Acta Mech 227, 1753–1766 (2016)
DOI 10.1007/s00707-016-1580-y

ORIGINAL PAPER

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A receding contact problem between a functionally graded layer and two homogeneous quarter planes

Received: 12 July 2015 / Revised: 1 December 2015 / Published online: 10 March 2016

Abstract In this paper, the plane problem of a frictionless receding contact between an elastic functionally graded layer and two homogeneous quarter planes is considered when the graded layer is pressed against the quarter planes. The top of the layer is subjected to normal tractions over a finite segment. The graded layer is modeled as a non-homogeneous medium with a constant Poisson's ratio and exponentially varying shear modules. The problem is converted into the solution of a Cauchy-type singular integral equation in which the contact pressure and the receding contact half-length are the unknowns using integral transforms. The singular integral equation is solved numerically using Gauss–Jacobi integration. The corresponding receding contact half-length that satisfies the global equilibrium condition is obtained using an iterative procedure. The effect of the material non-homogeneity parameter on the contact pressure and on the length of the receding contact is investigated.

1 Introduction

The materials research community has recently been exploring the possibility of using new concepts in coating or layer design such as functionally graded materials (FGMs) in which material properties vary smoothly along a spatial direction, as an alternative to the conventional homogeneous coating and layer [1]. As the application of FGMs has increased in modern industries, new methodologies have been developed to analyze the mechanical behavior of functionally graded elements. Also, there are many engineering applications where the stress analysis at the interface between two bodies in contact is principal in the structural design as the response of the structure depends on it. Examples of these applications in mechanical engineering are railways, foundation grillages, connecting rods, joint and support elements, rolling mills and pavements of highways and airfields

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[2,3]. So, problems involving the contact of two separate bodies pressed against each other have widely been studied by many researchers. Although the contact area increases after the application of the load in many cases, there are others where the contact area becomes smaller. This kind of problem is called receding in the literature. In other words, a contact can be named receding if the contact area in the loaded configuration is contained within the initial contact area [4].

Among the analytical studies on receding contact, the following are recorded in the literature. Keer et al. [5] solved the smooth receding contact problem between an elastic layer and a half-space when two bodies were pressed against each other by considering both plane and axisymmetric cases.

The same problem was solved treating the layer as a simple beam by Gladwell [6]. The frictionless contact problem for an elastic layer resting on two quarter planes and loaded compressively was solved by Erdogan and Ratwani [7]. Civelek and Erdogan [8] investigated the general axisymmetric double frictionless contact problem for an elastic layer resting on a half-space and pressed by an elastic stamp. The smooth receding contact problem for an elastic layer pressed against a half-space by frictionless semi-infinite elastic was examined by Gecit [9]. Aksogan et al. [10,11] studied a contact problem for an elastic layer supported by two elastic quarter planes with both symmetrical loading and axisymmetric loading. Comez et al. [12] solved a double receding contact problem for two elastic layers having different elastic constants and heights and pressed by a rigid stamp. Kahya et al. [13] considered a frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half plane, when the two bodies were pressed together by means of a rigid circular stamp. Yaylacı and Birinci [14] studied a receding contact problem of two elastic layers supported by two elastic quarter planes. The solution of a receding contact problem using an analytical method and a finite element method was examined by Oner et al. [15].

A receding contact plane problem for a functionally graded layer pressed against a homogeneous half-space was analyzed by El-Borgi et al. [1]. A multilayered model for sliding frictional contact analysis of functionally graded materials (FGMs) with arbitrarily varying shear modulus under plane strain-state deformation has been developed by Ke and Wang [16]. The two-dimensional frictionless contact problem of a coating structure consisting of a surface coating, a functionally graded layer and a substrate under a rigid cylindrical punch was investigated by Yang and Ke [17]. Barik et al. [18] studied the stationary plane contact of a functionally graded heat conducting punch and a rigid insulated half-space. The frictionless contact problem of a functionally graded piezoelectric layered half-plane in-plane strain state under the action of a rigid flat or cylindrical punch was examined by Ke et al. [19]. Sliding frictional contact between a rigid punch and a laterally graded elastic medium was studied by Dag et al. [20]. Rhimi et al. [21,22] considered the axisymmetric problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space when the two bodies were pressed together and double receding contact between a rigid stamp of axisymmetric profile, an elastic functionally graded layer and a homogeneous half-space. Chen and Chen [23] studied the contact behavior of a graded layer resting on a homogeneous half-space and pressed by a rigid stamp. Comez [24] considered a contact problem for a functionally graded layer loaded by means of a rigid stamp and supported by a Winkler foundation. The plane problem of a frictional receding contact formed between an elastic functionally graded layer and a homogeneous half-space, when they were pressed against each other, was investigated by El-Borgi et al. [25].

Although the receding contact problem of a homogeneous layer resting on two quarter planes has been studied by Erdogan and Ratwani [7] and Aksogan et al. [10], the problem has not been investigated in case of a functionally graded layer yet. In this paper, the plane problem of a frictionless receding contact between an elastic functionally graded layer and two homogeneous quarter planes is considered when the graded layer is pressed against the quarter planes. The problem is reduced to a Cauchy-type singular integral equation in which the unknowns are the receding contact half-length and contact pressure by using Fourier and Mellin integral transforms. The contact pressures and the length of the receding contact are calculated for various values of the material non-homogeneity parameter, loading and distance between the quarter planes by solving the resulting singular integral equation.

2 Formulation of the problem

As shown in Fig. 1, consider the symmetric plane strain problem consisting of an infinitely long functionally graded (FG) layer of thickness h resting on two quarter planes. For the layer, the Poisson's ratio ν_1 is taken as constant and the shear modulus μ_1 depends on the y-coordinate only as follows:

$$\mu_1(y) = \mu_0 \exp(\beta y), \qquad -h \le y \le 0, \tag{1}$$

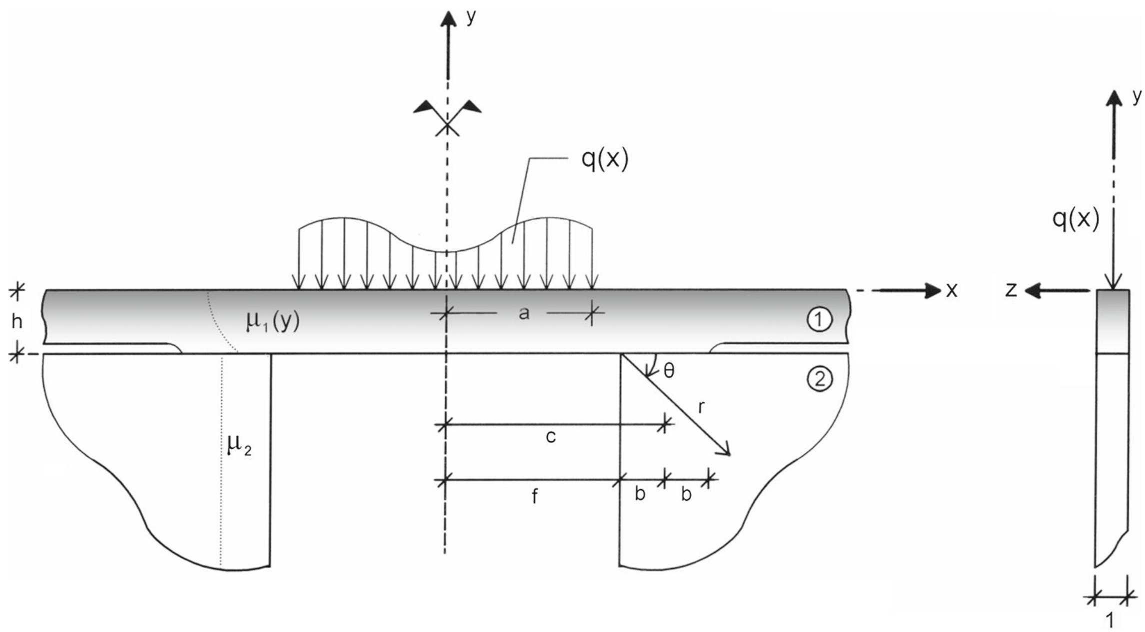

Fig. 1 Geometry and loading of the receding contact problem

where μ_0 is the shear modulus of the graded layer at $y = 0$ and β is the non-homogeneity parameter controlling the variation of the shear modulus in the graded layer. The quarter planes have constant Poisson's ratio ν_2 and shear modulus μ_2.

The top of the layer is subjected to a distributed load $q(x)$ over the segment $|x| \leq a$. The main unknowns of the problem are the contact pressure, denoted $p(x)$, over the contact area $c-b \leq x \leq c+b$ and the receding contact half-length, namely b.

It is assumed that the contact surfaces are frictionless and only compressive traction can be transmitted through the contact surfaces. In addition, $x = 0$ is to be the plane of symmetry with respect to external loads as well as geometry, for simplicity. Clearly, it is sufficient to consider one half (i.e., $x \geq 0$) of the medium only.

Assuming that the FG layer is isotropic at every point, equilibrium equations with body forces neglected, the strain–displacement relationships and the linear elastic stress–strain law, respectively, given by:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{2a,b}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{3a,b,c}$$

$$\sigma_x = \frac{\mu_1}{\kappa_1 - 1}\left[(1+\kappa_1)\varepsilon_{xx} + (3-\kappa_1)\varepsilon_{yy}\right], \tag{4a}$$

$$\sigma_y = \frac{\mu_1}{\kappa_1 - 1}\left[(3-\kappa_1)\varepsilon_{xx} + (1+\kappa_1)\varepsilon_{yy}\right], \tag{4b}$$

$$\tau_{xy} = 2\mu_1\varepsilon_{xy}, \tag{4c}$$

where u and v are the x and y components of the displacement field, respectively; σ_x, σ_y and τ_{xy} are the components of the stress field in the same coordinate system; ε_x, ε_y and ε_{xy} are the corresponding components of the strain field; and κ_1 is a material property defined as $\kappa_1 = 3 - 4\nu_1$ for plane strain problems. Combining Eqs. (1)–(4), the following two-dimensional Navier equations are obtained:

$$(\kappa_1+1)\frac{\partial^2 u}{\partial x^2} + (\kappa_1-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x\partial y} + \beta(\kappa_1-1)\frac{\partial u}{\partial y} + \beta(\kappa_1-1)\frac{\partial v}{\partial x} = 0, \tag{5a}$$

$$(\kappa_1-1)\frac{\partial^2 v}{\partial x^2} + (\kappa_1+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x\partial y} + \beta(3-\kappa_1)\frac{\partial u}{\partial x} + \beta(\kappa_1+1)\frac{\partial v}{\partial y} = 0. \tag{5b}$$

The boundary conditions for the graded layer can be defined as follows:

$$\sigma_x(x,0) = -q(x)H(a-|x|), \qquad \tau_{xy}(x,0) = 0, \qquad 0 \le x < \infty, \tag{6a,b}$$

$$\sigma_x(x,-h) = -p(x)H(b-|x-c|), \qquad \tau_{xy}(x,-h) = 0, \qquad 0 \le x < \infty, \tag{6c,d}$$

where H is the Heaviside function.

For the homogeneous quarter plane, using the Airy stress function provides convenience, and the stress components and displacement components can be written in polar coordinates $(r,\ \theta)$ assuming zero body forces as follows:

$$\varphi = \varphi(r,\theta), \tag{7}$$

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}, \qquad \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}, \tag{8a,b,c}$$

$$2Gu_r = -\frac{\partial\varphi}{\partial r} + (1-\upsilon)r\frac{\partial\Psi}{\partial\theta}, \qquad 2Gu_\theta = -\frac{\partial\varphi}{\partial\theta} + (1-\upsilon)r^2\frac{\partial\Psi}{\partial r}, \tag{9a,b}$$

where φ is the Airy stress function; σ_r, σ_θ and $\tau_{r\theta}$ are the components of the stress field; u_r and u_θ are the components of the displacement field; and Ψ is a known function defined as:

$$\Delta\Psi = 0, \qquad \frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial\theta}\right) = \Delta\varphi. \tag{10a,b}$$

Equations (8a,b,c) satisfy the equilibrium equations automatically, and the compatibility equation becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \varphi = 0. \tag{11}$$

The boundary conditions for the quarter plane can be defined as follows:

$$\sigma_r(r,0) = p_r(r)H(b-|r-b|), \qquad \tau_{r\theta}(r,0) = 0, \qquad 0 \le r < \infty, \tag{12a,b}$$

$$\sigma_r\left(r,\frac{\pi}{2}\right) = 0, \qquad \tau_{r\theta}\left(r,\frac{\pi}{2}\right) = 0, \qquad 0 \le r < \infty, \tag{12c,d}$$

where $p_r(r)$ is the contact pressure over the contact surface $0 \le r \le 2b$ and equals to $p(r) = -p(x)$. In addition, it is assumed that the stress field goes to zero at infinity,

$$\sigma_x(x,y) = 0, \qquad \tau_{zy}(x,y) = 0, \qquad x^2 + y^2 \to \infty, \tag{13a,b}$$

$$\sigma_r(r,\theta) = 0, \qquad \tau_{r\theta}(r,\theta) = 0, \qquad r^2 \to \infty. \tag{13c,d}$$

The global equilibrium of the FG layer can be expressed as:

$$\int_0^a q(x)\mathrm{d}x = \int_{c-b}^{c+b} p(x)\mathrm{d}x. \tag{14}$$

In order to ensure continuity of the vertical displacement and eliminate rigid-body motion through the contact surface, the displacement field is subjected to the following boundary condition:

$$\frac{\partial}{\partial x}\left[v_1(x,-h) - v_2(x,-h)\right] = 0, \qquad c-b \le x \le c+b, \tag{15}$$

where v_1 is the vertical displacement of the FG layer, whereas v_2 is the vertical displacement of quarter plane in Cartesian coordinates (x,y) given by:

$$v_2(x,-h) = u_\theta(r,0), \qquad c-b \le x \le c+b, \qquad 0 \le r \le 2b. \tag{16}$$

3 Solution of the contact problem

Using symmetry considerations and Fourier transforms, the displacement components for the FG layer may be written as

$$u(x, y) = \frac{2}{\pi}\int_0^\infty \phi(\xi, y)\sin(\xi x)\mathrm{d}\xi, \qquad v(x, y) = \frac{2}{\pi}\int_0^\infty \psi(\xi, y)\cos(\xi x)\mathrm{d}\xi, \tag{17a,b}$$

where $\phi(\xi, y)$and $\psi(\xi, y)$are the Fourier sine and Fourier cosine transforms of u and v with respect to the x-coordinate, respectively. Substituting Eqs. (17a,b) into the plane elasticity equations (5a,b), the following ordinary differential equations are obtained:

$$-(\kappa_1+1)\xi^2\phi + (\kappa_1-1)\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - 2\xi\frac{\mathrm{d}\psi}{\mathrm{d}y} + \beta(\kappa_1-1)\left[\frac{\mathrm{d}\phi}{\mathrm{d}y} - \xi\psi\right] = 0, \tag{18a}$$

$$-(\kappa_1-1)\xi^2\psi + (\kappa_1+1)\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2} + 2\xi\frac{\mathrm{d}\phi}{\mathrm{d}y} + \beta\left[(3-\kappa_1)\xi\phi + (\kappa_1+1)\frac{\mathrm{d}\psi}{\mathrm{d}y}\right] = 0, \tag{18b}$$

where

$$\phi = \sum_{j=1}^{4} A_j \exp\left(n_j y\right), \qquad \psi = \sum_{j=1}^{4} A_j m_j \exp\left(n_j y\right). \tag{19a,b}$$

The unknown functions A_j $(j = 1, 2, 3, 4)$ are determined from the boundary conditions, and $n_1, \ldots, n_4$ are the four complex roots of the characteristic equation associated with Eqs. (18a,b), which may be written as:

$$n_j^4 + 2\beta n_j^3 + (\beta^2 - 2\xi^2)n_j^2 - 2\xi^2\beta n_j + \xi^2\left(\xi^2 + \beta^2\frac{3-\kappa_1}{\kappa_1+1}\right) = 0. \tag{20}$$

The roots of Eq. (20) are obtained as

$$n_1 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right), \qquad n_2 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 - 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right), \tag{21a,b}$$

$$n_3 = -\frac{1}{2}\left(\beta + \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right), \qquad n_4 = -\frac{1}{2}\left(\beta - \sqrt{4\xi^2 + \beta^2 + 4\xi\beta i\sqrt{\frac{3-\kappa_1}{\kappa_1+1}}}\right). \tag{21c,d}$$

The known function m_j in Eq. (19b) may be expressed as follows:

$$m_j = \frac{\left(3\beta + 2n_j - \beta\kappa_1\right)\left[n_j\left(\beta + n_j\right)(\kappa_1+1) - \xi^2(\kappa_1+3)\right]}{\xi\left[4\xi^2 - \beta^2(\kappa_1-3)(\kappa_1+1)\right]}, \qquad (j = 1, 2, 3, 4). \tag{22}$$

Substituting Eqs. (17,19) into Eqs. (4), the stress field for the graded layer is obtained as

$$\sigma_y = \frac{2\mu_0\exp(\beta y)}{\pi(\kappa_1-1)}\int_0^\infty \sum_{j=1}^{4} A_j C_j \exp\left(n_j y\right)\cos(\xi x)\,\mathrm{d}\xi, \tag{23a}$$

$$\tau_{xy} = \frac{2\mu_0\exp(\beta y)}{\pi}\int_0^\infty \sum_{j=1}^{4} A_j D_j \exp\left(n_j y\right)\sin(\xi x)\,\mathrm{d}\xi, \tag{23b}$$

in which the known functions C_j and $D_j (j = 1, 2, 3, 4)$ are given by:

$$C_j = (3 - \kappa_1)\,\xi + (\kappa_1 + 1)\,m_j n_j, \qquad D_j = n_j - \xi m_j. \tag{24a,b}$$

Applying boundary conditions (6a-d) to the stress field (23a,b), the following linear algebraic system of equations is obtained:

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \\ C_1 \exp(-n_1 h) & C_2 \exp(-n_2 h) & C_3 \exp(-n_3 h) & C_4 \exp(n_4 h) \\ D_1 \exp(-n_1 h) & D_2 \exp(-n_2 h) & D_3 \exp(-n_3 h) & D_4 \exp(n_4 h) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} -q^t \\ 0 \\ -p^t \\ 0 \end{Bmatrix}, \tag{25}$$

where $\overline{q}$ and $\overline{p}$ are known functions defined as:

$$q^t = \frac{\kappa_1 - 1}{\mu_0} \int_0^{\infty} q\,(x) \cos(\xi x)\, \mathrm{d}\xi = \frac{\kappa_1 - 1}{\mu_0} \int_0^{a} q\left(t_q\right) \cos\left(\xi t_q\right) \mathrm{d}\xi, \tag{26a}$$

$$p^t = \frac{\kappa_1 - 1}{\mu_0 \exp(-\beta h)} \int_0^{\infty} p\,(x) \cos(\xi x)\, \mathrm{d}\xi = \frac{\kappa_1 - 1}{\mu_0 \exp(-\beta h)} \int_{c-b}^{c+b} p\,(t) \cos(\xi t)\, \mathrm{d}\xi. \tag{26b}$$

The unknown functions A_j $(j = 1, 2, 3, 4)$ may be obtained in terms of q^t and p^t solving Eq. (25) analytically and can be expressed as

$$A_j = A_{j1} p^t + A_{j2} q^t, \tag{26}$$

where A_{j1} and A_{j2} $(j = 1, \ldots, 4)$ are shown in Appendix 1. Equation (11) can be solved using the Mellin transform with respect to the r-coordinate:

$$\left(\frac{\partial^2}{\partial \theta^2} + (s+2)^2\right)\left(\frac{\partial^2}{\partial \theta^2} + s^2\right)\varphi^M = 0, \tag{27}$$

in which φ^M is the Mellin transform of the Airy stress function:

$$\varphi^M = \int_0^{\infty} \varphi r^{s-1} \mathrm{d}r. \tag{28}$$

The solution of the ordinary differential equation (27) may be defined as

$$\varphi^M = B_1 e^{is\theta} + B_2 e^{-is\theta} + B_3 e^{i(s+2)\theta} + B_4 e^{-i(s+2)\theta}, \tag{29}$$

where B_j $(j = 1, 2, 3, 4)$ are the unknown functions that will be determined from the boundary conditions (12). Using the Mellin transform after multiplying r^2, the stress field (8) and the displacement field (9) are obtained:

$$\left(r^2 \sigma_r\right)^M = \left(\frac{\partial^2}{\partial \theta^2} - s\right)\varphi^M, \tag{30a}$$

$$\left(r^2 \tau_{r\theta}\right)^M = (s+1)\frac{\partial}{\partial \theta}\varphi^M, \tag{30b}$$

$$2G\left(r^2 \frac{\partial u_\theta}{\partial r}\right)^M = is(s+1)\left(B_1 e^{is\theta} - B_2 e^{-is\theta}\right) + i\left[(s+2)(s+1) + (1-\upsilon)(-4s-4)\right]\left[B_3 e^{i(s+2)\theta} - B_4 e^{-i(s+2)\theta}\right]. \tag{31}$$

Applying the boundary conditions (12a-d) to the stress field (30), the following linear algebraic system of equations is obtained:

$$\begin{bmatrix} s(s+1) & s(s+1) & s(s+1) & s(s+1) \\ s & -s & s+2 & -(s+2) \\ \exp\left(\frac{\pi}{2}is\right) & \exp\left(-\frac{\pi}{2}is\right) & \exp\left(\frac{\pi}{2}i(s+2)\right) & \exp\left(-\frac{\pi}{2}i(s+2)\right) \\ s\exp\left(\frac{\pi}{2}is\right) & -s\exp\left(-\frac{\pi}{2}is\right) & (s+2)\exp\left(\frac{\pi}{2}i(s+2)\right) & -(s+2)\exp\left(-\frac{\pi}{2}i(s+2)\right) \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} = \begin{Bmatrix} p_{sr} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{32}$$

where $\overline{p}_r$ is a known function defined as:

$$p_{sr} = \int_0^\infty p_r(r) r^{s+1} \mathrm{d}r = \int_0^{2b} p_r(\tau)\tau^{s+1}\mathrm{d}\tau. \tag{33}$$

The unknown functions B_j $(j = 1, 2, 3, 4)$ may be obtained in terms of p_{sr} solving Eq. (32) analytically. To switch from polar coordinates (r, θ) to Cartesian coordinates (x, y) along the direction $\theta = 0$, the following conversions are considered:

$$r = x - (c - b), \qquad \tau = t - (c - b), \qquad u_\theta(r, 0) = v_2(x, -h), \qquad p_r(r) = -p(x). \tag{34a-d}$$

Applying the remaining boundary condition (16) and using Eqs. (25), (26), (32), (33) and (34) yields the following singular integral equation, in which the unknowns are the contact pressure $p(t)$ and the receding contact half-length b:

$$\int_{c-b}^{c+b} \left\{ \frac{\kappa_1+1}{8}\left(\frac{1}{t-x} - \frac{1}{t+x}\right) + N_1(t,x) \right.$$
$$\left. + \frac{\kappa_2+1}{8}\frac{\mu_0 \exp(-\beta h)}{\mu_2}\frac{1}{x-(c-b)}\left(\frac{1}{\alpha} + N_2(t,x) - \frac{\pi^2}{\pi^2-4}\right)\right\} p(t)\mathrm{d}t$$
$$= \exp(-\beta h)\int_0^a q(t_q)\cos\left(\xi t_q\right) N_3(x)\mathrm{d}t_q, \tag{35}$$

in which α, N_1, N_2 and N_3 are given in Appendix 2.

To complete the solution of the problem, the obtained contact pressures must satisfy the equilibrium condition of the graded layer given by Eq. (14).

4 Numerical solution of the singular integral equation

Using the following dimensionless quantities, the numerical solution of the problem can be simplified:

$$t = c + br, \qquad \mathrm{d}t = b\mathrm{d}r, \qquad x = c + bs, \qquad z = \xi h, \qquad \mathrm{d}z = h\mathrm{d}\xi, \tag{36a-e}$$

$$p(t) = \frac{Q\phi(r)}{h}, \qquad Q = \int_0^a q(t_p)\mathrm{d}t_p, \tag{37a,b}$$

where Q is the resultant force.

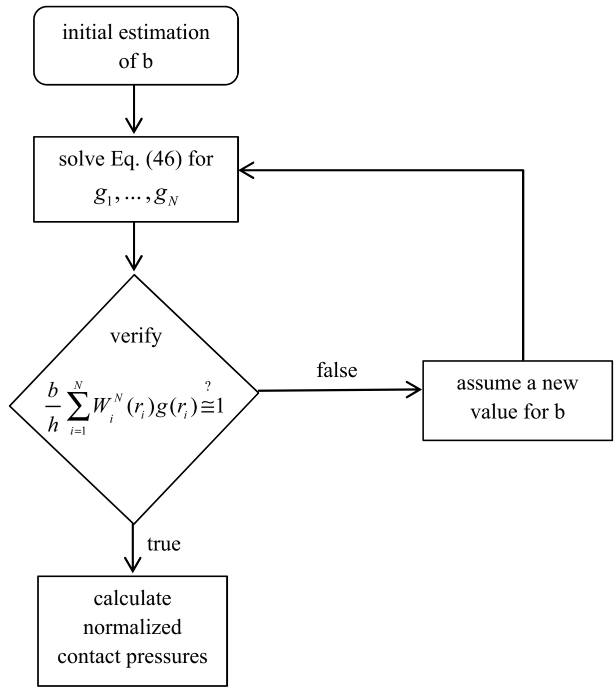

Fig. 2 Flowchart of the iterative algorithm

The singular equation (35) and the equilibrium condition (14) become:

$$\int_{-1}^{1}\left\{\frac{\kappa_1+1}{8}\left(\frac{1}{r-s}-\frac{1}{r+s+2\frac{c/h}{b/h}}\right)+\frac{b}{h}k_1(r,s)\right.$$
$$\left.+\frac{\kappa_2+1}{8}\frac{\mu_0\exp(-\beta h)}{\mu_2}\frac{1}{s+1}\left(\frac{1}{\alpha}+k_2(r,s)-\frac{\pi^2}{\pi^2-4}\right)\right\}\phi(r)\mathrm{d}r$$
$$=\frac{\exp(-\beta h)}{Q/h}\int_{0}^{a}q(t_q)\cos\left(\xi t_q\right)k_3(s)\mathrm{d}t_q, \tag{38}$$

$$k_1(r,s)=hN_1(r,s),\qquad k_2(r,s)=N_2(r,s),\qquad k_3(r,s)=N_3(r,s), \tag{39a-c}$$

$$\frac{b}{h}\int_{-1}^{1}\phi(r)\mathrm{d}r=1. \tag{40}$$

It is clear to notice that the contact surface has a smooth contact at the right end $(c+b)$ and a stress singularity at the edge of the quarter plane $(c-b)$ (i.e., $\phi(-1)\to\infty$, $\phi(1)=0$). As a result, the integral equation has a generalized Cauchy kernel, which will influence the singular behavior of the solution at $x=c-b$. Hence, the solution may be sought as described in [26]:

$$\phi(r)=w(r)g(r),\qquad w(r)=(1-r)^a(1+r)^b,\qquad -1\le r\le 1,\qquad (i=1,\ \ldots\ ,N), \tag{41}$$

where a is equal to 0.5 because of smooth contact and b can be calculated from the following expressions

$$\frac{\mu_2(1+\kappa_1)}{\mu_o\exp(-\beta h)(1+\kappa_2)}\left(2\lambda^2-1+\cos\pi\lambda\right)\cos\pi\lambda-\sin^2\pi\lambda=0,\qquad b=\lambda-1, \tag{42a,b}$$

where λ is the root of Eq. (42a) with the smallest positive real part. Using the appropriate Gauss–Jacobi integration formulas, the solution of Eqs. (38, 40) may be expressed as a system of algebraic equations:

$$\sum_{i=1}^{N}\left[\frac{\kappa_1+1}{8}\left(\frac{1}{r_i-s_k}-\frac{1}{r_i+s_k+2\frac{c/h}{b/h}}\right)+\frac{b}{h}k_1(r_i,s_k)\right.$$
$$\left.+\frac{\kappa_2+1}{8}\frac{\mu_0\exp(-\beta h)}{\mu_2}\frac{1}{s_k+1}\left(\frac{1}{\alpha}+k_2(r_i,s_k)-\frac{\pi^2}{\pi^2-4}\right)\right]W_i^N g(r_i)$$
$$=\frac{\exp(-\beta h)}{Q/h}\int_0^a q(t_q)\cos\left(\xi t_q\right)k_3(s_k)\mathrm{d}t_q,\qquad (k=1,\ \ldots\ ,N), \tag{43}$$

$$\frac{b}{h}\sum_{i=1}^{N}W_i^N(r_i)g(r_i)=1, \tag{44}$$

where r_i and s_k are the roots of the corresponding Jacobi polynomials and W_i^N are the weighting constants as shown in Appendix 3.

The system of algebraic equations (43) is consist of (N) equations with $(N+1)$ unknowns, namely $g_1,\ \ldots\ ,g_N$ and b. In order to solve for $(N+1)$ unknowns, in addition to the system of equations given by (43), the global equilibrium condition (44) is also used. When analyzed, it is observed that the system of equations given by (43) and (44) is nonlinear in terms of the variable b, and an iterative procedure given as a flow chart in Fig. 2 can be used to find the unknowns. As it is seen in Fig. 2, firstly an initial estimate of the variable b is assumed and the solution of Eq. (43) is performed using this value for the unknowns $g_1,\ \ldots\ ,g_N$. Equation (44) represents global equilibrium condition of the graded layer, and it should be verified using the calculated $g_1,\ \ldots\ ,g_N$ values. If the left-hand side of Eq. (44) is less than one, the variable b increases by a certain amount or vice versa. After the value of the left-hand side of Eq. (44) crosses one, it means that b should be between the last two assumed values. These last two assumptions can be taken as initial upper and lower limits, namely b_u and b_l, respectively, and the correct value of b can be found using a half-step approach. Each time a new value for b is taken in the middle of the b_u and b_l. If the assumed variable b satisfies Eq. (44) with in an acceptable error, correct values of contact pressures and on the half contact length are obtained. Otherwise, b is assigned as b_l if the value of the left-hand side of Eq. (44) is less than one, else b is assigned as b_u for the next iteration and the procedure continues.

5 Numerical results

The geometry and loading of the problem are given in Fig. 1. The load applied on the FG layer, i.e., $q(x)$, is uniformly distributed load with $a/h=0.01, 0.5, 1.0, 2.0$ such that the resultant force, Q is always equal to 1. The load can be considered as concentrated force for $a/h=0.01$. The shear modulus of the graded layer at $y=-h$, μ_h, is defined as:

$$\mu_h=\mu_0\exp(-\beta h). \tag{45}$$

As can be seen from Eq. (45), the top of the layer becomes stiffer if the non-homogeneity parameter increases or vice versa. Note that all quantities are normalized. The height of the graded layer h is taken as 1, whereas the Poison's ratios of the graded layer and the quarter plane ν_1 and υ_2, respectively, are taken as 0.25. In addition, for Table 3 and Figs. 3, 4, 5, 6 and 7, iterations are continued until the resultant force tolerance, i.e., the absolute value of $(Q-1)$, is less than 10^{-5} for $N=20$ and the shear modulus of the FG layer at $y=-h$, μ_h, is taken equal to the shear modulus of quarter plane μ_2, $\mu_h=\mu_2$.

Table 1 shows the comparison of the half contact length b/h for a homogeneous layer, i.e., $\beta=0.001$, between the values reported in the literature and obtained in this study for various μ_h/μ_2. It can be seen that the b/h values of this study are approximately the same as given by Aksogan et al. [10] and close to the values reported by Erdogan and Ratwani [7].

The comparison of the half contact length b/h and the resultant force, i.e., Q, each step of the iterative procedure described at the end of Sect. 4 for various values of the non-homogeneity parameter β is given in Table 2. As can be seen from the table, firstly b/h is increased until Q becomes greater than one. Then, a half-step approach is applied in order to find the solution of the problem. Iterations are continued until the

Table 1 Comparison of the half contact length between the values reported in the literature and obtained in this study for various μ_h/μ_2 ($f/h = 1.0,\ a/h = 1.0$)

μ_h/μ_2	1/99	1/7	3/7
Erdogan and Ratwani [7]	0.108	0.123	0.156
Aksogan et al. [10]	0.111	0.140	0.192
This study	0.113	0.142	0.196

Table 2 Variation of resultant force, i.e., Q, at each iteration for various values of the non-homogeneity parameter β ($f/h = 1.0,\ a/h = 1.0, \mu_h/\mu_2 = 1$)

$\beta = -1.0$			$\beta = 0.001$			$\beta = 1.0$		
Iter. no.	b/h	Q	Iter. no.	b/h	Q	Iter. no.	b/h	Q
1	0.100000	0.991071	1	0.100000	0.982432	1	0.100000	0.969057
2	0.200000	0.999510	2	0.200000	0.996077	2	0.200000	0.990529
3	0.300000	1.002177	3	0.300000	1.000501	3	0.300000	0.997527
4	0.250000	1.001150	4	0.250000	0.998765	4	0.400000	1.000808
5	0.200000	0.999510	…	…………	………	5	0.350000	0.999437
6	0.225000	1.000430	8	0.284375	1.000032	6	0.375000	1.000178
7	0.212500	0.999998	9	0.282813	0.999981	7	0.362500	0.999822
			10	0.283594	1.000007	8	0.368750	1.000004

absolute value of $(Q - 1)$ is less than 10^{-5}. As given in Table 2, at most 10 iterations are required to solve the problem with an acceptable error.

The variation of the half contact length b/h for various values of the non-homogeneity parameter β and load amplitude a/h is given in Table 3. It is seen from the table that for a fixed value of β, b/h increases for increasing a/h and the smallest contact length is obtained in case of concentrated load ($a/h = 0.01$). Also, the half contact length becomes larger for increasing values of β.

Figures 3, 4 and 5 illustrate the effect of the non-homogeneity parameter β on the normalized contact pressure $p(x)/(Q/h)$ for various load cases ($a/h = 0.01, 1.0, 2.0$). It is seen from these figures that for a fixed value of a/h, decreasing β results in the reduction of the contact zone in addition to a decrease the change of pressure approaching infinity at the edge of the quarter plane.

The effect of the load amplitude a/h on the normalized contact pressure $p(x)/(Q/h)$ for the homogeneous layer, $\beta = 0.001$, is given in Fig. 6. It is seen that b/h increases while the change of pressure approaching infinity decreases at the edge of the quarter plane for increasing values of a/h.

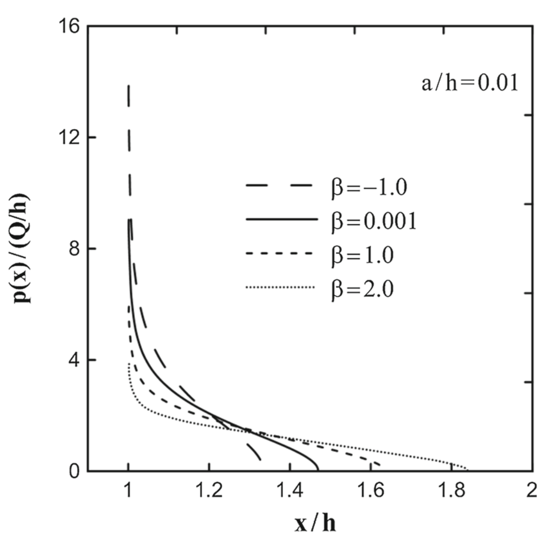

Fig. 3 Effect of the non-homogeneity parameter βon the normalized contact pressure $p(x)/(Q/h)$ for the concentrated load case, $a/h = 0.01$ ($f/h = 1.0$, $\mu_h/\mu_2 = 1$)

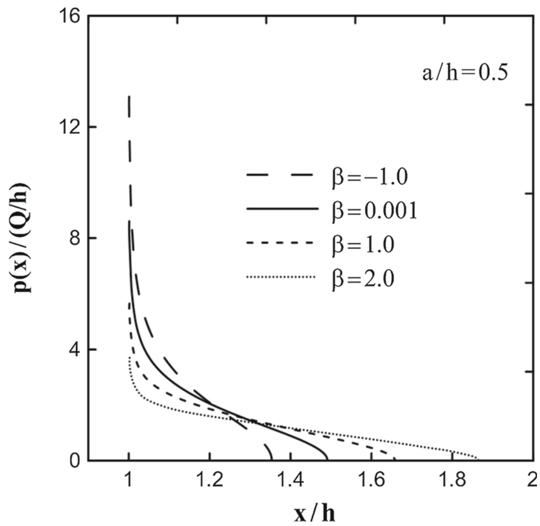

Fig. 4 Effect of the non-homogeneity parameter βon the normalized contact pressure $p(x)/(Q/h)$ for the uniformly distributed load case, $a/h = 0.5$ ($f/h = 1.0$, $\mu_h/\mu_2 = 1$)

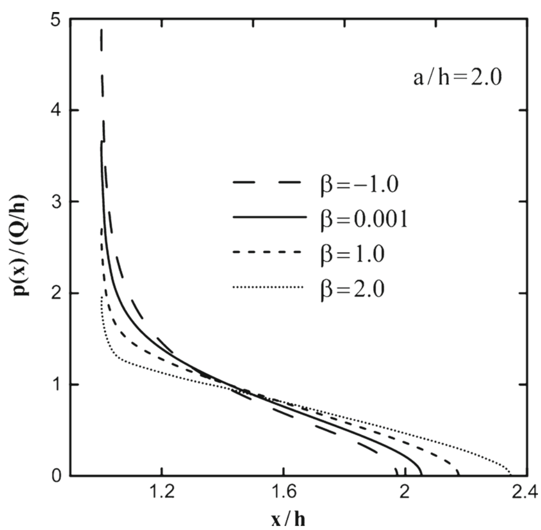

Fig. 5 Effect of the non-homogeneity parameter β on the normalized contact pressure $p(x)/(Q/h)$ for the uniformly distributed load case, $a/h = 2.0$ ($f/h = 1.0$, $\mu_h/\mu_2 = 1$)

Table 3 Variation of the half contact length b/h for the various values of the non-homogeneity parameter β and load amplitude a/h ($f/h = 1.0$, $\mu_h/\mu_2 = 1$)

Parameters	$\beta = -2.0$	$\beta = -1.0$	$\beta = 0.001$	$\beta = 1.0$	$\beta = 2.0$
$a/h = 0.01$	0.1135	0.1669	0.2350	0.3178	0.4204
$a/h = 0.5$	0.1212	0.1768	0.2457	0.3295	0.4327
$a/h = 1.0$	0.1541	0.2125	0.2835	0.3688	0.4725
$a/h = 2.0$	0.4715	0.4871	0.5267	0.5879	0.6731

Figure 7 shows the effect of the normalized half distance between the quarter planes f/h on the normalized contact pressure $p(x)/(Q/h)$. From the figure, it can be concluded that there is a reduction in the half contact length b/h if f/h increases. On the other hand, the values of $p(x)/(Q/h)$ increases for increasing values of f/h.

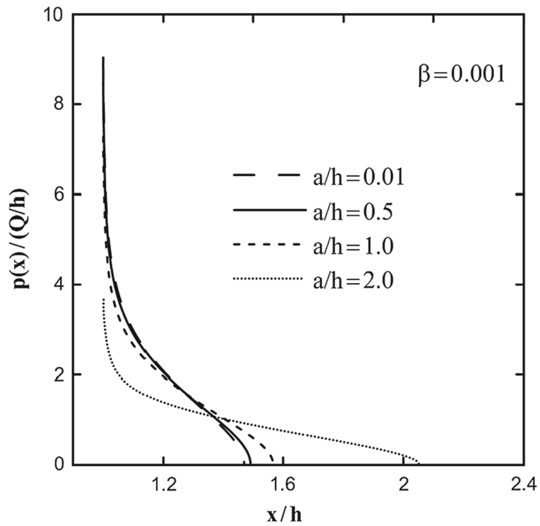

Fig. 6 Effect of the load amplitude a/h on the normalized contact pressure $p(x)/(Q/h)$ for the homogeneous layer, $\beta = 0.001$ $(f/h = 1.0, \mu_h/\mu_2 = 1)$

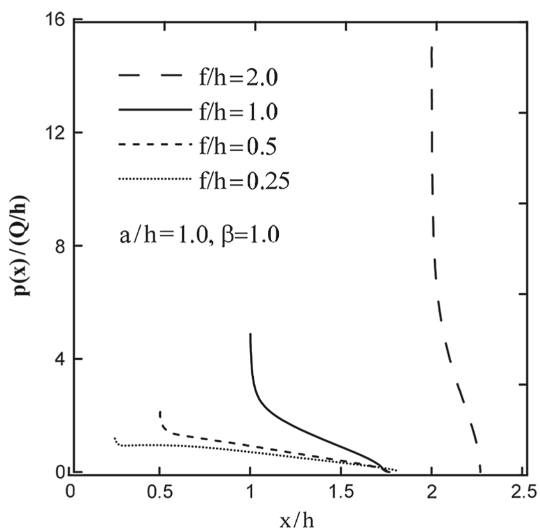

Fig. 7 Effect of the half distance between the quarter planes f/h on the normalized contact pressure $p(x)/(Q/h)$ $(f/h = 1.0,$ $\mu_h/\mu_2 = 1)$

6 Conclusions

In this paper, a frictionless receding contact of a functionally graded layer resting on homogeneous quarter planes was considered. The layer was subjected to a load distributed over a finite region of its top surface. Using Fourier cosine and Fourier sine transforms for the graded layer and Mellin transform for the quarter plane, the problem was converted into the solution of a Cauchy-type singular integral equation in which the contact pressure and the receding contact half-length are the unknowns. The singular integral equation was solved numerically using the Gauss–Jacobi integration formulation. An iterative procedure was employed to obtain the correct receding contact half-length that satisfies the global equilibrium condition. The effect of the non-homogeneity parameter β, loading and distance between the two quarter planes on the contact pressure and on the half contact length was investigated for different loading cases using a parametric study.

The obtained results show that the contact zone between the graded layer and the quarter plane becomes smaller if β increases. However, decreasing β results in a larger contact zone.

Appendix 1

$$A_{11} = -\frac{1}{\Delta A}\Big[e^{-n_2 h}\left(C_3D_2D_4 - C_4D_2D_3\right) + e^{-n_3 h}\left(-C_2D_3D_4 + C_4D_2D_3\right) + e^{-n_4 h}\left(C_2D_3D_4 - C_3D_2D_4\right)\Big]$$

$$A_{12} = -\frac{1}{\Delta A}\Big[e^{-(n_2+n_3)h}\left(C_2D_3D_4 - C_3D_2D_4\right) + e^{-(n_2+n_4)h}\left(-C_2D_3D_4 + C_4D_2D_3\right) + e^{-(n_3+n_4)h}\left(C_3D_2D_4 - C_4D_2D_3\right)\Big]$$

$$A_{21} = \frac{1}{\Delta A}\Big[e^{-n_1 h}\left(C_3D_1D_4 - C_4D_1D_3\right) + e^{-n_3 h}\left(-C_1D_3D_4 + C_4D_1D_3\right) + e^{-n_4 h}\left(C_1D_3D_4 - C_3D_1D_4\right)\Big]$$

$$A_{22} = \frac{1}{\Delta A}\Big[e^{-(n_1+n_3)h}\left(C_1D_3D_4 - C_3D_1D_4\right) + e^{-(n_1+n_4)h}\left(-C_1D_3D_4 + C_4D_1D_3\right) + e^{-(n_3+n_4)h}\left(C_3D_1D_4 - C_4D_1D_3\right)\Big]$$

$$A_{31} = -\frac{1}{\Delta A}\Big[e^{-n_1 h}\left(C_2D_1D_4 - C_4D_1D_2\right) + e^{-n_2 h}\left(-C_1D_2D_4 + C_4D_1D_2\right) + e^{-n_4 h}\left(C_1D_2D_4 - C_2D_1D_4\right)\Big]$$

$$A_{32} = -\frac{1}{\Delta A}\Big[e^{-(n_1+n_2)h}\left(C_1D_2D_4 - C_2D_1D_4\right) + e^{-(n_1+n_4)h}\left(-C_1D_2D_4 + C_4D_1D_2\right) + e^{-(n_2+n_4)h}\left(C_2D_1D_4 - C_4D_1D_2\right)\Big]$$

$$A_{41} = \frac{1}{\Delta A}\Big[e^{-n_1 h}\left(C_2D_1D_3 - C_3D_1D_2\right) + e^{-n_2 h}\left(-C_1D_2D_3 + C_3D_1D_2\right) + e^{-n_3 h}\left(C_1D_2D_3 - C_2D_1D_3\right)\Big]$$

$$A_{42} = \frac{1}{\Delta A}\Big[e^{-(n_1+n_2)h}\left(C_1D_2D_3 - C_2D_1D_3\right) + e^{-(n_1+n_3)h}\left(-C_1D_2D_3 + C_3D_1D_2\right) + e^{-(n_2+n_3)h}\left(C_2D_1D_3 - C_3D_1D_2\right)\Big]$$

$$\begin{aligned}\Delta A = {} & e^{-(n_1+n_2)h}\left(C_1C_3D_2D_4 - C_1C_4D_2D_3 - C_2C_3D_1D_4 + C_2C_4D_1D_3\right)\\ & + e^{-(n_1+n_3)h}\left(-C_1C_2D_3D_4 + C_1C_4D_2D_3 + C_2C_3D_1D_4 - C_3C_4D_1D_2\right)\\ & + e^{-(n_1+n_4)h}\left(C_1C_2D_3D_4 - C_1C_3D_2D_4 - C_2C_4D_1D_3 + C_3C_4D_1D_2\right)\\ & + e^{-(n_2+n_3)h}\left(C_1C_2D_3D_4 - C_1C_3D_2D_4 - C_2C_4D_1D_3 + C_3C_4D_1D_2\right)\\ & + e^{-(n_2+n_4)h}\left(-C_1C_2D_3D_4 + C_1C_4D_2D_3 + C_2C_3D_1D_4 - C_3C_4D_1D_2\right)\\ & + e^{-(n_3+n_4)h}\left(C_1C_3D_2D_4 - C_1C_4D_2D_3 - C_2C_3D_1D_4 + C_2C_4D_1D_3\right)\end{aligned}$$

Appendix 2

$$\alpha = \log\left(\frac{t-(c-b)}{x-(c-b)}\right)$$

$$N_1(t,x) = \int_0^\infty \left[-\xi(\kappa_1-1)\sum_{j=1}^{4}\left(A_{j1}m_j\exp(-n_jh)\right) - S\right]\sin(\xi x)\cos(\xi t)\,\mathrm{d}\xi$$

$$N_2(t,x) = \int_0^\infty \left[\frac{\sinh(\pi\lambda)}{-2\lambda^2-1+\cosh(\pi\lambda)}\right]\sin(\alpha\lambda)\,\mathrm{d}\lambda$$

$$N_3(x,t) = \int_0^\infty \xi\,(\kappa_1-1)\left[\sum_{j=1}^{4}A_{j2}m_j\exp\left(-n_jh\right)\right]\sin(\xi x)\,\mathrm{d}\xi$$

$$S = \lim_{\xi\to\infty}\left[-\xi(\kappa_1-1)\sum_{j=1}^{4}\left(A_{j1}m_j\exp(-n_jh)\right)\right] = -\frac{\kappa_1+1}{4}$$

Appendix 3

$$P_N^{(a,\,b)}(r_i) = 0, \qquad P_N^{(a-1,\,b)}(s_k) = 0, \qquad (i = 1,\ \ldots\ , N), \qquad (k = 1,\ \ldots\ , N)$$

$$W_i^N = -\frac{2N+2+a+b}{(N+1)!(N+1+a+b)} \frac{\Gamma(N+1+a)\Gamma(N+1+b)}{\Gamma(N+1+a+b)} x \frac{2^{a+b}}{P_N^{(a,b)}(r_i) P_{N+1}^{(a,b)}(r_i)}$$

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