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ORIGINAL PAPER

Xu Wang

An isotropic circular inhomogeneity partially bonded to an anisotropic medium

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Abstract We study the generalized plane strain deformations of an elastically isotropic circular inhomogeneity partially bonded to an unbounded generally anisotropic elastic matrix. The two-phase composite is subjected to a uniform loading at infinity, and meanwhile a line force and a line dislocation are applied both in the inhomogeneity and in the matrix. An elegant closed-form solution is obtained by reducing the original boundary value problem to a non-homogeneous Riemann–Hilbert problem of vector form which can be analytically solved by using a decoupling method and evaluating the Cauchy integrals. Surface traction on the bonded part of the interface, displacement jump across the debonded part of the interface, and the complex and real stress intensity factors at the crack tips are explicitly derived when the composite is only subjected to a remote uniform loading.

1 Introduction

Problems involving interfacial cracks between two dissimilar isotropic or anisotropic elastic materials are fascinating and have attracted considerable interest from theoreticians in the field of solid mechanics [1–13]. The plane strain and anti-plane shear problems of circular arc-shaped cracks lying along the interface between a circular elastic inhomogeneity and an infinite matrix have also been discussed in detail in [14–19] by using Muskhelishvili's complex variable method [20]. In [14–19], both the circular inhomogeneity and the surrounding matrix were taken to be elastically isotropic. The problem of an interfacial arc-shaped crack becomes rather complicated when the matrix is elastically anisotropic because Muskhelishvili's formulation does not apply to anisotropic materials. In a recent study [21], the present author derived an exact solution to the problem of an infinite anisotropic matrix containing a perfectly bonded isotropic elastic circular inhomogeneity. An Eshelby inclusion of arbitrary shape or a line dislocation exists inside the circular inhomogeneity.

In this study, we endeavor to attack the challenging and interesting two-dimensional problem of an isotropic elastic circular inhomogeneity partially bonded to an infinite anisotropic elastic matrix. The practical importance of the considered problem lies in that examples of an elastically anisotropic matrix are abundant [10]. The elegant and powerful Stroh's sextic formalism [22] is employed to handle the two-dimensional (or generalized plane strain) deformations of the anisotropic matrix, whereas Muskhelishvili's formulation is used to address the plane strain and anti-plane shear deformations of the isotropic circular inhomogeneity. The considered loadings in the present discussion include uniform in-plane and anti-plane stresses at infinity, and an isolated singularity due to a line force and a line dislocation located anywhere in the two-phase composite. A concise and elegant closed-form solution is derived to the aforementioned problem. Fracture parameters such as stress intensity factors and energy release rate can then be extracted from the obtained complete solution.

X. Wang (✉)
School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai 200237, China
E-mail: xuwang_sun@hotmail.com

2 Complex variable formulations

In this Section, the Stroh's sextic formalism for anisotropic elasticity and Muskhelishvili's formulation for isotropic elasticity will be briefly reviewed.

2.1 Stroh's sextic formalism for anisotropic elastic materials

In a fixed rectangular coordinate system $x_i (i = 1, 2, 3)$ let σ_{ij} and u_i be the stresses and displacements in an anisotropic elastic material. The equations of equilibrium and the stress–strain law are given by

$$\sigma_{ij,j} = 0, \sigma_{ij} = C_{ijkl} u_{k,l} \tag{1}$$

where C_{ijkl} are the elastic stiffnesses.

For the generalized plane strain problems in which all the physical quantities depend only on the plane coordinates x_1 and x_2, the general solution can be expressed as [22]

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\mathrm{T}} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}(z)}, \\ \boldsymbol{\varphi} &= \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix}^{\mathrm{T}} = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}\mathbf{f}(z)} \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}, \\ &\mathbf{f}(z) = \begin{bmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) \end{bmatrix}^{\mathrm{T}}, \\ &z_i = x_1 + p_i x_2, \operatorname{Im}\{p_i\} > 0, (i = 1, 2, 3), \end{aligned} \tag{3}$$

with

$$\begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} = p_i \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix}, (i = 1, 2, 3) \tag{4}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \mathbf{N}_2 = \mathbf{T}^{-1}, \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}, \tag{5}$$

and

$$Q_{ik} = C_{i1k1}, R_{ik} = C_{i1k2}, T_{ik} = C_{i2k2}. \tag{6}$$

The stress function vector $\boldsymbol{\varphi}$ is defined, in terms of the stresses, as follows:

$$\sigma_{i1} = -\varphi_{i,2}, \sigma_{i2} = \varphi_{i,1}, (i = 1, 2, 3). \tag{7}$$

It is stressed that the general solution in Eq. (2) is valid when the 6×6 fundamental elastic matrix $\mathbf{N}$ is simple, i.e., $p_1 \neq p_2 \neq p_3$ or semisimple [22]. Due to the fact that the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following orthogonality relations [22]:

$$\begin{aligned} \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{I} = \overline{\mathbf{B}}^{\mathrm{T}}\overline{\mathbf{A}} + \overline{\mathbf{A}}^{\mathrm{T}}\overline{\mathbf{B}}, \\ \mathbf{B}^{\mathrm{T}}\overline{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}} = \mathbf{0} = \overline{\mathbf{B}}^{\mathrm{T}}\mathbf{A} + \overline{\mathbf{A}}^{\mathrm{T}}\mathbf{B}, \end{aligned} \tag{8}$$

the following three real Barnett–Lothe tensors $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{L}$ can then be introduced [22]:

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^{\mathrm{T}}, \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^{\mathrm{T}}. \tag{9}$$

Furthermore, the two matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite, while $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$, $\mathbf{SL}^{-1}$ are anti-symmetric. The following identities are also valid [22]

$$\begin{aligned} 2\mathbf{A}\langle p_\alpha \rangle \mathbf{B}^{\mathrm{T}} &= \mathbf{N}_1 + \mathrm{i}(\mathbf{N}_2\mathbf{L} - \mathbf{N}_1\mathbf{S}), \\ 2\mathbf{A}\langle p_\alpha \rangle \mathbf{A}^{\mathrm{T}} &= \mathbf{N}_2 - \mathrm{i}(\mathbf{N}_1\mathbf{H} + \mathbf{N}_2\mathbf{S}^{\mathrm{T}}), \\ 2\mathbf{B}\langle p_\alpha \rangle \mathbf{B}^{\mathrm{T}} &= \mathbf{N}_3 + \mathrm{i}(\mathbf{N}_1^{\mathrm{T}}\mathbf{L} - \mathbf{N}_3\mathbf{S}), \\ 2\mathbf{B}\langle p_\alpha \rangle \mathbf{A}^{\mathrm{T}} &= \mathbf{N}_1^{\mathrm{T}} - \mathrm{i}(\mathbf{N}_3\mathbf{H} + \mathbf{N}_1^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}) \end{aligned} \tag{10}$$

where $\langle * \rangle$ is a 3×3 diagonal matrix in which each component is varied according to the Greek index α.

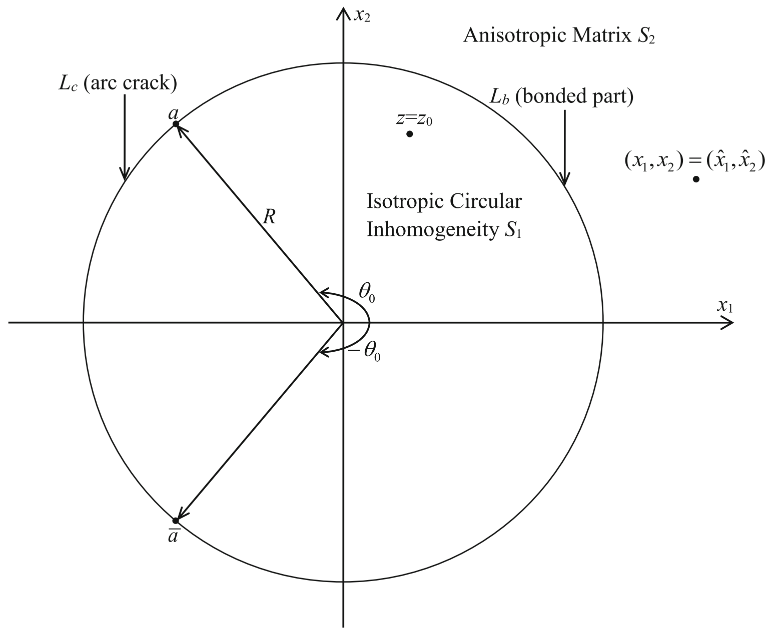

Fig. 1 An isotropic elastic circular inhomogeneity partially bonded to an anisotropic elastic matrix

2.2 Muskhelishvili's formulation for isotropic elastic materials

For plane strain deformation of an isotropic elastic material, the relevant stresses, displacements, and stress functions can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2 = r\exp(\mathrm{i}\theta)$ as [20]

$$\begin{aligned}\sigma_{11}+\sigma_{22} &= 2\left[\phi'(z)+\overline{\phi'(z)}\right],\\ \sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12} &= 2\left[\bar{z}\phi''(z)+\psi'(z)\right],\end{aligned} \tag{11}$$

$$\begin{aligned}2\mu(u_1+\mathrm{i}u_2) &= \kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},\\ \varphi_1+\mathrm{i}\varphi_2 &= \mathrm{i}\left[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}\right]\end{aligned} \tag{12}$$

where $\kappa = 3-4\nu$; μ and ν, where $\mu > 0$ and $0 \le \nu \le 0.5$, are the shear modulus and Poisson's ratio, respectively.

For the anti-plane shear deformation of an isotropic elastic material, the shear stresses, out-of-plane displacement, and stress function φ_3 can be expressed in terms of a single analytic function $\eta(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [20]

$$\sigma_{32}+\mathrm{i}\sigma_{31}=\eta'(z),\ \varphi_3+\mathrm{i}\mu u_3=\eta(z). \tag{13}$$

3 The complete solution

Let a generally anisotropic infinite matrix contain a partially bonded isotropic elastic circular inhomogeneity, as shown in Fig. 1. The center of the circular inhomogeneity of radius R is at origin, and an interfacial arc crack, whose surface is traction free, is made along the arc L_c of the interface while along the remaining arc L_b the inhomogeneity is still perfectly bonded to the matrix. Let the center of the arc L_b lie on the positive x_1-axis and the central angle subtended by the arc L_b be $2\theta_0$. $a = R\,e^{\mathrm{i}\theta_0}$ and $\overline{a} = R\,e^{-\mathrm{i}\theta_0}$ are the positions of the two crack tips. We represent the matrix by the domain S_2 and assume that the circular inhomogeneity occupies the region S_1. The two-phase composite is subjected to remote uniform stresses $(\sigma_{11}^{\infty}, \sigma_{12}^{\infty}, \sigma_{22}^{\infty}, \sigma_{31}^{\infty}, \sigma_{32}^{\infty})$. In addition, let a line force $\mathbf{q}_1$ and a line dislocation with Burgers vector $\mathbf{b}_1$ be applied at $z = z_0$ in the isotropic inhomogeneity, and meanwhile let a line force $\mathbf{q}_2$ and a line dislocation with Burgers vector $\mathbf{b}_2$ be applied at

$(x_1, x_2) = (\hat{x}_1, \hat{x}_2)$ in the anisotropic matrix. In what follows, the subscripts 1 and 2 refer to the regions S_1 and S_2, respectively.

Consider the following mapping functions for the anisotropic matrix:

$$z_\alpha = x_1 + p_\alpha x_2 = \omega_\alpha(\xi_\alpha) = \frac{1}{2}(1 - \mathrm{i}p_\alpha)\xi_\alpha + \frac{R^2}{2}(1 + \mathrm{i}p_\alpha)\frac{1}{\xi_\alpha}, (\alpha = 1, 2, 3). \tag{14}$$

By using the above mapping functions, the exterior of an elliptical region in the z_α-plane is mapped onto the exterior of the circle $|\xi_\alpha| > R$ in the ξ_α-plane (the circular interface $|z| = R$ is mapped to the boundary of the elliptical region in the z_α-plane, where $z_\alpha = x_1^\alpha + \mathrm{i}x_2^\alpha = x_1 + p'_\alpha x_2 + \mathrm{i}p''_\alpha x_2$ with p'_α and p''_α being the real and imaginary parts of p_α). By considering the fact that $\xi_1 = \xi_2 = \xi_3 = z$ on $|z| = R$ [23], we can first replace ξ_α by the common variable z. When the analysis is finished, the complex variable z shall be changed back to the corresponding complex variables ξ_α. As a result, we can write $f_\alpha(z) = f_\alpha(\xi_\alpha) = f_\alpha(\omega_\alpha(\xi_\alpha)) = f_\alpha(z_\alpha)$, and $\mathbf{f}_2(z)$ can be temporarily interpreted as an analytic function vector of the complex variable z.

Now, we introduce the following analytic function vector $\mathbf{f}_1(z)$ for the isotropic circular inhomogeneity:

$$\mathbf{f}_1(z) = \begin{bmatrix} \phi(z) \\ \psi(z) + \frac{R^2}{z}\phi'(z) \\ \eta(z) \end{bmatrix}, z \in S_1. \tag{15}$$

Consequently the displacement and stress function vectors along the circular interface $|z| = R$ on the inhomogeneity side can be expressed in terms of $\mathbf{f}_1(z)$ as follows:

$$\mathbf{u}_1 = \mathbf{A}_1\mathbf{f}_1(z) + \overline{\mathbf{A}}_1\overline{\mathbf{f}_1(z)}, \boldsymbol{\varphi}_1 = \mathbf{B}_1\mathbf{f}_1(z) + \overline{\mathbf{B}}_1\overline{\mathbf{f}_1(z)}, |z| = R \tag{16}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} \frac{\kappa}{4\mu} & -\frac{1}{4\mu} & 0 \\ -\frac{\mathrm{i}\kappa}{4\mu} & -\frac{\mathrm{i}}{4\mu} & 0 \\ 0 & 0 & -\frac{\mathrm{i}}{2\mu} \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{17}$$

It can be easily verified that instead of Eq. (8) the two matrices $\mathbf{A}_1$ and $\mathbf{B}_1$ satisfy the following orthogonality relations:

$$\mathbf{B}_1^{\mathrm{T}}\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}\mathbf{B}_1 = -\overline{\mathbf{B}}_1^{\mathrm{T}}\overline{\mathbf{A}}_1 - \overline{\mathbf{A}}_1^{\mathrm{T}}\overline{\mathbf{B}}_1 = -\frac{\mathrm{i}}{4\mu}\begin{bmatrix} 0 & \kappa+1 & 0 \\ \kappa+1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$
$$\mathbf{B}_1^{\mathrm{T}}\overline{\mathbf{A}}_1 + \mathbf{A}_1^{\mathrm{T}}\overline{\mathbf{B}}_1 = \overline{\mathbf{B}}_1^{\mathrm{T}}\mathbf{A}_1 + \overline{\mathbf{A}}_1^{\mathrm{T}}\mathbf{B}_1 = \mathbf{0}.$$

The continuity condition of tractions across the circular interface $|z| = R$ can then be expressed in terms of $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ as follows:

$$\mathbf{B}_1\mathbf{f}_1^+(z) + \overline{\mathbf{B}}_1\overline{\mathbf{f}}_1^-(R^2/z) = \mathbf{B}_2\mathbf{f}_2^-(z) + \overline{\mathbf{B}}_2\overline{\mathbf{f}}_2^+(R^2/z), \quad |z| = R, \tag{18}$$

or equivalently

$$\mathbf{B}_1\mathbf{f}_1^+(z) - \overline{\mathbf{B}}_2\overline{\mathbf{f}}_2^+(R^2/z) = \mathbf{B}_2\mathbf{f}_2^-(z) - \overline{\mathbf{B}}_1\overline{\mathbf{f}}_1^-(R^2/z), \quad |z| = R. \tag{19}$$

By applying Liouville's theorem, we can finally arrive at the following relationship:

$$\overline{\mathbf{B}}_2\overline{\mathbf{f}}_2(R^2/z) - \mathbf{B}_1\mathbf{f}_1(z) = \overline{\mathbf{B}}_1\overline{\mathbf{f}}_1(R^2/z) - \mathbf{B}_2\mathbf{f}_2(z) = \mathbf{g}(z) \tag{20}$$

where

$$\mathbf{g}(z) = \mathbf{g}_1(z) + \mathbf{g}_2(z) + \mathbf{g}_3(z), \tag{21}$$

with $\mathbf{g}_1(z)$, $\mathbf{g}_2(z)$ and $\mathbf{g}_3(z)$ being defined by

$$\begin{aligned}
\mathbf{g}_1(z) &= -\ln(z-z_0)\mathbf{B}_1(\mathbf{Y}_1\mathbf{q}_1+\mathbf{Y}_2\mathbf{b}_1)+\ln(z-R^2/\bar{z}_0)\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_1\mathbf{q}_1+\overline{\mathbf{Y}}_2\mathbf{b}_1)+\frac{\ln z}{2\pi \mathrm{i}}(\overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_1+\overline{\mathbf{B}}_2\overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_1)\\
&\quad -\frac{1}{z-z_0}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1+\mathbf{Y}_4\mathbf{b}_1)-\frac{R^2\bar{z}_0^{-2}}{z-R^2/\bar{z}_0}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1+\overline{\mathbf{Y}}_4\mathbf{b}_1),\\
\mathbf{g}_2(z) &= -\frac{1}{2\pi \mathrm{i}}\mathbf{B}_2\left\langle \ln(z-\hat{\xi}_\alpha)\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2+\mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2)-\frac{1}{2\pi \mathrm{i}}\overline{\mathbf{B}}_2\langle \ln\frac{z-R^2/\bar{\hat{\xi}}_\alpha}{z}\rangle(\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_2+\overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_2),\\
\mathbf{g}_3(z) &= \left[\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}-\mathbf{B}_2\mathbf{k}\right]z+R^2\left[\overline{\mathbf{B}}_2\bar{\mathbf{k}}-\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]z^{-1}
\end{aligned} \tag{22}$$

where $\mathbf{i}_2=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^{\mathrm{T}}$ and $\hat{\xi}_\alpha=\omega_\alpha^{-1}(\hat{z}_\alpha)=\frac{\hat{z}_\alpha+\sqrt{\hat{z}_\alpha^2-R^2(1+p_\alpha^2)}}{1-\mathrm{i}p_\alpha}$ with $\hat{z}_\alpha=\hat{x}_1+p_\alpha\hat{x}_2$.

In Eq. (22), the four matrices $\mathbf{Y}_j$ ($j=1,2,3,4$) and the vector $\mathbf{k}$ are given by

$$\begin{aligned}
\mathbf{Y}_1 &= \begin{bmatrix} -\frac{1}{2\pi(\kappa+1)} & -\frac{\mathrm{i}}{2\pi(\kappa+1)} & 0\\ \frac{\kappa}{2\pi(\kappa+1)} & -\frac{\mathrm{i}\kappa}{2\pi(\kappa+1)} & 0\\ 0 & 0 & -\frac{\mathrm{i}}{2\pi}\end{bmatrix},\ \mathbf{Y}_2=\begin{bmatrix} \frac{\mu}{\pi \mathrm{i}(\kappa+1)} & \frac{\mu}{\pi(\kappa+1)} & 0\\ -\frac{\mu}{\pi \mathrm{i}(\kappa+1)} & \frac{\mu}{\pi(\kappa+1)} & 0\\ 0 & 0 & \frac{\mu}{2\pi}\end{bmatrix},\\
\mathbf{Y}_3 &= -\frac{R^2-|z_0|^2}{2\pi z_0(\kappa+1)}\begin{bmatrix}0&0&0\\1&\mathrm{i}&0\\0&0&0\end{bmatrix},\ \mathbf{Y}_4=\frac{\mu(R^2-|z_0|^2)}{\pi z_0(\kappa+1)}\begin{bmatrix}0&0&0\\-\mathrm{i}&1&0\\0&0&0\end{bmatrix},
\end{aligned} \tag{23}$$

$$\mathbf{k}=\frac{1}{2}\langle 1-\mathrm{i}p_\alpha\rangle(\mathbf{B}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_1^\infty+\mathbf{A}_2^{\mathrm{T}}\mathbf{t}_2^\infty) \tag{24}$$

where

$$\begin{aligned}
\mathbf{t}_1^\infty &= \begin{bmatrix}\sigma_{11}^\infty & \sigma_{12}^\infty & \sigma_{31}^\infty\end{bmatrix}^{\mathrm{T}},\ \mathbf{t}_2^\infty=\begin{bmatrix}\sigma_{12}^\infty & \sigma_{22}^\infty & \sigma_{32}^\infty\end{bmatrix}^{\mathrm{T}},\\
\boldsymbol{\varepsilon}_1^\infty &= \begin{bmatrix}\varepsilon_{11}^\infty & 0 & 2\varepsilon_{31}^\infty\end{bmatrix}^{\mathrm{T}}=-\mathbf{N}_3^{-1}\mathbf{t}_1^\infty-\mathbf{N}_3^{-1}\mathbf{N}_1^{\mathrm{T}}\mathbf{t}_2^\infty,
\end{aligned} \tag{25}$$

with $\mathbf{N}_3^{-1}$ being the pseudoinverse of $\mathbf{N}_3$ [22].

In Eq. (22), $\mathbf{g}_1(z)$ is solely induced by the line force and line dislocation applied in the inhomogeneity, $\mathbf{g}_2(z)$ is solely induced by the line force and line dislocation applied in the matrix, $\mathbf{g}_3(z)$ is induced by the remote uniform loading and by the first-order pole at z=0 with unknown strength $R^2\phi'(0)\mathbf{i}_2$ appearing in $\mathbf{f}_1(z)$ defined by Eq. (15).

The continuity condition of displacements across the bonded part of the interface can be expressed in terms of $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ as follows:

$$\mathbf{A}_1\mathbf{f}_1^+(z)+\overline{\mathbf{A}}_1\bar{\mathbf{f}}_1^-(R^2/z)=\mathbf{A}_2\mathbf{f}_2^-(z)+\overline{\mathbf{A}}_2\bar{\mathbf{f}}_2^+(R^2/z),\ z\in L_b. \tag{26}$$

It follows from Eq. (20) that

$$\begin{aligned}
\bar{\mathbf{f}}_2(R^2/z) &= \overline{\mathbf{B}}_2^{-1}\mathbf{B}_1\mathbf{f}_1(z)+\overline{\mathbf{B}}_2^{-1}\mathbf{g}(z),\\
\bar{\mathbf{f}}_1(R^2/z) &= \overline{\mathbf{B}}_1^{-1}\mathbf{B}_2\mathbf{f}_2(z)+\overline{\mathbf{B}}_1^{-1}\mathbf{g}(z).
\end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (26), we obtain

$$\mathbf{M}_*\mathbf{B}_1\mathbf{f}'^+_1(z)-\overline{\mathbf{M}}_*\mathbf{B}_2\mathbf{f}'^-_2(z)=(\overline{\mathbf{M}}_1^{-1}-\overline{\mathbf{M}}_2^{-1})\mathbf{g}'(z),\ z\in L_b \tag{28}$$

where

$$\mathbf{M}_1^{-1}=\mathrm{i}\mathbf{A}_1\mathbf{B}_1^{-1}=\mathbf{L}_1^{-1}-\mathrm{i}\mathbf{S}_1\mathbf{L}_1^{-1}=\begin{bmatrix}\frac{\kappa+1}{4\mu} & \frac{\mathrm{i}(\kappa-1)}{4\mu} & 0\\ -\frac{\mathrm{i}(\kappa-1)}{4\mu} & \frac{\kappa+1}{4\mu} & 0\\ 0 & 0 & \frac{1}{\mu}\end{bmatrix}, \tag{29}$$

$$\mathbf{M}_2^{-1}=\mathrm{i}\mathbf{A}_2\mathbf{B}_2^{-1}=\mathbf{L}_2^{-1}-\mathrm{i}\mathbf{S}_2\mathbf{L}_2^{-1}, \tag{30}$$

$$\mathbf{M}_* = \mathbf{M}_1^{-1} + \overline{\mathbf{M}}_2^{-1} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1} - \mathrm{i}(\mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}). \tag{31}$$

It is seen from Eqs. (29) and (30) that both $\mathbf{M}_1^{-1}$ and $\mathbf{M}_2^{-1}$ are positive definite Hermitian matrices. In view of Eq. (28), we introduce a sectionally holomorphic function vector $\mathbf{h}(z)$ defined by

$$\begin{aligned}\mathbf{h}(z) =& \mathbf{B}_1\mathbf{f}'_1(z) - \frac{1}{z-z_0}\mathbf{B}_1(\mathbf{Y}_1\mathbf{q}_1 + \mathbf{Y}_2\mathbf{b}_1) + \frac{1}{(z-z_0)^2}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1 + \mathbf{Y}_4\mathbf{b}_1)\\ &- \frac{1}{z-R^2/\bar{z}_0}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_1\mathbf{q}_1 + \overline{\mathbf{Y}}_2\mathbf{b}_1) - \frac{R^2\bar{z}_0^{-2}}{(z-R^2/\bar{z}_0)^2}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1 + \overline{\mathbf{Y}}_4\mathbf{b}_1)\\ &- \frac{1}{\pi\mathrm{i}}\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\langle\frac{1}{z-\hat{\xi}_\alpha}\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2 + \mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2) - \mathbf{M}_*^{-1}\left[(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)} + 2\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}\right]\\ &+ R^2\mathbf{B}_1\mathbf{i}_2\phi'(0)z^{-2},\\ &|z| < R;\end{aligned} \tag{32.1}$$

$$\begin{aligned}\mathbf{h}(z) =& \mathbf{M}_*^{-1}\overline{\mathbf{M}}_*\mathbf{B}_2\mathbf{f}'_2(z) - \frac{2}{z-z_0}\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_1\mathbf{q}_1 + \mathbf{Y}_2\mathbf{b}_1) + \frac{2}{(z-z_0)^2}\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1 + \mathbf{Y}_4\mathbf{b}_1)\\ &- \frac{1}{2\pi\mathrm{i}}\mathbf{M}_*^{-1}\overline{\mathbf{M}}_*\mathbf{B}_2\left\langle\frac{1}{z-\hat{\xi}_\alpha}\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2 + \mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2) - \frac{1}{2\pi\mathrm{i}}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_2\left\langle\frac{1}{z-R^2/\bar{\hat{\xi}}_\alpha}\right\rangle(\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_2 + \overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_2)\\ &+ \frac{1}{2\pi\mathrm{i}}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\left[\overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^{\mathrm{T}}(\mathbf{q}_1 + \mathbf{q}_2) + \overline{\mathbf{B}}_2\overline{\mathbf{B}}_2^{\mathrm{T}}(\mathbf{b}_1 + \mathbf{b}_2)\right]z^{-1} - \mathbf{M}_*^{-1}\overline{\mathbf{M}}_*\mathbf{B}_2\mathbf{k}\\ &- R^2\mathbf{M}_*^{-1}\left[(\overline{\mathbf{M}}_1^{-1} - \overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_2\overline{\mathbf{k}} - 2\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]z^{-2},\\ &|z| > R.\end{aligned} \tag{32.2}$$

It is observed that $\mathbf{h}(z)$ is analytic in $|z| < R$ and $|z| > R$, and is continuous across the bonded part of the interface. In addition, $\mathbf{h}(z) \cong O(z^{-2})$ as $z \to \infty$. Thus, the imposition of the traction-free condition on the debonded part of the interface will yield the following non-homogeneous Riemann–Hilbert problem of vector form:

$$\begin{aligned}\overline{\mathbf{M}}_*\mathbf{h}^+(z) + \mathbf{M}_*\mathbf{h}^-(z) = \mathbf{v}(z), z \in L_c,\\ \mathbf{h}^+(z) - \mathbf{h}^-(z) = \mathbf{0}, z \in L_b\end{aligned} \tag{33}$$

where the superscripts "+" and "-" mean the limiting values by approaching the circular interface from S_1 and S_2, respectively, and the vector $\mathbf{v}(z)$ is given by

$$\begin{aligned}\mathbf{v}(z) =& -\frac{2}{z-z_0}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_1\mathbf{q}_1 + \mathbf{Y}_2\mathbf{b}_1) + \frac{2}{(z-z_0)^2}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1 + \mathbf{Y}_4\mathbf{b}_1)\\ &- \frac{2}{z-R^2/\bar{z}_0}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_1\mathbf{q}_1 + \overline{\mathbf{Y}}_2\mathbf{b}_1) - \frac{2R^2\bar{z}_0^{-2}}{(z-R^2/\bar{z}_0)^2}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1 + \overline{\mathbf{Y}}_4\mathbf{b}_1)\\ &- \frac{1}{\pi\mathrm{i}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\left\langle\frac{1}{z-\hat{\xi}_\alpha}\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2 + \mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2) + \frac{1}{\pi\mathrm{i}}\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\left\langle\frac{1}{z-R^2/\bar{\hat{\xi}}_\alpha}\right\rangle(\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_2 + \overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_2)\\ &- 2\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k} + \mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right] - \frac{1}{\pi\mathrm{i}}\mathbf{L}_2^{-1}\left[\overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^{\mathrm{T}}(\mathbf{q}_1 + \mathbf{q}_2) + \overline{\mathbf{B}}_2\overline{\mathbf{B}}_2^{\mathrm{T}}(\mathbf{b}_1 + \mathbf{b}_2)\right]z^{-1}\\ &+ 2R^2\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}} + \mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]z^{-2}.\end{aligned} \tag{34}$$

We consider the following eigenvalue problem:

$$\overline{\mathbf{M}}_*\mathbf{w} = \mathrm{e}^{2\pi\varepsilon}\mathbf{M}_*\mathbf{w}. \tag{35}$$

Three distinct eigenpairs $(\varepsilon, \mathbf{w})$, $(-\varepsilon, \overline{\mathbf{w}})$, $(0, \mathbf{w}_3)$ with ε and $\mathbf{w}_3$ being real and $\mathbf{w}$ being complex can be found such that [10]

$$\overline{\mathbf{M}}_*\mathbf{w} = \mathrm{e}^{2\pi\varepsilon}\mathbf{M}_*\mathbf{w}, \overline{\mathbf{M}}_*\mathbf{w}_3 = \mathbf{M}_*\mathbf{w}_3. \tag{36}$$

In addition, the real number ε or the oscillatory index can be explicitly given by [22]

$$\varepsilon = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta}, \beta = \left\{-\frac{1}{2}\mathrm{tr}(\breve{\mathbf{S}}^2)\right\}^{1/2} \tag{37}$$

where

$$\breve{\mathbf{S}} = \mathbf{D}^{-1}\mathbf{W}, \mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}, \mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}. \tag{38}$$

If $\mathbf{h}(z)$ is represented by

$$\mathbf{h}(z) = h_1(z)\mathbf{w} + h_2(z)\overline{\mathbf{w}} + h_3(z)\mathbf{w}_3, \tag{39}$$

Eq. (33) can be decoupled into

$$\left.\begin{array}{l} h_1^+(z) + \mathrm{e}^{-2\pi\varepsilon}h_1^-(z) = \frac{\overline{\mathbf{w}}^{\mathrm{T}}\mathbf{v}(z)}{\overline{\mathbf{w}}^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}}, \\ h_2^+(z) + \mathrm{e}^{2\pi\varepsilon}h_2^-(z) = \frac{\mathbf{w}^{\mathrm{T}}\mathbf{v}(z)}{\mathbf{w}^{\mathrm{T}}\overline{\mathbf{M}}_*\overline{\mathbf{w}}}, \\ h_3^+(z) + h_3^-(z) = \frac{\mathbf{w}_3^{\mathrm{T}}\mathbf{v}(z)}{\mathbf{w}_3^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_3}, \end{array}\right\} z \in L_c; \\ h_1^+(z) - h_1^-(z) = 0, h_2^+(z) - h_2^-(z) = 0, h_3^+(z) - h_3^-(z) = 0, z \in L_b \tag{40}$$

whose solution can be expediently given by [20]

$$\begin{aligned} h_1(z) &= \frac{1}{\overline{\mathbf{w}}^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}}\frac{\chi_1(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\overline{\mathbf{w}}^{\mathrm{T}}\mathbf{v}(t)\mathrm{d}t}{\chi_1^+(t)(t-z)}, \\ h_2(z) &= \frac{1}{\mathbf{w}^{\mathrm{T}}\overline{\mathbf{M}}_*\overline{\mathbf{w}}}\frac{\chi_2(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\mathbf{w}^{\mathrm{T}}\mathbf{v}(t)\mathrm{d}t}{\chi_2^+(t)(t-z)}, \\ h_3(z) &= \frac{1}{\mathbf{w}_3^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_3}\frac{\chi_3(z)}{2\pi\mathrm{i}}\int_{L_c}\frac{\mathbf{w}_3^{\mathrm{T}}\mathbf{v}(t)\mathrm{d}t}{\chi_3^+(t)(t-z)} \end{aligned} \tag{41}$$

where

$$\begin{aligned} \chi_1(z) &= (z-a)^{-\frac{1}{2}-\mathrm{i}\varepsilon}(z-\overline{a})^{-\frac{1}{2}+\mathrm{i}\varepsilon}, \\ \chi_2(z) &= (z-a)^{-\frac{1}{2}+\mathrm{i}\varepsilon}(z-\overline{a})^{-\frac{1}{2}-\mathrm{i}\varepsilon}, \\ \chi_3(z) &= (z-a)^{-\frac{1}{2}}(z-\overline{a})^{-\frac{1}{2}}. \end{aligned} \tag{42}$$

The branch cuts for the Plemelj functions $\chi_1(z)$, $\chi_2(z)$ and $\chi_3(z)$ are chosen as the debonded part of the interface, i.e., $z \in L_c$ such that $\chi_1(z), \chi_2(z), \chi_3(z) \cong z^{-1}$ as $z \to \infty$. It is seen from Eq. (41) that $h_j(z) \cong O(z^{-2})$ as $z \to \infty$. Consequently, the asymptotic condition that $\mathbf{h}(z) \cong O(z^{-2})$ as $z \to \infty$ has been satisfied. By analytically evaluating the Cauchy integrals in Eq. (41), we can finally obtain the following expressions of $h_1(z)$, $h_2(z)$ and $h_3(z)$:

$$\begin{aligned} &\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})h_j(z) = -2\left[\frac{1}{z-z_0} - \frac{\chi_j(z)}{\chi_j(z_0)(z-z_0)} - \chi_j(z)\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y_1q}_1 + \mathbf{Y}_2\mathbf{b}_1) \\ &+2\left[\frac{1}{(z-z_0)^2} - \frac{\chi_j(z)}{\chi_j(z_0)(z-z_0)^2} + \frac{\chi_j(z)\chi_j'(z_0)}{[\chi_j(z_0)]^2(z-z_0)}\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1 + \mathbf{Y}_4\mathbf{b}_1) \\ &-2\left[\frac{1}{z-R^2/\overline{z}_0} - \frac{\chi_j(z)}{\chi_j(R^2/\overline{z}_0)(z-R^2/\overline{z}_0)} - \chi_j(z)\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_\mathbf{1}\mathbf{q}_1 + \overline{\mathbf{Y}}_2\mathbf{b}_1) \\ &-2R^2\overline{z}_0^{-2}\left[\frac{1}{(z-R^2/\overline{z}_0)^2} - \frac{\chi_j(z)}{\chi_j(R^2/\overline{z}_0)(z-R^2/\overline{z}_0)^2} + \frac{\chi_j(z)\chi_j'(R^2/\overline{z}_0)}{[\chi_j(R^2/\overline{z}_0)]^2(z-R^2/\overline{z}_0)}\right] \\ &\times\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1 + \overline{\mathbf{Y}}_4\mathbf{b}_1) \\ &-\frac{1}{\pi\mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\left\langle\frac{1}{z-\hat{\xi}_\alpha} - \frac{\chi_j(z)}{\chi_j(\hat{\xi}_\alpha)(z-\hat{\xi}_\alpha)} - \chi_j(z)\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2 + \mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2) \\ &+\frac{1}{\pi\mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\left\langle\frac{1}{z-R^2/\overline{\hat{\xi}}_\alpha} - \frac{\chi_j(z)}{\chi_j(R^2/\overline{\hat{\xi}}_\alpha)(z-R^2/\overline{\hat{\xi}}_\alpha)} - \chi_j(z)\right\rangle(\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_2 + \overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_2) \\ &-2\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k} + \mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\left[1-\chi_j(z)\left[z - \mathrm{Re}\left\{a(1+2\mathrm{i}\varepsilon_j)\right\}\right]\right] \end{aligned} \tag{43}$$

$$-\frac{1}{\pi \mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_2^{-1}\left[\overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^{\mathrm{T}}(\mathbf{q}_1+\mathbf{q}_2)+\overline{\mathbf{B}}_2\overline{\mathbf{B}}_2^{\mathrm{T}}(\mathbf{b}_1+\mathbf{b}_2)\right]\left[\frac{1}{z}-\frac{\chi_j(z)}{\chi_j(0)z}-\chi_j(z)\right]$$
$$+2R^2\overline{\mathbf{w}}_j^{\mathrm{T}}\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}}+\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]\left[\frac{1}{z^2}-\frac{\chi_j(z)}{\chi_j(0)z^2}+\frac{\chi_j(z)\chi_j'(0)}{[\chi_j(0)]^2 z}\right],\ j=1,2,3$$

where $\mathbf{w}_1=\mathbf{w}$, $\mathbf{w}_2=\overline{\mathbf{w}}$, $\varepsilon_1=\varepsilon$, $\varepsilon_2=-\varepsilon$, $\varepsilon_3=0$.

Up to now, there still remains an unknown complex constant $\phi'(0)$ to be determined. It is derived from Eq. (32.1) that

$$\begin{aligned}&\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\left[h_1(0)\mathbf{w}+h_2(0)\overline{\mathbf{w}}+h_3(0)\mathbf{w}_3\right]=\phi'(0)+\frac{1}{z_0}\mathbf{i}_1^{\mathrm{T}}(\mathbf{Y}_1\mathbf{q}_1+\mathbf{Y}_2\mathbf{b}_1)\\&+\bar{z}_0R^{-2}\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1}-\overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_1\mathbf{q}_1+\overline{\mathbf{Y}}_2\mathbf{b}_1)+R^{-2}\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_1^{-1}-\overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1+\overline{\mathbf{Y}}_4\mathbf{b}_1)\\&+\frac{1}{\pi \mathrm{i}}\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\left\langle\frac{1}{\hat{z}_\alpha}\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2+\mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2)-\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\mathbf{M}_*^{-1}\left[(\overline{\mathbf{M}}_1^{-1}-\overline{\mathbf{M}}_2^{-1})\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}+2\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}\right]\end{aligned}\tag{44}$$

where $\mathbf{i}_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, and $h_1(0), h_2(0), h_3(0)$ are explicitly given by

$$\begin{aligned}&\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})h_j(0)=2\left[\frac{1}{z_0}-\frac{\chi_j(0)}{z_0\chi_j(z_0)}+\chi_j(0)\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_1\mathbf{q}_1+\mathbf{Y}_2\mathbf{b}_1)\\&+2\left[\frac{1}{z_0^2}-\frac{\chi_j(0)}{\chi_j(z_0)z_0^2}-\frac{\chi_j(0)\chi_j'(z_0)}{z_0[\chi_j(z_0)]^2}\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_1^{-1}\mathbf{B}_1(\mathbf{Y}_3\mathbf{q}_1+\mathbf{Y}_4\mathbf{b}_1)\\&+2\left[\frac{1}{R^2/\bar{z}_0}-\frac{\chi_j(0)}{R^2/\bar{z}_0\chi_j(R^2/\bar{z}_0)}+\chi_j(0)\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_1\mathbf{q}_1+\overline{\mathbf{Y}}_2\mathbf{b}_1)\\&-2\left[\frac{1}{R^2}-\frac{\chi_j(0)}{R^2\chi_j(R^2/\bar{z}_0)}-\frac{\chi_j(0)\chi_j'(R^2/\bar{z}_0)}{\bar{z}_0[\chi_j(R^2/\bar{z}_0)]^2}\right]\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1(\overline{\mathbf{Y}}_3\mathbf{q}_1+\overline{\mathbf{Y}}_4\mathbf{b}_1)\\&+\frac{1}{\pi \mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\left\langle\frac{1}{\hat{\xi}_\alpha}-\frac{\chi_j(0)}{\hat{\xi}_\alpha\chi_j(\hat{\xi}_\alpha)}+\chi_j(0)\right\rangle(\mathbf{A}_2^{\mathrm{T}}\mathbf{q}_2+\mathbf{B}_2^{\mathrm{T}}\mathbf{b}_2)\\&-\frac{1}{\pi \mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\left\langle\frac{1}{R^2/\bar{\hat{\xi}}_\alpha}-\frac{\chi_j(0)}{R^2/\bar{\hat{\xi}}_\alpha\chi_j(R^2/\bar{\hat{\xi}}_\alpha)}+\chi_j(0)\right\rangle(\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{q}_2+\overline{\mathbf{B}}_2^{\mathrm{T}}\mathbf{b}_2)\\&-2\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\left[1+\chi_j(0)\,\mathrm{Re}\left\{a(1+2\mathrm{i}\varepsilon_j)\right\}\right]\\&+\frac{1}{\pi \mathrm{i}}\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{L}_2^{-1}\left[\overline{\mathbf{B}}_2\overline{\mathbf{A}}_2^{\mathrm{T}}(\mathbf{q}_1+\mathbf{q}_2)+\overline{\mathbf{B}}_2\overline{\mathbf{B}}_2^{\mathrm{T}}(\mathbf{b}_1+\mathbf{b}_2)\right]\left[\frac{\chi_j'(0)}{\chi_j(0)}+\chi_j(0)\right]\\&+R^2\overline{\mathbf{w}}_j^{\mathrm{T}}\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}}+\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]\frac{2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)}{[\chi_j(0)]^2},\ j=1,2,3.\end{aligned}\tag{45}$$

$\phi'(0)$ can then be uniquely determined by solving the linear algebraic equation for $\phi'(0)$ in Eq. (44). For example, if the composite is only subject to a remote uniform loading by setting $\mathbf{q}_1=\mathbf{q}_2=\mathbf{b}_1=\mathbf{b}_2=\mathbf{0}$, the linear algebraic equation for $\phi'(0)$ can be explicitly obtained as follows:

$$\begin{aligned}&\left[1-R^2\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})[\chi_j(0)]^2}\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\right]\phi'(0)\\&+\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\left[\mathbf{M}_*^{-1}(\overline{\mathbf{M}}_2^{-1}-\overline{\mathbf{M}}_1^{-1})+2\sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[1+\chi_j(0)\mathrm{Re}\left\{a(1+2\mathrm{i}\varepsilon_j)\right\}\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\mathbf{L}_1^{-1}\right]\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\end{aligned}$$

$$= \mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\left[\mathbf{I} - \sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[1+\chi_j(0)\mathrm{Re}\left\{a(1+2\mathrm{i}\varepsilon_j)\right\}\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})}\overline{\mathbf{M}}_*\right]\mathbf{M}_*^{-1}\mathbf{L}_2^{-1}\mathbf{B}_2\langle 1-\mathrm{i}p_\alpha\rangle\left(\mathbf{B}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_1^{\infty}+\mathbf{A}_2^{\mathrm{T}}\mathbf{t}_2^{\infty}\right)$$
$$+\frac{R^2}{2}\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1}\sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[2[\chi_j'(0)]^2-\chi_j(0)\chi_j''(0)\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})[\chi_j(0)]^2}\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\langle 1+\mathrm{i}\overline{p}_\alpha\rangle(\overline{\mathbf{B}}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_1^{\infty}+\overline{\mathbf{A}}_2^{\mathrm{T}}\mathbf{t}_2^{\infty}) \quad (46)$$

where

$$\mathbf{i}_1^{\mathrm{T}}\mathbf{B}_1^{-1} = \begin{bmatrix}-\mathrm{i} & 1 & 0\end{bmatrix}, \mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2 = \frac{\kappa+1}{8\mu}\begin{bmatrix}-\mathrm{i} & 1 & 0\end{bmatrix}^{\mathrm{T}}, \mathbf{B}_1\mathbf{i}_2 = \frac{1}{2}\begin{bmatrix}-\mathrm{i} & 1 & 0\end{bmatrix}^{\mathrm{T}},$$
$$\chi_j(0) = \frac{1}{R\mathrm{e}^{2\varepsilon_j(\pi-\theta_0)}}, \chi_j'(0) = \frac{\cos\theta_0 + 2\varepsilon_j\sin\theta_0}{R^2\mathrm{e}^{2\varepsilon_j(\pi-\theta_0)}}, \quad (47)$$
$$\chi_j''(0) = \frac{1+4\varepsilon_j^2+(3-4\varepsilon_j^2)\cos 2\theta_0 + 8\varepsilon_j\sin 2\theta_0}{2R^3\mathrm{e}^{2\varepsilon_j(\pi-\theta_0)}}.$$

In addition, the matrix products $\mathbf{B}_2\mathbf{B}_2^{\mathrm{T}}, \mathbf{B}_2\mathbf{A}_2^{\mathrm{T}}, \mathbf{B}_2\langle p_\alpha\rangle\mathbf{B}_2^{\mathrm{T}}, \mathbf{B}_2\langle p_\alpha\rangle\mathbf{A}_2^{\mathrm{T}}$ and their conjugates appearing on the right-hand side of Eq. (46) can be determined by using the identities in Eqs. (9) and (10). In other words, the solution of $\phi'(0)$ will not contain the Stroh eigenvalues p_1, p_2, p_3 for the anisotropic matrix when only subjected to a remote uniform loading. As a result, the expression of $\phi'(0)$ for this loading case is still valid for any mathematically degenerate materials in which the 6×6 fundamental elasticity matrix $\mathbf{N}$ is nonsemisimple [22]. It is deduced from Eqs. (11) and (12) that $\phi'(0)$ is related to the mean stress $(\sigma_{11}+\sigma_{22})$ and the rigid body rotation $\varpi_{21} = \frac{1}{2}(u_{2,1}-u_{1,2})$ at the center of the circular inhomogeneity through

$$\phi'(0) = \frac{\sigma_{11}+\sigma_{22}}{4} + \frac{2\mathrm{i}\mu\varpi_{21}}{\kappa+1}, \text{ at } z = 0, \quad (48)$$

which gives the physical meaning of $\phi'(0)$.

Once $\phi'(0)$ is solved, $\mathbf{h}(z)$ can be considered as uniquely determined. Consequently, $\mathbf{f}'_1(z)$ (or equivalently $\phi'(z), \psi'(z), \eta'(z)$) defined in the inhomogeneity and $\mathbf{f}'_2(z)$ defined in the matrix can be further obtained from Eqs. (32.1,2). Thus, the elastic fields in the inhomogeneity and in the matrix can then be determined by using the analytic function vectors in the two-phase composite. In particular, along the circular interface, we have

$$\frac{\mathrm{d}(\mathbf{u}_1-\mathbf{u}_2)}{\mathrm{d}\theta} = z\mathbf{M}_*[\mathbf{h}^+(z)-\mathbf{h}^-(z)], z\in L_c,$$
$$\mathbf{t} = -\mathrm{i}\frac{z}{R}\overline{\mathbf{M}}_*^{-1}\left[\overline{\mathbf{M}}_*\mathbf{h}^+(z)+\mathbf{M}_*\mathbf{h}^-(z)-\mathbf{v}(z)\right] \quad (49)$$
$$= -\mathrm{i}\frac{z}{R}\overline{\mathbf{M}}_*^{-1}\left[2(\mathbf{L}_1^{-1}+\mathbf{L}_2^{-1})\mathbf{h}^+(z)-\mathbf{v}(z)\right], \quad z\in L_b$$

where $\mathbf{t} = -\frac{\mathrm{d}\boldsymbol{\varphi}_1}{R\mathrm{d}\theta}$ is the surface traction on the bonded part of the interface. Furthermore, the interfacial normal and shear stress components along L_b are

$$\mathbf{t}_r = \begin{bmatrix}-\sigma_{\theta r} & \sigma_{rr} & \sigma_{3r}\end{bmatrix}^{\mathrm{T}} = \boldsymbol{\Omega}(\theta)\mathbf{t}, \quad z = R\mathrm{e}^{\mathrm{i}\theta}, \quad -\theta_0 < \theta < \theta_0 \quad (50)$$

where the orthogonal matrix $\boldsymbol{\Omega}(\theta)$ is defined by

$$\boldsymbol{\Omega}(\theta) = \begin{bmatrix}\sin\theta & -\cos\theta & 0\\ \cos\theta & \sin\theta & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (51)$$

When the composite is only subjected to a uniform loading at infinity, the surface traction along L_b can be concisely presented as

$$\mathbf{t} = \frac{4}{R}\mathrm{Im}\left\{\sum_{j=1}^{3}\frac{\mathrm{e}^{2\pi\varepsilon_j}\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\mathbf{M}_*\mathbf{w}_j}z\chi_j(z)\left[z-\mathrm{Re}\left\{a(1+2\mathrm{i}\varepsilon_j)\right\}\right]\right\}, \quad z\in L_b. \quad (52)$$

It is impossible to obtain an analytic expression of $\int \mathbf{h}(z)\mathrm{d}z$ when the composite is subjected to general loadings. However, if the composite is only subject to a remote uniform loading, $\int \mathbf{h}(z)\mathrm{d}z$ can be exactly derived as

$$\begin{aligned}\int \mathbf{h}(z)\mathrm{d}z = &-2\sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{M}_*^{-1}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})}\left[z-\mathrm{X}_j(z)\right]\\ &-2R^2\sum_{j=1}^{3}\frac{\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}}+\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j(1+\mathrm{e}^{-2\pi\varepsilon_j})}\left[\frac{1}{z}-\frac{\mathrm{X}_j(z)}{\mathrm{X}_j(0)z}\right]\end{aligned} \tag{53}$$

where

$$\mathrm{X}_j(z)=(z-a)^{\frac{1}{2}-\mathrm{i}\varepsilon_j}(z-\overline{a})^{\frac{1}{2}+\mathrm{i}\varepsilon_j},\quad j=1,2,3. \tag{54}$$

The branch cuts for $\mathrm{X}_j(z)$ are also chosen along the arc crack L_c such that $\mathrm{X}_j(z)\cong z$ as $z\to\infty$. By using Eq. (53), the displacements everywhere in the two-phase composite can be arrived at. In particular, the jump in displacements across the faces of the interface arc crack can be finally derived as

$$\mathbf{u}_1-\mathbf{u}_2=4\mathrm{Im}\left\{\sum_{j=1}^{3}\frac{\mathrm{e}^{4\pi\varepsilon_j}\mathbf{M}_*\mathbf{w}_j\overline{\mathbf{w}}_j^{\mathrm{T}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]}{\overline{\mathbf{w}}_j^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_j}\mathrm{X}_j^+(z)\right\},\quad z\in L_c. \tag{55}$$

It can be easily proved that the expressions of surface traction and displacement jump in Eqs. (52) and (55) are both valid for any mathematically degenerate materials. Although the matrix is generally anisotropic, Eqs. (52) and (55) are strikingly simple and concise. We point out that it is not a simple task to arrive at Eqs. (52) and (55). During the derivation of these two expressions, we have used the following identities:

$$\begin{aligned}&\mathrm{Re}\{a(1+2\mathrm{i}\varepsilon_1)\}=\frac{R^2\chi_2'(0)}{\chi_2(0)},\ \mathrm{Re}\{a(1+2\mathrm{i}\varepsilon_2)\}=\frac{R^2\chi_1'(0)}{\chi_1(0)},\mathrm{Re}\{a(1+2\mathrm{i}\varepsilon_3)\}=\frac{R^2\chi_3'(0)}{\chi_3(0)};\\ &\overline{\chi_1(z)}=\frac{z\chi_2(z)}{R^2\chi_2(0)},\ \overline{\chi_2(z)}=\frac{z\chi_1(z)}{R^2\chi_1(0)},\ \overline{\chi_3(z)}=\frac{z\chi_3(z)}{R^2\chi_3(0)},z\in L_b;\\ &\overline{\mathrm{X}_1^+(z)}=-\frac{R^2\mathrm{e}^{2\pi\varepsilon_2}\mathrm{X}_2^+(z)}{z\mathrm{X}_2(0)},\ \overline{\mathrm{X}_2^+(z)}=-\frac{R^2\mathrm{e}^{2\pi\varepsilon_1}\mathrm{X}_1^+(z)}{z\mathrm{X}_1(0)},\ \overline{\mathrm{X}_3^+(z)}=-\frac{R^2\mathrm{X}_3^+(z)}{z\mathrm{X}_3(0)},\quad z\in L_c.\end{aligned} \tag{56}$$

The stress intensity factors can be extracted from the present complete solution by comparison with the asymptotic solution derived by Suo [10]. When the composite is only subjected to a remote uniform loading, the complex and real stress intensity factors K and K_3 can be extracted as follows:

$$\begin{aligned}K&=2(1+2\mathrm{i}\varepsilon)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\varepsilon}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\varepsilon}\\ &\times\frac{\mathbf{w}^{\mathrm{T}}\left\{\mathrm{e}^{\varepsilon(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}}+\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]+\mathrm{e}^{\varepsilon(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\right\}}{\overline{\mathbf{w}}^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}},\\ K_3&=\frac{4(\pi R)^{\frac{1}{2}}(\sin\theta_0)^{\frac{1}{2}}\mathbf{w}_3^{\mathrm{T}}\mathrm{Re}\left\{\mathrm{e}^{\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\right\}}{\mathbf{w}_3^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_3},\end{aligned} \tag{57}$$

at the upper crack tip $z=a$, and

$$\begin{aligned}K&=-2(1+2\mathrm{i}\varepsilon)(\pi R)^{\frac{1}{2}}(2R)^{-\mathrm{i}\varepsilon}(\sin\theta_0)^{\frac{1}{2}-\mathrm{i}\varepsilon}\\ &\times\frac{\overline{\mathbf{w}}^{\mathrm{T}}\left\{\mathrm{e}^{\varepsilon(\pi-\theta_0)+\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\overline{\mathbf{B}}_2\overline{\mathbf{k}}+\mathbf{L}_1^{-1}\mathbf{B}_1\mathbf{i}_2\phi'(0)\right]+\mathrm{e}^{\varepsilon(\pi+\theta_0)-\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\right\}}{\overline{\mathbf{w}}^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}},\\ K_3&=-\frac{4(\pi R)^{\frac{1}{2}}(\sin\theta_0)^{\frac{1}{2}}\mathbf{w}_3^{\mathrm{T}}\mathrm{Re}\left\{\mathrm{e}^{-\frac{\theta_0}{2}\mathrm{i}}\left[\mathbf{L}_2^{-1}\mathbf{B}_2\mathbf{k}+\mathbf{L}_1^{-1}\overline{\mathbf{B}}_1\mathbf{i}_2\overline{\phi'(0)}\right]\right\}}{\mathbf{w}_3^{\mathrm{T}}\overline{\mathbf{M}}_*\mathbf{w}_3},\end{aligned} \tag{58}$$

at the lower crack tip $z = \overline{a}$.

The normal and shear stresses along L_b are singularly distributed near the two crack tips as follows:

$$\begin{aligned}\mathbf{t}_r &= (2\pi r)^{-\frac{1}{2}}\boldsymbol{\Omega}(\theta_0)\left[Kr^{\mathrm{i}\varepsilon}\mathbf{w} + \overline{K}r^{-\mathrm{i}\varepsilon}\overline{\mathbf{w}} + K_3\mathbf{w}_3\right],\ r = |z-a| \to 0, \quad z \in L_b;\\ \mathbf{t}_r &= (2\pi r)^{-\frac{1}{2}}\boldsymbol{\Omega}(-\theta_0)\left[Kr^{\mathrm{i}\varepsilon}\mathbf{w} + \overline{K}r^{-\mathrm{i}\varepsilon}\overline{\mathbf{w}} + K_3\mathbf{w}_3\right],\ r = |z-\overline{a}| \to 0, \quad z \in L_b.\end{aligned} \tag{59}$$

By assuming that the crack propagates along the interface, the energy release rate can then be conveniently obtained as

$$G = \frac{\overline{\mathbf{w}}^{\mathrm{T}}(\mathbf{L}_1^{-1} + \mathbf{L}_2^{-1})\mathbf{w}\,|K|^2}{2\cosh^2\pi\varepsilon} + \frac{1}{4}\mathbf{w}_3^{\mathrm{T}}(\mathbf{L}_1^{-1} + \mathbf{L}_2^{-1})\mathbf{w}_3 K_3^2. \tag{60}$$

Note that the fracture criterion (11.2) in [10] is formulated in terms of the positive real energy release rate.

4 An example

If the matrix is an orthotropic material with its principal axes along the x_1, x_2, and x_3 directions, the real matrices $\mathbf{H}_2, \mathbf{L}_2, \mathbf{S}_2, \mathbf{N}_1, \mathbf{N}_3$ for the matrix and the Hermitian matrix $\mathbf{M}_*$ for the bimaterial are explicitly given by [10,22,24]

$$\begin{aligned}\mathbf{H}_2 &= \begin{bmatrix} H_{11} & 0 & 0\\ 0 & H_{22} & 0\\ 0 & 0 & H_{33}\end{bmatrix}, \mathbf{L}_2 = \begin{bmatrix} L_{11} & 0 & 0\\ 0 & L_{22} & 0\\ 0 & 0 & L_{33}\end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} 0 & S_{12} & 0\\ S_{21} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\\ \mathbf{N}_3 &= -\begin{bmatrix} 1/s'_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1/s'_{55}\end{bmatrix}, \mathbf{N}_1 = \begin{bmatrix} 0 & -1 & 0\\ s'_{12}/s'_{11} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\end{aligned} \tag{61}$$

$$\mathbf{M}_* = \begin{bmatrix} Y_{11} & -\mathrm{i}\beta(Y_{11}Y_{22})^{\frac{1}{2}} & 0\\ \mathrm{i}\beta(Y_{11}Y_{22})^{\frac{1}{2}} & Y_{22} & 0\\ 0 & 0 & Y_{33}\end{bmatrix} \tag{62}$$

where s'_{ij} are the reduced elastic compliances, and

$$\begin{aligned}\sqrt{\lambda}H_{11} &= \frac{H_{22}}{\sqrt{\lambda}} = \sqrt{2s'_{11}s'_{22}(1+\rho)} - \frac{\left(\sqrt{s'_{11}s'_{22}} + s'_{12}\right)^2}{\sqrt{2s'_{11}s'_{22}(1+\rho)}},\ H_{33} = \sqrt{s'_{44}s'_{55}},\\ \frac{L_{11}}{\sqrt{\lambda}} &= \sqrt{\lambda}L_{22} = \frac{1}{\sqrt{2s'_{11}s'_{22}(1+\rho)}},\ L_{33} = \frac{1}{\sqrt{s'_{44}s'_{55}}},\ \frac{S_{21}}{\sqrt{\lambda}} = -\sqrt{\lambda}S_{12} = \frac{\sqrt{s'_{11}s'_{22}} + s'_{12}}{\sqrt{2s'_{11}s'_{22}(1+\rho)}} > 0,\\ Y_{11} &= \frac{\kappa+1}{4\mu} + \sqrt{\frac{2s'_{11}s'_{22}(1+\rho)}{\lambda}},\ Y_{22} = \frac{\kappa+1}{4\mu} + \sqrt{2\lambda s'_{11}s'_{22}(1+\rho)},\\ \beta &= \frac{\sqrt{s'_{11}s'_{22}} + s'_{12} - \frac{\kappa-1}{4\mu}}{\sqrt{Y_{11}Y_{22}}},\ Y_{33} = \frac{1}{\mu} + \sqrt{s'_{44}s'_{55}},\end{aligned} \tag{63}$$

with λ and ρ being two dimensionless parameters defined as

$$\lambda = \sqrt{\frac{s'_{22}}{s'_{11}}},\ \rho = \frac{2s'_{12} + s'_{66}}{2\sqrt{s'_{11}s'_{22}}} > -1. \tag{64}$$

In this case, the three eigenvectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are [10]

$$\mathbf{w}_1 = \begin{bmatrix} -\frac{1}{2}\mathrm{i} \\ \frac{1}{2}(Y_{11}/Y_{22})^{\frac{1}{2}} \\ 0 \end{bmatrix}, \ \mathbf{w}_2 = \begin{bmatrix} \frac{1}{2}\mathrm{i} \\ \frac{1}{2}(Y_{11}/Y_{22})^{\frac{1}{2}} \\ 0 \end{bmatrix}, \ \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{65}$$

The oscillatory index is still given by Eq. (37). By using the explicit results in Eqs. (61), (62) and (65), all the terms appearing in Eq. (46) for the determination of $\phi'(0)$, Eq. (52) for the surface traction, Eq. (55) for displacement jump, Eqs. (57) and (58) for the stress intensity factors at the two crack tips are analytically given.

5 Conclusions

In this work, a rigorous closed-form solution has been obtained to the generalized plane strain problem of a partially debonded isotropic circular inhomogeneity embedded in a generally anisotropic infinite matrix by using Muskhelishvili's complex variable formulation for isotropic elasticity and Stroh's sextic formalism for anisotropic elasticity. Interestingly, existing closed-form solutions to perfectly bonded circular elastic inhomogeneities [21,25–30] and to arc-shaped cracks [14–16,18–20] can all be treated as limiting cases of the present solution. Finally, we point out that Dundurs' solution of an edge dislocation near a perfectly bonded circular inhomogeneity [30] has been applied as Green's function to study an interfacial arc-shaped Zener–Stroh crack [31] and a matrix crack near a circular piezoelectric inhomogeneity [32].

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