Acta Mech 227, 1493–1514 (2016)
DOI 10.1007/s00707-016-1575-8

ORIGINAL PAPER

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A general Fourier formulation for vibration analysis of functionally graded sandwich beams with arbitrary boundary condition and resting on elastic foundations

Received: 25 August 2015 / Revised: 9 November 2015 / Published online: 17 February 2016

Abstract Free vibration analysis of functionally graded sandwich beams with general boundary conditions and resting on a Pasternak elastic foundation is presented by using strong form formulation based on modified Fourier series. Two types of common sandwich beams, namely beams with functionally graded face sheets and isotropic core and beams with isotropic face sheets and functionally graded core, are considered. The bilayered and single-layered functionally graded beams are obtained as special cases of sandwich beams. The effective material properties of functionally graded materials are assumed to vary continuously in the thickness direction according to power-law distributions in terms of volume fraction of constituents and are estimated by Voigt model and Mori–Tanaka scheme. Based on the first-order shear deformation theory, the governing equations and boundary conditions can be obtained by Hamilton's principle and can be solved using the modified Fourier series method which consists of the standard Fourier cosine series and several supplemented functions. A variety of numerical examples are presented to demonstrate the convergence, reliability and accuracy of the present method. Numerous new vibration results for functionally graded sandwich beams with general boundary conditions and resting on elastic foundations are given. The influence of the power-law indices and foundation parameters on the frequencies of the sandwich beams is also investigated.

1 Introduction

Sandwich structures are widely used in a variety of engineering applications including transportation, constructions, aerospace and marine engineering due to their high specific stiffness and strength for a low-weight consideration. Availability of a wide selection of face sheet and core materials makes it possible to obtain multifunctional benefits. However, this composition has several disadvantages that restrict the structure's usage and reduce its reliability due to the abrupt changes in materials. In order to improve the sandwich structure performance, a new class of composite materials called functionally graded materials (FGMs) is utilized for the face sheets or core in the sandwich structures. Since the FGMs possess continuous and smooth spatial variations of material properties in the desired direction, the functionally graded (FG) sandwich structures have more extensive potential applications.

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Beams are one of the most fundamental structural elements, and the analysis of beam structures is of high interest. Compared with isotropic beams [1–6], FG beams [7–19] and sandwich beams [20–26], the literature on FG sandwich beams is limited. Among those available, Amirani et al. [27] analyzed the free vibration behavior of a sandwich beam with FG core using the element free Galerkin method in conjunction with penalty method used for satisfaction of essential boundary condition and continuity of the beam. Apetre et al. [28] described the behavior of the sandwich beams with FG core using four available models. Rahmani et al. [29] investigated the free vibration of sandwich beams with syntactic foam as a functionally graded flexible core using a new model based on high-order sandwich panel theory. The formulation uses the classical beam theory for the face sheets and an elasticity theory for the FG core. A new beam finite element is developed by Chakraborty et al. [30] to study the thermoelastic behavior of FG beams and FG sandwich beams based on the first-order shear deformation theory. Pradhan and Murmu [31] presented thermomechanical vibration analysis of FG beams and FG sandwich beams resting on elastic foundations in which the modified differential quadrature method (MDQM) is used to solve the governing differential equations and modified weighting coefficient matrix (MWCM) is used to implement the applied kinematic boundary conditions. Chakraborty et al. [32] analyzed the wave propagation behavior in functionally graded beam structures subjected to high-frequency impulse loading using a new spectrally formulated element method based on the first-order shear deformation theory. The investigation of the bending response of a simply supported functionally graded (FG) viscoelastic sandwich beam resting on Pasternak's elastic foundation is presented by Zenkour et al. [33] on the basis of a refined sinusoidal shear deformation beam theory. Vo et al. [34] presented vibration and buckling analysis of functionally graded sandwich beams based on a refined shear deformation theory and finite element model. The core of sandwich beam is fully metallic or ceramic, and skins are composed of a functionally graded material. Bui et al. [35] proposed a novel truly mesh-free method to analyze the transient responses and free vibration of sandwich beams with functionally graded (FG) core, in which the penalty technique is adopted to treat the material discontinuities at the interface between the core and the two face sheets.

The above review indicates that there exists some literature on the free vibration of a functionally graded sandwich beam, but the numerical solutions for free vibration of an FG sandwich beam resting on elastic foundations seem to be limited. Moreover, the aforementioned work on FG sandwich beams focuses on classical boundary conditions, and most of the existing methods require modifications of the solution procedures and corresponding computation codes to adapt to different boundary cases. Therefore, the establishment of a unified, efficient and accurate formulation for free vibration of functionally graded sandwich beams with general boundary conditions and resting on elastic foundations is necessary and significant.

This paper presents an accurate solution for the free vibration of functionally graded sandwich beams with general boundary conditions and resting on a Pasternak elastic foundation. Two types of common sandwich beams, namely beams with functionally graded face sheets and isotropic core and beams with isotropic face sheets and functionally graded core are considered. The bilayered and single-layered functionally graded beams are obtained as special cases of sandwich beams. The effective material properties of functionally graded materials are assumed to vary continuously in the thickness direction according to power-law distributions in terms of the volume fraction of the constituents and are estimated by Voigt model and Mori–Tanaka scheme. Based on the first-order shear deformation theory, the governing equations and boundary conditions can be obtained by Hamilton's principle and can be solved using the modified Fourier series method which consists of the standard Fourier cosine series and several supplemented functions. A variety of numerical examples are presented to demonstrate the convergence, reliability and accuracy of the present method. Numerous new vibration results for functionally graded sandwich beams with general boundary conditions and resting on elastic foundations are given. The influence of the power-law indices and foundation parameters on the frequencies of the sandwich beams is also investigated.

2 Theoretical formulation

2.1 Description of the model

Consider an FG sandwich beam with length L, width b and thickness h, as depicted in Fig. 1a. The coordinate system composed of x and z is used to describe the dimensions of the sandwich beam. The middle surface of the sandwich beam is defined by $z = 0$. The vertical positions of the bottom and top, and of the two interfaces between the layers are denoted by $z_0 = -h/2$, z_1, z_2, $z_3 = h/2$. The beam is resting on an elastic foundation with foundation moduli of k_s and k_w. The u_0 and w_0 are the displacements of the beam in x and z direction, respectively.

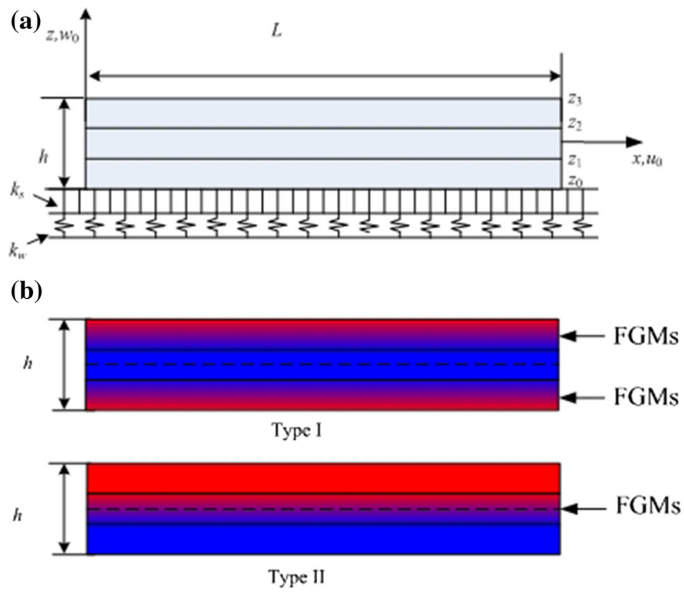

Fig. 1 Schematic diagram of an FG sandwich beam resting on an elastic foundation: **a** geometry and coordinates; **b** materials profile

It is assumed that the FGM layers of the sandwich beam are made of a mixture of two material constituents M_1 and M_2. Both Voigt model and Mori–Tanaka scheme are employed to evaluate the effective material properties. The Voigt model assumes that the material properties including Young's modulus E_f, Poisson's ratio μ_f and mass density ρ_f are proportional to the volume fraction according to

$$E_f^n = (E_1^n - E_2^n)V_1^n + E_2^n, \quad \mu_f^n = (\mu_1^n - \mu_2^n)V_1^n + \mu_2^n, \quad \rho_f^n = (\rho_1^n - \rho_2^n)V_1^n + \rho_2^n \tag{1. 1–3}$$

where V_1^n is the volume fraction of the M_1 of the nth layer.

According to the Mori–Tanaka scheme, the effective local bulk modulus K_f and the shear modulus G_f of the nth FGM layer of the sandwich beams can be given by

$$\frac{K_f^n - K_2^n}{K_1^n - K_2^n} = \frac{V_1^n}{1 + (1 - V_1^n)(K_1^n - K_2^n)/(K_2^n + 4G_2^n/3)}, \tag{2}$$

$$\frac{G_f^n - G_2^n}{G_1^n - G_2^n} = \frac{V_1^n}{1 + (1 - V_1^n)(G_1^n - G_2^n)/\left\{G_2^n + G_2^n(9K_2^n + 8G_2^n)/\left[6(K_2^n + 2G_2^n)\right]\right\}} \tag{3}$$

where $K_i^n = E_i^n/[3(1 - 2\mu_i^n)]$, $G_i^n = E_i^n[2(1 + \mu_i^n)]$ $(i = 1, 2)$. The effective mass density defined by Eq. (1.2) is also used in the Mori–Tanaka scheme. The effective Young's modulus E_f and Poisson's ratio μ_f can be expressed as

$$E_f^n = \frac{9K_f^n G_f^n}{3K_f^n + G_f^n}, \quad \mu_f^n = \frac{3K_f^n - 2G_f^n}{6K_f^n + 2G_f^n}. \tag{4}$$

Two common types of FG sandwich beams are considered in this study, as shown in Fig. 1b. The volume fraction V_1 of the FG sandwich beam is defined as:

$$\text{Type I} \begin{cases} V_1^1 = \left(\frac{z - z_0}{z_1 - z_0}\right)^{k_1} & z \in [z_0, z_1] \\ V_1^2 = 1 & z \in [z_1, z_2] \\ V_1^3 = \left(\frac{z - z_3}{z_2 - z_3}\right)^{k_2} & z \in [z_2, z_3] \end{cases}$$

$$\text{Type II} \begin{cases} V_1^1 = 0 & z \in [z_0, z_1] \\ V_1^2 = \left(\frac{z - z_1}{z_2 - z_1}\right)^{k} & z \in [z_1, z_2] \\ V_1^3 = 1 & z \in [z_2, z_3] \end{cases}$$

where the k, k_1, and k_2 are the power-law indices which can determine the material profile in FGMs. A type I sandwich beam is the beam with FG face sheets and isotropic core, whereas Type II sandwich beam is the beam with isotropic face sheets and FG core. The ratio of thickness of each layer from bottom to top is denoted by the combination of three numbers, for example, "1–2–1" denotes that $h_1{:}h_2{:}h_3 = 1{:}2{:}1$ ($h_i = z_i - z_{i-1}$).

2.2 Energy functional

Within the context of first-order shear deformation beam theory, the displacement field can be expressed as

$$u_0 = u(x,t) + z\phi(x,t), \quad w_0 = w(x,t) \tag{5}$$

where the u and w are the displacement components of the mid-surface in x and z directions, respectively. ϕ is the rotation of cross section. t is the time

The linear strain–displacement relations in the beam space are expressed as

$$\varepsilon_x = \varepsilon_x^0 + z\chi_x, \gamma_{xz} = \phi + \frac{\partial w}{\partial x} \tag{6}$$

where ε_x^0 is the membrane strain of the mid-surface defined as $\varepsilon_x^0 = \partial u/\partial x$; χ_x is the curvature change of the beam and defined as $\chi_x = \partial\phi/\partial x$. γ_{xz} is the transverse shear strain.

The constitutive relation in the nth layer of the FG sandwich beam can be given as

$$\begin{Bmatrix} \sigma_x \\ \sigma_{xz} \end{Bmatrix}_n = \begin{bmatrix} Q_{11}^n & 0 \\ 0 & Q_{55}^n \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix}_n \tag{7}$$

where σ_x and σ_{xz} are the normal and shear stress, respectively. The elastic stiffness coefficients Q_{11}^n and Q_{55}^n are defined as

$$Q_{11}^n = E_f^n, \quad Q_{55}^n = \frac{E_f^n}{2(1+\mu_f^n)}. \tag{8}$$

The force and moment resultants are obtained by integrating the stresses over the beam thickness and can be written as

$$\{N_x, M_x, Q_{xz}\} = \int_{-h/2}^{h/2} \{\sigma_x, z\sigma_x, \kappa\sigma_{xz}\}\mathrm{d}z \tag{9}$$

where N_x, M_x and Q_{xz} are the in-plane force resultant, bending moment resultant and transverse shear force resultant. The shear correction factor κ is taken as 5/6 in this study. Substituting Eqs. (5–8) into Eq. (9), the force and moment resultants can be rewritten as

$$\begin{Bmatrix} N_x \\ M_x \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \chi_x \\ \gamma_{xz} \end{Bmatrix} \tag{10}$$

where the coefficients A_{11}, B_{11}, D_{11} and A_{55} can be expressed as

$$\{A_{11}, B_{11}, D_{11}\} = \sum_{n=1}^{3} \int_{z_{n-1}}^{z_n} Q_{11}^n \left\{1, z, z^2\right\} dz, \quad A_{55} = \kappa \sum_{n=1}^{3} \int_{z_{n-1}}^{z_n} Q_{55}^n dz. \tag{11}$$

The strain energy (U) of the FG sandwich beams can be expressed in integral form as

$$U = \frac{b}{2} \int_0^L \left\{N_x\varepsilon_x^0 + M_x\chi_x + Q_{xz}\gamma_{xz}\right\} \mathrm{d}x. \tag{12}$$

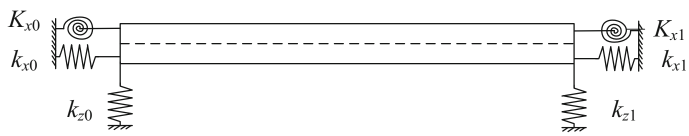

Fig. 2 General boundary conditions of a beam

Substituting Eqs. (5–6) and (10–11) into Eq. (12), the strain energy (U) can be rewritten as

$$U = \frac{b}{2}\int_0^L \left\{ A_{11}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2B_{11}\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x} + D_{11}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x} + A_{55}\phi\phi + A_{55}\frac{\partial w}{\partial x}\frac{\partial w}{\partial x} + 2A_{55}\phi\frac{\partial w}{\partial x} \right\} \mathrm{d}x. \tag{13}$$

The kinetic energy (T) of the beam can be given as

$$T = \frac{b}{2}\int_0^L \int_{-h/2}^{h/2} \rho_f \left[\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w_0}{\partial t}\right)^2 \right] \mathrm{d}z\mathrm{d}x. \tag{14}$$

Substituting Eq. (5) into Eq. (14), the kinetic energy can be expressed as

$$T = \frac{b}{2}\int_0^L \left[I_0\left(\frac{\partial u}{\partial t}\right)^2 + 2I_1\frac{\partial u}{\partial t}\frac{\partial \phi}{\partial t} + I_2\left(\frac{\partial \phi}{\partial t}\right)^2 + I_0\left(\frac{\partial w}{\partial t}\right)^2 \right] \mathrm{d}x \tag{15}$$

where the inertia terms I_0, I_1 and I_2 are defined as

$$\{I_0, I_1, I_2\} = \sum_{n=1}^{3} \int_{z_{n-1}}^{z_n} \rho_f^n \left\{1, z, z^2\right\} \mathrm{d}z. \tag{16}$$

In this paper, the boundary conditions are represented as two sets of independent linear springs (k_x and k_z) and one set of rotational spring (K_x) arranged at both ends of the beam. k_x and k_z are used to restrain the displacements at the ends in x and z direction, respectively. The rotation of the cross section is restrained by K_x. The general boundary conditions can be easily obtained by setting proper springs' stiffness, as shown in Fig. 2. k_{x0}, k_{z0}, K_{x0} and k_{x1}, k_{z1}, K_{x1} are used to indicate the rigidities (per unit length) of the boundary springs at the boundary $x = 0$ and L, respectively. The potential energy (U_{sp}) stored in the boundary springs can be obtained as

$$U_{sp} = \frac{b}{2}\left[\left(k_{x0}u^2 + k_{z0}w^2 + K_{x0}\phi^2\right)\big|_{x=0} + \left(k_{x1}u^2 + k_{z1}w^2 + K_{x1}\phi^2\right)\big|_{x=L} \right]. \tag{17}$$

Since the FG sandwich beams are resting on an elastic foundation, the potential energy (U_{ef}) stored in the elastic foundation needs to be considered and is given as

$$U_{\mathrm{ef}} = \frac{b}{2}\int_0^L \left[k_w (w)^2 + k_s\left(\frac{\partial w}{\partial x}\right)^2 \right] \mathrm{d}x \tag{18}$$

where k_w is the Winkler foundation stiffness, while k_s is the shear stiffness of the elastic foundation.

The total energy functional of the FG sandwich beam resting on an elastic foundation can be given as

$$L = T - U - U_{sp} - U_{\mathrm{ef}}. \tag{19}$$

2.3 Equations of motion

The equations of motion can be obtained by taking a differential element of the beam and requiring the sum of the external and internal body force and moments each to be zero, and they also can be derived by using Hamilton's principle. In this work, Hamilton's principle is employed, and we can obtain:

$$\delta \int_0^t \left(T - U - U_{sp} - U_{\mathrm{ef}}\right)\mathrm{d}t = 0. \tag{20}$$

Using Eqs. (13), (15), (17) and (18), Eq. (20) can be rewritten as

$$\begin{aligned}
&\int_0^t \int_0^L \left(I_0 \frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + I_1 \frac{\partial u}{\partial t}\frac{\partial \delta \phi}{\partial t} + I_1 \frac{\partial \phi}{\partial t}\frac{\partial \delta u}{\partial t} + I_2 \frac{\partial \phi}{\partial t}\frac{\partial \delta \phi}{\partial t} + I_0 \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right)\mathrm{d}x\mathrm{d}t \\
&= \int_0^t \int_0^L \left(\begin{array}{l} A_{11}\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x} + B_{11}\frac{\partial u}{\partial x}\frac{\partial \delta \phi}{\partial x} + B_{11}\frac{\partial \phi}{\partial x}\frac{\partial \delta u}{\partial x} + D_{11}\frac{\partial \phi}{\partial x}\frac{\partial \delta \phi}{\partial x} + \\ A_{55}\phi\delta\phi + A_{55}\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + A_{55}\phi\frac{\partial \delta w}{\partial x} + A_{55}\frac{\partial w}{\partial x}\delta\phi \end{array}\right)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^t \left[(k_{x0}u\delta u + k_{z0}w\delta w + K_{x0}\phi\delta\phi)|_{x=0} + (k_{x1}u\delta u + k_{z1}w\delta w + K_{x1}\phi\delta\phi)|_{x=L}\right]\mathrm{d}t \\
&+ \int_0^t \int_0^L \left(k_w w\delta w + k_s \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{21}$$

Integrating by parts to relieve the virtual displacements δu, δw and $\delta\phi$, Eq. (21) can be expressed as

$$\begin{aligned}
0 &= \int_0^t \int_0^L \left(A_{11}\frac{\partial^2 u}{\partial x^2} + B_{11}\frac{\partial^2 \phi}{\partial x^2} - I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 \phi}{\partial t^2}\right)\delta u \mathrm{d}x\mathrm{d}t \\
&+ \int_0^t \int_0^L \left(B_{11}\frac{\partial^2 u}{\partial x^2} + D_{11}\frac{\partial^2 \phi}{\partial x^2} - A_{55}\phi - A_{55}\frac{\partial w}{\partial x} - I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^2 \phi}{\partial t^2}\right)\delta\phi \mathrm{d}x\mathrm{d}t \\
&+ \int_0^t \int_0^L \left(A_{55}\frac{\partial^2 w}{\partial x^2} + A_{55}\frac{\partial \phi}{\partial x} - k_w w + k_s \frac{\partial^2 w}{\partial x^2} - I_0 \frac{\partial^2 w}{\partial t^2}\right)\delta w \mathrm{d}x\mathrm{d}t \\
&+ \int_0^t \left\{\begin{array}{l} \left(A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial \phi}{\partial x} - k_{x0}u\right)\delta u\Big|_{x=0} + \left(B_{11}\frac{\partial u}{\partial x} + D_{11}\frac{\partial \phi}{\partial x} - K_{x0}\phi\right)\delta\phi\Big|_{x=0} + \\ \left(A_{55}\frac{\partial w}{\partial x} + A_{55}\phi + k_s\frac{\partial w}{\partial x} - k_{z0}w\right)\delta w\Big|_{x=0} - \left(A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial \phi}{\partial x} + k_{x1}u\right)\delta u\Big|_{x=L} - \\ \left(B_{11}\frac{\partial u}{\partial x} + D_{11}\frac{\partial \phi}{\partial x} + K_{x1}\phi\right)\delta\phi\Big|_{x=L} - \left(A_{55}\frac{\partial w}{\partial x} + A_{55}\phi + k_s\frac{\partial w}{\partial x} - k_{z1}w\right)\delta w\Big|_{x=L} \end{array}\right\}\mathrm{d}t.
\end{aligned} \tag{22}$$

The equations of motion for the FG sandwich beam resting on an elastic foundation can be obtained as

$$\begin{aligned}
&A_{11}\frac{\partial^2 u}{\partial x^2} + B_{11}\frac{\partial^2 \phi}{\partial x^2} - I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 \phi}{\partial t^2} = 0, \\
&B_{11}\frac{\partial^2 u}{\partial x^2} + D_{11}\frac{\partial^2 \phi}{\partial x^2} - A_{55}\phi - A_{55}\frac{\partial w}{\partial x} - I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^2 \phi}{\partial t^2} = 0, \\
&A_{55}\frac{\partial^2 w}{\partial x^2} + A_{55}\frac{\partial \phi}{\partial x} - k_w w + k_s \frac{\partial^2 w}{\partial x^2} - I_0 \frac{\partial^2 w}{\partial t^2} = 0.
\end{aligned} \tag{23.1–3}$$

The boundary conditions can be expressed as

$$x=0\begin{cases}A_{11}\frac{\partial u}{\partial x}+B_{11}\frac{\partial \phi}{\partial x}-k_{x0}u=0,\\ B_{11}\frac{\partial u}{\partial x}+D_{11}\frac{\partial \phi}{\partial x}-K_{x0}\phi=0,\\ A_{55}\frac{\partial w}{\partial x}+A_{55}\phi+k_s\frac{\partial w}{\partial x}-k_{z0}w=0\end{cases}\qquad x=L\begin{cases}A_{11}\frac{\partial u}{\partial x}+B_{11}\frac{\partial \phi}{\partial x}+k_{x1}u=0,\\ B_{11}\frac{\partial u}{\partial x}+D_{11}\frac{\partial \phi}{\partial x}+K_{x1}\phi=0,\\ A_{55}\frac{\partial w}{\partial x}+A_{55}\phi+k_s\frac{\partial w}{\partial x}+k_{z1}w=0\end{cases}\qquad (24.1\text{–}6)$$

2.4 Admissible functions and solution procedure

Constructing of appropriate admissible displacement functions is of crucial importance. The admissible functions are common in the forms of the polynomials. However, when using only lower-order polynomials, the solution becomes less definitive in regard to its convergence due to the fact that those polynomials do not form a complete set [36]. The higher-order polynomials trend to become numerically unstable owing to the computer round-off errors [36,37]. Those difficulties can be avoided by the Fourier series due to the fact that the Fourier series constitute a complete set and exhibit an excellent numerical stability. However, conventional Fourier series has a convergence problem along the boundary conditions except for a few simple boundary edges. Therefore, a modified Fourier series is employed in this paper. Unlike traditional Fourier series, the modified Fourier series consists of standard Fourier cosine series and supplemented functions introduced to ensure and accelerate the convergence of series representation. The displacement and rotation components of the FG sandwich beam can be expressed as [36,38–40]:

$$u(x,t)=\left\{\sum_{m=0}^{M}A_m\cos\lambda_m x+\sum_{p=1}^{P}a_p f_p(x)\right\}e^{j\omega t},$$

$$\phi(x,t)=\left\{\sum_{m=0}^{M}B_m\cos\lambda_m x+\sum_{p=1}^{P}b_p f_p(x)\right\}e^{j\omega t},$$

$$\omega(x,t)=\left\{\sum_{m=0}^{M}C_m\cos\lambda_m x+\sum_{p=1}^{P}c_p f_p(x)\right\}e^{j\omega t}\qquad (25.1\text{–}3)$$

where $\lambda_m=m\pi/L$. ω is the angular frequency of the beam. A_m, a_p, B_m, b_p, C_m and c_p are the unknown coefficients of the modified Fourier series. All of them need to be determined in future. P is the number of supplemented functions. According to Eqs. (23) and (24), it is required that at least two-order derivatives of the displacement and rotation components exist and are continuous for the beam. Therefore, two closed-form auxiliary functions are introduced, namely $P=2$. The functions are defined as follows:

$$f_1(x)=x\left(\frac{x}{L}-1\right)^2,\quad f_2(x)=\frac{x^2}{L}\left(\frac{x}{L}-1\right).$$

It is easy to verify that

$$f_1(0)=f_1(L)=f_2(0)=f_2(L)=f_1'(L)=f_2'(0)=0,\quad f_1'(0)=f_2'(L)=1.$$

The purpose of introducing the auxiliary terms is to remove any discontinuities potentially of the original displacement functions and their derivatives at the edges. All the unknown expansion coefficients are sought in a strong form by letting the solution satisfy both the boundary conditions and governing equations. Substituting Eq. (25) into Eq. (23), the governing equations of the beam can be written as

$$A_{11}\left(\sum_{m=0}^{M}-\lambda_m^2A_m\cos\lambda_m x+\sum_{p=1}^{2}a_p f_p''(x)\right)+B_{11}\left(\sum_{m=0}^{M}-\lambda_m^2B_m\cos\lambda_m x+\sum_{p=1}^{2}b_p f_p''(x)\right)$$
$$+I_0\omega^2\left(\sum_{m=0}^{M}A_m\cos\lambda_m x+\sum_{p=1}^{2}a_p f_p(x)\right)+I_1\omega^2\left(\sum_{m=0}^{M}B_m\cos\lambda_m x+\sum_{p=1}^{2}b_p f_p(x)\right)=0,\qquad (26.1)$$

$$B_{11}\left(\sum_{m=0}^{M}-\lambda_m^2 A_m\cos\lambda_m x+\sum_{p=1}^{2}a_p f_p''(x)\right)+D_{11}\left(\sum_{m=0}^{M}-\lambda_m^2 B_m\cos\lambda_m x+\sum_{p=1}^{2}b_p f_p''(x)\right)$$

$$-A_{55}\left(\sum_{m=0}^{M}B_m\cos\lambda_m x+\sum_{p=1}^{2}b_p f_p(x)\right)-A_{55}\left(\sum_{m=0}^{M}-\lambda_m C_m\sin\lambda_m x+\sum_{p=1}^{2}c_p f_p'(x)\right)$$

$$+I_1\omega^2\left(\sum_{m=0}^{M}A_m\cos\lambda_m x+\sum_{p=1}^{2}a_p f_p(x)\right)+I_2\omega^2\left(\sum_{m=0}^{M}B_m\cos\lambda_m x+\sum_{p=1}^{2}b_p f_p(x)\right)=0, \quad (26.2)$$

$$(A_{55}+k_s)\left(\sum_{m=0}^{M}-\lambda_m^2 C_m\cos\lambda_m x+\sum_{p=1}^{2}c_p f_p''(x)\right)+A_{55}\left(\sum_{m=0}^{M}-\lambda_m B_m\sin\lambda_m x+\sum_{p=1}^{2}b_p f_p'(x)\right)$$

$$-k_w\left(\sum_{m=0}^{M}C_m\cos\lambda_m x+\sum_{p=1}^{2}c_p f_p(x)\right)+I_0\omega^2\left(\sum_{m=0}^{M}C_m\cos\lambda_m x+\sum_{p=1}^{2}c_p f_p(x)\right)=0. \quad (26.3)$$

In order to derive the constraint equations for the unknown expansion coefficients, all the sine terms, the supplementary terms and their derivatives in Eq. (26) should be expanded into cosine series, and by collecting the coefficients for the similar cosine terms, the following equations can be obtained:

$$\sum_{m=0}^{M}\left\{\begin{array}{l}-A_{11}\lambda_m^2 A_m-B_{11}\lambda_m^2 B_m+\sum\limits_{p=1}^{2}\left(A_{11}\alpha_p^2 a_p+B_{11}\alpha_p^2 b_p\right)+\\ \omega^2\left(I_0 A_m+I_1 B_m+\sum\limits_{p=1}^{2}\left(I_0\alpha_p^0 a_p+I_1\alpha_p^0 b_p\right)\right)\end{array}\right\}\cos\lambda_m x=0, \quad (27.1)$$

$$\sum_{m=0}^{M}\left\{\begin{array}{l}-B_{11}\lambda_m^2 A_m-D_{11}\lambda_m^2 B_m-A_{55}B_m+A_{55}\lambda_m C_m\sum\limits_{i=0}^{\infty}\beta_i^m+\\ \sum\limits_{p=1}^{2}\left(A_{11}\alpha_p^2 a_p+B_{11}\alpha_p^2 b_p-A_{55}\alpha_p^0 b_p-A_{55}\alpha_p^1 c_p\right)+\\ \omega^2\left(I_1 A_m+I_2 B_m+\sum\limits_{p=1}^{2}\left(I_1\alpha_p^0 a_p+I_2\alpha_p^0 b_p\right)\right)\end{array}\right\}\cos\lambda_m x=0, \quad (27.2)$$

$$\sum_{m=0}^{M}\left\{\begin{array}{l}-\left(A_{55}+k_s\right)\lambda_m^2 A_m-A_{55}\lambda_m B_m\sum\limits_{i=0}^{\infty}\beta_i^m-k_w C_m\\ \sum\limits_{p=1}^{2}\left(\left(A_{55}+k\right)\chi_p^2 c_p-A_{55}\alpha_p^1 b_p-k_w\alpha_p^0 c_p\right)+\\ \omega^2\left(I_0 C_m+\sum\limits_{p=1}^{2}I_0\alpha_p^0 c_p\right)\end{array}\right\}\cos\lambda_m x=0 \quad (27.3)$$

where α_p^0, α_p^1 and α_p^2 $(p=1,2)$ are the coefficients of Fourier cosine series for the two supplementary terms and their derivatives. β_i^m are the coefficients of Fourier cosine series for sine terms. The detailed expressions for the above symbols are given in "Appendix A". Multiplying Eq. (27) with cos$\lambda_m x$ in the left side and integrating it from 0 to L with respect to x, we have

$$\mathbf{K}^c\begin{Bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{Bmatrix}+\mathbf{K}^s\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{Bmatrix}-\omega^2\left\{\mathbf{M}^c\begin{Bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{Bmatrix}+\mathbf{M}^s\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{Bmatrix}\right\}=0 \quad (28)$$

where

$$\mathbf{K}^c=\begin{bmatrix}\mathbf{K}_{11}^c & \mathbf{K}_{12}^c & \mathbf{K}_{13}^c\\ \mathbf{K}_{21}^c & \mathbf{K}_{22}^c & \mathbf{K}_{23}^c\\ \mathbf{K}_{31}^c & \mathbf{K}_{32}^c & \mathbf{K}_{33}^c\end{bmatrix},\quad \mathbf{K}^c=\begin{bmatrix}\mathbf{K}_{11}^s & \mathbf{K}_{12}^s & \mathbf{K}_{13}^s\\ \mathbf{K}_{21}^s & \mathbf{K}_{22}^s & \mathbf{K}_{23}^s\\ \mathbf{K}_{31}^s & \mathbf{K}_{32}^s & \mathbf{K}_{33}^s\end{bmatrix},$$

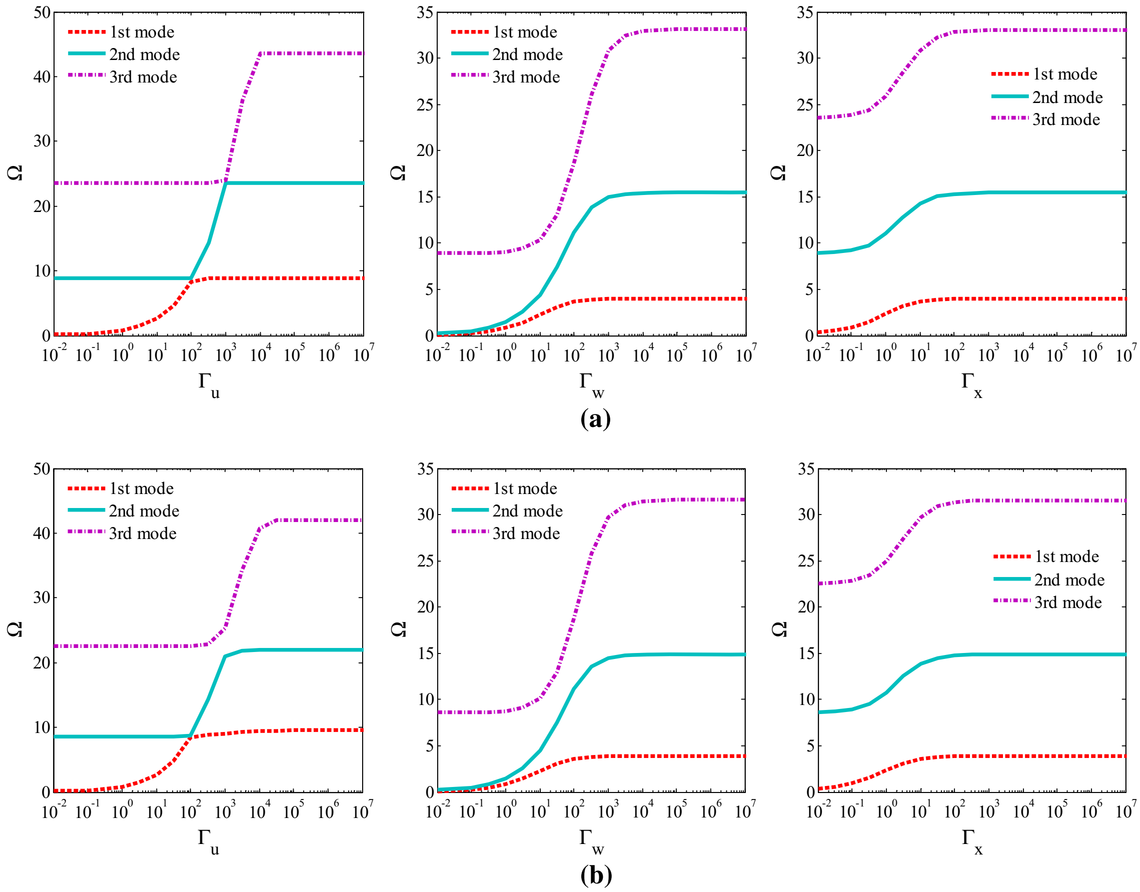

Fig. 3 Influence of spring parameters on the first three frequency parameters Ω for sandwich beams ($h/L = 10$, $k = 1$): **a** Type I and Voigt model; **b** Type II and Mori–Tanaka scheme

$$\mathbf{M}^c = \begin{bmatrix} \mathbf{M}_{11}^c & \mathbf{M}_{12}^c & \mathbf{M}_{13}^c \\ \mathbf{M}_{21}^c & \mathbf{M}_{22}^c & \mathbf{M}_{23}^c \\ \mathbf{M}_{31}^c & \mathbf{M}_{32}^c & \mathbf{M}_{33}^c \end{bmatrix}, \quad \mathbf{K}^c = \begin{bmatrix} \mathbf{M}_{11}^s & \mathbf{M}_{12}^s & \mathbf{M}_{13}^s \\ \mathbf{M}_{21}^s & \mathbf{M}_{22}^s & \mathbf{M}_{23}^s \\ \mathbf{M}_{31}^s & \mathbf{M}_{32}^s & \mathbf{M}_{33}^s \end{bmatrix}.$$

The superscripts c and s denominate the matrices obtained from the terms related to the coefficients of standard cosine series and supplemented functions, respectively. The elements of the matrices $\mathbf{K}^c$, $\mathbf{K}^s$, $\mathbf{M}^c$ and $\mathbf{M}^s$ can be directly obtained from Eq. (27).

Substituting Eq. (25) into the boundary conditions, as shown in Eq. (24), yields

$$-\mathbf{L}_c \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{Bmatrix} + \mathbf{L}_s \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{Bmatrix} = 0 \tag{29}$$

where

$$\mathbf{L}_c = \begin{bmatrix} k_{x0}\mathbf{H}_0 & \mathbf{0} & \mathbf{0} \\ -k_{x1}\mathbf{H}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K_{x0}\mathbf{H}_0 & \mathbf{0} \\ \mathbf{0} & -K_{x1}\mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & -A_{55}\mathbf{H}_0 & k_{z0}\mathbf{H}_0 \\ \mathbf{0} & A_{55}\mathbf{H}_1 & -k_{z1}\mathbf{H}_1 \end{bmatrix}, \quad L_s = \begin{bmatrix} A_{11} & 0 & B_{11} & 0 & 0 & 0 \\ 0 & A_{11} & 0 & B_{11} & 0 & 0 \\ B_{11} & 0 & D_{11} & 0 & 0 & 0 \\ 0 & B_{11} & 0 & D_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55}+k_s & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{55}+k_s \end{bmatrix}$$

where $\mathbf{H}_0 = [1, \ldots, 1]_{1\times(M+1)}$, $\mathbf{H}_1 = [\cos 0, \cos\pi, \ldots, \cos m\pi, \ldots, \cos M\pi]_{1\times(M+1)}$.

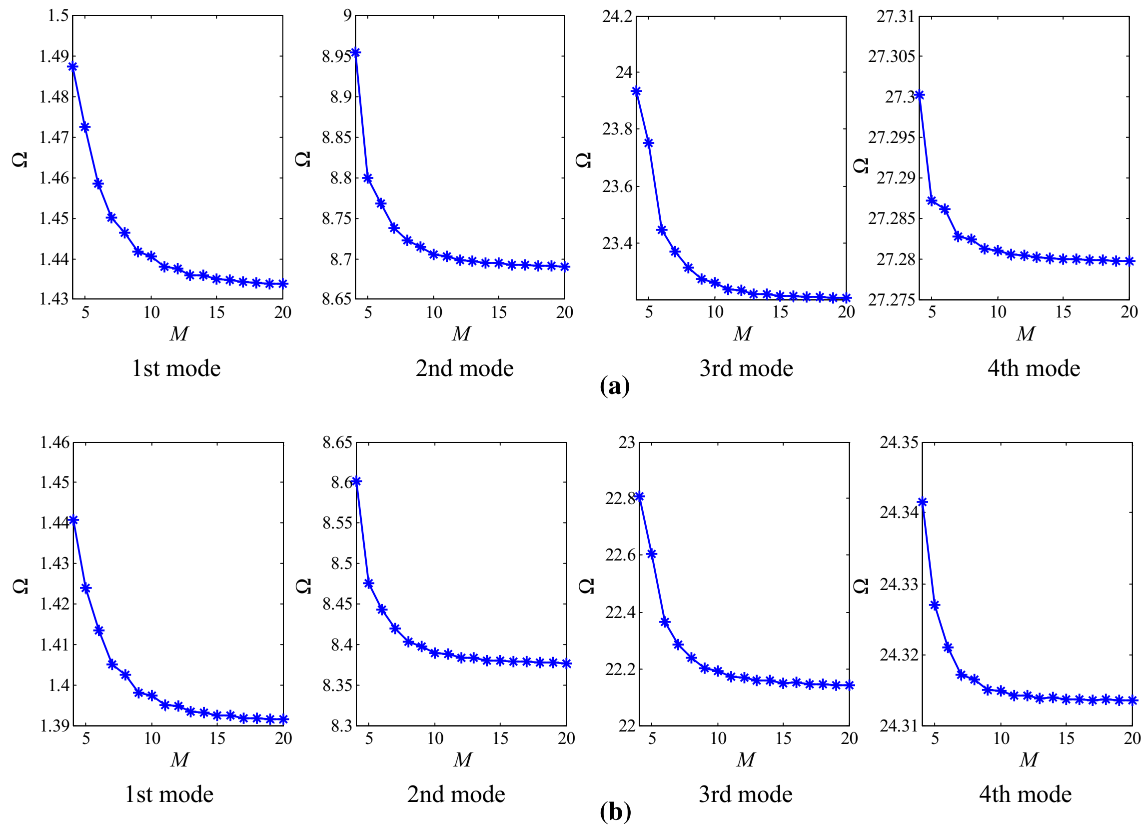

Fig. 4 Convergence of the frequency parameters Ω versus the truncated number M for sandwich beams ($h/L = 10$, $k = 1$; Boundary conditions: F–C) **a** Type I and Voigt model; **b** Type II and Mori–Tanaka scheme

Table 1 Comparison of fundamental frequency parameter Ω for FG beams with different power-law exponent (Voigt model)

BC	L/h	$k = 0$	$k = 0.2$	$k = 0.5$	$k = 1$	$k = 2$	$k = 5$	$k = 10$
C–C	20							
	Ref. [14]	12.2235	11.3850	10.4263	9.4314	8.6040	8.1699	7.9128
	Present	12.2217	11.3807	10.4242	9.4305	8.6032	8.1687	7.9113
	Diff (%)	0.0143	0.0377	0.0199	0.0095	0.0090	0.0153	0.0188
	5							
	Ref. [14]	10.0344	9.4176	8.7005	7.9253	7.2113	6.6676	6.3406
	Present	9.9986	9.3836	8.6715	7.9007	7.1889	6.6436	6.3155
	Diff (%)	0.3563	0.3616	0.3333	0.3102	0.3113	0.3600	0.3960
C–F	20							
	Ref. [14]	1.9496	1.8146	1.6604	1.5010	1.3697	1.3038	1.2650
	Present	1.9554	1.8197	1.6658	1.5063	1.3744	1.3075	1.2682
	Diff (%)	0.2954	0.2835	0.3229	0.3503	0.3417	0.2856	0.2586
	5							
	Ref. [14]	1.8948	1.7655	1.6174	1.4630	1.3338	1.2645	1.2240
	Present	1.8948	1.7652	1.6173	1.4631	1.3339	1.2645	1.2239
	Diff (%)	0.0022	0.0181	0.0044	0.0100	0.0062	0.0019	0.0028
S–S	20							
	Ref. [14]	5.4603	5.0827	4.6514	4.2051	3.8368	3.6509	3.5416
	Present	5.4790	5.0995	4.6684	4.2216	3.8517	3.6631	3.5523
	Diff (%)	0.3429	0.3303	0.3652	0.3917	0.3879	0.3349	0.3040
	5							
	Ref. [14]	5.1525	4.8066	4.4083	3.9902	3.6344	3.4312	3.3134
	Present	5.1540	4.8069	4.4093	3.9916	3.6356	3.4322	3.3143
	Diff (%)	0.0284	0.0054	0.0227	0.0347	0.0330	0.0282	0.0255

Table 2 Comparison of fundamental frequency parameter Ω for Type I FG sandwich beams with various boundary conditions ($L/R = 20$; Voigt model)

BC	k	1–0–1	2–1–2	2–1–1	1–1–1	2–2–1	1–2–1	1–8–1
C–C	0.5							
	Ref. [34]	9.6942	9.9501	10.1001	10.1800	10.3668	10.5460	11.4459
	Present	9.6840	9.9402	10.0913	10.1710	10.3588	10.5388	11.4434
	1							
	Ref. [34]	8.3594	8.7241	8.9474	9.0722	9.3550	9.6411	11.0421
	Present	8.3482	8.7131	8.9376	9.0621	9.3458	9.6329	11.0391
	2							
	Ref. [34]	7.1563	7.5417	7.8293	7.9727	8.3430	8.7262	10.6336
	Present	7.1445	7.5306	7.8191	7.9625	8.3334	8.7176	10.6303
	5							
	Ref. [34]	6.4064	6.6116	6.9389	7.0170	7.4461	7.8692	10.2298
	Present	6.3931	6.5998	6.9282	7.0064	7.4361	7.8603	10.2261
S–S	0.5							
	Ref. [34]	4.3148	4.4290	4.4970	4.5324	4.6170	4.6979	5.1067
	Present	4.3321	4.4468	4.5149	4.5505	4.6353	4.7162	5.1254
	1							
	Ref. [34]	3.7147	3.8768	3.9774	4.0328	4.1602	4.2889	4.9233
	Present	3.7313	3.8943	3.9952	4.0508	4.1784	4.3073	4.9421
	2							
	Ref. [34]	3.1764	3.3465	3.4754	3.5389	3.7049	3.8769	4.7382
	Present	3.1918	3.3636	3.4929	3.5567	3.7231	3.8954	4.7571
	5							
	Ref. [34]	2.8439	2.9310	3.0773	3.1111	3.3028	3.4921	4.5554
	Present	2.8568	2.9470	3.0940	3.1284	3.3208	3.5105	4.5744
C–F	0.5							
	Ref. [34]	1.5397	1.5805	1.6048	1.6175	1.6477	1.6766	1.8229
	Present	1.5454	1.5863	1.6106	1.6233	1.6536	1.6825	1.8288
	1							
	Ref. [34]	1.3253	1.3831	1.4191	1.4388	1.4844	1.5304	1.7573
	Present	1.3308	1.3889	1.4250	1.4448	1.4904	1.5364	1.7633
	2							
	Ref. [34]	1.1330	1.1937	1.2398	1.2623	1.3217	1.3831	1.6911
	Present	1.1383	1.1995	1.2457	1.2684	1.3278	1.3893	1.6971
	5							
	Ref. [34]	1.0145	1.0453	1.0977	1.1096	1.1781	1.2456	1.6257
	Present	1.0189	1.0509	1.1034	1.1156	1.1842	1.2519	1.6318

Table 3 Comparison of the first seven natural frequencies (Hz) for Type II FG sandwich beam subjected to C–F boundary conditions ($L = 1$ m, $h = 0.02$ m, h_1:h_2:h_3 = 3:14:3)

	$f = 1$	$f = 2$	$f = 3$	$f = 4$	$f = 5$	$f = 6$	$f = 7$
M-T scheme							
Ref. [27]	18.31	114.61	320.42	626.54	1033.2	1287.7	1538.9
Present	18.69	115.95	323.28	629.23	1034.6	1263.7	1531.4
Voigt model							
Ref. [27]	18.32	114.66	320.57	626.81	1033.7	1290.1	1539.6
Present	18.76	116.37	324.50	631.64	1038.7	1283.2	1537.7

Combining Eqs. (28) and (29), the final system equation can be obtained as

$$[\mathbf{K}^c + \mathbf{K}^s \mathbf{L}_s^{-1} \mathbf{L}_c] \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \\ C \end{Bmatrix} - \omega^2 \left[\mathbf{M}^c + \mathbf{M}^s \mathbf{L}_s^{-1} \mathbf{L}_c\right] \begin{Bmatrix} \mathbf{A} \\ \mathbf{B} \\ C \end{Bmatrix} = 0. \qquad (30)$$

All the natural frequencies of the FG sandwich beams can be determined easily by solving the standard characteristic equation (30).

Table 4 Comparison of fundamental frequency parameter for the isotropic beams resting on an elastic foundation (μ = 0.3)

L/h	K_w	K_p/π^2							
		0		0.5		1.0		2.5	
		Ref. [6]	Present	Ref. [6]	Present	Ref. [6]	Present	Ref. [6]	Present
C–C									
120	0	4.7314	4.7351	4.8683	4.8715	4.9938	4.9976	5.3195	5.3243
	10^2	4.9515	4.9541	5.0718	5.0755	5.1834	5.1868	5.4783	5.4817
	10^4	10.1227	10.1232	10.1373	10.1377	10.1517	10.1524	10.1942	10.1949
15	0	4.6655	4.6602	4.8039	4.7998	4.9303	4.9273	5.2567	5.2559
	10^2	4.8927	4.8889	5.0135	5.0107	5.1254	5.1234	5.4198	5.4196
	10^4	10.0490	10.1060	10.0640	10.1204	10.0788	10.1346	10.1223	10.1762
S–S									
120	0	3.1414	3.1879	3.4766	3.5113	3.7359	3.7641	4.2969	4.3159
	10^2	3.7482	3.7759	3.9607	3.9843	4.1436	4.1644	4.5823	4.5980
	10^4	10.0240	10.0256	10.0361	10.0376	10.0481	10.0497	10.0839	10.0855
15	0	3.1302	3.1332	3.4667	3.4696	3.7266	3.7294	4.2881	4.2911
	10^2	3.7389	3.7417	3.9517	3.9545	4.1347	4.1376	4.5735	4.5766
	10^4	9.9958	10.0151	10.0078	10.0272	10.0197	10.0392	10.0552	10.0750

Table 5 Fundamental frequency parameter Ω for Type I FG sandwich beam with classical boundary conditions ($L/R = 10$)

BC	$k_1 = k_2$	Rule of mixture				M-T scheme			
		1–1–1	1–2–1	1–3–1	1–4–1	1–1–1	1–2–1	1–3–1	1–4–1
C–C	0	11.653	11.653	11.653	11.653	11.653	11.653	11.653	11.653
	0.6	9.5365	9.9254	10.204	10.409	8.2001	8.8110	9.2646	9.6020
	1	8.7640	9.2984	9.6836	9.9663	7.5834	8.2930	8.8304	9.2316
	5	6.8298	7.6460	8.2905	8.7747	6.4913	7.3003	7.9780	8.4977
C–S	0	8.2487	8.2487	8.2487	8.2487	8.2487	8.2487	8.2487	8.2487
	0.6	6.7114	6.9906	7.1919	7.3404	5.7550	6.1897	6.5146	6.7573
	1	6.1555	6.5376	6.8146	7.0187	5.3142	5.8177	6.2016	6.4895
	5	4.7765	5.3534	5.8125	6.1591	4.5383	5.1082	5.5898	5.9609
S–S	0	5.3988	5.3988	5.3988	5.3988	5.3988	5.3988	5.3988	5.3988
	0.6	4.3706	4.5555	4.6894	4.7885	3.7388	4.0246	4.2394	4.4004
	1	4.0017	4.2539	4.4376	4.5734	3.4480	3.7782	4.0314	4.2220
	5	3.0937	3.4708	3.7728	4.0017	2.9387	3.3101	3.6263	3.8709
F–F	0	12.007	12.007	12.007	12.007	12.007	12.0067	12.0067	12.007
	0.6	9.7344	10.145	10.441	10.661	8.3276	8.9629	9.4400	9.7971
	1	8.9159	9.4765	9.8842	10.185	7.6822	8.4170	8.9797	9.4028
	5	6.8953	7.7368	8.4091	8.9179	6.5496	7.3784	8.0824	8.6264
S–F	0	8.3562	8.3562	8.3562	8.3562	8.3562	8.3562	8.3562	8.3562
	0.6	6.7705	7.0563	7.2631	7.4159	5.7924	6.2346	6.5667	6.8155
	1	6.2004	6.5905	6.8745	7.0842	5.3428	5.8540	6.2457	6.5404
	5	4.7949	5.3795	5.8471	6.2012	4.5546	5.1305	5.6200	5.9986
C–F	0	1.9396	1.9396	1.9396	1.9396	1.9396	1.9396	1.9396	1.9396
	0.6	1.5674	1.6341	1.6825	1.7183	1.3398	1.4426	1.5200	1.5781
	1	1.4342	1.5251	1.5914	1.6405	1.2350	1.3537	1.4449	1.5136
	5	1.1074	1.2427	1.3514	1.4339	1.0518	1.1850	1.2987	1.3868

3 Numerical examples and discussion

In this section, a variety of numerical examples are presented to demonstrate the convergence, reliability and accuracy of the present method. Numerous new vibration results for functionally graded sandwich beams with general boundary conditions and resting on elastic foundations are given, which can serve as benchmark solutions. In addition, the influence of the power-law indices and foundation parameters on the frequencies of the sandwich beams is also investigated. In order to simplify the presentation, F, S and C denominate free, simply supported and clamped boundary conditions which are defined as $N_x = M_x = Q_{xz} = 0$ for F; $N_x = M_x = w = 0$ for S; $u = w = \phi = 0$ for C; unless stated otherwise, the nondimensional frequency parameter is defined as $\Omega = \omega L^2/h\sqrt{\rho_2/E_2}$. And the material properties for M_1 and

Table 6 Fundamental frequency parameter Ω for Type II FG sandwich beam with classical boundary conditions ($L/R = 10$)

BC	k	Rule of mixture				M-T scheme			
		1–1–1	1–2–1	1–3–1	1–4–1	1–1–1	1–2–1	1–3–1	1–4–1
C–C	0	8.6374	9.2071	9.6191	9.9191	8.6374	9.2071	9.6191	9.9191
	0.6	8.2663	8.5364	8.7370	8.8848	8.0190	8.1263	8.2099	8.2700
	1	8.1503	8.3221	8.4490	8.5413	7.9522	8.0045	8.0438	8.0683
	5	7.9558	7.9970	8.0193	8.0260	7.9185	7.9516	7.9594	7.9476
C–S	0	6.0636	6.4711	6.7675	6.9842	6.0636	6.4711	6.7675	6.9842
	0.6	5.8049	5.9992	6.1442	6.2512	5.6341	5.7149	5.7776	5.8228
	1	5.7246	5.8495	5.9420	6.0094	5.5892	5.6322	5.6640	5.6843
	5	5.5955	5.6337	5.6557	5.6650	5.5725	5.6083	5.6226	5.6205
S–S	0	3.9406	4.2095	4.4061	4.5503	3.9406	4.2095	4.4061	4.5503
	0.6	3.7736	3.9025	3.9989	4.0703	3.6643	3.7198	3.7628	3.7938
	1	3.7221	3.8057	3.8677	3.9129	3.6362	3.6677	3.6908	3.7057
	5	3.6426	3.6727	3.6907	3.6993	3.6295	3.6599	3.6743	3.6764
F–F	0	8.7634	9.3678	9.8073	10.129	8.7634	9.3678	9.8073	10.1287
	0.6	8.3799	8.6698	8.8862	9.0460	8.1315	8.2561	8.3528	8.4225
	1	8.2606	8.4479	8.5869	8.6883	8.0647	8.1342	8.1862	8.2199
	5	8.0695	8.1312	8.1693	8.1878	8.0388	8.1015	8.1323	8.1375
S–F	0	6.1016	6.5193	6.8240	7.0471	6.1016	6.5193	6.8240	7.0471
	0.6	5.8398	6.0400	6.1896	6.3003	5.6692	5.7553	5.8220	5.8700
	1	5.7588	5.8884	5.9845	6.0546	5.6245	5.6729	5.7087	5.7318
	5	5.6317	5.6765	5.7037	5.7166	5.6110	5.6562	5.6778	5.6810
C–F	0	1.4125	1.5092	1.5801	1.6322	1.4125	1.5092	1.5801	1.6322
	0.6	1.3530	1.3994	1.4342	1.4600	1.3141	1.3344	1.3500	1.3613
	1	1.3347	1.3650	1.3874	1.4037	1.3043	1.3160	1.3246	1.3301
	5	1.3070	1.3185	1.3255	1.3289	1.3026	1.3144	1.3202	1.3214

Table 7 Fundamental frequency parameter Ω for Type I FG sandwich beam with elastic boundary conditions ($L/R = 10$)

k	Γ	Rule of mixture				M-T scheme			
		1–1–1	1–2–1	1–3–1	1–4–1	1–1–1	1–2–1	1–3–1	1–4–1
$E^1 - E^1$									
0.6	10^0	0.8188	0.8101	0.8051	0.8017	0.8188	0.8101	0.8051	0.8017
	10^1	2.5873	2.5600	2.5440	2.5335	2.5870	2.5598	2.5439	2.5334
	10^2	8.1183	8.0363	7.9881	7.9564	8.1069	8.0292	7.9830	7.9524
1	10^0	0.8309	0.8188	0.8119	0.8073	0.8308	0.8188	0.8119	0.8073
	10^1	2.6251	2.5873	2.5654	2.5511	2.6247	2.5871	2.5652	2.5510
	10^2	8.2309	8.1183	8.0525	8.0094	7.6822	8.1109	8.0474	8.0055
5	10^0	0.8657	0.8434	0.8309	0.8228	0.8657	0.8434	0.8309	0.8228
	10^1	2.7345	2.6646	2.6251	2.5998	2.7342	2.6645	2.6251	2.5997
	10^2	6.8953	7.7368	8.2309	8.1554	6.5496	7.3784	8.0824	8.1534
$E^2 - E^2$									
0.6	10^0	0.8075	0.8000	0.7957	0.7928	0.8034	0.7972	0.7936	0.7912
	10^1	2.2726	2.2755	2.2783	2.2804	2.1781	2.2079	2.2271	2.2398
	10^2	3.9250	4.0466	4.1324	4.1947	3.4511	3.6637	3.8175	3.9292
1	10^0	0.8168	0.8069	0.8011	0.7974	0.8120	0.8037	0.7989	0.7957
	10^1	2.2451	2.2576	2.2659	2.2713	2.1415	2.1850	2.2123	2.2295
	10^2	3.6623	3.8405	3.9658	4.0558	3.2255	3.4818	3.6702	3.8069
5	10^0	0.8394	0.8239	0.8150	0.8091	0.8366	0.8220	0.8137	0.8082
	10^1	2.1043	2.1674	2.2063	2.2293	2.0565	2.1297	2.1786	2.2083
	10^2	2.9430	3.2503	3.4853	3.6562	2.8088	3.1177	3.3694	3.5563
$E^3 - E^3$									
0.6	10^0	2.3784	2.3884	2.3959	2.4013	2.2589	2.3011	2.3289	2.3476
	10^1	3.9266	4.0505	4.1384	4.2022	3.4479	3.6625	3.8183	3.9317
	10^2	4.3145	4.4916	4.6195	4.7139	3.7015	3.9785	4.1860	4.3410
1	10^0	2.3358	2.3592	2.3746	2.3850	2.2088	2.2676	2.3058	2.3306
	10^1	3.6603	3.8410	3.9684	4.0601	3.2208	3.4787	3.6689	3.8074
	10^2	3.9582	4.2015	4.3782	4.5084	3.4181	3.7396	3.9852	4.1694
5	10^0	2.1525	2.2347	2.2878	2.3210	2.0981	2.1899	2.2539	2.2946
	10^1	2.9370	3.2453	3.4819	3.6544	2.8027	3.1122	3.3652	3.5536
	10^2	3.0716	3.4408	3.7351	3.9573	2.9190	3.2832	3.5920	3.8301

Table 8 Fundamental frequency parameter Ω for Type II FG sandwich beam with elastic boundary conditions ($L/R = 10$)

k	Γ	Rule of mixture				M-T scheme			
		1–1–1	1–2–1	1–3–1	1–4–1	1–1–1	1–2–1	1–3–1	1–4–1
$E^1 - E^1$									
0.6	10^0	0.8500	0.8467	0.8448	0.8435	0.8500	0.8467	0.8447	0.8434
	10^1	2.6839	2.6739	2.6680	2.6640	2.6832	2.6730	2.6669	2.6629
	10^2	8.3600	8.3410	8.3296	8.3219	8.2306	8.3109	8.2960	8.2864
1	10^0	0.8566	0.8566	0.8566	0.8566	0.8566	0.8566	0.8566	0.8566
	10^1	2.7045	2.7047	2.7048	2.7049	2.7038	2.7037	2.7037	2.7037
	10^2	8.3541	8.4190	8.4243	8.4281	8.1759	8.2410	8.2874	8.3164
5	10^0	0.8752	0.8849	0.8909	0.8949	0.8752	0.8849	0.8908	0.8949
	10^1	2.7617	2.7920	2.8107	2.8234	2.7615	2.7916	2.8102	2.8229
	10^2	8.2000	8.2664	8.3032	8.3188	8.1743	8.2399	8.2663	8.2653
$E^2 - E^2$									
0.6	10^0	0.8330	0.8310	0.8299	0.8292	0.8320	0.8295	0.8280	0.8271
	10^1	2.2414	2.2589	2.2716	2.2806	2.2201	2.2253	2.2300	2.2335
	10^2	3.4981	3.5972	3.6705	3.7241	3.4109	3.4535	3.4864	3.5101
1	10^0	0.8388	0.8396	0.8401	0.8405	0.8380	0.8383	0.8385	0.8387
	10^1	2.2433	2.2595	2.2711	2.2792	2.2258	2.2323	2.2371	2.2400
	10^2	3.4610	3.5275	3.5765	3.6119	3.3920	3.4173	3.4359	3.4479
5	10^0	0.8554	0.8648	0.8706	0.8745	0.8553	0.8647	0.8705	0.8743
	10^1	2.2587	2.2816	2.2956	2.3043	2.2559	2.2788	2.2920	2.2991
	10^2	3.4068	3.4362	3.4539	3.4629	3.3962	3.4258	3.4405	3.4442
$E^3 - E^3$									
0.6	10^0	2.3205	2.3437	2.3605	2.3724	2.2945	2.3024	2.3091	2.3139
	10^1	3.4955	3.5953	3.6691	3.7231	3.4086	3.4515	3.4847	3.5085
	10^2	3.7395	3.8648	3.9584	4.0276	3.6336	3.6877	3.7296	3.7597
1	10^0	2.3194	2.3392	2.3533	2.3633	2.2983	2.3061	2.3118	2.3154
	10^1	3.4584	3.5253	3.5745	3.6101	3.3896	3.4153	3.4341	3.4461
	10^2	3.6902	3.7717	3.8322	3.8762	3.6069	3.6379	3.6606	3.6752
5	10^0	2.3294	2.3525	2.3665	2.3751	2.3260	2.3491	2.3622	2.3690
	10^1	3.4046	3.4344	3.4523	3.4613	3.3942	3.4242	3.4391	3.4428
	10^2	3.6149	3.6455	3.6637	3.6725	3.6024	3.6333	3.6480	3.6503

M_2 are given as: $E_1 = 380$ GPa, $\rho_1 = 3960$ kg/m^3, $\mu_1 = 0.3$ and $E_2 = 70$ GPa, $\rho_2 = 2702$ kg/m^3, $\mu_2 = 0.3$.

3.1 Determination of spring stiffness

In this study, the boundary conditions are simulated by boundary springs. Therefore, the influence of spring stiffness on the frequencies for sandwich beams is studied to determine the proper spring stiffness for different boundary conditions. Figure 3 depicts variations of the first three frequency parameters Ω for sandwich beams with different spring parameters. Type I beams with Voigt model and Type II beams with Mori–Tanaka scheme are considered. The geometry parameters of the FG sandwich beam are given as: $h/L = 10$, $h_1 : h_2 : h_3 = 1 : 1 : 1$. The power-law index is taken as one. The restraint parameters are defined as $\Gamma_u = k_x/E_1\mathrm{I}$, $\Gamma_w = k_z/E_1\mathrm{I}$ and $\Gamma_x = K_x/E_1\mathrm{I}$ ($I = bh^3/12$). Both ends of the beam are restrained by only one kind of boundary springs whose restraint parameters vary from 10^{-2} to 10^7. It is obvious that the frequency parameters increase rapidly in a certain range and beyond the range the frequency parameters have little change.

3.2 Convergence and comparison studies

Figure 4 shows the variations of the first four frequency parameters Ω of the FG sandwich beams with different truncated numbers M. The geometry parameters and material types are the same as in the above example. The boundary conditions C–F are considered. It is observed that the results show a monotonic convergence trend, and in the following examples the truncated numbers will be uniformly selected as $M = 17$.

Table 9 Fundamental frequency parameter Ω for Type I FG sandwich beam resting on elastic foundation with various foundation parameters ($L/R = 10$, 1–1–1)

BC	K_w	K_p/π^2	Rule of mixture				M-T scheme			
			$k = 0.6$	$k = 1$	$k = 2$	$k = 5$	$k = 0.6$	$k = 1$	$k = 2$	$k = 5$
C–C	0	0	9.5365	8.7640	7.7356	6.8298	8.2001	7.5834	6.9387	6.4913
		0.5	10.507	9.8421	8.9866	8.2743	9.3129	8.8087	8.3105	7.9962
		1.0	11.383	10.799	10.064	9.4755	10.291	9.8643	9.4620	9.2308
	10^2	0	11.145	10.539	9.7726	9.1562	10.025	9.5790	9.1546	8.9065
		0.5	11.986	11.451	10.790	10.279	10.954	10.576	10.234	10.057
		1.0	12.762	12.284	11.703	11.269	11.797	11.470	11.189	11.064
C–S	0	0	6.7114	6.1555	5.4198	4.7765	5.7550	5.3142	4.8552	4.5383
		0.5	7.9711	7.5455	7.0133	6.5905	7.1838	6.8749	6.5845	6.4182
		1.0	9.0434	8.6995	8.2839	7.9744	8.3536	8.1196	7.9190	7.8294
	10^2	0	8.8491	8.4936	8.0645	7.7463	8.1473	7.9050	7.6964	7.6017
		0.5	9.8390	9.5494	9.2117	8.9797	9.2126	9.0288	8.8895	8.8540
		1.0	10.726	10.485	10.213	10.040	10.151	10.009	9.9189	9.9250
S–S	0	0	4.3706	4.0017	3.5159	3.0937	3.7388	3.4480	3.1462	2.9387
		0.5	5.9427	5.7192	5.4540	5.2632	5.4947	5.3464	5.2233	5.1736
		1.0	7.1784	7.0289	6.8652	6.7699	6.8121	6.7290	6.6833	6.7004
	10^2	0	7.2381	7.0917	6.9321	6.8406	6.8750	6.7946	6.7520	6.7719
		0.5	8.2828	8.1846	8.0894	8.0576	7.9674	7.9285	7.9356	7.9993
		1.0	9.2097	9.1479	9.1007	9.1134	8.9270	8.9195	8.9643	9.0619
C–F	0	0	1.5674	1.4342	1.2592	1.1074	1.3398	1.2350	1.1263	1.0518
		0.5	2.9902	2.9303	2.8546	2.7965	2.8389	2.7981	2.7663	2.7593
		1.0	3.7913	3.7439	3.6850	3.6451	3.6407	3.6119	3.5970	3.6082
	10^2	0	5.9891	6.0384	6.1161	6.2116	5.9336	5.9941	6.0901	6.2019
		0.5	6.5102	6.5592	6.6338	6.7246	6.4425	6.5016	6.5965	6.7093
		1.0	6.9156	6.9615	7.0318	7.1199	6.8344	6.8916	6.9861	7.1012

Table 10 Fundamental frequency parameter Ω for Type II FG sandwich beam resting on the elastic foundation with various foundation parameters ($L/R = 10$, 1–1–1)

BC	K_w	K_p/π^2	Rule of mixture				M-T scheme			
			$k = 0.6$	$k = 1$	$k = 2$	$k = 5$	$k = 0.6$	$k = 1$	$k = 2$	$k = 5$
C–C	0	0	8.2663	8.1503	8.0236	7.9558	8.0190	7.9522	7.9112	7.9185
		0.5	9.4480	9.3631	9.2756	9.2399	9.2304	9.1890	9.1765	9.2064
		1.0	10.481	10.418	10.358	10.346	10.282	10.259	10.267	10.314
	10^2	0	10.204	10.137	10.073	10.059	10.004	9.9779	9.9837	10.029
		0.5	11.183	11.136	11.097	11.103	10.999	10.989	11.014	11.075
		1.0	12.068	12.036	12.017	12.039	11.895	11.899	11.938	12.012
C–S	0	0	5.8049	5.7246	5.6383	5.5955	5.6341	5.5892	5.5635	5.5725
		0.5	7.3187	7.2754	7.2357	7.2313	7.1821	7.1674	7.1760	7.2126
		1.0	8.5488	8.5281	8.5167	8.5363	8.4296	8.4338	8.4642	8.5195
	10^2	0	8.3349	8.3124	8.2991	8.3179	8.2160	8.2189	8.2479	8.3022
		0.5	9.4522	9.4479	9.4577	9.4962	9.3461	9.3644	9.4116	9.4818
		1.0	10.434	10.443	10.470	10.524	10.336	10.366	10.427	10.510
S–S	0	0	3.7736	3.7221	3.6676	3.6426	3.6643	3.6362	3.6214	3.6295
		0.5	5.6287	5.6183	5.6155	5.6337	5.5554	5.5612	5.5850	5.6251
		1.0	7.0088	7.0197	7.0438	7.0857	6.9497	6.9737	7.0193	7.0788
	10^2	0	7.0746	7.0863	7.1117	7.1546	7.0160	7.0408	7.0873	7.1477
		0.5	8.2153	8.2418	8.2860	8.3461	8.1645	8.2022	8.2648	8.3400
		1.0	9.2159	9.2540	9.3133	9.3874	9.1703	9.2185	9.2943	9.3820
C–F	0	0	1.3530	1.3347	1.3155	1.3070	1.3141	1.3043	1.2993	1.3026
		0.5	2.9205	2.9238	2.9319	2.9476	2.8939	2.9027	2.9203	2.9441
		1.0	3.7522	3.7617	3.7783	3.8026	3.7251	3.7400	3.7662	3.7987
	10^2	0	6.1485	6.1899	6.2480	6.3106	6.1395	6.1829	6.2443	6.3095
		0.5	6.6744	6.7179	6.7791	6.8455	6.6625	6.7084	6.7739	6.8439
		1.0	7.0787	7.1234	7.1866	7.2557	7.0641	7.1118	7.1801	7.2536

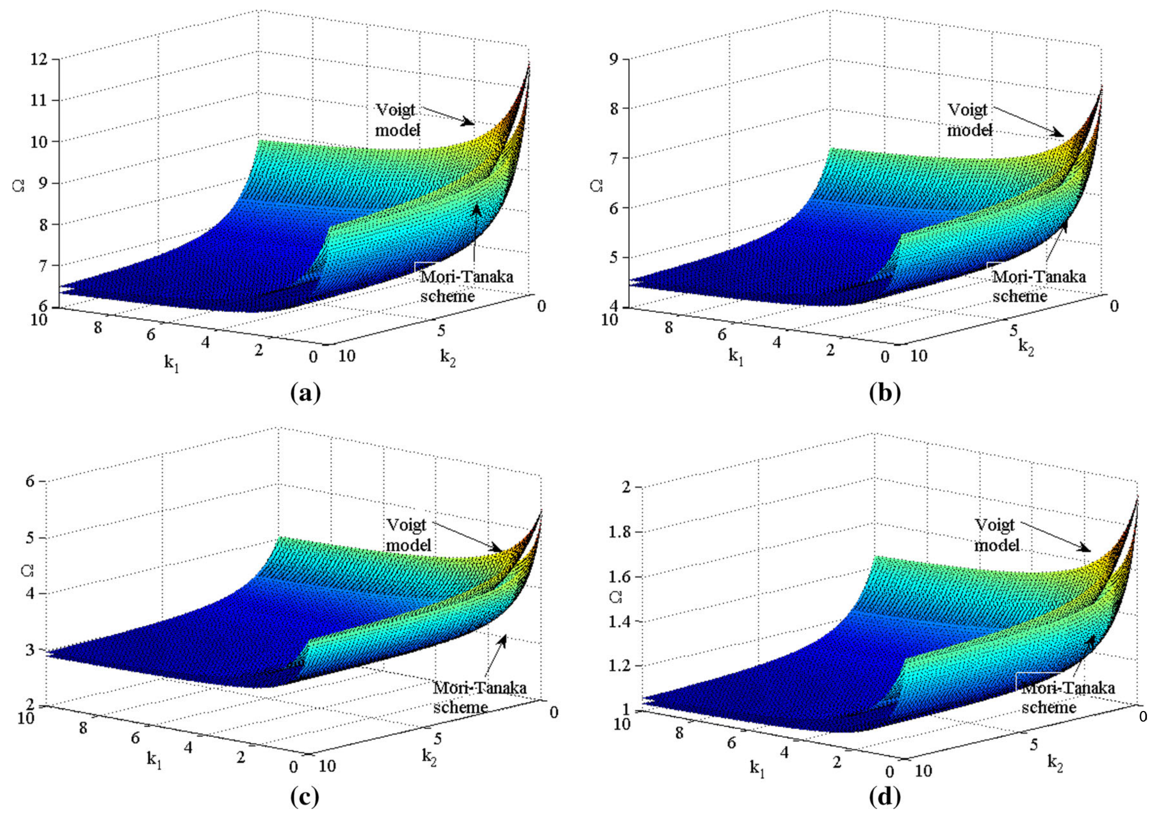

Fig. 5 Variations of the fundamental frequency parameters of the Type I FG sandwich beam with different boundary conditions: **a** C–C; **b** C–S; **c** S–S; **d** C–F

Table 1 shows the fundamental frequency parameter Ω for an FG beam with different power-law exponents based on Voigt model. Three types of boundary conditions, namely C–C, C–F and S–S, are considered. The length to thickness is taken as $L/h = 20$ and 5. The results are compared with other published solutions by Şimşek [14] using the first-order shear deformation theory. Table 2 presents the fundamental frequency parameter Ω for a Type I FGM sandwich beam with various boundary conditions based on the Voigt model. The geometry parameters used in the analysis are given as: $L/h = 20$, h_1:h_2:h_3 = 1:0:1, 2:1:2, 2:1:1, 1:1:1, 2:2:1, 1:2:1 and 1:8:1. The power-law index is the taken as $k_1 = k_2 = 0.5$, 1, 2 and 5. The same vibration problem has been studied by Vo et al. [34] using the finite element method on the basis of a refined shear deformation theory. Table 3 presents the first seven natural frequencies for a Type II FG sandwich beam with C–F boundary conditions. Both Voigt model and Mori–Tanaka scheme are considered. The geometry parameters are given as: $L = 1$m, $h = 0.02$m, $h_1 : h_2 : h_3 = 3 : 14 : 3$. The material properties for M_1 and M_2 are given as: $E_1 = 151$ GPa, $\rho_1 = 5700$ kg/m^3, $\mu_1 = 0.3$ and $E_2 = 70$ GPa, $\rho_2 = 2700$ kg/m^3, $\mu_2 = 0.3$. The properties of the face sheets are considered as: $E = 210$ GPa, $\rho = 7860$ kg/m^3, $\mu = 0.3$. The reference results are given by Amirani et al. [27] using the element free Galerkin method. Table 4 presents the fundamental frequency parameter for the isotropic beams resting on an elastic foundation. Different foundation parameters are considered and defined as $K_w = k_w L^4/EI$, $K_s = k_s L^2/EI$. The comparison of the present results with an available solution given by Chen et al. [6] is presented. It is seen that a good agreement of the results is achieved from Tables 1, 2, 3 and 4.

3.3 FG sandwich beams with general boundaries and elastic foundations

Tables 5 and 6 present the fundamental frequency parameters of the Types I and II FG sandwich beams with classical boundary conditions including C–C, C–S, S–S, F–F, S–F and C–F. The geometry parameters of the sandwich beam used in the study are given as $L/R = 10$, and h_1:h_2:h_3 = 1:1:1, 1:2:1, 1:3:1 and 1:4:1. Both Voigt model and Mori–Tanaka scheme are considered. The power-law indices are taken as

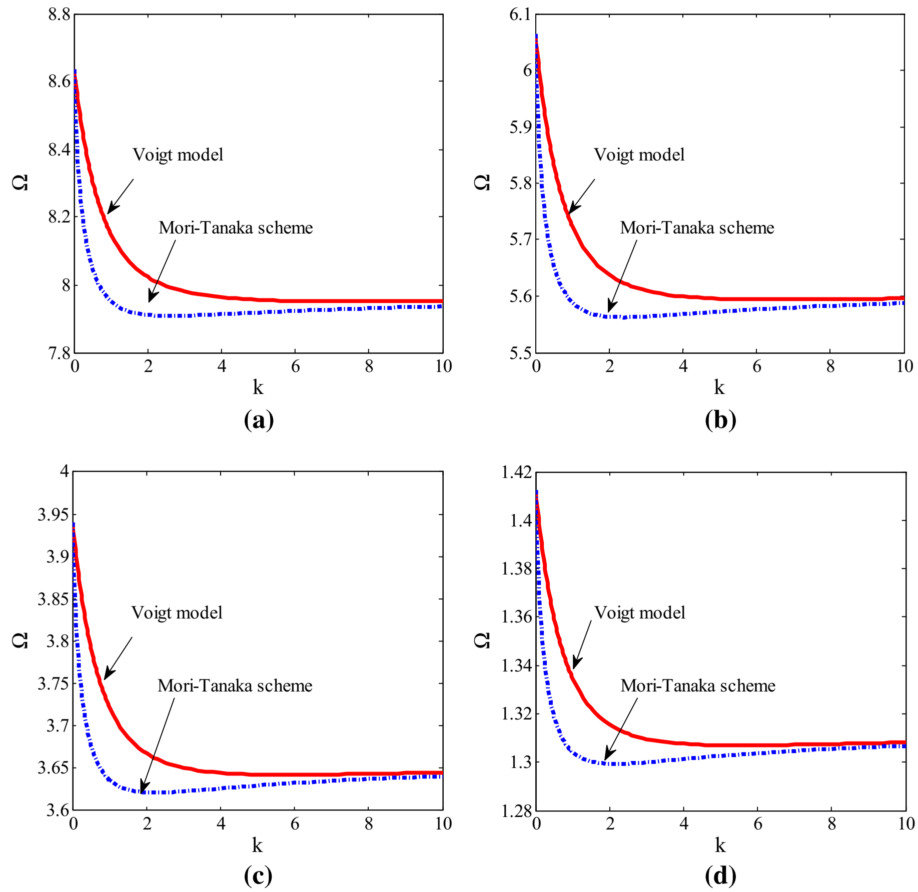

Fig. 6 Variations of the fundamental frequency parameters of the Type II FG sandwich beam with different boundary conditions: **a** C–C; **b** C–S; **c** S–S; **d** C–F

$k_1 = k_2 = 0$, 0.6, 1 and 5. It is found that the fundamental frequencies of the sandwich beam are strongly influenced by the boundary conditions and power-law indices. The increase of the power-law indices leads to the decrease of the fundamental frequency parameters. The thickness ratios have also a significant effect on the frequency parameters of the beams. It is worth noting that when $k_1 = k_2 = 0$ the Type I sandwich beam becomes isotropic, thus the fundamental frequencies have no change with different thickness ratios.

Tables 7 and 8 present the fundamental frequency parameter Ω for a Type I and Type II FG sandwich beam with elastic boundary conditions. The geometry parameters used are the same as in the above example. The E^1, E^2, E^3 denominate three types of elastic boundary conditions which are defined as $\Gamma_u \neq 0$, $\Gamma_w = \Gamma_x = 0$ for E^1; $\Gamma_w \neq 0$, $\Gamma_u = \Gamma_x = 0$ for E^2; $\Gamma_x \neq 0$, $\Gamma_u = \Gamma_w = 0$ for E^3. It is obvious that the increase of the spring parameters leads to an increase of the fundamental frequency parameters of the beam. It is interesting that for FG sandwich beams with elastic boundary conditions some of the fundamental frequency parameters decrease as the power-law index increases, while some increase as the power-law index increases.

Tables 9 and 10 present the fundamental frequency parameter Ω for Type I and Type II FG sandwich beams resting on an elastic foundation with various foundation parameters. The thicknesses of each layer are considered to be the same. Four types of boundary conditions (i.e., C–C, C–S, S–S and C–F) are considered. It is observed that the increase of the two foundation parameters leads to an increase of the fundamental frequency parameters. It is seen from the tables that for an FG sandwich beam of fixed foundation parameters and boundary conditions, in most cases, the fundamental frequency parameters decrease as the power-law index increases. There exist some cases where the fundamental frequency parameters increase as the power-law index increases.

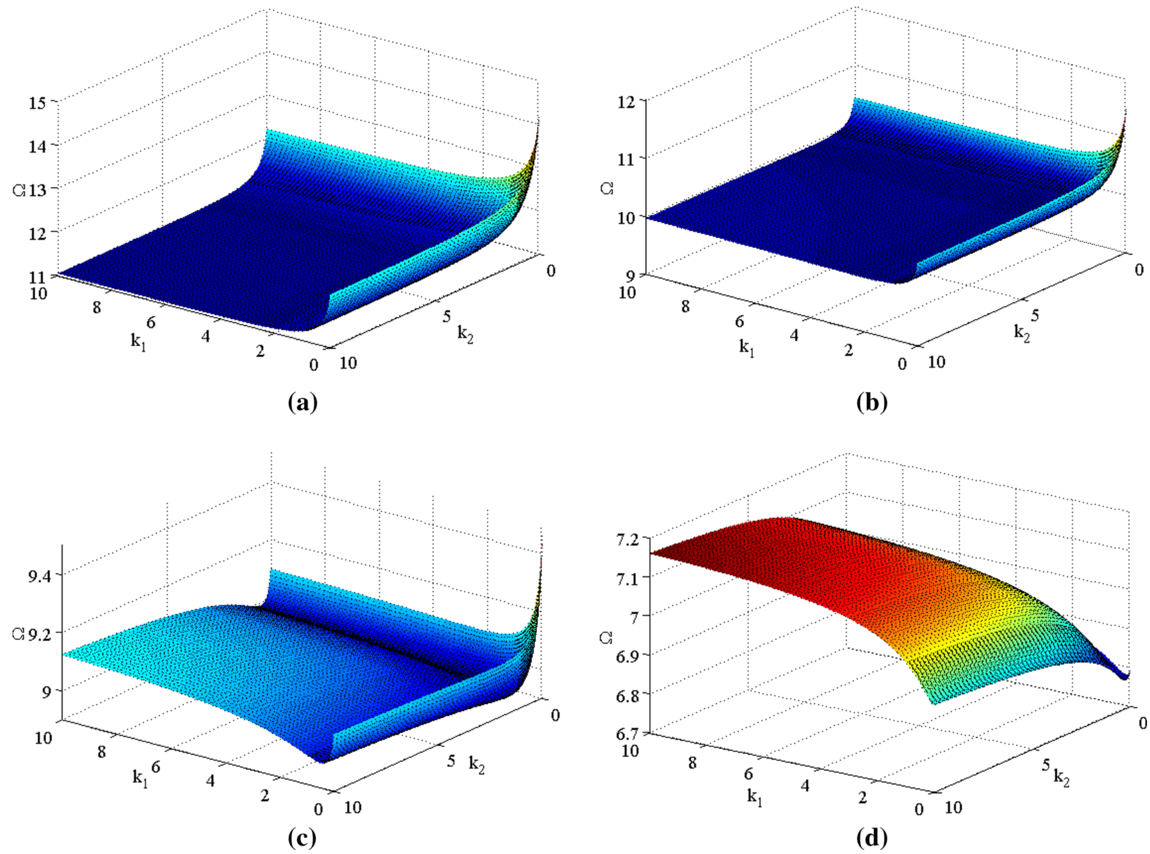

Fig. 7 Variations of the fundamental frequency parameters of the Type I FG sandwich beam resting on an elastic foundation with $K_w = 100$ and $K_s/\pi^2 = 1$: **a** C–C; **b** C–S; **c** S–S; **d** C–F

3.4 Parameter study

Figures 5 and 6 depict the variations of the fundamental frequency parameters for FG sandwich beams with different power-law indices. Both Voigt model and Mori–Tanaka model are used. Four types of boundary conditions including C–C, C–S, S–S and C–F are considered. It is obvious from the figures that the Voigt solution is always larger than the M-T solution for the same condition. It is observed that, on the whole, the increase of the power-law indices leads to a decrease of the frequency parameters. It is worth noting that for the Type II beam the frequency parameters increase slowly as the power-law index increases when the power-law index is larger than some certain value.

Figures 7, 8, 9 and 10 show the variation of fundamental frequency parameters for FG sandwich beams resting on an elastic foundation. Only the Mori–Tanaka scheme is used in this study. It is seen that the variations of fundamental frequency parameters versus power-law index are more complex. Those variations of frequency parameters are strongly influenced by the boundary conditions and foundation parameters. When the foundation parameters are taken as $K_w = 100$ and $K_s/\pi^2 = 1$, for Type I beams with C–C and C–S boundary conditions, the frequency parameters decrease as the power-law indices increase; for the Type I beams with S–S and C–F boundary conditions, the frequency parameters decrease and then increase as the power-law indices increase. When the foundation parameters are taken as $K_w = 1000$ and $K_s/\pi^2 = 2$, for Type I beams with C–C and C–S boundary conditions the frequency parameters decrease and then increase as the power-law indices increase; for the Type I beams with S–S and C–F boundary conditions, the frequency parameters increase as the power-law indices increase. There exist similar conditions for the Type II sandwich beams.

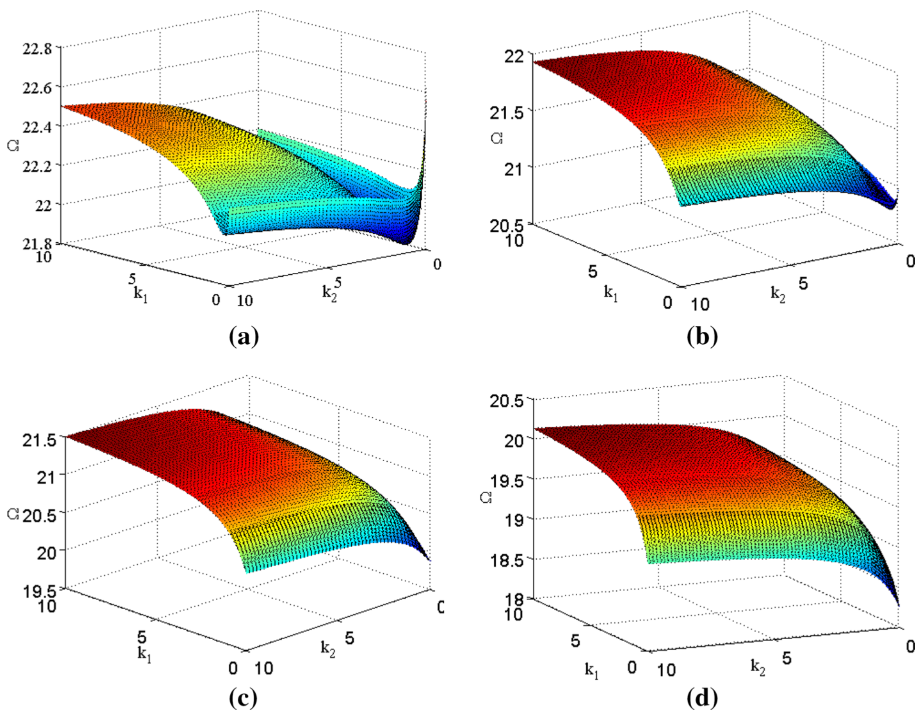

Fig. 8 Variations of the fundamental frequency parameters of the Type I FG sandwich beam resting on an elastic foundation with $K_w = 1000$ and $K_s/\pi^2 = 2$: **a** C–C; **b** C–S; **c** S–S; **d** C–F

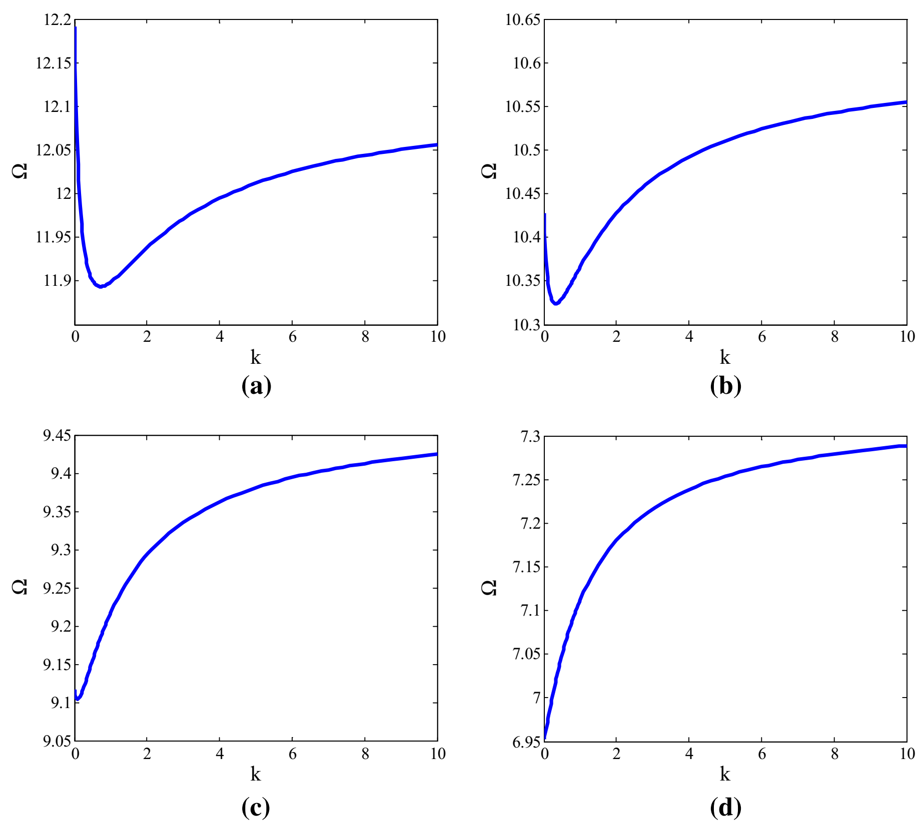

Fig. 9 Variations of the fundamental frequency parameters of the Type II FG sandwich beam resting on an elastic foundation with $K_w = 100$ and $K_s/\pi^2 = 1$: **a** C–C; **b** C–S; **c** S–S; **d** C–F

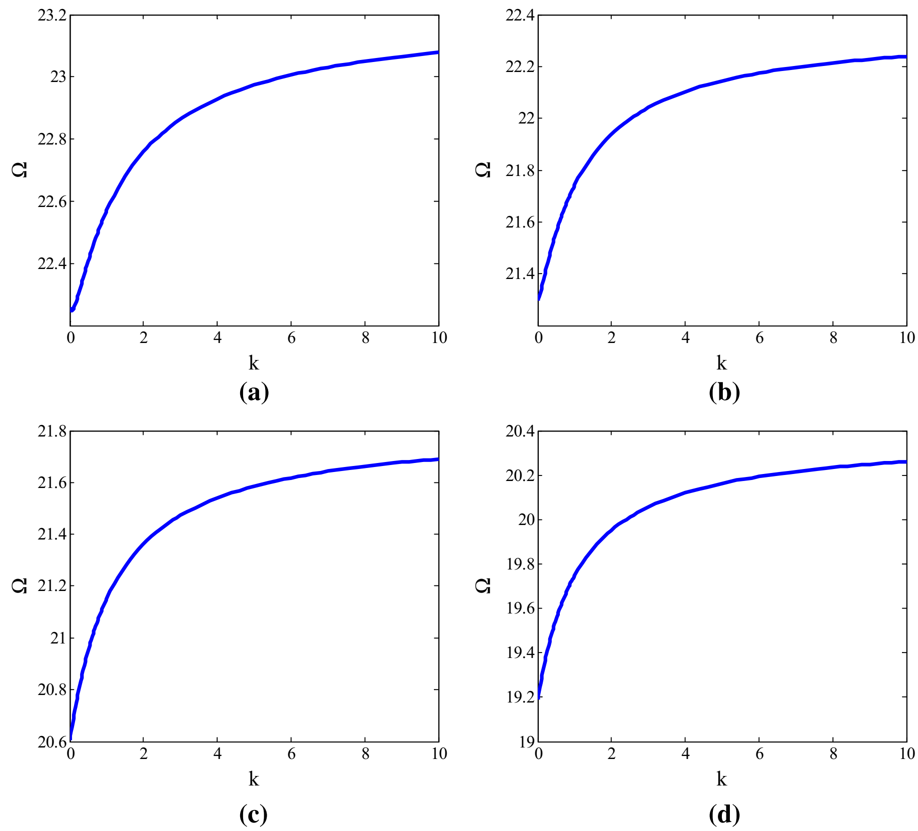

Fig. 10 Variations of the fundamental frequency parameters of the Type II FG sandwich beam resting on an elastic foundation with $K_w = 1000$ and $K_s/\pi^2 = 2$: **a** C–C; **b** C–S; **c** S–S; **d** C–F

4 Conclusions

An accurate solution is presented for free vibration of functionally graded sandwich beams with general boundary conditions and resting on a Pasternak elastic foundation. Two types of common sandwich beams, namely beams with functionally graded face sheets and isotropic core and beams with isotropic face sheets and functionally graded core, are considered. The bilayered and single-layered functionally graded beams are obtained as special cases of sandwich beams. The effective material properties of functionally graded materials are assumed to vary continuously in the thickness direction according to simple power-law distributions in terms of the volume fraction of constituents and are estimated by Voigt's model and Mori–Tanaka's scheme. Based on the first-order shear deformation theory, the governing equations and boundary conditions can be obtained by Hamilton's principle and can be solved using the modified Fourier series method which consists of the standard Fourier cosine series and several supplemented functions. A variety of numerical examples are presented to demonstrate the convergence, reliability and accuracy of the present method. Numerous new vibration results for functionally graded sandwich beams with general boundary conditions and resting on elastic foundations are given, which can serve as benchmark solutions. In addition, the influence of the power-law indices and foundation parameters on the frequencies of the sandwich beams is also investigated.

Acknowledgments The authors gratefully acknowledge the financial support from the National Natural Science Foundation of China (Nos. 51175098 and 51279035) and the Fundamental Research Funds for the Central Universities of China (No. HEUCFQ1401).

Appendix A

$$f_1(x) = \sum_{m=0}^{M} \alpha_1^0 \cos\lambda_m x \Rightarrow \alpha_1^0 = \begin{cases} \frac{1}{12} & m = 0 \\ \frac{2L\left(6-6(-1)^m-m^2\pi^2\right)}{m^4\pi^4} & m \neq 0 \end{cases}, \tag{A.1}$$

$$f_2(x) = \sum_{m=0}^{M} \alpha_2^0 \cos\lambda_m x \Rightarrow \alpha_2^0 = \begin{cases} -\frac{1}{12} & m = 0 \\ \frac{2L\left(6-6(-1)^m+m^2\pi^2(-1)^m\right)}{m^4\pi^4} & m \neq 0 \end{cases}, \tag{A.2}$$

$$f_1'(x) = \sum_{m=0}^{M} \alpha_1^1 \cos\lambda_m x \Rightarrow \alpha_1^1 = \begin{cases} 0 & m = 0 \\ \frac{8+4(-1)^m}{m^2\pi^2} & m \neq 0 \end{cases}, \tag{A.3}$$

$$f_2'(x) = \sum_{m=0}^{M} \alpha_2^1 \cos\lambda_m x \Rightarrow \alpha_2^1 = \begin{cases} 0 & m = 0 \\ \frac{4+8(-1)^m}{m^2\pi^2} & m \neq 0 \end{cases}, \tag{A.4}$$

$$f_1''(x) = \sum_{m=0}^{M} \alpha_1^2 \cos\lambda_m x \Rightarrow \alpha_1^2 = \begin{cases} -\frac{1}{L} & m = 0 \\ \frac{-12+12(-1)^m}{Lm^2\pi^2} & m \neq 0 \end{cases}, \tag{A.5}$$

$$f_2''(x) = \sum_{m=0}^{M} \alpha_2^2 \cos\lambda_m x \Rightarrow \alpha_2^2 = \begin{cases} \frac{1}{L} & m = 0 \\ \frac{-12+12(-1)^m}{Lm^2\pi^2} & m \neq 0 \end{cases}, \tag{A.6}$$

$$\sin\lambda_m x = \sum_{i=0}^{M} \beta_i^m \cos\lambda_i x \Rightarrow \beta_i^m = \begin{cases} 0 & i = 0 \\ \frac{1-(-1)^i}{i\pi} & i \neq 0, m = 0 \\ \frac{2i\left((-1)^{m+i}-1\right)}{(m^2-i^2)\pi} & i \neq 0, m \neq 0 \end{cases}. \tag{A.7}$$

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