Acta Mech 227, 2439–2449 (2016)
DOI 10.1007/s00707-016-1622-5

ORIGINAL PAPER

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Noether's theorem for fractional Birkhoffian systems of variable order

Received: 8 August 2015 / Revised: 27 February 2016 / Published online: 4 May 2016

Abstract The Noether symmetries and conserved quantities of the Birkhoffian systems in terms of fractional derivatives of variable order are studied. Firstly, the Pfaff–Birkhoff–d'Alembert principle within fractional derivatives of variable order is obtained, and corresponding variable order fractional Birkhoff's equations are deduced. Secondly, the invariance of the fractional Pfaff action of variable order is studied under the one-parameter group of infinitesimal transformations, and the definition of the variable order fractional conserved quantity is given. Finally, the Noether's theorem for the fractional Birkhoffian system of variable order is established. At the end of this paper, an example is given to illustrate the application of the results.

1 Introduction

In recent years, the problems of application of the variable order fractional models have become a frontier subject in the research of fractional calculus [1]. The versions of the fractional derivative of variable order and the fractional integral of variable order were first presented by Samko and Ross in 1993 [2]. Afterward, according to the results that Samko and Ross gave, Lorenzo et al. [3] studied some important properties of the variable order fractional operators. Coimbra [4] utilized the variable order model for examining the dynamical behaviors of a mechanical system with frictional force of variable order. Then, Diaz and Coimbra [5] used the variable order differential equation to reveal some dynamical control behaviors of the nonlinear oscillator. Sun et al. [6] introduced a class of variable order fractional models, which can vary with space, time, viscosity, and other independent variables. Today, the advantages of the variable order fractional models have been recognized by many scholars. And the fractional models of variable order have many wide application contexts in mathematics, mechanics, nonlinear viscoelastic oscillators, and other dynamical systems [3–14].

Symmetries and conserved quantities not only exhibit the important mathematical properties but also are useful to understand some inherent physical properties of dynamical systems [15]. The definitions of the fractional conserved quantity and the fractional Noether's theorem were proposed by Frederico and Torres [16]. Besides, according to the definition of the classical conserved quantity, Atanacković [17] gave another form of the fractional conserved quantity and the corresponding fractional Noether's theorem for a fractional Lagrangian system. Later, the study of the fractional Noether's theorem has been extended to fractional Hamiltonian systems [18–20], nonconservative fractional Lagrangian systems with time delay [21], and fractional Birkhoffian systems [22–24].

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However, the study of the fractional Noether's theorem of variable order has only just begun. The fractional Noether's theorem of variable order for a Lagrangian system under the special one-parameter group of infinitesimal transformations was established by Torres and his coworkers [25]. As it is well known, Birkhoffian mechanics is a natural generalization of Lagrangian mechanical systems and Hamiltonian mechanical systems [26,27]. Birkhoffian mechanical systems have been an important direction in modern analytical mechanics [28]. Moreover, the basic theory of Birkhoffian mechanics not only can be applied to holonomic, nonholonomic, and other constrained mechanical systems [29] but also has a significant point in hadronic physics [27].

Meanwhile, the theory of symmetries and conserved quantities is a valuable aspect for the investigation of Birkhoffian mechanics, see [30]. In recent decades, a series of research results on Noether symmetries and conserved quantities for Birkhoffian systems have been obtained [29–33].

In this paper, we will investigate the Noether symmetries and conserved quantities for the Birkhoffian system in terms of Caputo fractional derivatives of variable order. The model of the Caputo fractional derivatives is usually applicable to natural science and engineering. The initial conditions with integer order cannot be used in the Riemann–Liouville fractional model, but can be adequate for the Caputo fractional model. Thus, we believe that the model of Caputo fractional derivatives will provide a convenient mathematical method for the study of Birkhoffian mechanics.

This paper is organized as follows: in Sect. 2, the definitions and some properties of the Caputo fractional derivatives of variable order and Riemann–Liouville fractional derivatives of variable order are given. In Sect. 3, the fractional Pfaff–Birkhoff–d'Alembert principle of variable order and the fractional Birkhoff's equations of variable order are established. In Sect. 4, the inherent relationship between the fractional Noether symmetries of variable order and the fractional conserved quantities of variable order is studied. In Sect. 5, an example is proposed to explain the results.

2 Definitions and properties of fractional derivatives of variable order

In this section, we will briefly introduce the definitions of the Caputo fractional derivatives of variable order and the Riemann–Liouville fractional integrals and derivatives of variable order, and the corresponding formula of variable order fractional integration by parts is given. For detailed proofs and discussions, see [25].

The left Caputo fractional derivative of variable order is defined as

$$ {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t) = \int_{t_1}^{t} \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\,\mathrm{d}\tau. \tag{1} $$

and the right Caputo fractional derivative of variable order is defined as

$$ {}_{t}^{C}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \int_{t}^{t_2} \frac{1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right) f(\tau)\,\mathrm{d}\tau. \tag{2} $$

where $\Gamma(\cdot)$ denotes the Euler Gamma function and $f(t)$ is an absolutely continuous function in the interval $[t_1,t_2]$, i.e., $f(t) \in AC[t_1,t_2]$. $\alpha(\cdot,\cdot)$ is the variable order of the fractional derivative such that $0<\alpha(\cdot,\cdot)<1$. In this paper, we let $\alpha(t,\tau)=\alpha(t-\tau)$, $\alpha(\tau,t)=\alpha(\tau-t)$, $t,\tau\in[t_1,t_2]$.

Remark 1 If $\alpha(\cdot,\cdot)=\alpha$,$\alpha\in(0,1)$, then Eqs. (1) and (2) become [34]

$$ {}_{t_1}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} (t-\tau)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\,\mathrm{d}\tau, \tag{3} $$

$$ {}_{t}^{C}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t}^{t_2} (t-\tau)^{-\alpha} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right) f(\tau)\,\mathrm{d}\tau. \tag{4} $$

Remark 2 If $\alpha(\cdot,\cdot)=1$, then these derivatives can be defined as [34]

$$ {}_{t_1}^{C}D_t^{1} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t), \tag{5} $$

$$ {}_{t}^{C}D_{t_2}^{1} f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} f(t). \tag{6} $$

The left Riemann–Liouville fractional integral of variable order is defined as

$$ {}_{t_1}I_t^{\alpha(\cdot,\cdot)} f(t) = \int_{t_1}^{t} \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} f(\tau)\,\mathrm{d}\tau, \tag{7} $$

and the right Riemann–Liouville fractional integral of variable order is defined as

$$ {}_{t}I_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \int_{t}^{t_2} \frac{1}{\Gamma(\alpha(\tau,t))} (\tau-t)^{\alpha(\tau,t)-1} f(\tau)\,\mathrm{d}\tau, \tag{8} $$

where $f(t)$ is a Lebesgue integrable function in the interval $[t_1, t_2]$.

The left Riemann–Liouville fractional derivative of variable order is defined as

$$ {}_{t_1}D_t^{\alpha(\cdot,\cdot)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_1}^{t} \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} f(\tau)\,\mathrm{d}\tau, \tag{9} $$

and the right Riemann–Liouville fractional derivative of variable order is defined as

$$ {}_{t}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} {}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{t_2} \frac{1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} f(\tau)\,\mathrm{d}\tau, \tag{10} $$

where ${}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t) \in AC[t_1, t_2]$, ${}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} f(t) \in AC[t_1, t_2]$.

Next we give the formula of variable order fractional integration by parts used in the following sections [25],

$$ \int_{t_1}^{t_2} g(t)\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t)\,\mathrm{d}t = \left(f(t)\, {}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} g(t) \right)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} f(t)\, {}_{t}D_{t_2}^{\alpha(\cdot,\cdot)} g(t)\,\mathrm{d}t, \tag{11} $$

where $g(t)$ is a continuous function in the interval $[t_1, t_2]$, ${}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} g(t) \in AC[t_1, t_2]$.

3 Fractional Birkhoff's equations of variable order

Assume that $B(t, a^{\mu})$ is a Birkhoffian, $R_{\upsilon}(t, a^{\mu})$, $(\upsilon, \mu = 1, 2, \ldots, 2n)$ are the Birkhoff's functions, a^{μ} is the Birkhoff variable. The fractional Pfaffian action of variable order is

$$ A\left[a^{\upsilon}(t)\right] = \int_{t_1}^{t_2} \left\{ R_{\upsilon}\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} - B \right\} \mathrm{d}t. \tag{12} $$

The fractional Pfaff–Birkhoff principle of variable order is

$$ \delta A\left[a^{\upsilon}(t)\right] = \delta \int_{t_1}^{t_2} \left\{ R_{\upsilon}\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} - B \right\} \mathrm{d}t = 0, \tag{13} $$

with the terminal conditions

$$ \delta a^{\upsilon}\big|_{t=t_1} = \delta a^{\upsilon}\big|_{t=t_2} = 0, \quad (\upsilon = 1, 2, \ldots, 2n) \tag{14} $$

and the commutative relation

$$ \delta\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} = {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \delta a^{\upsilon}. \tag{15} $$

Actually, since

$$ \delta\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} = {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \bar{a}^{\upsilon}(t) - {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon}(t) = {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \left[\bar{a}^{\upsilon}(t) - a^{\upsilon}(t)\right] = {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \delta a^{\upsilon}. \tag{16} $$

Now the fractional Birkhoff's equations of variable order can be derived from the variable order fractional Pfaff–Birkhoff principle. According to the identity (13), we have

$$ \delta A = \int_{t_1}^{t_2} \left\{ \left[\left(\frac{\partial R_{\upsilon}}{\partial a^{\mu}}\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} \right) - \frac{\partial B}{\partial a^{\mu}} \right] \delta a^{\mu} + \left(R_{\upsilon}\, \delta\, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^{\upsilon} \right) \right\} \mathrm{d}t = 0. \tag{17} $$

Using the formulae (11), the commutative relation (15) and the terminal conditions (14), the formula (17) gives

$$\delta A=\int_{t_1}^{t_2}\left\{\left[\left(\frac{\partial R_\upsilon}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^\upsilon\right)-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^{\alpha(\cdot,\cdot)}R_\mu\right]\delta a^\mu\right\}\mathrm{d}t=0. \tag{18}$$

Due to the arbitrariness of the integral interval $[t_1, t_2]$, the identity (18) can be written as follows:

$$\left(\frac{\partial R_\upsilon}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^\upsilon-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^{\alpha(\cdot,\cdot)}R_\mu\right)\delta a^\mu=0. \tag{19}$$

The formula (19) is the fractional Pfaff–Birkhoff–d'Alembert principle of variable order, and noting the independence of δa^μ, we have

$$\frac{\partial R_\upsilon}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^\upsilon-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^{\alpha(\cdot,\cdot)}R_\mu=0,\quad(\mu=1,2,\ldots,2n). \tag{20}$$

The formulae (20) are the fractional Birkhoff's equations of variable order.

If $\alpha(\cdot,\cdot)=\alpha$, α is a fractional constant order, then the formulae (20) become

$$\frac{\partial R_\upsilon}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha}a^\upsilon-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^{\alpha}R_\mu=0,\quad(\mu=1,2,\ldots,2n). \tag{21}$$

The formulae (21) are the fractional Birkhoff's equations within Caputo derivatives [23].

If $\alpha(\cdot,\cdot)=1$, then by using the formulae (5) and (6), Eq. (20) can be written as

$$\frac{\partial R_\upsilon}{\partial a^\mu}\frac{\mathrm{d}}{\mathrm{d}t}a^\upsilon-\frac{\partial B}{\partial a^\mu}-\frac{\mathrm{d}}{\mathrm{d}t}R_\mu=0,\quad(\mu=1,2,\ldots,2n), \tag{22}$$

i.e.,

$$\left(\frac{\partial R_\upsilon}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\upsilon}\right)\dot{a}^\upsilon-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0,\quad(\mu=1,2,\ldots,2n). \tag{23}$$

The formulae (23) are the classical Birkhoff's equations [29].

4 Noether's theorem for the fractional Birkhoffian system of variable order

Let us introduce the infinitesimal transformations of the one-parameter finite transformation group, i.e.,

$$\bar{t}=t+\Delta t,\quad \bar{a}^\mu\left(\bar{t}\right)=a\left(t\right)+\Delta a^\mu,\quad(\mu=1,2,\ldots,2n) \tag{24}$$

and their expansion formulae

$$\bar{t}=t+\varepsilon\xi_0\left(t,a^\upsilon\right),\quad \bar{a}^\mu\left(\bar{t}\right)=a^\mu\left(t\right)+\varepsilon\xi_\mu\left(t,a^\upsilon\right),\quad(\mu,\upsilon=1,2,\ldots,2n), \tag{25}$$

where ε is the infinitesimal parameter, ξ_0, ξ_μ are the generating functions or generators of the infinitesimal transformations.

Definition 1 For the fractional Birkhoffian system of variable order, the variable order fractional Pfaffian action (12) is invariant under the infinitesimal transformations of the one-parameter group (24), if and only if

$$\int_{\bar{T}_1}^{\bar{T}_2}\left\{R_\upsilon\left(\bar{t},\bar{a}^\mu\right){}_{\bar{t}_1}^{C}D_{\bar{t}}^{\bar{\alpha}(\cdot,\cdot)}\bar{a}^\upsilon-B\left(\bar{t},\bar{a}^\mu\right)\right\}\mathrm{d}\bar{t}=\int_{T_1}^{T_2}\left\{R_\upsilon\left(t,a^\mu\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^\upsilon-B\left(t,a^\mu\right)\right\}\mathrm{d}t \tag{26}$$

for every subinterval $[T_1, T_2]\subseteq[t_1, t_2]$.

According to Definition 1, the infinitesimal transformations of the one-parameter group (24) can be called the invariable symmetric transformations under the meaning of the Noether theory [15].

Theorem 1 *For the fractional Birkhoffian system of variable order* (20), *if the variable order fractional Pfaffian action* (12) *is invariant under the infinitesimal transformations of one-parameter group* (24), *then the following condition*

$$\left(R_{\upsilon}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\right)\dot{\xi}_0+\left(\frac{\partial R_{\upsilon}}{\partial a^{\mu}}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{\partial B}{\partial a^{\mu}}\right)\xi_{\mu}+\left(\frac{\partial R_{\upsilon}}{\partial t}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{\partial B}{\partial t}\right)\xi_0$$
$$+R_{\upsilon}\left[{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\bar{\xi}_{\upsilon}+\left(\frac{d}{dt}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}\right)\cdot\xi_0\left(t,a^{\upsilon}(t)\right)\right]$$
$$-\frac{R_{\upsilon}}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^{\upsilon}\left(t_1\right)\xi_0\left(t_1,a^{\upsilon}\left(t_1\right)\right)=0 \tag{27}$$

holds.

Proof The difference of the fractional Pfaff action (12) before and after the transformation is

$$\int_{\bar{T}_1}^{\bar{T}_2}\left[R_{\upsilon}\left(\bar{t},\bar{a}^{\mu}\right){}_{\bar{t}_1}^{C}D_{\bar{t}}^{\bar{\alpha}(\cdot,\cdot)}\bar{a}^{\upsilon}-B\left(\bar{t},\bar{a}^{\mu}\right)\right]\mathrm{d}\bar{t}-\int_{T_1}^{T_2}\left[R_{\upsilon}\left(t,a^{\mu}\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\left(t,a^{\mu}\right)\right]\mathrm{d}t$$
$$=\int_{T_1}^{T_2}\left[R_{\upsilon}\left(\bar{t},\bar{a}^{\mu}\right){}_{\bar{t}_1}^{C}D_{\bar{t}}^{\bar{\alpha}(\cdot,\cdot)}\bar{a}^{\upsilon}-B\left(\bar{t},\bar{a}^{\mu}\right)\right]\left(1+\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t$$
$$-\int_{T_1}^{T_2}\left[R_{\upsilon}\left(t,a^{\mu}\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\left(t,a^{\mu}\right)\right]\mathrm{d}t$$
$$=\int_{T_1}^{T_2}\left[R_{\upsilon}\left(\bar{t},\bar{a}^{\mu}\right){}_{\bar{t}_1}^{C}D_{\bar{t}}^{\bar{\alpha}(\cdot,\cdot)}\bar{a}^{\upsilon}-B\left(\bar{t},\bar{a}^{\mu}\right)\right]\mathrm{d}t-\int_{T_1}^{T_2}\left[R_{\upsilon}\left(t,a^{\mu}\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\left(t,a^{\mu}\right)\right]\mathrm{d}t$$
$$+\int_{T_1}^{T_2}\left[R_{\upsilon}\left(t,a^{\mu}\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\left(t,a^{\mu}\right)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t$$
$$=\int_{T_1}^{T_2}\left\{\left(\frac{\partial R_{\upsilon}}{\partial a^{\mu}}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{\partial B}{\partial a^{\mu}}\right)\Delta a^{\mu}+\left(\frac{\partial R_{\upsilon}}{\partial t}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{\partial B}{\partial t}\right)\Delta t+R_{\upsilon}\Delta{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}\right\}\mathrm{d}t$$
$$+\int_{T_1}^{T_2}\left[R_{\upsilon}\left(t,a^{\mu}\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\left(t,a^{\mu}\right)\right]\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t=0, \tag{28}$$

where

$${}_{\bar{t}_1}^{C}D_{\bar{t}}^{\bar{\alpha}(\cdot,\cdot)}\bar{a}^{\upsilon}\left(\bar{t}\right)=\int_{\bar{t}_1}^{\bar{t}}\frac{1}{\Gamma\left(1-\alpha\left(\bar{t},\bar{\tau}\right)\right)}\left(\bar{t}-\bar{\tau}\right)^{-\alpha(\bar{t},\bar{\tau})}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\bar{a}^{\upsilon}\left(\bar{\tau}\right)\mathrm{d}\bar{\tau}$$
$$=\int_{t_1}^{t}\frac{1}{\Gamma\left(1-\alpha\left(t+\Delta t,\tau+\Delta\tau\right)\right)}\left(t+\Delta t-\tau-\Delta\tau\right)^{-\alpha(t+\Delta t,\tau+\Delta\tau)}$$
$$\times\frac{\mathrm{d}}{\mathrm{d}\tau+\mathrm{d}\Delta\tau}\left(a^{\upsilon}+\Delta a^{\upsilon}\right)\left(1+\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\mathrm{d}\tau$$
$$=\int_{t_1}^{t}\left[\frac{1}{\Gamma\left(1-\alpha\left(t,\tau\right)\right)}-\frac{\Delta t\frac{\mathrm{d}}{\mathrm{d}t}\Gamma\left(1-\alpha\left(t,\tau\right)\right)+\Delta\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\Gamma\left(1-\alpha\left(t,\tau\right)\right)}{\left(\Gamma\left(1-\alpha\left(t,\tau\right)\right)\right)^2}\right]$$
$$\times\left[\left(t-\tau\right)^{-\alpha(t,\tau)}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}\left(t-\tau\right)^{-\alpha(t,\tau)}+\Delta\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\left(t-\tau\right)^{-\alpha(t,\tau)}\right]$$
$$\times\left(1-\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a^{\upsilon}+\Delta a^{\upsilon}\right)\right)\left(1+\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\mathrm{d}\tau$$
$$={}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}+{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\delta a^{\upsilon}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{1}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^{\upsilon}\left(t_1\right)\Delta t_1 \tag{29}$$

and

$$\Delta t = \varepsilon \xi_0, \quad \Delta a^\upsilon = \varepsilon \xi_\upsilon, \tag{30}$$

$$\delta a^\upsilon = \Delta a^\upsilon - \dot{a}^\upsilon \Delta t = \varepsilon \left(\xi_\upsilon - \dot{a}^\upsilon \xi_0\right) \triangleq \varepsilon \bar{\xi}_\upsilon. \tag{31}$$

Because of the pre-conditions of Theorem 1 and noting Definition 1, we have

$$\Delta A = 0. \tag{32}$$

Here, the variation ΔA of the action A is the principal linear part for ε in the difference (28), that is, the part which is accurate to first-order infinitesimal. Using the formulae (28)–(31), we obtain

$$\begin{aligned}\Delta A = \int_{T_1}^{T_2} \varepsilon \Bigg\{ & \left(R_\upsilon {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - B\right) \dot{\xi}_0 + \left(\frac{\partial R_\upsilon}{\partial a^\mu} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu \\ & + \left(\frac{\partial R_\upsilon}{\partial t} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - \frac{\partial B}{\partial t}\right) \xi_0 + R_\upsilon \left[{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \bar{\xi}_\upsilon + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right) \cdot \xi_0 \left(t, a^\upsilon(t)\right)\right] \\ & - \frac{R_\upsilon}{\Gamma\left(1 - \alpha(t, t_1)\right)} (t - t_1)^{-\alpha(t,t_1)} \dot{a}^\upsilon(t_1) \xi_0 \left(t_1, a^\upsilon(t_1)\right) \Bigg\} \mathrm{d}t. \end{aligned} \tag{33}$$

From (32) and (33), we obtain the formula (27); thus, the theorem is proved.

Next we give the definition of variable order fractional conserved quantity for the fractional Birkhoffian system with variable order derivatives.

Definition 2 The function $I\left(t, a^\mu, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\mu, {}_{t_1}D_t^{\alpha(\cdot,\cdot)} a^\mu\right)$ is said to be a variable order fractional conserved quantity, if

$$\frac{\mathrm{d}}{\mathrm{d}t} I\left(t, a^\mu, {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\mu, {}_{t_1}D_t^{\alpha(\cdot,\cdot)} a^\mu\right) = 0, \quad (\mu = 1, 2, \ldots, 2n) \tag{34}$$

hold, along all the solution curves of the variable order fractional Birkhoff's equations of motion (20).

Theorem 2 *For the fractional Birkhoffian system of variable order (20), if the infinitesimal transformations of one-parameter group (24) are the invariable symmetric transformations under Definition 1, then the system exists with variable order fractional conserved quantity as follows*

$$\begin{aligned} I = & \left(R_\upsilon {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - B\right) \xi_0 + \int_{t_1}^{t} \left(R_\upsilon {}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)} \bar{\xi}_\upsilon - \bar{\xi}_\upsilon {}_s D_{t_2}^{\alpha(\cdot,\cdot)} R_\upsilon\right) \mathrm{d}s \\ & - \int_{t_1}^{t} R_\upsilon \frac{1}{\Gamma\left(1 - \alpha(s, t_1)\right)} (s - t_1)^{-\alpha(s,t_1)} \dot{a}^\upsilon(t_1) \xi_0 \left(t_1, a^\upsilon(t_1)\right) \mathrm{d}s = \text{const}. \end{aligned} \tag{35}$$

Proof Since the infinitesimal transformations of the one-parameter group (24) are the invariable symmetric transformations under Definition 1, by using Theorem 1, we have

$$\begin{aligned} 0 = & \left(R_\upsilon {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - B\right) \dot{\xi}_0 + \left(\frac{\partial R_\upsilon}{\partial a^\mu} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - \frac{\partial B}{\partial a^\mu}\right) \xi_\mu + \left(\frac{\partial R_\upsilon}{\partial t} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - \frac{\partial B}{\partial t}\right) \xi_0 \\ & + R_\upsilon \left[{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} \bar{\xi}_\upsilon + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right) \cdot \xi_0 \left(t, a^\upsilon(t)\right)\right] \\ & - \frac{R_\upsilon}{\Gamma\left(1 - \alpha(t, t_1)\right)} (t - t_1)^{-\alpha(t,t_1)} \dot{a}^\upsilon(t_1) \xi_0 \left(t_1, a^\upsilon(t_1)\right) \\ = & \left(R_\upsilon {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon - B\right) \dot{\xi}_0 + \left[\left(\frac{\partial R_\upsilon}{\partial a^\mu} \dot{a}^\mu + \frac{\partial R_\upsilon}{\partial t}\right) {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right] \xi_0 \\ & + R_\upsilon \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right) \xi_0 - \left(\frac{\partial B}{\partial a^\mu} \dot{a}^\mu + \frac{\partial B}{\partial t}\right) \xi_0 + \left(\frac{\partial B}{\partial a^\mu} \dot{a}^\mu\right) \xi_0 \\ & - \frac{\partial B}{\partial a^\mu} \xi_\mu - \left[\left(\frac{\partial R_\upsilon}{\partial a^\mu} \dot{a}^\mu\right) {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right] \xi_0 + \left(\frac{\partial R_\upsilon}{\partial a^\mu} {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} a^\upsilon\right) \xi_\mu \end{aligned}$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1}^{t}\left(R_{\upsilon}{}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)}\bar{\xi}_{\upsilon}\right)\mathrm{d}s\right]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1}^{t}\left(\bar{\xi}_{\upsilon s}D_{t_2}^{\alpha(\cdot,\cdot)}R_{\upsilon}\right)\mathrm{d}s\right]+\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1}^{t}\left(\bar{\xi}_{\upsilon s}D_{t_2}^{\alpha(\cdot,\cdot)}R_{\upsilon}\right)\mathrm{d}s\right]$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1}^{t}R_{\upsilon}\frac{1}{\Gamma\left(1-\alpha\left(s,t_1\right)\right)}\left(s-t_1\right)^{-\alpha(s,t_1)}\dot{a}^{\upsilon}\left(t_1\right)\xi_0\left(t_1,a^{\upsilon}\left(t_1\right)\right)\mathrm{d}s\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(R_{\upsilon}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\right)\xi_0+\int_{t_1}^{t}\left(R_{\upsilon}{}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)}\bar{\xi}_{\upsilon}-\bar{\xi}_{\upsilon s}D_{t_2}^{\alpha(\cdot,\cdot)}R_{\upsilon}\right)\mathrm{d}s\right.$$
$$\left.-\int_{t_1}^{t}R_{\upsilon}\frac{1}{\Gamma\left(1-\alpha\left(s,t_1\right)\right)}\left(s-t_1\right)^{-\alpha(s,t_1)}\dot{a}^{\upsilon}\left(t_1\right)\xi_0\left(t_1,a^{\upsilon}\left(t_1\right)\right)\mathrm{d}s\right]$$
$$+\left(\frac{\partial R_{\upsilon}}{\partial a^{\mu}}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-\frac{\partial B}{\partial a^{\mu}}+{}_tD_{t_2}^{\alpha(\cdot,\cdot)}R_{\mu}\right)\bar{\xi}_{\mu}. \tag{36}$$

Substituting the formulae (20) into the identity (36), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(R_{\upsilon}{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}-B\right)\xi_0+\int_{t_1}^{t}\left(R_{\upsilon}{}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)}\bar{\xi}_{\upsilon}-\bar{\xi}_{\upsilon s}D_{t_2}^{\alpha(\cdot,\cdot)}R_{\upsilon}\right)\mathrm{d}s\right.$$
$$\left.-\int_{t_1}^{t}R_{\upsilon}\frac{1}{\Gamma\left(1-\alpha\left(s,t_1\right)\right)}\left(s-t_1\right)^{-\alpha(s,t_1)}\dot{a}^{\upsilon}\left(t_1\right)\xi_0\left(t_1,a^{\upsilon}\left(t_1\right)\right)\mathrm{d}s\right]=0. \tag{37}$$

Using Definition 2 and the formula (37), we can verify the identity (35) is the variable order fractional conserved quantity. Therefore, the theorem is proved.

Theorem 2 is called the Noether's theorem of the fractional Birkhoffian system with variable order derivatives. The Noether's theorem shows that we will obtain the corresponding variable order fractional conserved quantities if we can find the invariable symmetric transformations of the fractional Birkhoffian system with variable order derivatives.

Specifically, if $\alpha(\cdot,\cdot)=1$, then for the formulae (5), (6) and noting that $\Gamma(0)=\infty$ [34], the identity (35) gives

$$I=\left(R_{\upsilon}\dot{a}^{\upsilon}-B\right)\xi_0+\int_{t_1}^{t}\left(R_{\upsilon}\dot{\bar{\xi}}_{\upsilon}+\bar{\xi}_{\upsilon}\dot{R}_{\upsilon}\right)\mathrm{d}s=\left(R_{\upsilon}\dot{a}^{\upsilon}-B\right)\xi_0+R_{\upsilon}\bar{\xi}_{\upsilon}=\mathrm{const.} \tag{38}$$

Substituting the formula (31) into the identity (38), we have

$$I=R_{\upsilon}\xi_{\upsilon}-B\xi_0=\mathrm{const.} \tag{39}$$

The identity (39) is the conserved quantity of the classical Birkhoffian system [29].

5 Example

Now let us study the Hénon–Heiles problem. The Hénon–Heiles problem is a typical example of a nonintegrable Hamiltonian system, which can result in Chaotic solutions [35–37]. Besides, the Hénon–Heiles differential equations of motion have nonlinear character, which has become a hot subject in the research of modern nonlinear science [29].

The Birkhoff functions and Birkhoffian of the Hénon–Heiles problem can be expressed in the following form [29]:

$$R_1=R_2=0,\quad R_3=-a^1,\quad R_4=-a^2, \tag{40}$$
$$B=\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2-\frac{2}{3}\left(a^2\right)^3\right]. \tag{41}$$

Using the formulae (40) and (41), the fractional Pfaff action of variable order (12) gives

$$A=\int_{t_1}^{t_2}\left\{-a^1{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3-a^2{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4-\frac{1}{2}\right.$$
$$\left.\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2-\frac{2}{3}\left(a^2\right)^3\right]\right\}\mathrm{d}t. \tag{42}$$

According to the formulae (40)–(42) and the formula (12), the identity (20) gives

$$
\begin{aligned}
&-{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3 - a^1 - 2a^2a^1 = 0, \quad -{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4 - a^2 + \left(a^2\right)^2 - \left(a^1\right)^2 = 0,\\
&-a^3 - {}_tD_{t_2}^{\alpha(\cdot,\cdot)}a^1 = 0, \quad -a^4 - {}_tD_{t_2}^{\alpha(\cdot,\cdot)}a^2 = 0.
\end{aligned}
\tag{43}
$$

The formulae (43) can be called the fractional Birkhoff's equations of variable order for the Hénon–Heiles problem.

Next we study the fractional Noether symmetry and conserved quantity of variable order for the Hénon–Heiles problem. For Eqs. (40) and (41), identity (27) gives

$$
\begin{aligned}
&\left\{-a^1{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3 - a^2{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4 - \frac{1}{2}\left[\left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 + \left(a^4\right)^2 + 2a^2\left(a^1\right)^2 - \frac{2}{3}\left(a^2\right)^3\right]\right\}\dot{\xi}_0\\
&+\left(-{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3 - a^1 - 2a^2a^1\right)\xi_1 + \left(-{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4 - a^2 + \left(a^2\right)^2 - \left(a^1\right)^2\right)\xi_2\\
&+\left(-a^3\right)\xi_3 + \left(-a^4\right)\xi_4 + \left(-a^1\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\bar{\xi}_3 + \left(-a^1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3\right)\cdot\xi_0,\\
&+\left(-a^2\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\bar{\xi}_4 + \left(-a^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4\right)\cdot\xi_0\\
&-\frac{\left(-a^1\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^3\left(t_1\right)\xi_0\left(t_1, a^\upsilon\left(t_1\right)\right)\\
&-\frac{\left(-a^2\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^4\left(t_1\right)\xi_0\left(t_1, a^\upsilon\left(t_1\right)\right) = 0.
\end{aligned}
\tag{44}
$$

Suppose the infinitesimal generators of the system have the following form:

$$
\xi_0 = c_0, \xi_\mu = f_\mu\left(a^\upsilon\right), \quad (\mu, \upsilon = 1, 2, 3, 4),
\tag{45}
$$

where c_0 is an arbitrary constant. Substituting the generators (45) into the identity (44) and noting that $\bar{\xi}_\mu = \xi_\mu - \dot{a}^\mu\xi_0$, we have

$$
\begin{aligned}
&\left(-{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3 - a^1 - 2a^2a^1\right)f_1 + \left(-{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4 - a^2 + \left(a^2\right)^2 - \left(a^1\right)^2\right)f_2\\
&+\left(-a^3\right)f_3 + \left(-a^4\right)f_4 + \left(-a^1\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\left(f_3 - \dot{a}^3c_0\right) + \left(-a^1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3\right)\cdot c_0\\
&+\left(-a^2\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\left(f_4 - \dot{a}^4c_0\right) + \left(-a^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4\right)\cdot c_0\\
&-\frac{\left(-a^1\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^3\left(t_1\right)c_0\\
&-\frac{\left(-a^2\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^4\left(t_1\right)c_0 = 0.
\end{aligned}
\tag{46}
$$

According to Eq. (43), the identity (46) holds for arbitrary functions $f_1\left(a^\upsilon\right), f_2\left(a^\upsilon\right)$, and substituting Eq. (43) into the identity (46), we have

$$
\begin{aligned}
&\left(-a^3\right)f_3 + \left(-a^4\right)f_4 + \left(-a^1\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\left(f_3 - \dot{a}^3c_0\right) + \left(-a^1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3\right)\cdot c_0\\
&+\left(-a^2\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\left(f_4 - \dot{a}^4c_0\right) + \left(-a^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4\right)\cdot c_0\\
&-\frac{\left(-a^1\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^3\left(t_1\right)c_0\\
&-\frac{\left(-a^2\right)}{\Gamma\left(1-\alpha\left(t,t_1\right)\right)}\left(t-t_1\right)^{-\alpha(t,t_1)}\dot{a}^4\left(t_1\right)c_0 = 0.
\end{aligned}
\tag{47}
$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^{\upsilon}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_1}^{t}\frac{1}{\Gamma\left(1-\alpha\left(t,\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t,\tau)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\upsilon}\right)\mathrm{d}\tau \\ &= \int_{t_1}^{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma\left(1-\alpha\left(t-\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t-\tau)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\upsilon}\right)\mathrm{d}\tau \\ &\quad + \left[\frac{1}{\Gamma\left(1-\alpha\left(t-\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t-\tau)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\upsilon}\right)\right]\Bigg|_{\tau=t} \\ &= -\int_{t_1}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\Gamma\left(1-\alpha\left(t-\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t-\tau)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\upsilon}\right)\mathrm{d}\tau \\ &\quad + \left[\frac{1}{\Gamma\left(1-\alpha\left(t-\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t-\tau)}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\upsilon}\right)\right]\Bigg|_{\tau=t} \\ &= {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\dot{a}^{\upsilon} + \left[\frac{1}{\Gamma\left(1-\alpha\left(t,\tau\right)\right)}\left(t-\tau\right)^{-\alpha(t,\tau)}\dot{a}^{\upsilon}\right]\Bigg|_{\tau=t_1},\end{aligned} \tag{48}$$

where $\alpha\left(t,\tau\right)=\alpha\left(t-\tau\right)$, $\frac{\mathrm{d}}{\mathrm{d}t}\alpha\left(t-\tau\right)=-\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha\left(t-\tau\right)$, $t,\tau\in[t_1,t_2]$, substituting the formula (48) into the identity (47), we have

$$\left(-a^3\right)f_3+\left(-a^4\right)f_4+\left(-a^1\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}f_3+\left(-a^2\right){}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}f_4=0. \tag{49}$$

The identity (49) has a solution, i.e.,

$$f_3=f_4=0. \tag{50}$$

Thus, we find a solution of the identity (44), which is

$$\xi_0=c_0,\quad \xi_1=f_1\left(a^{\upsilon}\right),\quad \xi_2=f_2\left(a^{\upsilon}\right),\quad \xi_3=0,\quad \xi_4=0, \tag{51}$$

where $f_1\left(a^{\upsilon}\right)$, $f_2\left(a^{\upsilon}\right)$ are arbitrary functions, c_0 is an arbitrary constant. The generator (51) corresponds to a fractional Noether symmetry of the system we discussed. And for simplicity, we take

$$\xi_0=1,\quad \xi_1=1,\quad \xi_2=1,\quad \xi_3=0,\quad \xi_4=0. \tag{52}$$

According to Theorem 2, the generator (52) can lead to a conserved quantity as follows:

$$\begin{aligned}I &= \Bigg\{-a^1{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^3-a^2{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}a^4-\frac{1}{2}\Big[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2 \\ &\quad -\frac{2}{3}\left(a^2\right)^3\Big]\Bigg\}+\int_{t_1}^{t}\left(a^1{}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)}\dot{a}^3-\dot{a}^3{}_sD_{t_2}^{\alpha(\cdot,\cdot)}a^1+a^2{}_{t_1}^{C}D_s^{\alpha(\cdot,\cdot)}\dot{a}^4-\dot{a}^4{}_sD_{t_2}^{\alpha(\cdot,\cdot)}a^2\right)\mathrm{d}s \\ &\quad +\int_{t_1}^{t}\left[\left(-a^1\right)\frac{1}{\Gamma\left(1-\alpha\left(s,t_1\right)\right)}\left(s-t_1\right)^{-\alpha(t,t_1)}\dot{a}^3\left(t_1\right)\right]\mathrm{d}t \\ &\quad +\int_{t_1}^{t}\left[\left(-a^2\right)\frac{1}{\Gamma\left(1-\alpha\left(s,t_1\right)\right)}\left(s-t_1\right)^{-\alpha(t,t_1)}\dot{a}^4\left(t_1\right)\right]\mathrm{d}t=\text{const.}\end{aligned} \tag{53}$$

If $\alpha\left(\cdot,\cdot\right)=1$, then for the formulae (5), (6) and noting $\Gamma\left(0\right)=\infty$ [34], the identity (53) becomes

$$\begin{aligned}I &= \left\{-a^1\dot{a}^3-a^2\dot{a}^4-\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2-\frac{2}{3}\left(a^2\right)^3\right]\right\} \\ &\quad +\int_{t_1}^{t}\left(a^1\frac{\mathrm{d}}{\mathrm{d}s}\dot{a}^3+\dot{a}^3\frac{\mathrm{d}}{\mathrm{d}s}a^1+a^2\frac{\mathrm{d}}{\mathrm{d}s}\dot{a}^4+\dot{a}^4\frac{\mathrm{d}}{\mathrm{d}s}a^2\right)\mathrm{d}s=\text{const,}\end{aligned} \tag{54}$$

i.e.,

$$I=-\frac{1}{2}\left[\left(a^1\right)^2+\left(a^2\right)^2+\left(a^3\right)^2+\left(a^4\right)^2+2a^2\left(a^1\right)^2-\frac{2}{3}\left(a^2\right)^3\right]=\text{const.} \tag{55}$$

The identity (55) is a conserved quantity for the Hénon–Heiles problem of classical Birkhoffian system [29].

6 Conclusions

Variable order fractional calculus is a generalization of the fractional calculus. Variable order fractional integrals and derivatives have become a significant model alongside the fractional calculus and standard calculus. Meanwhile, some fractional differential equations of motion of variable order have been achieved. In this paper, we demonstrated the variable order fractional Birkhoff's equations and studied the Noether symmetry and conserved quantity. By using the fractional Noether's theorem of variable order we obtained in this paper, one can find a corresponding conserved quantity through Noether symmetry of the fractional Birkhoffian system with variable order. Besides, the Noether theorems for the classical Birkhoffian system (see [29]) and the fractional Birkhoffian system (see [23,24]) are the special case of this paper. Therefore, the methods and results in the paper are expected to be a contribution in the field. We expect that the theory presented in this paper can be further extended to constrained mechanical systems, optimal control systems, and other kinds of systems.

Acknowledgments This work was supported by the National Natural Science Foundation of China (Grant Nos. 10972151, 11272227 and 11572212). The authors are grateful to the anonymous reviewers for their valuable comments and suggestions.

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