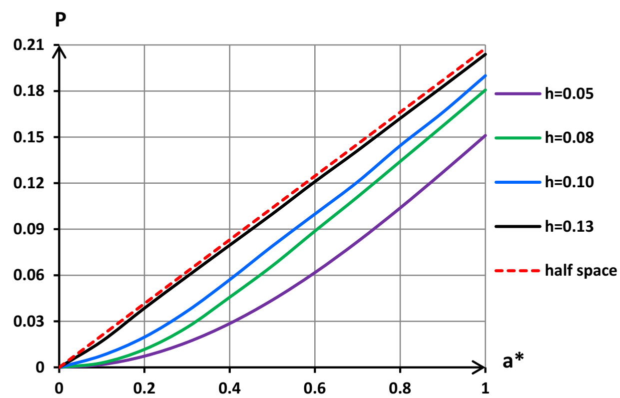


Fig. 10 Effect of layer width h on applied load P

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