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ORIGINAL PAPER

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Perturbations of Lagrangian systems based on the preservation of subalgebras of Noether symmetries

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Abstract Systems of second-order ordinary differential equations admitting a Lagrangian formulation are perturbed requiring that the extended Lagrangian preserves a fixed subalgebra of Noether symmetries of the original system. For the simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, this provides nonlinear systems with two independent constants of the motion quadratic in the velocities. Pinney-type equations are characterized as the most general $\mathfrak{sl}(2, \mathbb{R})$-preserving perturbation of the time-dependent (damped) harmonic oscillator. The procedure is generalized naturally to higher dimensions. In particular, it is shown that any perturbation of the time-dependent harmonic oscillator in two dimensions that preserves an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra of Noether symmetries is equivalent to a generalized Ermakov–Ray–Reid system that satisfies the Helmholtz conditions of the Inverse Problem of Lagrangian Mechanics. Application of the method to determine perturbations of the free Lagrangian in $\mathbb{R}^N$ is illustrated for the canonical chain of subalgebras of the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(N)$.

1 Introduction

The Lie symmetry analysis of differential equations, originally applied to physical problems mainly in the context of (quantum) mechanical systems, constitutes nowadays a standard method in a wide spectrum of physical situations, ranging from classical mechanics over quantum phenomena or nonlinear optics to cosmological problems (see, e.g., [1–7] and references therein). Among the systems of ordinary differential equations (ODEs) analyzed for their symmetry properties and relevant to physical applications, (generalized) Ermakov systems occupy a distinguished position for their various interesting structural properties, such as the existence of a nonlinear superposition principle. This has motivated intensive studies of such systems [8–14]. Besides the classical Ermakov systems, related to the time-dependent harmonic oscillator, various types of generalizations have been proposed, nowadays known as Ermakov–Ray–Reid systems or ERR systems in short, along with their multidimensional analogues, which have been proven to be of interest in soliton theory [15]. In the context of the symmetry analysis, it has been shown that point symmetries of ERR systems are closely related to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ [16]. The existence of a Hamiltonian formalism for ERR systems has been further studied in [17], showing that the Noether approach is not yet exhausted. The problem of analyzing dynamical systems possessing a (given) Noether symmetry has already been considered by some authors in the context of the geometric reformulation of the Noether theorem [18,19], suggesting a symmetry-based classification

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of dynamical systems. All these approaches connect with recent work on generic symmetries of systems of ODEs and their relation to certain types of integrable systems [3,5,20–26].

In this work, we develop a somewhat inverse procedure, basing on a "symmetry-preservation" procedure applied to Lie algebras of Noether symmetries, and formally related to the geometric approaches proposed in [18,19]. Starting with a Lagrangian L associated to a generic linear homogeneous second-order ODE, we determine the most general forcing term $G(t,\mathbf{x})$ such that the extended Lagrangian preserves a subalgebra of Noether symmetries with identical generators. This enables to write a constant of the motion as a combination of the invariant of the original equation and a part corresponding to the forcing term. It follows from this approach that the Pinney-type equation $\ddot{x}+p(t)\dot{x}+q(t)x+C\left(\exp(\int p(t)\mathrm{d}t)\right)^{-2}x^{-3}=0$ can be characterized as the most general perturbation of the ODE $\ddot{x}+p(t)\dot{x}+q(t)x=0$ that preserves a subalgebra of Noether symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This implies that for such nonlinear equations, point symmetries are always Noether symmetries. The special case $g_1(t)=0$ provides an additional explanation for the relation between the time-dependent harmonic oscillator and the Pinney equation [8,12,27], hence suggesting a connection with Ermakov systems. The procedure is then considered for systems in two dimensions, starting with an uncoupled system of damped oscillators. Here, forcing terms can depend on the velocities, giving rise to a more ample class of perturbed nonlinear systems. It follows in particular that the most general perturbation of the time-dependent harmonic oscillator in two dimensions preserving an $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries corresponds to a Ermakov–Ray–Reid system that satisfies the Helmholtz conditions of the Inverse Problem in Lagrangian Mechanics [28,29]. Finally, the case of perturbations of the free Lagrangian in $\mathbb{R}^N$ is studied for the canonical chain of subalgebras of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(N)$, enabling us to find a relation between perturbations of free Lagrangians in different dimensions.

We stress the fact that, in contrast to the geometrical constructions of dynamical systems possessing a given Noether symmetry, as developed, e.g., in [18,19], we are interested in determining nonlinear systems obtained by perturbation of a given Lagrangian system and preserving a certain subalgebra of Noether symmetries, in order to compare the constants of the motion of the resulting system with respect to the original one. The requirement on the subalgebra is imposed in order to obtain information on the possible integrability of nonlinear systems.

1.1 Point symmetries of second-order ordinary differential equations

To describe point symmetries of differential equations, we use the standard formulation in terms of differential operators [30]. It is well known that a system of $N\geq 1$ second-order ordinary differential equations

$$\ddot{x}_i=\omega_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right),\quad 1\leq i\leq N \tag{1}$$

is formulated in equivalent form in terms of the partial differential equation

$$\mathbf{A}f=\left(\frac{\partial}{\partial t}+\dot{x}_i\frac{\partial}{\partial x_i}+\omega_i\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{x}_i}\right)f=0. \tag{2}$$

We call a vector field $X=\xi\left(t,\mathbf{x}\right)\frac{\partial}{\partial t}+\eta_j\left(t,\mathbf{x}\right)\frac{\partial}{\partial x_j}\in\mathfrak{X}\left(\mathbb{R}^{N+1}\right)$ a Lie point symmetry of the equation(s) (1) if the prolongation $\dot{X}=X+\dot{\eta}_j\left(t,\mathbf{x},\dot{\mathbf{x}}\right)\frac{\partial}{\partial \dot{x}_j}$ satisfies the commutator

$$\left[\dot{X},\mathbf{A}\right]=-\frac{\mathrm{d}\xi}{\mathrm{d}t}\mathbf{A} \tag{3}$$

where $\dot{\eta}_j=-\frac{\mathrm{d}\xi}{\mathrm{d}t}\dot{x}_j+\frac{\mathrm{d}\eta_j}{\mathrm{d}x}$.

For $N=1$, given arbitrary functions $g_1(t)$, $g_2(t)$, it is straightforward to verify that a second-order linear homogeneous differential equation

$$\ddot{x}+g_1\left(t\right)\dot{x}+g_2\left(t\right)x=0 \tag{4}$$

possesses an algebra of point symmetries $\mathcal{L}$ isomorphic to $\mathfrak{sl}\left(3,\mathbb{R}\right)$ (see, e.g., [30]). Three of the symmetry generators of $\mathcal{L}$ are immediate and can be taken as

$$Y_1=x\frac{\partial}{\partial x},\quad Y_2=U_1\left(t\right)\frac{\partial}{\partial x},\quad Y_3=U_2\left(t\right)\frac{\partial}{\partial x} \tag{5}$$

where the general solution of (4) is given by

$$x(t) = \lambda_1 U_1(t) + \lambda_2 U_2(t); \quad \lambda_1, \lambda_2 \in \mathbb{R}. \tag{6}$$

1.2 Noether symmetries and constants of the motion

An important problem in dynamics is whether a given system of second-order ODEs (1) follows from a variational principle, i.e., if there exists a Lagrangian function $L(t, \mathbf{x}, \dot{\mathbf{x}})$ such that the so-called Helmholtz conditions

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = f_{ij}(t, \mathbf{x}, \dot{\mathbf{x}})\,(\ddot{x}_i - \omega_i(t, \mathbf{x}, \dot{\mathbf{x}})), \quad 1 \le i, j \le N \tag{7}$$

are satisfied [28]. The Inverse Problem in Lagrangian Mechanics, for both holonomic and non-holonomic systems, has been studied in detail by many authors, and many interesting problems emerge in this context, like the non-uniqueness of the Lagrangian formalism and the arising ambiguities in the interpretation of the correspondence between constants of the motion and symmetries [29,31–33].

For the scalar case $N = 1$ the answer is always in the affirmative, and we can find functions $f(t, x, \dot{x})$ and $L(t, x, \dot{x})$ such that (7) is satisfied. For an ODE of type (4), an admissible Lagrangian L is given by

$$L(t, x, \dot{x}) = \varphi(t)\left(\dot{x}^2 - g_2(t)\,x^2\right)/2, \tag{8}$$

where $\varphi(t)$ is defined in terms of the Wronskian $\mathbf{W} = W\{U_1(t), U_2(t)\}$ of (4) as $\varphi(t) = -1/\mathbf{W}$.

The resulting ODE can be seen as the equation of motion of a one-dimensional time-dependent damped oscillator [34].

An obvious advantage of having a dynamical system derived from a Lagrangian resides in the possibility of studying point symmetries arising from a variational principle [14]. Recall that a point symmetry X is a Noether symmetry of a Lagrangian $L(t, \mathbf{x}, \dot{\mathbf{x}})$ if there exists a function $V(t, \mathbf{x})$ such that the identity

$$\dot{X}(L) + A(\xi)\,L - A(V) = 0 \tag{9}$$

is satisfied. As a consequence, the quantity

$$\psi = \xi(t, \mathbf{x})\left[\dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L\right] - \eta_i(t, \mathbf{x})\frac{\partial L}{\partial \dot{x}_i} + V(t, \mathbf{x}) \tag{10}$$

is always a constant of the motion of the system [5,18,31].

As the ODE (4) possesses maximal symmetry $\mathfrak{sl}(3, \mathbb{R})$, it has exactly five independent Noether symmetries [30]. It is immediate to verify that the symmetries Y_2 and Y_3 of (5) satisfy the Noether symmetry condition (9) for the function $V(t, x) = \varphi(t)\,x\dot{U}_k(t)$, $k = 1, 2$. The conserved quantities deduced from them are $I_k = \varphi(t)\left(x\dot{U}_k(t) - \dot{x}U_k(t)\right)$. As shown in [35], the Lie point symmetries of linear equations of type (4), as well as of linear n-systems, can be described generically in terms of the general solution of the equations, leading to realizations of $\mathfrak{sl}(n+2, \mathbb{R})$ in terms of special functions. In particular, this approach can be applied to the subalgebra of Noether symmetries. For the equation under discussion (4), it follows that the three remaining Noether symmetries are described in terms of the general solution (6) of the ODE. To this extent, we observe that the function φ defined above satisfies the first-order equation

$$R_2 := \frac{d\varphi}{\mathrm{d}t} - g_1(t)\,\varphi(t) = 0. \tag{11}$$

Proposition 1 *For arbitrary functions $g_1(t), g_2(t)$, the vector fields*

$$X_k = \varphi(t)\,U_k^2(t)\,\frac{\partial}{\partial t} + x\,\varphi(t)\,U_k(t)\dot{U}_k(t)\,\frac{\partial}{\partial x}, \quad k = 1, 2 \tag{12}$$

are independent Noether symmetries of the linear homogeneous ODE (4).

The detailed proof of this property can be found in [35]. It further follows that the vector field

$$X_3 := [X_1, X_2] = -2\varphi(t)U_1(t)U_2(t)\frac{\partial}{\partial t} - \varphi(t)\left(U_1(t)\dot{U}_2(t) + U_2(t)\dot{U}_1(t)\right)x\frac{\partial}{\partial x} \tag{13}$$

is also a Noether symmetry of the equation. These three symmetries generate a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, with the two additional Noether symmetries Y_2, Y_3 transforming according to the two-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. We observe that the structure constants of $\mathcal{L}_{NS}$ are independent of the particular form of the solutions $U_k(t)$ of (4). The constants of the motion associated to the symmetries X_1, X_2, given, respectively, by

$$J_{\alpha\beta} = \frac{1}{2}I_\alpha I_\beta, \quad \alpha, \beta = 1, 2; \tag{14}$$

are functionally dependent on the invariants associated with Y_2 and Y_3.

1.3 Perturbations by means of symmetry-preserving forcing terms

The close relation between the time-dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)\,x = 0 \tag{15}$$

and the Pinney equation

$$\ddot{\rho} + \omega^2(t)\,\rho = \frac{\alpha}{\rho^3} \tag{16}$$

has been discussed extensively in the literature (see [9,25,36] and references therein), from a geometrical point of view, as well as in the context of the physical interpretation of the (generalized) Lewis invariant [8,12,37] and its generalization to higher dimensions [34].

We provide a complementary approach to the problem, namely considering perturbed Lagrangians that preserve exactly a fixed subalgebra of Noether symmetries. For a differential equation of type (4),[1] we compute the most general forcing or perturbation term $G_\varepsilon(t,x)$ such that the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries is preserved, implying that the latter nonlinear equation has an algebra of point symmetries coincident with that of Noether symmetries. Two of the resulting constants of the motion quadratic in the velocities will be independent and not obtainable from linear invariants, as these correspond to Noether symmetries that are not preserved.

The starting point for the ansatz is to consider the extended Lagrangian

$$\widetilde{L}(t,x,\dot{x}) = L_0(t,x,\dot{x}) + \varepsilon \quad \Phi(t,x) = \varphi(t)\left(\frac{\dot{x}^2}{2} - \frac{g_2(t)}{2}y^2 - G_\varepsilon(t,x)\right) \tag{17}$$

where $L_0(t,x,\dot{x})$ is the Lagrangian given in (8) and $\Phi(t,x) = -\varphi(t)\,G_\varepsilon(t,x)$ for some function $G_\varepsilon(t,x)$ such that $\lim_{\varepsilon\to 0} G_\varepsilon(t,x) = 0$. The corresponding equation of motion is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{L}}{\partial\dot{x}}\right) - \frac{\partial\widetilde{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} - \frac{\partial\Phi}{\partial x} = \varphi(t)\left(\ddot{x} + g_1(t)\,\dot{x} + g_2(t)\,x + \frac{\partial G_\varepsilon}{\partial x}\right) = 0.$$

Discarding the common term, the equation

$$\ddot{x} + g_1(t)\,\dot{x} + g_2(t)\,x + \frac{\partial G_\varepsilon}{\partial x} = 0 \tag{18}$$

describes the motion of a particle with both damping and forcing terms.

Having in mind the relation (11) defined previously, we impose the conservation of the Noether symmetries X_1 and X_2 for the (generally nonlinear) Eq. (18).

[1] More precisely, for the Lagrangian (8) associated with the equation.

Proposition 2 *For $k = 1, 2$ the vector fields*

$$X_k = \varphi(t)\, U_k^2(t)\, \frac{\partial}{\partial t} + x\, \varphi(t)\, U_k(t)\, \dot{U}_k(t) \frac{\partial}{\partial x} \tag{19}$$

are Noether symmetries of the equation of motion (18) only if the forcing term has the form

$$G_\varepsilon(t, x) = \frac{\varepsilon}{\varphi(t)^2\, x^2}, \quad \varepsilon \in \mathbb{R}. \tag{20}$$

We prove the assertion evaluating directly the symmetry condition (9). For the Lagrangian $\widetilde{L}$ and the prolongation $\dot{Y}_k$, the evaluation of the symmetry condition (9) reduces to the following expression:

$$\begin{aligned} &\frac{\varphi(t)}{2} U_k^2(t)\, R_2\, \dot{x}^2 + \dot{x} \left\{ x\, \varphi^2(t) \left(\dot{U}_k(t)^2 + \ddot{U}_k(t) U_k(t) + g_1(t)\, U_k(t)\, \dot{U}_k(t) \right) - \frac{\partial V}{\partial x} \right\} \\ &- \frac{x^2}{2} U_k(t)\, \varphi^2(t) \left\{ 4 g_2(t)\, \dot{U}_k(t) + \left(2 g_1(t)\, g_2(t) + \frac{\mathrm{d} g_2}{\mathrm{d}t} \right) U_k(t) \right\} - \varphi^2(t)\, U_k^2(t)\, \frac{\partial G_\varepsilon}{\partial t} \\ &- x\, \varphi^2(t)\, U_k(t)\, \dot{U}_k(t) \frac{\partial G_\varepsilon}{\partial x} - \frac{\partial V}{\partial t} - 2\varphi^2(t)\, U_k(t) \left(g_1(t)\, U_k(t) + \dot{U}_k(t) \right) G_\varepsilon(t, x), \end{aligned} \tag{21}$$

using the relation (11). As $R_2 = 0$, the term in $\dot{x}^2$ vanishes. The function $V(t, x)$ is obtained from the term in $\dot{x}$ as

$$V(t, x) = \frac{x^2}{2} \varphi(t)^2 \left(\ddot{U}_k(t) U_k(t) + \dot{U}_k^2(t) + g_1(t) U_k(t)\, \dot{U}_k(t) \right). \tag{22}$$

Inserting this expression into (21) and simplifying, the Noether symmetry condition reduces to

$$-\varphi^2(t)\, U_k(t) \left(U_k(t) \frac{\partial G_\varepsilon}{\partial t} + x \dot{U}_k(t) \frac{\partial G_\varepsilon}{\partial x} + 2 \left(U_k(t)\, g_1(t) + \dot{U}_k(t) \right) G_\varepsilon(t, x) \right) \tag{23}$$

because of $R_2 = 0$ and $U_k(t)$ being a solution of (4). This surviving term corresponds to the partial differential equation that must be satisfied by the forcing term $G_\varepsilon(t, x)$ if it preserves the symmetry X_k:

$$U_k(t) \frac{\partial G_\varepsilon}{\partial t} + x \dot{U}_k(t) \frac{\partial G_\varepsilon}{\partial x} + 2 \left(U_k(t)\, g_1(t) + \dot{U}_k(t) \right) G_\varepsilon(t, x) = 0. \tag{24}$$

As the latter equation should be satisfied simultaneously for the independent solutions $U_1(t)$ and $U_2(t)$, we can rewrite the condition in terms of the Wronskian as

$$\mathbf{W} \begin{pmatrix} \frac{\partial G_\varepsilon}{\partial t} + 2 g_1(t)\, G_\varepsilon(t, x) \\ x \frac{\partial G_\varepsilon}{\partial x} + 2 G_\varepsilon(t, x) \end{pmatrix} = 0. \tag{25}$$

As $\mathbf{W} \neq 0$, $G_\varepsilon(t, x)$ must satisfy the equations

$$\left(\frac{\partial G_\varepsilon}{\partial t} + 2 g_1(t)\, G_\varepsilon(t, x) \right) = 0, \quad \left(x \frac{\partial G_\varepsilon}{\partial x} + 2\, G_\varepsilon(t, x) \right) = 0. \tag{26}$$

The solution to this system is easily found to be

$$G_\varepsilon(t, x) = \frac{\varepsilon}{\varphi^2(t)\, x^2}, \quad \varepsilon \in \mathbb{R}. \tag{27}$$

Therefore, the nonlinear ODE

$$\ddot{x} + g_1(t)\, \dot{x} + g_2(t)\, x - \frac{2\varepsilon}{\varphi^2(t)\, x^3} = 0 \tag{28}$$

possesses at least the three Noether symmetries X_1, X_2 and X_3 inherited from the associated homogeneous equation (4).[2] We observe that no forcing terms $G(t,x,\dot{x})$ with $\frac{\partial G_\varepsilon}{\partial \dot{x}} \neq 0$ can exist, as follows at once from the symmetry condition (9).

Lemma 1 *For arbitrary functions $g_1(t)$ and $g_2(t)$, the Lie algebra $\mathcal{L}$ of point symmetries of the ODE (28) is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and coincides with the algebra of Noether symmetries.*

From the symmetry condition (3), a routine computation shows that a symmetry generator X must have the shape

$$X = \xi(t)\frac{\partial}{\partial t} + \frac{1}{2}\frac{\dot{\xi}(t)\varphi(t) - \xi(t)\dot{\varphi}(t)}{\varphi(t)} x \frac{\partial}{\partial x}. \tag{29}$$

In order to satisfy the symmetry condition, the function $\xi(t)$ must be a solution to the third-order ODE

$$\frac{\mathrm{d}^3\xi}{\mathrm{d}t^3} + \left(4g_2(t) - g_1^2(t) - 2\frac{\mathrm{d}g_1}{\mathrm{d}t}\right)\frac{\mathrm{d}\xi}{\mathrm{d}t} + \left(2\frac{\mathrm{d}g_2}{\mathrm{d}t} - \left(\frac{\mathrm{d}g_1}{\mathrm{d}t}\right)^2 - g_1(t)\frac{\mathrm{d}g_1}{\mathrm{d}t}\right)\xi = 0. \tag{30}$$

Now, as the vector fields X_1, X_2 and $[X_1, X_2]$ are point symmetries of (28) for being Noether symmetries, for any constants $\lambda_1, \lambda_2, \lambda_3$ the function

$$\xi(t) = \varphi(t)\left(\lambda_1 U_1^2(t) + \lambda_2 U_1(t)U_2(t) + \lambda_3 U_2^2(t)\right) \tag{31}$$

is a solution of (30), and since $U_1(t)$ and $U_2(t)$ are independent, it follows that (31) is the general solution of the equation, proving the assertion.

As the homogeneous ODE (4) and the nonlinear equation (28) share the same subalgebra of Noether symmetries with identical generators (and function $V(t,x)$), this implies that the corresponding constant of the motion satisfies

$$\psi = \xi\left[\dot{x}\frac{\partial \widetilde{L}}{\partial \dot{x}} - \widetilde{L}\right] - \eta\frac{\partial \widetilde{L}}{\partial \dot{x}} + V(t,x) = \xi\left[\dot{x}\frac{\partial L_0}{\partial \dot{x}} - L_0\right] - \eta\frac{\partial L_0}{\partial \dot{x}} + V(t,x) + \frac{\varepsilon\xi}{\varphi(t)\,x^2}. \tag{32}$$

Now $\psi_0 = \xi\left[\dot{x}\frac{\partial L_0}{\partial \dot{x}} - L_0\right] - \eta\frac{\partial L_0}{\partial \dot{x}} + V(t,x)$ corresponds to the constant of the motion of the homogeneous equation (4) with the Lagrangian L_0 of (8), while the last term of (32) is the genuine contribution of the forcing term. We can also use the well-known Lewis invariant [8,12,36] to express (32) in compact form. As x is a solution of equation (28), ψ can be simplified to

$$\psi = \varphi^2(t)(\dot{\rho}x - \rho\dot{x})^2 + \frac{\beta x^2}{\rho^2} - \frac{2\varepsilon\rho^2}{x^2}. \tag{33}$$

In particular, for $g_1(t) = 0$ and $g_2(t) = \omega(t)^2$, this method shows how the Pinney equation arises as the only perturbation of the (time-dependent) harmonic oscillator that preserves a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries.

It is to be observed that there is a certain ambiguity in the notion of perturbation, as the latter depends essentially on the Lagrangian L and not on the resulting equations of motion. An alternative Lagrangian is likely to provide different symmetry-preserving forcing terms and hence different perturbations. However, whenever there is no ambiguity on the Lagrangian L used, we will simply use the term perturbation.

2 Perturbations in $N = 2$ dimensions and generalized Ermakov–Ray–Reid systems

As a natural generalization of the scalar case, we can consider the perturbation problem for (linear) systems of ODEs possessing at least an $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries. Here forcing terms depending on the velocities are possible, providing a more ample class of nonlinear systems. Considering first the time-dependent (damped) harmonic oscillator in two dimensions, we determine those perturbations that preserve an $\mathfrak{sl}(2,\mathbb{R})$-subalgebra. It is further shown that these perturbations actually correspond to a subclass of (generalized) ERR systems that admit a Hamiltonian formalism [17].

[2] Equation (28) should not be confused with the so-called damped Pinney equation.

Let $g_1(t)$, $g_2(t)$ be arbitrary functions and consider the uncoupled two-dimensional damped oscillator

$$\ddot{x}_i + g_1(t)\,\dot{x}_i + g_2(t)\,x_i = 0, \quad i = 1, 2 \tag{34}$$

obtained from the time-dependent Lagrangian

$$L = \frac{1}{2}\varphi(t)\left(\dot{x}_1^2 + \dot{x}_2^2 - g_2(t)\left(x_1^2 + x_2^2\right)\right). \tag{35}$$

Using the symmetry condition (9), a routine computation shows that a Noether symmetry $X = \xi(t, \mathbf{x})\frac{\partial}{\partial t} + \eta_j(t, \mathbf{x})\frac{\partial}{\partial x_j}$ has the following form ($1 \leq j \leq 2, k \neq j$):

$$\begin{aligned} &\xi(t, \mathbf{x}) = \xi(t), \\ &\eta_j(t, \mathbf{x}) = \tfrac{1}{2}\left(\dot{\xi}(t) - g_1(t)\,\xi(t)\right)x_j + \lambda_j^k x_k + \psi_j(t) \end{aligned} \tag{36}$$

where $\xi(t)$ satisfies Eq. (30) and $\psi_j(t)$ is a solution of (4) for $j = 1, 2$. The Lie algebra of Noether symmetries $\mathcal{L}_{NS}$ has thus dimension 8. A basis of $\mathcal{L}_{NS}$ can be easily chosen as

$$\begin{aligned} &X_k = \varphi(t)\,U_k^2(t)\frac{\partial}{\partial t} + \varphi(t)\,U_k(t)\,\dot{U}_k(t)\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right), \quad k = 1, 2; \\ &X_3 = [X_1, X_2]; \;\; X_{12} = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}; \;\; Y_{kj} = U_k(t)\frac{\partial}{\partial x_j}, \quad 1 \leq j, k \leq 2. \end{aligned} \tag{37}$$

Clearly, the Levi subalgebra of $\mathcal{L}_{NS}$ is isomorphic to $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2)$, while the generators Y_{ij} transform according to the representation $\Lambda \otimes \Gamma_{\frac{1}{2}}$ of $\mathfrak{s}$, where Λ is the standard representation of $\mathfrak{so}(N)$ and $\Gamma_{\frac{1}{2}}$ the two-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$. From this we easily conclude that $\mathcal{L}_{NS}$ is isomorphic to the unextended Schrödinger algebra $S(2)$.[3]

For systems of this type, the computation of the most general forcing term $G(t, \mathbf{x}, \dot{\mathbf{x}})$ that can be added to the Lagrangian L in (35) and preserving the $\mathfrak{sl}(2, \mathbb{R})$-subalgebra of Noether symmetries is quite similar to the previous case. For technical reasons, it is, however, convenient to separate the case of forcing terms independent and dependent on the velocities.

2.1 Velocity-independent forcing terms

We require that the extended Lagrangian

$$\widetilde{L} = \varphi(t)\left(\frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 - g_2(t)\,(x_1^2 + x_2^2)\right) - G_\varepsilon(t, \mathbf{x})\right) \tag{38}$$

preserves the Noether symmetries X_1, X_2, X_3 of (37). In analogy with the scalar case, the symmetry condition (9) for X_k is only satisfied if the forcing term $G_\varepsilon(t, \mathbf{x})$ is a solution of the PDE

$$U_k(t)\frac{\partial G_\varepsilon}{\partial t} + \dot{U}_k(t)\left(x_1\frac{\partial G_\varepsilon}{\partial x_1} + x_2\frac{\partial G_\varepsilon}{\partial x_2}\right) + 2\left(\dot{U}_k(t) + g_1(t)\,U_k(t)\right)G_\varepsilon(t, \mathbf{x}) = 0. \tag{39}$$

Imposing that the latter PDE is satisfied simultaneously for $k = 1, 2$, and using again that the Wronskian $\mathbf{W}$ does not vanish, it follows after some computation that the general solution is given by

$$G_\varepsilon(t, \mathbf{x}) = \varepsilon\, F\left(\frac{x_2}{x_1}\right)x_1^{-2}\varphi^{-2}(t). \tag{40}$$

As before, it follows that for the nonlinear system

$$\ddot{x}_i + g_1(t)\,\dot{x}_i + g_2(t)\,x_i + \frac{\partial G_\varepsilon}{\partial x_i} = 0, \quad 1 \leq i \leq 2 \tag{41}$$

[3] This clearly follows from the fact that the equivalence class of (34) is that of the free particle system and hence possesses $\mathfrak{sl}(4, \mathbb{R})$-symmetry [38].

the Lie algebra of point symmetries $\mathcal{L}$ coincides with that of Noether symmetries $\mathcal{L}_{NS}$, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. We further observe that for generic choices of F, the equations are coupled non-trivially.

If we now introduce two functions $F_1\left(\frac{x_2}{x_1}\right)$, $F_2\left(\frac{x_1}{x_2}\right)$ that satisfy the constraint $x_1\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)+x_2F\left(\frac{x_2}{x_1}\right)=0$, $F\left(\frac{x_2}{x_1}\right)$ being the generic function from (40), the solution of (39) can be written as

$$G_\varepsilon\left(x_1,x_2\right)=-\frac{\varepsilon}{x_1x_2\varphi^2\left(t\right)}\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right). \tag{42}$$

This enables to write the partial derivatives as

$$\frac{\partial G_\varepsilon}{\partial x_1}=\varepsilon\frac{x_1x_2\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)-x_1^2F_2'\left(\frac{x_1}{x_2}\right)+x_2^2F_1'\left(\frac{x_2}{x_1}\right)}{x_1^3x_2^2\varphi^2(t)}=\frac{\varepsilon}{x_1^2x_2\varphi^2(t)}H_1\left(\frac{x_2}{x_1}\right), \tag{43}$$

$$\frac{\partial G_\varepsilon}{\partial x_2}=\varepsilon\frac{x_1x_2\left(F_1\left(\frac{x_2}{x_1}\right)+F_2\left(\frac{x_1}{x_2}\right)\right)+x_1^2F_2'\left(\frac{x_1}{x_2}\right)-x_2^2F_1'\left(\frac{x_2}{x_1}\right)}{x_1^2x_2^3\varphi^2(t)}=\frac{\varepsilon}{x_1x_2^2\varphi^2(t)}H_2\left(\frac{x_1}{x_2}\right). \tag{44}$$

The system possesses two independent constants of the motion quadratic in the velocities. Either using formula (10) for the X_i or the preceding equations of motion, it is easily shown that one invariant is

$$J_1=\frac{\varphi^2\left(t\right)}{2}\left(x_2\dot{x}_1-x_1\dot{x}_2\right)^2+\int^{x_2/x_1}H_1\left(z\right)\,\mathrm{d}z+\int^{x_1/x_2}H_2\left(z\right)\,\mathrm{d}z, \tag{45}$$

while the second can be expressed as[4]

$$J_2=\varphi^2\left(t\right)\left(\dot{x}_1\dot{x}_2+\int^{x_1x_2}g_2(z)\mathrm{d}z\right)-\int\left(\varphi^2\left(t\right)g_1\left(t\right)\int^{x_1x_2}g_2(z)\mathrm{d}z\right)\mathrm{d}t+\frac{K\varepsilon}{x_1^2}$$
$$-\frac{\varepsilon}{x_1^2}\int^{x_2/x_1}\frac{H_1(z)}{z}\mathrm{d}z. \tag{46}$$

The independence of J_1 and J_2 is straightforward, as J_2 depends explicitly on $g_1\left(t\right)$ and $g_2\left(t\right)$.

In particular, for equations with $g_1\left(t\right)=0$ the function $\varphi(t)$ reduces to a constant, and the preceding constants of the motion are not explicitly dependent on time. This situation generalizes the systems obtained from the particular choice $g_2\left(t\right)=\omega^2\left(t\right)$ and $\varepsilon=1$, where the system (41) constitutes a special case of the generalized Ermakov systems introduced in [10].[5] This shows that perturbations of the two-dimensional time-dependent harmonic oscillator [with respect to the Lagrangian (35)] that preserve the $\mathfrak{sl}\left(2,\mathbb{R}\right)$-subalgebra correspond to a subclass of the generalized Ermakov systems with velocity-independent potentials (hence, this subclass admits a Hamiltonian).

The two constants of the motion (45) and (46) reduce to

$$J_1=\frac{1}{2}\left(x_2\dot{x}_1-x_1\dot{x}_2\right)^2+\int^{x_2/x_1}H_1\left(z\right)\,\mathrm{d}z+\int^{x_1/x_2}H_2\left(z\right)\,\mathrm{d}z \tag{47}$$

and

$$J_2=\dot{x}_1\dot{x}_2+\frac{K}{x_1^2}+\int^{x_1x_2}\omega(z)^2\mathrm{d}z-\frac{1}{x_1^2}\int^{x_2/x_1}\frac{H_1(z)}{z}\mathrm{d}z. \tag{48}$$

The first integral J_1 corresponds to the well-known ERR invariant, while J_2 coincides with the additional invariant found in [39].

[4] This expression follows using the relation (11), as well as from the fact that H_1 and H_2 are actually dependent functions, as given in (43)–(44).

[5] In fact, the constraint satisfied by F_1 and F_2 corresponds to the sufficiency condition for the ERR system to arise from a Lagrangian in two dimensions, i.e., to satisfy the Helmholtz conditions (7).

2.2 Velocity-dependent forcing terms

We now analyze the more general case of perturbations of the Lagrangian (35) with forcing terms of the form $G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})$, starting from

$$\widetilde{L} = L - \varphi(t)\; G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})\,. \tag{49}$$

Evaluating the Noether symmetry condition for the vector fields X_1, X_2 leads, after some algebraic manipulation and simplification, to the following PDEs:

$$U_k(t)\dot{U}_k(t)\left(\sum_{l=1}^{2}\left(x_l\frac{\partial G_\varepsilon}{\partial x_l}-\dot{x}_l\frac{\partial G_\varepsilon}{\partial \dot{x}_l}\right)+2G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})\right)+\dot{U}_k^2(t)\sum_{l=1}^{2}x_l\frac{\partial G_\varepsilon}{\partial \dot{x}_l}+U_k^2(t)\frac{\partial G_\varepsilon}{\partial t}$$
$$+U_k^2(t)\left(2g_1(t)\;G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})-g_1(t)\sum_{l=1}^{2}\dot{x}_l\frac{\partial G_\varepsilon}{\partial \dot{x}_l}-g_2(t)\sum_{l=1}^{2}x_l\frac{\partial G_\varepsilon}{\partial \dot{x}_l}\right)=0,\quad k=1,2. \tag{50}$$

Basing on the fact that the Wronskian $\mathbf{W}$ does not vanish, we can consider linear combinations of the latter equations that reduce them to an equivalent form that, however, do not involve the U_k explicitly. Proceeding like this, it can be shown that the solution to the system (50) is equivalent to the solution of the following system:

$$x_1\frac{\partial G_\varepsilon}{\partial \dot{x}_1}+x_2\frac{\partial G_\varepsilon}{\partial \dot{x}_2}=0, \tag{51}$$
$$x_1\frac{\partial G_\varepsilon}{\partial x_1}+x_2\frac{\partial G_\varepsilon}{\partial x_2}-\dot{x}_1\frac{\partial G_\varepsilon}{\partial \dot{x}_1}-\dot{x}_2\frac{\partial G_\varepsilon}{\partial \dot{x}_2}+2G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})=0, \tag{52}$$
$$\frac{\partial G_\varepsilon}{\partial t}+2g_1(t)\;G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})-g_1(t)\left(\dot{x}_1\frac{\partial G_\varepsilon}{\partial \dot{x}_1}+\dot{x}_2\frac{\partial G_\varepsilon}{\partial \dot{x}_2}\right)=0. \tag{53}$$

Solving successively these equations, after a lengthy but routine computation, we find as general solution to this system

$$G_\varepsilon(t,\mathbf{x},\dot{\mathbf{x}})=\varepsilon\;F\left(\frac{x_2}{x_1},\varphi(t)\left(\dot{x}_2x_1-\dot{x}_1x_2\right)\right)x_1^{-2}\varphi^{-2}(t)\,. \tag{54}$$

If $g_1(t)=0$, the function G simplifies to

$$G_\varepsilon(\mathbf{x},\dot{\mathbf{x}})=\varepsilon\;F\left(\frac{x_2}{x_1},\left(\dot{x}_2x_1-\dot{x}_1x_2\right)\right)x_1^{-2}. \tag{55}$$

Introducing the auxiliary variables $r=x_2x_1^{-1}$ and $W=x_1\dot{x}_2-\dot{x}_1x_2$ as in [9], the equations of motion for the extended Lagrangian

$$\widetilde{L}=\frac{\varphi(t)}{2}\left(\dot{x}_1^2+\dot{x}_2^2-g_2(t)\left(x_1^2+x_2^2\right)\right)-\frac{\varepsilon}{x_1^2\varphi(t)}F\left(r,\varphi(t)\,W\right) \tag{56}$$

are explicitly given by

$$\ddot{x}_1+g_1(t)\,\dot{x}_1+g_2(t)\,x_1+\frac{2\varepsilon\dot{r}}{\varphi(t)\,x_1}\frac{\partial F}{\partial W}+\frac{\varepsilon\;r}{\varphi(t)^2\,x_1^3}\left(\varphi(t)\,W\frac{\partial^2F}{\partial r\partial W}-\frac{\partial F}{\partial r}\right)-\frac{2\varepsilon\;F}{\varphi(t)^2\,x_1^3}$$
$$+\frac{\varepsilon\;r\left(\dot{W}+g_1(t)\,W\right)}{x_1}\frac{\partial^2F}{\partial W^2}=0, \tag{57}$$
$$\ddot{x}_2+g_1(t)\,\dot{x}_2+g_2(t)\,x_2+\varepsilon\frac{\frac{\partial F}{\partial r}-\varphi(t)\,W\frac{\partial^2F}{\partial r\partial W}}{\varphi(t)^2\,x_1^3}-\varepsilon\frac{\left(\dot{W}+g_1(t)\,W\right)}{x_1}\frac{\partial^2F}{\partial W^2}=0. \tag{58}$$

It is straightforward to verify that this system is equivalent to the original two-dimensional oscillator if F is a linear function of W. For nonlinear functions in W, the Lie algebra $\mathcal{L}$ of point symmetries of (57), (58) coincides once more with that of Noether symmetries, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In this case, as expected from the presence of a damping term $g_1(t)$, the two constants of the motion derived from the Noether symmetries

X_1 and X_2 adopt in general a rather complicated (explicitly time-dependent) integral form, for which reason we skip their detailed expression.

As an elementary example illustrating the latter situation, let $g_1(t) = 1$ and $g_2(t) = 0$. We consider the forcing term given by the function $G_\varepsilon(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon W^2$. The equations of motion of the Lagrangian $L = \frac{1}{2}e^t\left(\dot{x}_1^2 + \dot{x}_2^2 + \varepsilon W^2 x_1^{-2}\right)$ can be brought to the form

$$\ddot{x}_1 + \dot{x}_1 - \frac{(1+\varepsilon)W^2}{x_1^3\left((1+\varepsilon)+\varepsilon r^2\right)} = 0, \quad \ddot{x}_2 + \dot{x}_2 - \frac{\varepsilon^2 r\, W^2}{x_1^3\left((1+\varepsilon)+\varepsilon r^2\right)} = 0. \tag{59}$$

A first invariant can be found easily. Multiplying the first equation by $x_2 W^{-1}$ and the second by $x_1 W^{-1}$, the difference of the equations leads, after integration, to the conserved quantity

$$I_1 = e^t\, W\sqrt{\left((1+\varepsilon)+\varepsilon r^2\right)} = e^t\,(\dot{x}_2 x_1 - x_2\dot{x}_1)\, x_1^{-2}\sqrt{\left((1+\varepsilon)x_1^2 + \varepsilon x_2^2\right)}. \tag{60}$$

A second independent invariant is more complicated to find, although for this purpose the explicit Noether symmetries can be used. With this method, and after some lengthy calculation, the following invariant can be found:

$$I_2 = \frac{e^t}{2}\left(\dot{x}_1^2 + (1+\varepsilon)\dot{x}_2^2 + x_1\dot{x}_1 + x_2\dot{x}_2 - \frac{\varepsilon x_2\dot{x}_1\,(2\dot{x}_2 x_1 - x_2\dot{x}_1)}{x_1^2}\right). \tag{61}$$

We further observe that $I_1 I_2^{-1}$ provides an invariant that does not explicitly depend on time.

At this stage, the natural question that arises in this context is whether for the case $g_1(t) = 0$ the perturbed system (57), (58) also corresponds to a generalized Ermakov–Ray–Reid system (with velocity-dependent potential) that allows a two-dimensional Lagrangian. We prove this assumption to be correct.

Such a generalized ERR system, as first introduced in [11], has the generic form:[6]

$$\ddot{x}_1 + \omega^2(t)\,x_1 - \frac{1}{x_2^3}\frac{\partial F_1}{\partial r} + \frac{W}{x_2^3}\frac{\partial^2 F_1}{\partial r\partial W} + \frac{\dot{W}}{x_2}\frac{\partial^2 F_1}{\partial W^2} = 0, \tag{62}$$

$$\ddot{x}_2 + \omega^2(t)\,x_2 + \frac{1}{x_2^2 x_1}\frac{\partial G_1}{\partial r} - \frac{W}{x_2^2 x_1}\frac{\partial^2 G_1}{\partial r\partial W} - \frac{\dot{W}}{x_1}\frac{\partial^2 G_1}{\partial W^2} = 0, \tag{63}$$

where F_1 and G_1 are arbitrary functions of r and W. Using the Helmholtz conditions (7) (see also [28,29]), a long but routine computation shows that (62), (63) correspond to the equations of motion of a two-dimensional Lagrangian if the following constraints are satisfied:

$$\frac{\partial^2 G_1}{\partial W^2} - r^2\frac{\partial^2 F_1}{\partial W^2} = 0, \tag{64}$$

$$3\frac{\partial F_1}{\partial r} + \frac{1}{r^2}\frac{\partial G_1}{\partial r} - 3w\frac{\partial^2 F_1}{\partial r\partial W} - \frac{W}{r^2}\frac{\partial^2 G_1}{\partial r\partial W} + r\frac{\partial^2 F_1}{\partial r^2} - \frac{1}{r}\frac{\partial^2 G_1}{\partial r^2} - rw\frac{\partial^3 F_1}{\partial r^2\partial W} + \frac{W}{r}\frac{\partial^3 G_1}{\partial r^2\partial W} = 0. \tag{65}$$

Integrating the first equation and using the method of characteristics [40], we can write the solution to this system as

$$F_1(r, W) = \frac{G_1(r, w)}{r^2} + f_1(r)\,w + \frac{C}{r^2} \tag{66}$$

where $G_1(r, W)$ is still an arbitrary function. Now, inserting the latter expression into (62)–(63) and simplifying, the equations of motion adopt the form

$$\ddot{x}_1 + \omega^2(t)\,x_1 + \frac{2G_1}{x_1^3} - \frac{2W}{x_1^3}\frac{\partial G_1}{\partial W} - \frac{1}{x_1^2 x_2}\frac{\partial G_1}{\partial r} + \frac{W}{x_1^2 x_2}\frac{\partial^2 G_1}{\partial r\partial W} + \frac{x_2\dot{W}}{x_1^2}\frac{\partial^2 G_1}{\partial W^2} = 0, \tag{67}$$

[6] The only formal difference with respect to [11] is that we have skipped the explicit use of the variable $\widetilde{r} = r^{-1}$ in the equations of motion.

$$\ddot{x}_2 + \omega^2(t)\, x_2 + \frac{1}{x_2^2 x_1}\frac{\partial G_1}{\partial r} - \frac{W}{x_2^2 x_1}\frac{\partial^2 G_1}{\partial r \partial W} - \frac{\dot{W}}{x_1}\frac{\partial^2 G_1}{\partial W^2} = 0. \tag{68}$$

These equations are rather similar to those in (57)–(58) with $g_1(t) = 0$ and $\varphi(t) = 1$. In fact, if we define the forcing term as $G_\varepsilon(t, \mathbf{x}, \dot{\mathbf{x}}) = -\varepsilon G_1\left(\frac{1}{r}, -W\right)$ and consider the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 - g_2(t)\left(x_1^2 + x_2^2\right)\right) + \frac{1}{x_1^2} G_\varepsilon\left(\frac{1}{r}, -W\right), \tag{69}$$

it is immediate to verify that the equations of motion are exactly those given in (67)–(68) with $\varepsilon = 1$.

Jointly with the result obtained in the preceding paragraph for the velocity-independent forcing terms, we conclude that perturbations of the two-dimensional time-dependent oscillator with respect to the Lagrangian (35) give rise to generalized ERR systems. This can be formulated in compact form as follows:

Proposition 3 *For $g_1(t) = 0$ and $g_2(t) = \omega^2(t)$, any $\mathfrak{sl}(2, \mathbb{R})$-preserving perturbation of the two-dimensional time-dependent oscillator (34) corresponds to a generalized ERR system (62)–(63) satisfying the constraints (64)–(65). Conversely, any ERR system admitting a Hamiltonian is obtained as a perturbation of the two-dimensional time-dependent oscillator.*

For $g_1(t) \neq 0$, the systems can be seen as a further possible generalization of ERR systems, albeit for generic choices of the forcing term, the Lagrangian and the constants of the motion of the system will generally be explicitly depending on time. The perturbations can provide, however, additional examples of integrable systems with dissipative terms.

3 Perturbations of the free Lagrangian in $\mathbb{R}^N$

There is no formal difficulty in generalizing the results to systems in N dimensions. The uncoupled system

$$\ddot{x}_i + g_1(t)\,\dot{x}_i + g_2(t)\, x_i = 0, \quad i = 1, \dots, N \tag{70}$$

obtained from the Lagrangian $L = \frac{1}{2}\varphi(t)\sum_{i=1}^N \left(\dot{x}_i^2 - g_2(t)\, x_i^2\right)$ is always linearizable, and hence, the subalgebra of Noether symmetries is isomorphic to the Schrödinger algebra $S(N)$ with Levi subalgebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(N)$. In particular, the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra is generated by the vector fields

$$X_k = \varphi(t)\, U_k^2(t)\frac{\partial}{\partial t} + \sum_{l=1}^N \varphi(t)\, U_k(t)\, \dot{U}_k(t)\, x_l \frac{\partial}{\partial x_l}, \quad k = 1, 2. \tag{71}$$

Like before, the most general forcing term $G_\varepsilon(t, \mathbf{x}, \dot{\mathbf{x}})$ preserving the $\mathfrak{sl}(2, \mathbb{R})$-symmetry is given by

$$G_\varepsilon(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon\, F\left(\frac{x_2}{x_1}, \dots, \frac{x_N}{x_1}, \varphi(t)\left(\dot{x}_2 x_1 - \dot{x}_1 x_2\right), \dots, \varphi(t)\left(\dot{x}_N x_1 - \dot{x}_1 x_N\right)\right) x_1^{-2}\varphi^{-2}(t). \tag{72}$$

In analogy to the low-dimensional cases, for generic choices of F, the point symmetry algebra of the system is also isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, and hence, the system is genuinely nonlinear.

Imposing invariance by additional symmetries (or subalgebras) further restricts the form of the functions $G_\varepsilon(t, \mathbf{x}, \dot{\mathbf{x}})$, allowing the existence of supplementary constants of the motion and eventually leading to a characterization of Lagrangians in terms of subalgebras of the Schrödinger algebra $S(N)$. The main difficulty in this approach is of technical nature, namely in obtaining the general solution of a large system of partial differential equations.

An illustrative case to the procedure is the restriction to perturbations of the free Lagrangian in N dimensions (such that $\varphi(t) = 1$ holds). It can be easily verified that the only Lagrangians that preserve the complete Levi subalgebra $\mathfrak{so}(N) \oplus \mathfrak{sl}(2, \mathbb{R})$ of the (unextended) Schrödinger algebra $S(N)$ are given by

$$\widetilde{L} = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \cdots + \dot{x}_N^2\right) - \varepsilon\, \frac{F\left(\left(\sum_{i=1}^N x_i^2\right)\left(\sum_{i=1}^N \dot{x}_i^2\right) - \left(\sum_{i=1}^N x_i \dot{x}_i\right)^2\right)}{\left(x_1^2 + x_2^2 + \cdots + x_N^2\right)} \tag{73}$$

where F is a an arbitrary function of the argument. We observe that invariance with respect to any other Noether symmetry of the free Lagrangian would imply the existence of cyclic coordinates. If the forcing term $G_{\varepsilon}(t, \mathbf{x}, \dot{\mathbf{x}})$ does not depend explicitly on the velocities, it reduces to

$$G_{\varepsilon}(t, \mathbf{x}) = \frac{\varepsilon}{\left(x_1^2 + x_2^2 + \cdots + x_N^2\right)}, \quad \varepsilon \in \mathbb{R}. \tag{74}$$

This fact suggests that the interesting subalgebras of $S(N)$ to be analyzed with respect to symmetry preservation are the subalgebras of the Levi factor $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(N)$. In particular, we have the canonical chain

$$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(N) \supset \cdots \supset \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(3) \supset \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2) \supset \mathfrak{sl}(2, \mathbb{R}). \tag{75}$$

Comparing (72) and (73), it stands to reason to suspect that the number of degrees of freedom in the forcing term $G_{\varepsilon}(t, \mathbf{x}, \dot{\mathbf{x}})$ is strongly dependent on the rank of the preserved semisimple algebra of Noether symmetries.[7]

To prove this assertion, for any $2 \leq m \leq N$ we define the auxiliary functions

$$J_1 = \sum_{i=1}^{m} x_i \dot{x}_i, \quad J_2 = \sum_{i=1}^{m} x_i^2, \quad J_3 = \sum_{i=1}^{m} \dot{x}_i^2 \tag{76}$$

and the variables

$$z_j = \frac{x_{m+j}}{\sqrt{J_2}}, \quad z_{N-m+j} = \frac{\dot{x}_{m+j} J_2 - x_{m+j} J_1}{\sqrt{J_2}}, \quad 1 \leq j \leq N - m. \tag{77}$$

We further define the family of Lagrangians

$$\widetilde{L}_m = \frac{1}{2}\left(\dot{x}_1^2 + \cdots + \dot{x}_N^2\right) + G_{\varepsilon,m}(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{78}$$

where

$$G_{\varepsilon,m}(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon \frac{F\left(J_2 J_3 - J_1^2, z_1, \ldots, z_{2N-2m}\right)}{J_2}. \tag{79}$$

Proposition 4 *For any $2 \leq m \leq N - 1$, $\widetilde{L}_m$ is a perturbation of the free Lagrangian in $\mathbb{R}^N$ possessing a Noether symmetry algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(m)$. In particular, if*

$$\frac{\partial G_{\varepsilon,m}}{\partial \dot{x}_k} = 0, \quad 1 \leq k \leq N,$$

then $\widetilde{L}_m$ reduces to

$$\widetilde{L}_m = \frac{1}{2}\left(\dot{x}_1^2 + \cdots + \dot{x}_N^2\right) + \varepsilon \frac{F(z_1, \ldots, z_{N-m})}{J_2}. \tag{80}$$

The Noether symmetries of the free Lagrangian in $\mathbb{R}^N$ corresponding to the generators of the Levi subalgebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(n)$ are easily seen to have the form

$$X_1 = t^2 \frac{\partial}{\partial t} + \sum_{i=1}^{N} t\, x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{N} (x_i - t\, \dot{x}_i) \frac{\partial}{\partial \dot{x}_i}, \quad X_3 = \frac{\partial}{\partial t},$$

$$X_2 = -[X_1, X_3], \quad X_{i,j} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \leq i, < j \leq N.$$

The vector fields form a basis of the Levi subalgebra, and hence, it follows at once that the system is complete [41]. If we want to determine the most general perturbation that preserves an $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(m)$ subalgebra for some $2 \leq m < N$, it is convenient to analyze first the symmetry condition (9) for the orthogonal Lie algebra $\mathfrak{so}(m)$. As the latter is generated by the vector fields

$$X_{l,l+1} = x_{l+1} \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_{l+1}}, \quad 1 \leq l \leq m - 1 \tag{81}$$

7 We recall that the rank of $\mathfrak{so}(N) \oplus \mathfrak{sl}(2, \mathbb{R})$ is given by $1 + \left[\frac{N}{2}\right]$.

it suffices to inspect the symmetry condition for these generators, as the remaining equations will be satisfied by commutators. It is routine to verify that (9) reduces to the homogeneous system

$$X_{l,l+1} G_{\varepsilon,m}\,(t, \mathbf{x}, \dot{\mathbf{x}}) = 0, \quad 1 \leq l \leq m-1$$

with general solution given by

$$G_{\varepsilon,m}\,(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon\, F\,(t, J_1, J_2, J_3, x_{N-m+1}, \ldots, x_N, \dot{x}_{N-m+1}, \ldots, \dot{x}_N)\,. \tag{82}$$

We now impose the invariance with respect to $\mathfrak{sl}\,(2, \mathbb{R})$. The symmetry condition (9) leads to the equations

$$2t\, G_{\varepsilon,m}\,(t, \mathbf{x}, \dot{\mathbf{x}}) + t^2 \frac{\partial G_{\varepsilon,m}}{\partial t} + \sum_{i=1}^{N} t\, x_i \frac{\partial G_{\varepsilon,m}}{\partial x_i} + \sum_{i=1}^{N} (x_i - t\,\dot{x}_i)\,\frac{\partial G_{\varepsilon,m}}{\partial \dot{x}_i} = 0; \quad \frac{\partial G_{\varepsilon,m}}{\partial t} = 0 \tag{83}$$

where we have omitted the equation corresponding to the generator X_2, as it is satisfied whenever a solution of (83) is given. Let further $\widetilde{X}_1$ denote the homogeneous part of the first equation in (83). Evaluating $\widetilde{X}_1$ successively for the variables of (82) leads to the reduced equation

$$t^2 \frac{\partial G_{\varepsilon,m}}{\partial t} + J_2 \frac{\partial G_{\varepsilon,m}}{\partial J_1} + 2t\, J_2 \frac{\partial G_{\varepsilon,m}}{\partial J_2} + 2\,(J_1 - t\, J_3)\,\frac{\partial G_{\varepsilon,m}}{\partial J_3} + \sum_{l=n-m+1}^{N} \left(t\, x_l \frac{\partial G_{\varepsilon,m}}{\partial x_l} + (x_l - t\,\dot{x}_l)\,\frac{\partial G_{\varepsilon,m}}{\partial \dot{x}_l} \right) = 0.$$

The general solution to this equation is determined by means of the method of characteristics and equals

$$G_{\varepsilon,m}(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon\, F \left(\frac{t}{\sqrt{J_2}}, J_1 - \frac{J_2}{t}, J_2 J_3 - \frac{2 J_1 J_2}{t} + \frac{J_2^2}{t^2}, \frac{x_l}{\sqrt{J_2}}, \frac{(x_l - t\,\dot{x}_l)\sqrt{J_2}}{t} \right)_{n-m+1 \leq l \leq N}.$$

By (83), we are only interested in the variables that do not explicitly depend on t. After some algebraic manipulation, we obtain that the variables $J_2 J_3 - J_1^2, z_1, \ldots, z_{2n-2m}$ with z_l defined as in (77) form an integrity basis for the homogeneous part of the system (83). Taking into account the non-homogeneous term in the system, the general solution is easily found, after some computation, to be given by the expression (79). Now, as the solution involves all variables $\{\mathbf{x}, \dot{\mathbf{x}}\}$, it is straightforward to verify that no other Noether symmetry of the free Lagrangian can satisfy the system (83), showing that the Lagrangian $\widetilde{L}_m$ has Noether symmetry algebra $\mathfrak{sl}\,(2, \mathbb{R}) \oplus \mathfrak{so}\,(m)$.

Finally, if the perturbation term $G_{\varepsilon,m}\,(t, \mathbf{x}, \dot{\mathbf{x}})$ does not explicitly depend on the velocities, then all variables involving J_1 and J_3 must be discarded, and the solution adopts the simple form

$$G_{\varepsilon,m}\,(t, \mathbf{x}, \dot{\mathbf{x}}) = \varepsilon\, \frac{F\,(z_1, \ldots, z_{N-m})}{J_2}. \tag{84}$$

These perturbation terms can thus be interpreted as those potentials for which the Lagrangian has the prescribed symmetry.

For $N = 3$ and $m = 2$, and considering only forcing terms of the latter type, the perturbations of the free Lagrangian possessing a Noether symmetry algebra isomorphic to $\mathfrak{sl}\,(2, \mathbb{R}) \oplus \mathfrak{so}\,(2)$ have the following form:

$$\widetilde{L} = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{\varepsilon}{(x_1^2 + x_2^2)} F \left(\frac{x_3}{x_1^2 + x_2^2} \right). \tag{85}$$

Introducing cylindrical coordinates $\{x_1 = \rho \cos\theta,\ x_2 = \rho \sin\theta\,,\ x_3 = \xi\}$ and setting $\varepsilon = 1$, these Lagrangians are rewritten as

$$\widetilde{L} = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \dot{\xi}^2\right) - \frac{\varepsilon}{\rho^2} F \left(\frac{\xi}{\rho} \right), \tag{86}$$

showing that θ is a cyclic coordinate. It is immediate to deduce that $J = \rho^2\dot{\theta}$ is a conserved quantity. Letting $\rho^2\dot{\theta} = \alpha$ and substituting in (86), we can reduce our analysis to the two-dimensional Lagrangian

$$\widetilde{L}' = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right) + \frac{\alpha^2}{2\rho^2} - \frac{\varepsilon}{\rho^2} F\left(\frac{\xi}{\rho}\right). \tag{87}$$

We observe that the potential $U(\rho, \xi) = \left(\frac{1}{2}\alpha^2 - \varepsilon F\left(\frac{\xi}{\rho}\right)\right)\rho^{-2}$ is of the form $\Psi\left(\frac{\xi}{\rho}\right)\rho^{-2}$, from which we conclude that (87) can be seen as a perturbation of the free planar Lagrangian $L_0 = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right)$. In particular, it preserves the $\mathfrak{sl}(2, \mathbb{R})$ -symmetry, and thus, two independent constants of the motion exist. These are given by

$$H_r = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right) + \frac{1}{\rho^2}\left(\varepsilon F\left(\frac{\xi}{\rho}\right) - \frac{\alpha^2}{2}\right), \tag{88}$$

$$I_1 = \frac{1}{2}\left(\xi\dot{\rho} - \dot{\xi}\rho\right)^2 + \int^{\xi\rho^{-1}}\left(z\left(2\varepsilon F(z) - \alpha^2\right) + \varepsilon\left(1 + z^2\right)\frac{dF}{\mathrm{d}z}\right)\mathrm{d}z. \tag{89}$$

We observe that the constants of the motion associated to the orthogonal Lie algebra can be used to reduce the number of degrees of freedom, and giving rise to symmetry-preserving perturbations in the reduced system. For this example, the invariants can be further used to obtain, jointly with J, three independent constants of the motion for the Lagrangian (86).

4 Conclusions

Basing on the comparison of the Noether symmetries of the time-dependent harmonic oscillator and the Pinney equation, it has been shown that the latter arises as the most general perturbation of the former that preserves a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Perturbations have been generalized to arbitrary linear homogeneous ordinary differential equations $\ddot{x} + g_1(t)\dot{x} + g_2(t)x = 0$, and perturbations preserving a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ have been obtained. The corresponding problem in two dimensions offers more general types of perturbations, specifically dependent on the velocities. In this context, it has been shown that $\mathfrak{sl}(2, \mathbb{R})$-preserving perturbations of the time-dependent harmonic oscillator are characterized as generalized Ermakov–Ray–Reid systems as introduced in [11], and admitting a Hamiltonian formalism. Within the classical interpretation of Ermakov systems, this result was to be expected. In the general case with damping terms, no such characterization is given, as the perturbed Lagrangians are usually explicitly time-dependent, a characteristic that is carried over to the invariants of the system. For arbitrary dimensions, and starting from the free Lagrangian in $\mathbb{R}^n$, the perturbations preserving the canonical chain of subalgebras of the Levi factor of the Schrödinger algebra have been computed, pointing out the relation existing between the rank of the symmetry algebra and the number of degrees of freedom in the perturbation term.

The perturbation procedure, as developed in this work, can be seen as a natural extension or enlargement of some recent work [23,35] concerning the generic structure of infinitesimal generators of Lie point symmetries of systems of ordinary differential equations, in the context of the linearization problem. As Noether symmetries constitute a particular type of point symmetries, the generic symmetry description can be adapted to the case of Lagrangian systems, providing an additional tool to analyze their properties and, in particular, the structure of their invariants. As scalar linear ODEs are derivable from a variational principle [29], this leads to a generalization of the Lewis invariant for arbitrary linear equations [35]. The next natural step in this analysis, developed here, resides in preserving a relevant number of Noether symmetries, however "breaking" the point symmetry algebra in order to ensure that the perturbation is no more a linearizable equation or system. This preservation of only a proper subgroup constitutes the main difference with respect to the previous ansatz of [23,35] and shows how techniques of linearizable systems can be applied, under certain assumptions, to the integrability problem of genuinely nonlinear systems.

Albeit our analysis has been mainly focused on Euclidean Lagrangians related to the time-dependent (damped) harmonic oscillator, the procedure is by no means restricted to these Lagrangians. Perturbations can be applied to any ODE or system and any subalgebra of Noether symmetries, in particular to non-trivially coupled systems possessing a sufficient number of such symmetries.

As an example of non-Euclidean Lagrangian, consider for instance the system

$$\ddot{x}_1 + \frac{\alpha}{x_1^3} = 0, \quad \ddot{x}_2 - \frac{3\alpha r}{x_1^3} = 0 \quad \left(r = x_2 x_1^{-1}\right). \tag{90}$$

An admissible Lagrangian is given by

$$L = \dot{x}_1 \dot{x}_2 - \alpha x_2 x_1^{-3}. \tag{91}$$

This Lagrangian can be seen as a perturbation of the free Lagrangian in the pseudo-Euclidean plane and admits an $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra of Noether symmetries generated by the vector fields $X = \xi(t) \frac{\partial}{\partial t} + \frac{1}{2} x_1 \dot{\xi}(t) \frac{\partial}{\partial x_1} + \frac{1}{2} x_2 \dot{\xi}(t) \frac{\partial}{\partial x_2}$, where $\xi^{(3)}(t) = 0$. In this case, the auxiliary function is given by $V(t, \mathbf{x}) = \frac{1}{2} \dot{x}_1 \dot{x}_2 \ddot{\xi}(t)$. Those potentials that preserve the $\mathfrak{sl}(2, \mathbb{R})$-symmetry have the generic form $G_\varepsilon(t, \mathbf{x}) = \varepsilon F\left(x_2 x_1^{-1}\right) x_1^{-2} = \varepsilon F(r) x_1^{-2}$. The equations of motion for the perturbed Lagrangian $\widehat{L} = \dot{x}_1 \dot{x}_2 - \alpha x_2 x_1^{-3} + F(r) x_1^{-2}$ are

$$\ddot{x}_1 + \frac{\alpha}{x_1^3} - \varepsilon \frac{F'(r)}{x_1^3} = 0, \quad \ddot{x}_2 - \frac{3\alpha r}{x_1^3} + \frac{2\varepsilon F(r)}{x_1^3} + \frac{r\,\varepsilon F'(r)}{x_1^3} = 0. \tag{92}$$

It is immediate to see that the Hamiltonian $H = \dot{x}_1 \dot{x}_2 + \frac{\alpha r}{x_1^2} - \varepsilon \frac{F(r)}{x_1^2}$ is a constant of the motion. The second invariant is given by

$$I_1 = \frac{1}{2} W^2 - 2\alpha r^2 + 2r\varepsilon F(r). \tag{93}$$

We observe that, incidentally, for $F(r) = \lambda$, the (perturbed) system is super-integrable, as it admits the additional constant of the motion $I_2 = \dot{x}_1^2 - \alpha x_1^{-2}$ [42].

Clearly, the perturbation problem is strongly dependent on the Lagrangian chosen, as well as the fixed subalgebra of Noether symmetries. The form of symmetry-preserving forcing terms for alternative Lagrangians giving rise to the same equations of motion may differ radically, as can be expected from the existing ambiguities in the Lagrangian formalism [31]. In this sense, it would be of interest to classify these perturbations according to their equivalence class as systems of ODEs [19,38]. The imposition of the generators of the subalgebra to remain unaltered by the perturbation, although not explicitly stated, constitutes a constraint that could be used for such a classification. From the physical perspective, however, it seems more relevant to determine a precise interpretation of the conservation laws of the resulting nonlinear system. For the case of systems admitting time-dependent invariants, their asymptotic behavior could provide useful information on the intrinsic structural properties of the system. Both problems present some interesting features that deserve to be inspected more closely.

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