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ORIGINAL PAPER

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Geometric theory on the dynamics of a position-dependent mass particle

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Abstract The connection between geometry and dynamics is a canonical subject of analytical mechanics. A very traditional issue of this topic is the transformation of the mechanical problem at hand into a shortest-path problem. This means the mathematical translation of the dynamical problem into a problem of finding the geodesic of a certain space. In the classical domain of conservative systems, especially following the famous book of Lanczos, this translating bridge is established by the usual condition of constant total energy. By nature, the motion of a particle with position-dependent mass is not a conservative problem. Therefore, the classical geometrical theory of mechanics is not straightforwardly applicable. Given that, we here aim at developing the geometrical theory for the mechanics of a position-dependent mass particle. This is our intended contribution. To our best knowledge, the content of our single investigation is original within this variable mass context. Our theory will be developed in the light of the inverse problem of Lagrangian mechanics, which will accordingly sets the variational framework. From that, we will demonstrate the proper generalization of Euler-Maupertuis' principle and the following generalization of Jacobi's principle, which, analogously to the classical procedure, can be seen as intermediate steps to enter geometrical arguments. Then, the corresponding geodesic will appear. Finally, as a closing result, a theorem on the mathematical equivalence between such geodesic and the equation of motion of a position-dependent mass particle will be proved. Our investigation aims at providing the reader with a fundamental contribution to the geometry of variable mass mechanics.

1 Introduction

The study of variable mass systems is recognized to be a particular research field of mechanics. The subject is relevant to the solution of engineering problems and also drives forward investigations of fundamental nature (see, e.g., [1–25]). The unfamiliar reader can find a comprehensive collection of these results in the recently published book [16].

In view of the fact that the basic equations of mechanics were primarily conceived under the hypothesis of constant mass, generalizing considerations are necessary for the treatment of variable mass systems. This issue has been addressed in a series of articles, and an appropriate formulation has been constructed (see, in particular, [3,4,9,12,15,19,22,23,25]). In the present contribution, we aim at giving an original step in this theoretical field: our objective is to situate the dynamics of a position-dependent mass particle in a geometrical level.

This type of problem is a canonical subject within analytical mechanics, but, as originally addressed, the corresponding results do not involve variable mass systems. The intrinsic interest in the subject rests on the elegant possibility of mathematically transforming a conservative mechanical problem into a geodesic problem. This meaningful connection is found to be treated, to some extent, in classical books of mechanics (see, e.g.,

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[26, Chap. I, Sect. 5 and Chap. V, Sect. 7], [27, Chap. XV, Sect. 186]). However, as commented by Lanczos [26, p. 17, footnote 1], "The most exhaustive investigation of the subject, based on the systematic use of tensor calculus, is due to Synge [28]."

To connect the dynamics of a position-dependent mass particle with geometry, we will adopt an approach which is classical and well established: we will first demonstrate the generalized version of Euler-Maupertuis' principle for a position-dependent mass particle and, in the next, the corresponding generalization of Jacobi's principle. This will give us a way toward our aimed geometric theory.

According to our best knowledge, such investigation has not been presented yet. We expect here to furnish a single and original contribution in the field of the analytical mechanics of variable mass systems. We further emphasize that our work essentially has a theoretical character.

The upcoming content is organized as explained in the following. In Sect. 2, we will introduce fundamental notions on the classical Euler-Maupertuis principle and on the classical Jacobi principle. In Sect. 3, we will set the mathematical groundworks of our article. First, in Sect. 3.1, we will briefly recover some of the results on the inverse problem of Lagrangian mechanics for a position-dependent mass particle. These results are found to be explained in detail in our previous articles [2,4]. This will be done with the intent of defining the initial variational basis, from which our theory will be developed. Second, in Sect. 3.2, we will bring out mathematical identities of the calculus of variations. These identities will be used to guide, in Sect. 4, the demonstration of the generalized Euler-Maupertuis principle for a position-dependent mass particle. Next, in Sect. 5, we will demonstrate the corresponding generalization of Jacobi's principle. In Sect. 6, we will use this form of Jacobi's principle to mathematically transform the dynamics of a position-dependent mass particle into a problem of geometry. Such discussion will be divided into two parts: in Sect. 6.1, we will derive the geodesic equation that is associated with the dynamics of a position-dependent mass particle; then, in Sect. 6.2, we will prove that this particular geodesic equation is mathematically equivalent to the equation of motion of such particle. This latter result will be presented via a theorem.

We remark that, in the course of the article, classical references in analytical mechanics will be repeatedly mentioned. With that, our intention is only to highlight the bases on which our results are being developed.

2 On the classical Euler-Maupertuis principle and on the classical Jacobi principle

According to Pars [29, Chap. XXVII], the Maupertuis principle (of least action) is such that "the action is stationary for the actual path in comparison with neighboring paths having the same endpoints (in the q-space) and the same energy." As explained by Whittaker [27, Chap. IX, Sect. 100], this is what states the essential difference with respect to Hamilton's principle, that is, while in Hamilton's principle it is the total time that is constant, in Maupertuis' principle it is the total energy that remains unaltered. Moreover, in the domain of the calculus of variations, while Hamilton's principle is associated with the well-known concept of synchronous variations, Maupertuis' principle, on the contrary, is written in terms of asynchronous variations (see, e.g., [30, Chap. 2], [31, Chap. 2 and Chap. 8.6]).

There is a very interesting historical fact involving the Maupertuis principle. It is the so-called "Euler-Maupertuis episode." In the words of Lanczos [26, pp. 345–346]: "The priority of Maupertuis' discovery was assailed by Koenig, who claimed that Leibniz expressed the same idea in a private letter (…). In the ensuing controversy, Euler defended most emphatically the priority rights of Maupertuis. The peculiar thing in this defense is that Euler himself had discovered the principle at least one year before Maupertuis, and in an entirely correct form (…). Although Euler must have seen the weakness of Maupertuis' argument, he refrained from any criticism, and refrained from so much as mentioning his own achievements in this field, putting all his authority in favor of proclaiming Maupertuis as the inventor of the principle of least action."

For this historical reason, we have adopted the terminology "Euler-Maupertuis' principle" instead of "Maupertuis' principle."

The Jacobi principle means a "geometrisation" of the Euler-Maupertuis principle. Dugas [32, p. 408] expounds that "this geometrisation, which is obtained by considering trajectories which correspond to the same *total energy*, and which explains the part played by this principle in many physical theories, was to be the concern, after Jacobi, of Liouville (1856), Lipschitz (1871), Thomson and Tait (1879), Levi-Civita (1896) and Darboux. The last-named devoted two chapters of his *Leçons sur la théorie générale des surfaces* to this topic." As explained by Lanczos [26, p. 135] and by Goldstein et al. [31, p. 361], Jacobi's form of the least action principle yields the path of the system in a certain space, not the motion in time.

A significant property of Jacobi's principle is that it overcomes an inconvenient aspect of the Euler-Maupertuis principle. Lanczos [26, p. 134] comments that "(…) Jacobi pointed out that this (i.e. the Euler-Maupertuis principle) is unsatisfactory because the time t cannot be used as an independent variable in the variational problem." The point in question is that, as emphasized by Whittaker [27, p. 248], "(…) time is correlated to the coordinates in such a way as to satisfy the same equation of energy." Therefore, in order to obtain the equation of motion from the Euler-Maupertuis principle, it appears the necessity of using the restriction of constant total energy as an auxiliary condition via the Lagrangian λ-method (see [26, pp. 136–138], [29, pp. 545–546]). Jacobi's principle then comes out as a modified form of the Euler-Maupertuis principle in which this restriction is discarded (see [29, p. 546]). Owing to this modification, Jacobi's principle acquires the significance of a simplified variational problem with a geometrical character (see [26, Chap. V, Sect. 7]). This noteworthy aspect, which is one of the mainstays of the geometry of the classical analytical dynamics (see, e.g., [28], [33, Chap. IV]), will be here extended to the dynamics of a position-dependent mass particle.

3 Mathematical groundworks

This section sets the mathematical groundworks of the article.

3.1 The inverse problem of Lagrangian mechanics for a position-dependent mass particle

The Meshchersky's equation

$$m\ddot{q} - Q - (w - \dot{q})\,\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \tag{1}$$

is the basic equation of motion of a variable mass particle, where m is the varying mass, q is the generalized coordinate, Q is the corresponding generalized force, and w is the absolute velocity at which mass is expelled (or joined).

Assuming that $m = m(q)$, $Q = -\mathrm{d}V(q)/\mathrm{d}q$, where $V = V(q)$ is the potential energy, and that $w = k\dot{q}$, where $k = \text{const.}$; Eq. (1) becomes

$$m(q)\ddot{q} + \frac{\mathrm{d}V(q)}{\mathrm{d}q} - \alpha\dot{q}^2\frac{\mathrm{d}m(q)}{\mathrm{d}q} = 0, \tag{2}$$

where $\alpha = k - 1 = \text{const}$.

Respecting such assumptions, Eq. (2) will be here called the equation of motion of a position-dependent mass particle.

The inverse problem of Lagrangian mechanics asserts that Eq. (2) comes from the variational principle

$$\delta\int_{t_1}^{t_2}\tilde{L}\mathrm{d}t = 0, \tag{3}$$

where t_1, t_2 are the limiting instants and

$$\tilde{L} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 - \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q. \tag{4}$$

Using Eq. (4) in the classical identity $\tilde{H} = (\partial\tilde{L}/\partial\dot{q})\dot{q} - \tilde{L}$, where $\tilde{H}$ is the Hamiltonian, we find

$$\tilde{H} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q. \tag{5}$$

The canonical momentum $\tilde{p}$ is obtained inserting Eq. (4) into the corresponding definition $\tilde{p} = \partial\tilde{L}/\partial\dot{q}$, that is,

$$\tilde{p} = m(q)^{-2\alpha}\dot{q}. \tag{6}$$

Substituting Eq. (6) in (5), we alternatively write the Hamiltonian $\tilde{H}$ in terms of the canonical variables q and $\tilde{p}$:

$$\tilde{H} = \frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q. \tag{7}$$

Once the Lagrangian $\tilde{L}$ as in Eq. (4) does not explicitly depend on time t, that is, $\tilde{L} = \tilde{L}(q, \dot{q})$, we have that $\tilde{H} = \tilde{E} = \text{const.}$, where $\tilde{E}$ means the generalized energy within the formulation of the inverse problem of Lagrangian mechanics.

Thus, looking at Eqs. (5) and (7), it is immediate that

$$\frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q = \tilde{E} = \text{const.} \tag{8}$$

and

$$\frac{1}{2}\frac{\tilde{p}^2}{m(q)^{-2\alpha}} + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q = \tilde{E} = \text{const.} \tag{9}$$

Still based on this formulation, we derive a next identity.

Take the virtual variation δ of $\tilde{L} = \tilde{L}(q, \dot{q})$:

$$\delta\tilde{L} = \frac{\partial\tilde{L}}{\partial q}\delta q + \frac{\partial\tilde{L}}{\partial\dot{q}}\delta\dot{q}. \tag{10}$$

The second term of the right-hand side of Eq. (10) can be expressed as

$$\frac{\partial\tilde{L}}{\partial\dot{q}}\delta\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{L}}{\partial\dot{q}}\delta q\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}}\right)\delta q. \tag{11}$$

Combining Eqs. (10) and (11), we have

$$\delta\tilde{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{L}}{\partial\dot{q}}\delta q\right) + \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}} + \frac{\partial\tilde{L}}{\partial q}\right)\delta q. \tag{12}$$

In virtue of the variational principle as in Eq. (3), the second term of the right-hand side of Eq. (12) identically vanishes, which renders

$$\delta\tilde{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{L}}{\partial\dot{q}}\delta q\right). \tag{13}$$

This is the required identity.

The results of this Section provide our starting basis. For a more detailed discussion on the formulation of variable mass systems from the perspective of the inverse problem of Lagrangian mechanics, see our previous articles [2–4].

3.2 Synchronous variations and asynchronous variations

Having in mind that q is the coordinate on the actual motion, we introduce q^* to represent a near coordinate on the varied motion. From that, two types of coordinate variations can be defined: (i) synchronous variations δq, which occur without variation of time, that is,

$$\delta q = q^*(t) - q(t); \tag{14}$$

and (ii) asynchronous variations Δq, which happen with variation of time, that is,

$$\Delta q = q^*(t^*) - q(t), \tag{15}$$

where, in accordance,

$$t^* = \Delta t + t. \tag{16}$$

Regarding the coordinate, synchronous variations and asynchronous variations are related through the expression

$$\Delta q = \delta q + \dot{q}\Delta t. \tag{17}$$

The mathematical form of Eq. (17) holds for an arbitrary function $h(q, \dot{q}, t)$, namely

$$\Delta h = \delta h + \frac{\mathrm{d}h}{\mathrm{d}t}\Delta t, \tag{18}$$

in which $\Delta h = h(q + \Delta q, \dot{q} + \Delta \dot{q}, t + \Delta t) - h(q, \dot{q}, t)$, $\delta h = h(q + \delta q, \dot{q} + \delta \dot{q}, t) - h(q, \dot{q}, t)$ and $\mathrm{d}h/\mathrm{d}t = (\partial h/\partial q)\dot{q} + (\partial h/\partial \dot{q})\ddot{q} + \partial h/\partial t$.

A next relation can be derived from applying the notion of Δ-variation on an arbitrary functional $\int_{t_1}^{t_2} H(q, \dot{q}, t)\mathrm{d}t$. By definition, we have

$$\Delta \int_{t_1}^{t_2} H(q, \dot{q}, t)\mathrm{d}t = \int_{t_1^*}^{t_2^*} H\left(q^*(t^*), \frac{dq^*(t^*)}{\mathrm{d}t^*}, t^*\right)\mathrm{d}t^* - \int_{t_1}^{t_2} H(q, \dot{q}, t)\mathrm{d}t, \tag{19}$$

where

$$\Delta \dot{q} \equiv \frac{\mathrm{d}q^*(t^*)}{\mathrm{d}t^*} - \dot{q}(t). \tag{20}$$

Using Eqs. (15), (16) and (20), the first term of the right-hand side of Eq. (19) can be expanded as

$$\begin{aligned}\int_{t_1^*}^{t_2^*} H\left(q^*(t^*), \frac{\mathrm{d}q^*(t^*)}{\mathrm{d}t^*}, t^*\right)\mathrm{d}t^* &= \int_{t_1^*}^{t_2^*} H(q + \Delta q, \dot{q} + \Delta \dot{q}, t + \Delta t)\mathrm{d}t^* \\ &= \int_{t_1^*}^{t_2^*} H(q, \dot{q}, t)\mathrm{d}t^* + \int_{t_1^*}^{t_2^*} \left(\frac{\partial H}{\partial q}\Delta q + \frac{\partial H}{\partial \dot{q}}\Delta \dot{q} + \frac{\partial H}{\partial t}\Delta t\right)\mathrm{d}t^*. \end{aligned}\tag{21}$$

Changing the variable of integration as $\int_{t_1^*}^{t_2^*}(.)\mathrm{d}t^* = \int_{t_1}^{t_2}(.)(\mathrm{d}t^*/\mathrm{d}t)\mathrm{d}t$, and also recognizing the notation $\Delta H = (\partial H/\partial q)\Delta q + (\partial H/\partial \dot{q})\Delta \dot{q} + (\partial H/\partial t)\Delta t$, Eq. (21) becomes

$$\int_{t_1^*}^{t_2^*} H\left(q^*(t^*), \frac{\mathrm{d}q^*(t^*)}{\mathrm{d}t^*}, t^*\right)\mathrm{d}t^* = \int_{t_1}^{t_2} H(q, \dot{q}, t)\frac{\mathrm{d}t^*}{\mathrm{d}t}\mathrm{d}t + \int_{t_1}^{t_2} \Delta H \frac{\mathrm{d}t^*}{\mathrm{d}t}\mathrm{d}t. \tag{22}$$

Now, putting Eq. (22) in (19), we find

$$\Delta \int_{t_1}^{t_2} H(q, \dot{q}, t)\mathrm{d}t = \int_{t_1}^{t_2} \Delta H \frac{\mathrm{d}t^*}{\mathrm{d}t}\mathrm{d}t + \int_{t_1}^{t_2} H(q, \dot{q}, t)\left(\frac{\mathrm{d}t^*}{\mathrm{d}t} - 1\right)\mathrm{d}t. \tag{23}$$

Seeing Eq. (16), we have

$$\frac{\mathrm{d}t^*}{\mathrm{d}t} = \frac{\mathrm{d}\Delta t}{\mathrm{d}t} + 1. \tag{24}$$

Introducing Eq. (24) into the right-hand side of (23), and retaining only first-order terms in $\Delta(.)$, we find the identity

$$\Delta \int_{t_1}^{t_2} H\mathrm{d}t = \int_{t_1}^{t_2} \Delta H\mathrm{d}t + \int_{t_1}^{t_2} H\frac{\mathrm{d}\Delta t}{\mathrm{d}t}\mathrm{d}t. \tag{25}$$

The content of this section, which assumes a fundamental role within the mathematical theory of the variational principles of mechanics, will be used to develop the demonstration of the next section. For a more complete explanation on this aspect of analytical mechanics, we indicate [27, Chap. IX], [30, Chap. 2].

4 The generalized Euler-Maupertuis principle for a position-dependent mass particle

The demonstration of such generalization of Euler-Maupertuis principle will be developed from the variational basis which is presented in Sect. 3.1. We will nearly follow the mathematical steps that are employed in [30, Chap. 2, Sect. 2.8], where the demonstration of the classical form of this principle is addressed. This procedure starts from the so-called Lagrange's central equation. In the present context, it is Eq. (13) that correspondingly plays such starting role.

For the sake of initiating with a simplified notation, we rewrite Eqs. (4) and (8) as

$$\tilde{L} = \tilde{T} - \tilde{V} \tag{26}$$

and

$$\tilde{T} + \tilde{V} = \tilde{E} = \text{const.}, \tag{27}$$

where

$$\tilde{T} = \frac{1}{2} m(q)^{-2\alpha} \dot{q}^2, \tag{28}$$

$$\tilde{V} = \int m(q)^{-2\alpha-1} \frac{\mathrm{d}V(q)}{\mathrm{d}q} \mathrm{d}q. \tag{29}$$

The first step is to express the left-hand side of Eq. (13) in terms of (26):

$$\delta\tilde{T} - \delta\tilde{V} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}} \delta q\right). \tag{30}$$

This equation is our starting point.

Applying $\delta(.)$ on both sides of Eq. (27), which is our generalized condition of constant energy $\tilde{E}$, we have

$$\delta\tilde{T} + \delta\tilde{V} = 0. \tag{31}$$

Using Eq. (31), we can write the left-hand side of Eq. (30) solely in terms of $\tilde{T}$:

$$2\delta\tilde{T} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}} \delta q\right). \tag{32}$$

Then, specifying Eq. (18) for $h = \tilde{T}$, we obtain

$$\Delta\tilde{T} = \delta\tilde{T} + \frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}\Delta t. \tag{33}$$

In view of Eqs. (17) and (33), Eq. (32) can be put in terms of Δ-variations:

$$2\left(\Delta\tilde{T} - \frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}\Delta t\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \tilde{L}}{\partial \dot{q}}(\Delta q - \dot{q}\Delta t)\right], \tag{34}$$

that is,

$$2\Delta\tilde{T} - 2\frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}\Delta t = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\Delta q\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\dot{q}\Delta t\right). \tag{35}$$

To continue from this point, we evoke the identity

$$\frac{\partial \tilde{L}}{\partial \dot{q}}\dot{q} = 2\tilde{T}, \tag{36}$$

which can be easily verified from manipulating Eqs. (26), (28) and (29).

Inserting Eq. (36) into the second parcel of the right-hand side of (35), we arrive at

$$2\Delta\tilde{T} - 2\frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}\Delta t = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{L}}{\partial\dot{q}}\Delta q\right) - 2\frac{\mathrm{d}(\tilde{T}\Delta t)}{\mathrm{d}t}. \tag{37}$$

Taking the identity

$$\frac{\mathrm{d}(\tilde{T}\Delta t)}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}\Delta t + \tilde{T}\frac{\mathrm{d}\Delta t}{\mathrm{d}t}, \tag{38}$$

Equation (37) can be simplified to

$$2\left(\Delta\tilde{T} + \tilde{T}\frac{\mathrm{d}\Delta t}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{L}}{\partial\dot{q}}\Delta q\right). \tag{39}$$

Now, we integrate both sides of Eq. (39) between instants t_1, t_2:

$$2\left(\int_{t_1}^{t_2}\Delta\tilde{T}\mathrm{d}t + \int_{t_1}^{t_2}\tilde{T}\frac{\mathrm{d}\Delta t}{\mathrm{d}t}\mathrm{d}t\right) = \left(\frac{\partial\tilde{L}}{\partial\dot{q}}\Delta q\right)\Bigg|_{t_1}^{t_2}. \tag{40}$$

Specifying Eq. (25) for $H = \tilde{T}$, we obtain the expression

$$\Delta\int_{t_1}^{t_2}\tilde{T}\mathrm{d}t = \int_{t_1}^{t_2}\Delta\tilde{T}\mathrm{d}t + \int_{t_1}^{t_2}\tilde{T}\frac{\mathrm{d}\Delta t}{\mathrm{d}t}\mathrm{d}t. \tag{41}$$

Last, we substitute Eq. (41) in (40):

$$2\Delta\int_{t_1}^{t_2}\tilde{T}\mathrm{d}t = \left(\frac{\partial\tilde{L}}{\partial\dot{q}}\Delta q\right)\Bigg|_{t_1}^{t_2}, \tag{42}$$

that is,

$$\Delta\int_{t_1}^{t_2}2\tilde{T}\mathrm{d}t = \left(\frac{\partial\tilde{L}}{\partial\dot{q}}\Delta q\right)\Bigg|_{t_1}^{t_2}. \tag{43}$$

If we suppose that the varied motion and the actual motion have the same limiting coordinates, where time and coordinate are accordingly related through Eq. (27), then $\Delta q(t_1) = \Delta q(t_2) = 0$. This simply transforms Eq. (43) into

$$\Delta\int_{t_1}^{t_2}2\tilde{T}\mathrm{d}t = 0. \tag{44}$$

The demonstration is concluded. Namely, Eq. (44) states the generalized Euler-Maupertuis principle for a position-dependent mass particle.

This result can be interpreted as: within the class of paths that satisfy the generalized energy equation $\tilde{E} =$ const. (see Eq. (27)) and that also have the same end-coordinates, in the sense that time is correlated to coordinate in such a way as to satisfy $\tilde{E} =$ const., it is the actual path that gives to the integral $\int_{t_1}^{t_2} 2\tilde{T}\mathrm{d}t$ a stationary value (see [27, Chap. IX, Sect. 100], [29, Chap. XXVII]).

Note also that Eq. (44) is totally analogous to the traditional form of the Euler-Maupertuis principle, that is, $\Delta\int_{t_1}^{t_2} 2T\mathrm{d}t = 0$, where T signifies kinetic energy (see, in particular, [30, p. 48], [31, p. 359]). The difference is that, instead of the real kinetic energy, Eq. (44) contains the generalized energy $\tilde{T} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2$. This is expected because the inverse problem of Lagrangian mechanics, which is being taken as the starting basis of our investigations, mathematically fits the dynamics of a position-dependent mass particle into the traditional standards of analytical mechanics.

5 The generalized Jacobi principle for a position-dependent mass particle

With the Euler-Maupertuis principle as in Eq. (44) at hand, we become ready to demonstrate the corresponding generalization of Jacobi principle. To achieve this result, we will proceed in order to change the variable of integration, in Eq. (44), from t to q. For that, the generalized condition of constant energy $\tilde{E}$ (see Eq. (8), or, equivalently, (27)) will be invoked.[1] This is the classical method, but, as originally conceived for conservative systems, it involves the condition of constant total (real) energy instead (see [26, Chap. V, Sect. 6], [31, pp. 360–361], [34, pp. 141–142]).

Due to the trivial identity $\dot{q} = \mathrm{d}q/\mathrm{d}t$, we can manipulate Eq. (8) to

$$\mathrm{d}t = \sqrt{\frac{m(q)^{-2\alpha}}{2\left(\tilde{E} - \tilde{V}(q)\right)}}\mathrm{d}q. \tag{45}$$

Next, we take Eq. (27), which is equivalent to (8), in the form of

$$\tilde{T} = \tilde{E} - \tilde{V}(q). \tag{46}$$

Now, we simply substitute Eqs. (45) and (46) in (44):

$$\Delta \int_{t_1}^{t_2} 2\tilde{T}\mathrm{d}t = 0 \Rightarrow \Delta \int_{q_1}^{q_2} 2\left(\tilde{E} - \tilde{V}(q)\right)\sqrt{\frac{m(q)^{-2\alpha}}{2\left(\tilde{E} - \tilde{V}(q)\right)}}\mathrm{d}q = 0, \tag{47}$$

that is,

$$\Delta \int_{q_1}^{q_2} \sqrt{2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right)}\mathrm{d}q = 0. \tag{48}$$

This concludes the demonstration: Eq. (48) states the generalized Jacobi's principle for a position-dependent mass particle.

Equation (48) is analogous to the classical form of Jacobi's principle

$$\Delta \int_{q_1}^{q_2} \sqrt{2m(E - V(q))}\mathrm{d}q = 0, \tag{49}$$

where m is the (constant) mass of a particle, $V(q)$ is the potential energy, and E is the total energy, for which $T + V = E = \text{const.}$, with $T = \frac{1}{2}m\dot{q}^2$ being the kinetic energy (see, e.g., [31, p. 361]).

In fact, since we are embedding our discussion in the formulation that follows from the inverse problem of Lagrangian mechanics, it is expected that the generalized energies $\tilde{E}$ and $\tilde{V}$ appear in the place of E and V (compare Eqs. (48) and (49)). To explain the substitution of m by $m(q)^{-2\alpha}$, which also can be observed when going from Eq. (49) to (48), we evoke an interesting remark of Lanczos [26, pp. 21–22]: "The principles of analytical mechanics have shown that the really fundamental quantity which characterizes the inertia of mass is not the momentum but the kinetic energy." Within the formulation of the inverse problem, this remark can be extended to a position-dependent mass particle through the definition of $\tilde{T}$, that is, $m(q)^{-2\alpha} = 2\tilde{T}/\dot{q}^2$ (see Eq. (28)). Given the direct analogy between the right-hand side of the expressions $m(q)^{-2\alpha} = 2\tilde{T}/\dot{q}^2$ and $m = 2T/\dot{q}^2$, the occurrence of $m(q)^{-2\alpha}$ in Eq. (48) can be then understood in terms of such energy arguments.

In the next section, we will use Eq. (48) to bring out the elegant perspective of considering the dynamics of a position-dependent mass particle as a problem of geometry.

[1] Emphatically, we remark that, to establish this bridge of understanding between the dynamics of a position-dependent mass particle and the theoretical framework of the classical analytical mechanics, we are harmoniously using the formulation of the inverse problem of Lagrangian mechanics (see Sect. 3.1).

6 The connection between the dynamics of a position-dependent mass particle and geometry

The result of Sect. 5 (see Eq. (48)) gives us the way to connect the dynamics of a position-dependent mass particle with geometry. This aspect begins to be revealed when, with the purpose of achieving a geometrical interpretation, we write Eq. (48) in the form of

$$\delta \int_{s_1}^{s_2} \mathrm{d}s = 0, \tag{50}$$

in which

$$\mathrm{d}s = \sqrt{2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right)}\mathrm{d}q \tag{51}$$

is understood, in the context of geometry, as the line-element associated with the dynamics of a position-dependent mass particle.

This idea becomes clearer when we remember that the geometry of a space is determined by postulating the line-element as[2]

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}q^i\mathrm{d}q^j, \tag{52}$$

that is,

$$\mathrm{d}s = \sqrt{g_{ij}\mathrm{d}q^i\mathrm{d}q^j}, \tag{53}$$

where g_{ij} are the metric coefficients and q_i are the generalized coordinates (see, e.g., [26, Chap. I, Sect. 5], [28], [33, p. 25], [35, Chap. 2.5]).

In the one-dimensional case, Eq. (53) becomes

$$\mathrm{d}s = \sqrt{g_{11}}\mathrm{d}q^1, \tag{54}$$

and, omitting the indexes,

$$\mathrm{d}s = \sqrt{g}\mathrm{d}q. \tag{55}$$

Comparing Eqs. (51) and (55), we obtain

$$g = 2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right). \tag{56}$$

This is the metric coefficient associated with the dynamics of a position-dependent mass particle.

Noticing that Eq. (50) represents the minimizing of the integral $\int_{s_1}^{s_2} \mathrm{d}s$, we have arrived at the geometrical problem of finding the shortest-path, in a certain space, between two fixed endpoints. It signifies that we have equivalently transformed the dynamical problem of a position-dependent mass particle into a problem of geometry.

In the light of this idea, we will in the following demonstrate a theorem asserting that the equation of motion of a position-dependent mass particle (see Eq. (2)) is mathematically equivalent to the geodesic equation of the metric $2m(q)^{-2\alpha}(\tilde{E} - \tilde{V}(q))$ (see Eq. (56)). Before proceeding to the theorem, we will first derive such geodesic equation. This derivation will be done by specifying the general geodesic equation for the metric $2m(q)^{-2\alpha}(\tilde{E} - \tilde{V}(q))$.

[2] The conventional nomenclature that appears in the textbooks of geometry (see, e.g., [33]) is being employed to write Eq. (52) and, accordingly, the related expressions.

6.1 The geodesic equation of the metric $2m(q)^{-2\alpha}(\tilde{E}-\tilde{V}(q))$

For the general line-element $\mathrm{d}s=\sqrt{g_{ij}\mathrm{d}q^i\mathrm{d}q^j}$ (see Eq. (53)), the corresponding minimum principle on the shortest-path, that is, $\delta\int_{s_1}^{s_2}\mathrm{d}s=0$, assumes the form of

$$\delta\int_{s_1}^{s_2}\sqrt{g_{ij}\frac{\mathrm{d}q^i}{\mathrm{d}s}\frac{\mathrm{d}q^j}{\mathrm{d}s}}\mathrm{d}s=0, \tag{57}$$

where the generalized coordinates are parameterized in terms of the arc-length s. The solution of this variational problem is given by the geodesic equations

$$\frac{\mathrm{d}^2q^i}{\mathrm{d}s^2}+\Gamma^i_{jk}\frac{\mathrm{d}q^j}{\mathrm{d}s}\frac{\mathrm{d}q^k}{\mathrm{d}s}=0, \tag{58}$$

where

$$\Gamma^i_{jk}=\frac{1}{2}g^{li}\left(\frac{\partial g_{jl}}{\partial q^k}+\frac{\partial g_{kl}}{\partial q^j}-\frac{\partial g_{jk}}{\partial q^l}\right) \tag{59}$$

is the Christoffel symbol, and

$$g^{il}g_{lk}=\delta^i_k, \tag{60}$$

with δ^i_k being the Kronecker delta symbol, namely $\delta^i_k=1$ when $i=k$, and $\delta^i_k=0$ when $i\neq k$. This is found to be shown in, for instance, [33, pp. 43–44], [35, pp. 106–108].

In the one-dimensional case, Eqs. (57)–(60) are simplified as

$$\delta\int_{s_1}^{s_2}\sqrt{g_{11}\frac{\mathrm{d}q^1}{\mathrm{d}s}\frac{\mathrm{d}q^1}{\mathrm{d}s}}\mathrm{d}s=0, \tag{61}$$

$$\frac{\mathrm{d}^2q^1}{\mathrm{d}s^2}+\Gamma^1_{11}\frac{\mathrm{d}q^1}{\mathrm{d}s}\frac{\mathrm{d}q^1}{\mathrm{d}s}=0, \tag{62}$$

$$\Gamma^1_{11}=\frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial q^1}, \tag{63}$$

$$g^{11}g_{11}=1. \tag{64}$$

Equations (63) and (64) can be used to write the one-dimensional geodesic equation (62) as

$$\frac{\mathrm{d}^2q^1}{\mathrm{d}s^2}+\frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial q^1}\frac{\mathrm{d}q^1}{\mathrm{d}s}\frac{\mathrm{d}q^1}{\mathrm{d}s}=0. \tag{65}$$

For the sake of simplicity, we now omit the indexes of Eqs. (61) and (65):

$$\delta\int_{s_1}^{s_2}\sqrt{g\frac{\mathrm{d}q}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}}\mathrm{d}s=0, \tag{66}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2}+\frac{1}{2g}\frac{\partial g}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}=0. \tag{67}$$

Equation (67) is the solution of the shortest-path problem $\delta\int_{s_1}^{s_2}\mathrm{d}s=0$ (or, equivalently, Eq. (66)) for the line-element $\mathrm{d}s=\sqrt{g}\mathrm{d}q$.

Seen that, we are ready to find the geodesic equation of the metric $2m(q)^{-2\alpha}(\tilde{E}-\tilde{V}(q))$. For this derivation, we transform Eq. (50) into

$$\delta\int_{s_1}^{s_2}\sqrt{2m(q)^{-2\alpha}\left(\tilde{E}-\tilde{V}(q)\right)\frac{\mathrm{d}q}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s}}\mathrm{d}s=0, \tag{68}$$

which means that, accordingly, we have parameterized the generalized coordinate of our variable mass problem in terms of the arc-length s.

Equation (68) means the shortest-path problem associated with the dynamics of a position-dependent mass particle.

Comparing Eqs. (66) and (68), we immediately recover the definition of the metric associated with the dynamical problem (see Eq. (56)).

Now, we insert Eq. (56) into (67):

$$\frac{d^2q}{\mathrm{d}s^2} - \frac{1}{2\left(\tilde{E} - \tilde{V}(q)\right)}\left[\frac{2\alpha}{m(q)}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\left(\tilde{E} - \tilde{V}(q)\right) + \frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q}\right]\frac{\mathrm{d}q}{\mathrm{d}s}\frac{\mathrm{d}q}{\mathrm{d}s} = 0. \tag{69}$$

This is the geodesic equation which is connected with the dynamics of a position-dependent mass particle.

6.2 A theorem

As a closing result, we announce the following theorem:

Theorem *the geodesic equation of the metric* $g = 2m(q)^{-2\alpha}(\tilde{E} - \tilde{V}(q))$ *(see Eq. (69)) is mathematically equivalent to the equation of motion of our problem (see Eq. (2)).*

Proof the metric $g = 2m(q)^{-2\alpha}(\tilde{E} - \tilde{V}(q))$ is such that it defines the line-element $\mathrm{d}s = \sqrt{2m(q)^{-2\alpha}(\tilde{E} - \tilde{V}(q))}\mathrm{d}q$ [see Eq. (51)]. Dividing both sides of Eq. (51) by $\mathrm{d}t$, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right)}\frac{\mathrm{d}q}{\mathrm{d}t}, \tag{70}$$

that is,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right)}\dot{q}. \tag{71}$$

Squaring both sides of Eq. (71), we find

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = 2m(q)^{-2\alpha}\left(\tilde{E} - \tilde{V}(q)\right)\dot{q}^2. \tag{72}$$

It is known that the generalized energy equation $\tilde{T} + \tilde{V}(q) = \tilde{E}$ (see Eq. (27)) holds. Thence, using the definition $\tilde{T} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2$ (see Eq. (28)) in Eq. (27), we get

$$\dot{q}^2 = \frac{2\left(\tilde{E} - \tilde{V}(q)\right)}{m(q)^{-2\alpha}}. \tag{73}$$

The substitution of Eq. (73) in the right-hand side of (72) gives

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = 4\left(\tilde{E} - \tilde{V}(q)\right)^2, \tag{74}$$

that is,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 2\left(\tilde{E} - \tilde{V}(q)\right). \tag{75}$$

Now, we consider Eq. (75) to transform the derivatives with respect to the arc-length s, which appear in the geodesic equation (69), into derivatives with respect to the time t. For that, the chain rule of differentiation will be applied.

First, we express $\mathrm{d}q/\mathrm{d}s$ as

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}q}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}. \tag{76}$$

In view of Eq. (75), Eq. (76) becomes

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\mathrm{d}q}{\mathrm{d}t} \frac{1}{2\left(\tilde{E} - \tilde{V}(q)\right)}. \tag{77}$$

This is the required transformation for $\mathrm{d}q/\mathrm{d}s$.
To find the transformation of $\mathrm{d}^2q/\mathrm{d}s^2$, we initially write the trivial identity

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}q}{\mathrm{d}s}\right). \tag{78}$$

Then, the right-hand side of Eq. (78) is conveniently rewritten by means of the chain rule:

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\right) = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}q}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}\right]. \tag{79}$$

Developing the derivative $\mathrm{d}[.]/\mathrm{d}t$ of the right-hand side of Eq. (79), we have

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}\left\{\frac{\mathrm{d}^2q}{\mathrm{d}t^2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1} - \frac{\mathrm{d}q}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-2}\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right\}. \tag{80}$$

Since the differentiation of Eq. (75) with respect to t produces

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -2\frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q}\frac{\mathrm{d}q}{\mathrm{d}t}, \tag{81}$$

we can finally put Eq. (80) in terms of (75) and (81):

$$\frac{\mathrm{d}^2q}{\mathrm{d}s^2} = \frac{1}{4\left(\tilde{E} - \tilde{V}(q)\right)^2}\left\{\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{\left(\tilde{E} - \tilde{V}(q)\right)}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2\frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q}\right\}. \tag{82}$$

This is the required transformation for $\mathrm{d}^2q/\mathrm{d}s^2$.
Then, we substitute Eqs. (77) and (82) into (69):

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{2\left(\tilde{E} - \tilde{V}(q)\right)}\frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{\alpha}{m(q)}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 = 0, \tag{83}$$

that is,

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{2\left(\tilde{E} - \tilde{V}(q)\right)}\frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q}\dot{q}^2 - \frac{\alpha}{m(q)}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\dot{q}^2 = 0. \tag{84}$$

Note that, inserting Eq. (28) into (27), we can write

$$\frac{\dot{q}^2}{2\left(\tilde{E} - \tilde{V}(q)\right)} = \frac{1}{m(q)^{-2\alpha}}. \tag{85}$$

Considering Eq. (85) in the second term of (84), we arrive at

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{m(q)^{-2\alpha}}\frac{\mathrm{d}\tilde{V}(q)}{\mathrm{d}q} - \frac{\alpha}{m(q)}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\dot{q}^2 = 0. \tag{86}$$

Finally, taking the definition of $\tilde{V}$ as in Eq. (29), Eq. (86) becomes

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{m(q)}\frac{\mathrm{d}V(q)}{\mathrm{d}q} - \frac{\alpha}{m(q)}\frac{\mathrm{d}m(q)}{\mathrm{d}q}\dot{q}^2 = 0. \tag{87}$$

In fact, multiplying Eq. (87) by $m(q)$ and writing $\mathrm{d}^2q/\mathrm{d}t^2 = \ddot{q}$, we obtain

$$m(q)\ddot{q} + \frac{\mathrm{d}V(q)}{\mathrm{d}q} - \alpha\frac{\mathrm{d}m(q)}{\mathrm{d}q}\dot{q}^2 = 0, \tag{88}$$

which is precisely Eq. (2).

This concludes the proof of our theorem. □

7 Conclusions

As the reader can verify in [26, Chap. I, Sect. 5], the idea of transforming a mechanical problem into a geometrical problem is canonical. Here, we have introduced this idea to the mechanics of a position-dependent mass particle, which represents the overall result of our work.

The inverse problem of Lagrangian mechanics has laid the variational basis for the demonstration of our results. The point in question is that, within the formulation following from the inverse problem, the motion of a position-dependent mass particle as in Eq. (2) is such that it obeys the generalized energy equation $\tilde{E} = \text{const.}$ (see Eq. (27)). This has motivated us to proceed toward the realm of geometry in accordance to the classical approach, that is, going from Euler-Maupertuis' principle to Jacobi's principle and, thence, arriving at the geodesic problem. Pursuing this way, we have demonstrated a chain of original results, which are specially written for the dynamics of a position-dependent mass particle:

1. The generalized Euler-Maupertuis principle (see Eq. (44));
2. The corresponding generalization of Jacobi's principle (see Eq. (48));
3. The particular line-element (see Eq. (51));
4. The corresponding metric (see Eq. (56));
5. The associated shortest-path problem (see Eq. (68));
6. The resulting geodesic equation (see Eq. (69));
7. The theorem of Sect. 6.2.

This chain of results is such that it establishes the translating bridge between the dynamics of a position-dependent mass particle and geometry. As a closure, the theorem of Sect. 6.2 mathematically proves such a translation.

In conclusion, the reader finds here an original contribution to the mathematical theory of the analytical mechanics of variable mass systems. This contribution is such that it situates the dynamics of a position-dependent mass particle in the level of geometry.

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