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Hossein Karimpour · Mostafa Pirmoradian · Mehdi Keshmiri

Instance of hidden instability traps in intermittent transition of moving masses along a flexible beam

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Abstract The problem of an elastic beam under the periodic loading of successive moving masses is investigated as a pragmatic case for studying dynamic stability of linear time-varying systems. This model serves to highlight the odds of multi-solutions coexistence, a form of hidden instability which reveals dangerous as it may be precipitated by the slightest disturbance or variation in the model. Since no engineering model perfectly represents a physical system, such situations for which Floquet theory naively predicts stability are potentially inevitable. The harmonic balancing method is used in order to thoroughly explore the stability diagrams for detecting these instability gaps. Although this phenomenon has also been described in other physical systems, it has not been addressed for beam–moving mass systems. This result may find particular importance in applications involving self-induced vibrations of elastic structures and hence also appears of practical relevance.

1 Introduction

Investigating dynamic stability of a flexible beam carrying a moving mass is an extensive problem embracing a wide variety of engineering applications. Examples are numerous, such as vehicles or trains transiting across bridges and rails [1], loads transported along cranes span [2], suspended conveying pipe systems [3], high-speed machining processes [4], to enumerate but a few. The presence of vibration instability induced by moving bodies travelling along the elastic structure can be taken into account depending on different aspects associated with the problem. Some of them include investigations by quasi-static methods [5,6], by Doppler waves propagation [7,8], along finite or infinite span [9–11]. The principal concern depends on the case and comprises problems like derailment [8], lateral buckling [12], improving cutting precision in high-speed machining [4], or preventing load swaying in crane transport [2].

The generation of instability is the result of the interaction between the elastic structure and the moving body, modeled in this case as a mass or oscillator instead of a simple load. Different instability phenomena which have been addressed and investigated in the problem of a beam under moving masses include external, internal [13,14] and parametric resonances [15,16]. External resonance arises from closeness of forced excitation

H. Karimpour (✉)
Department of Mechanical Engineering, Faculty of Engineering, University of Isfahan, Isfahan 81746-73441, Iran
E-mail: h.karimpour@eng.ui.ac.ir

M. Pirmoradian
Department of Mechanical Engineering, Khomeinishahr Branch, Islamic Azad University,
Khomeinishahr, Isfahan 84175-119, Iran

M. Keshmiri
Department of Mechanical Engineering, Isfahan University of Technology, Isfahan 84156-83111, Iran

frequency to the natural frequencies of the mass–beam combined system. In other circumstances, the energy transfer to the beam's transversal oscillations, which is ensued from the negative damping effect, leads to instability growth rather than resonance [17,18]. Internal resonance happens when the natural frequencies of the system become commensurable and conditions for strong interaction between different modes subsist. Consequently, energy is interchanged between involved modes, and the response becomes multi-modal. In another instance, parametric excitation appears as another internal factor leading to self-energizing vibrations and occurs when the frequency of excitation is close to twice one of the system's natural frequencies. It is important to clarify that this effect differs from regular resonance as it requires the initial amplitude to exhibit amplification, whereas the latter grows linearly in time regardless of the initial state [18].

Although there is practically no analytic solution for most categories of parametrically excited systems, useful properties can still be deduced for subclasses comprising periodically varying systems. Periodicity of the coefficients results in unique response properties that permit a simplifying analysis. The stability study of these systems is discussed with respect to parameters' variations, represented as stability diagrams in the plane of parameters. Among existing methods, Floquet's theory provides the essential tools to differentiate the multiple regions appearing in the stability diagram by numerically solving the problem for one single time period [19,20]. The boundaries separating those regions, referred as transition curves, represent parameter values for which periodic solutions occur (indeed an eigenvalue problem is resolved [21]). Instability regions usually illustrate in the form of resonance tongues which is a generic feature of differential equations with periodic coefficients [21,22].

As far as it concerns the present problem, it seems that only one article [23] has been published yet alluding to instability tongues existence, in contrast to all other studies which demonstrated just one transition curve, probably due to the insufficient sweeping resolution of the parameter plane by conventional methods. In fact, the process of numerical integration required by Floquet's method may change the system's inherent trait which makes detection of the tracked phenomena extremely difficult to capture numerically. Hence the necessity for adopting semi-analytical tools to enhance the identification process is evident, especially when dealing with structurally unstable systems (i.e., sensitive to the slightest changes).

The particular parametric phenomenon we seek to study is coexistence. Coexistence occurs when the tongues of instability cross or overlap, effectively closing the unstable region. The coexistence phenomenon has been treated from a theoretical and practical point of view in [24–28]. In [21,25], the authors addressed conditions under which a special case class of linear ODEs with periodic coefficients can exhibit the coexistence phenomenon. Such problems may be encountered when studying the stability of parametrically excited systems, but the phenomenon is sensitive to slight model variations. The influence of this factor may even be more pronounced when the system is structurally unstable. In regular problems, a change in the system's parameters will result in widening or retracting of the instability tongues, such as those appearing in Mathieu's case, for example. According to the continuous dependency of response characteristics with respect to parameters' variation in regular systems, there is no expectation for the appearance of suddenly born regions of instability or a morphology change in the middle of a quiescent part of the diagram. In contrast, systems for which a small change in parameters results in a severe change of behavior, like coexistence cases, are characterized to belong to the subgroup of "structurally" sensitive systems.

There are various physical systems which are known to exhibit coexistence, including the vibrating elastica [29], the elastic pendulum [30], rain–wind-induced vibrations [21], Josephson junctions [31], variable mass oscillators [28,32], and quasi-periodic potentials in the Schrödinger equation [33]. The present problem is nowadays relevant for the case of modern high-speed transportation trains transiting over bridges or fast lifting/moving cranes in which the dynamic stability of the structure under the sequential rapid loading is an issue of great concern. Any related engineering design process has to be concerned with the importance of recognizing the odds of coexistence occurrence in order to reach a sufficient safety margin, otherwise posing unexpected challenges if neglected. The purpose of diverse studies was to identify instability areas in the space of the system parameters. Nevertheless, the prediction of unattended dynamic behaviors due to the existence of hidden instability traps triggered by eventual modeling uncertainties has not been yet studied.

2 Governing equations

The model upon which the analysis is based may be looked as one of the simplest, nonetheless still including the essentials to take into account the interaction. The vehicle is assumed as a single inertia point mass travelling at constant speed along the bridge span, represented as an Euler–Bernoulli beam. The linear partial differential equation governing the vibrations of the beam excited by the moving mass sequences is derived as [15]

$$\rho A\ddot{v} + EIv^{(4)} + \left(m\ddot{v} + 2mU\dot{v}' + mU^2 v''\right)\big|_{x=Ut} = mg\delta\left(x - Ut\right) \tag{1}$$

where $v\left(x, t\right)$ represents the beam deflection from its static state at the interval x and instant t. Gravitational acceleration, mass density, cross-sectional area, length and flexural rigidity of the uniform beam are denoted by $g,\ \rho$, A, $l,\ EI$, respectively. m and U are the mass and velocity of the moving load, and δ is the Dirac delta function.

The partial differential Eq. (1) is converted to a set of ordinary differential equations on the modal coordinates by the Galerkin method according to the following solution expansion:

$$v\left(x, t\right) = \sum_{i=1}^{\infty} \varphi_i\left(x\right)\ q_i\left(t\right), \qquad \varphi_i\left(x\right) = \sqrt{\frac{2}{l}} \sin\left(i\pi\frac{x}{l}\right) \tag{2}$$

where $q_i(t)$ is the corresponding generalized coordinate of the ith free shape mode function $\varphi_{\mathrm{i}}(x)$. Considering the orthogonality condition between modes, the discretized set of differential equations is obtained as

$$\mathbf{M}\left(t\right)\frac{d^2\mathbf{q}}{dt^2} + \mathbf{B}\left(t\right)\frac{d\mathbf{q}}{dt} + \mathbf{K}\left(t\right)\mathbf{q} = \mathbf{f}\left(t\right) \tag{3}$$

where $\mathbf{q} = [q_1(t), q_2(t), \ldots, q_{\mathrm{n}}(t)]^{\mathrm{T}}$ is the array of the modal coordinates. The components of the defined matrices are expressed as

$$\begin{aligned} M_{ij} &= \delta_{ij} + \frac{m}{\rho A}\varphi_i\left(x_m\right)\varphi_j\left(x_m\right), \\ B_{ij} &= \frac{2mU}{\rho A}\varphi_i\left(x_m\right)\varphi_{j,x}\left(x_m\right), \\ K_{ij} &= \omega_i^2\delta_{ij} + \frac{mU^2}{\rho A}\varphi_i\left(x_m\right)\varphi_{j,xx}\left(x_m\right), \\ f_i &= \frac{mg}{\rho A}\varphi_i\left(x_m\right) \end{aligned} \tag{4}$$

where δ_{ij} and $\omega_i = \left(\frac{i\pi}{l}\right)^2\sqrt{EI/\rho A}$ are the Kronecker delta and the i-th natural frequency of a simply supported beam, respectively. The shape functions are evaluated at $x = x_m$, representing the particle's instant position along the beam. The equation governing the modal coordinate truncated to the first mode becomes

$$\begin{aligned} &\left(1 + 2\frac{m}{\rho Al}\sin^2\left(\frac{\pi Ut}{l}\right)\right)\frac{d^2q_1}{dt^2} + \left(4\frac{m\pi U}{\rho Al^2}\sin\left(\frac{\pi Ut}{l}\right)\cos\left(\frac{\pi Ut}{l}\right)\right)\frac{dq_1}{dt} \\ &+ \left(\omega_1^2 - \frac{2m\pi^2U^2}{\rho Al^3}\sin^2\left(\frac{\pi Ut}{l}\right)\right)q_1 = \sqrt{\frac{2}{l}}\frac{m}{\rho A}g\sin\left(\frac{\pi Ut}{l}\right). \end{aligned} \tag{5}$$

By defining the non-dimensional parameters

$$\alpha \triangleq \frac{m}{\rho Al}, \quad \tau \triangleq \frac{\pi Ut}{l}, \quad \Omega \triangleq \frac{\pi U}{l\omega_1}, \quad \bar{g} \triangleq \frac{g}{l\omega_1^2}, \quad Q \triangleq \frac{q_1}{l^{3/2}}, \quad \beta \triangleq \frac{1}{\Omega^2}, \tag{6}$$

the governing equation can be reformulated in dimensionless form,

$$\Omega^2\left(1 + \alpha\sin^2(\tau)\right)\ddot{Q} + 4\alpha\Omega^2\sin(\tau)\cos(\tau)\dot{Q} + \left(1 - 2\alpha\Omega^2\sin^2(\tau)\right)Q = \sqrt{2}\alpha\bar{g}\sin(\tau), \tag{7}$$

where a dot denotes derivation with respect to the dimensionless time τ.

While the moving mass has not left the beam span, the induced vibration is governed by the time-varying Eq. (7). As soon as it exits the beam, the mass will have no more influence on the beam, and consequently the governing equation will lose its time-varying character ($\alpha = 0$), resulting in the removal of any vibration amplification factor. By considering a repeating sequence of mass entrance and departure, the coefficients of Eq. (7) become periodical with period $T = l/U$ which necessitates to substitute the coefficients of Eq. (7) by their Fourier expansion of period T. In the sequel, the right-hand side of Eq. (7) can be disregarded assuming a negligible gravity effect while keeping the inertia of the transient masses in the dynamic stability analysis of the system. This leads to a homogeneous parametrically excited system with period $T = \pi$:

$$\Omega^2 \left(1 + \alpha \left(1 - \cos(2\tau)\right)\right) \ddot{Q} + 2\alpha\Omega^2 \sin(2\tau)\dot{Q} + \left(1 - \alpha\Omega^2 \left(1 - \cos(2\tau)\right)\right) Q = 0. \tag{8}$$

In linear time-varying systems, parametric as well as external resonance conditions can be met. According to the fact that the Fourier expansion of the right-hand side of Eq. (7) is reviving exciting terms of different frequencies, the authors have previously shown that if a solution becomes synchronized with the excitation, a generalized form of resonance for time-varying systems can occur [15,34].

3 Analysis of periodic systems by Harmonic Balance

In parametrically excited systems, conditions of internal stability have to be established according to the parameters involved in the equation. Regions corresponding to instability are identified in the space of system parameters. A well-known stability diagram is that established for the Mathieu equation [25]. In the present system, the parameters that are dealt with are (α, β) which respectively represent the mass ratio and transition frequency of moving loads. As alluded, the solutions to T-periodic differential equations are not generally periodic but just for certain parameters values which, according to Floquet's theory, delimit stable from unstable regions in the parameters' plane. This fact serves as the main principle for a majority of methods employed to study the stability of periodic differential equations.

A review of the studies accomplished on the dynamic stability of beam–moving mass systems reported the existence of just one such separating boundary with transition behavior except for one study [23], due to accuracy limitation caused by the process of numerical integration or the insufficient sweeping resolution of the parameter plane. So the necessity for employing a more efficient approach, stimulated by the expectation to find other transient curves, is justified. The incremental harmonic balance method was successfully applied as a semi-analytic alternative by the authors in order to explicitly recognize conditions for periodic solutions [34]. Although additional boundary curves were detected by this method, the expected transition behavior across them did not arise for all cases. In what follows, an explanation for this singular observation apparently contradictory with the theory will be held by applying a more subtle analysis.

Returning back to Eq. (8) and rewriting it in a more concise form in terms of perturbation parameters $\varepsilon \triangleq \alpha/(1+\alpha)$ and $\delta \triangleq 1/\beta^2$ will result in

$$(1 + a\cos(2\tau))\frac{d^2V}{d\tau^2} + b\sin(2\tau)\frac{dV}{d\tau} + (c + d\cos(2\tau))\,V = 0, \tag{9}$$

recognized as Ince's equation where

$$a \triangleq -\varepsilon, \quad b \triangleq 2\varepsilon, \quad c \triangleq \frac{\delta}{1+\alpha} - \varepsilon, \quad d \triangleq \varepsilon. \tag{10}$$

As mentioned, the boundary curves partitioning the regions of stability of Eq. (9) are T and 2T-periodic which, according to the period of the dimensionless equation, can be both expanded as a Fourier series of period $T = \pi$,

$$V(\tau) = \sum_{n=0}^{\infty} A_n \cos(n\tau) + \sum_{n=1}^{\infty} B_n \sin(n\tau). \tag{11}$$

By substituting in the main equation and applying some algebraic operations, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\left(-n^2(1+\alpha) + \delta - \alpha\right) A_n \cos(n\tau) + \frac{1}{2}\alpha (n+1)^2 A_n \cos(n+2)\tau \right.\\
&\quad \left. +\frac{1}{2}\alpha (n-1)^2 A_n \cos(n-2)\tau\right\}\\
&\quad +\sum_{n=1}^{\infty}\left\{\left(-n^2(1+\alpha) + \delta - \alpha\right) B_n \sin(n\tau) + \frac{1}{2}\alpha (n+1)^2 B_n \sin(n+2)\tau \right.\\
&\quad \left. +\frac{1}{2}\alpha (n-1)^2 B_n \sin(n-2)\tau\right\} = 0.
\end{aligned} \tag{12}$$

Noting that the first half of the equation is an even function and the second half is odd, the first summation can get decoupled from the second one, resulting in the two following equations:

$$\sum_{n=0}^{\infty}\left\{\left(-n^2(1+\alpha)+\delta-\alpha\right)A_n\cos(n\tau)+\frac{1}{2}\alpha(n+1)^2A_n\cos(n+2)\tau\right.$$
$$\left.+\frac{1}{2}\alpha(n-1)^2A_n\cos(n-2)\tau\right\}=0, \tag{13}$$
$$\sum_{n=1}^{\infty}\left\{\left(-n^2(1+\alpha)+\delta-\alpha\right)B_n\sin(n\tau)+\frac{1}{2}\alpha(n+1)^2B_n\sin(n+2)\tau\right.$$
$$\left.+\frac{1}{2}\alpha(n-1)^2B_n\sin(n-2)\tau\right\}=0. \tag{14}$$

Regarding that the trigonometric terms are independent functions, each coefficient has to be set equal to zero. Thus an infinite number of coupled equations in terms of A_n and B_n are obtained. By inspecting more attentively, it can be deduced that the odd-index terms of each expression are not interfering with the even-index terms and can further be decoupled into terms of similar parity. This can be explained by the fact that the even and odd terms correspond to the T-periodic and $2T$-periodic Floquet's solutions, respectively. The result is summarized in four independent sets of coupled homogeneous equations gathering A-even, A-odd, B-even and B-odd indices terms. The condition for existence of a nontrivial solution is that at least one of the determinants of those matrices be equal to zero:

$$A_{even}:\quad \det\begin{bmatrix} c & Q(-1) & 0 & 0 & 0 & \cdots \\ 2Q(0) & c-4 & Q(-2) & 0 & 0 & \cdots \\ 0 & Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & 0 & Q(2) & c-36 & Q(-4) & \cdots \\ 0 & 0 & 0 & Q(3) & c-64 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}=0, \tag{15}$$

$$B_{even}:\quad \det\begin{bmatrix} c-4 & Q(-2) & 0 & 0 & \cdots \\ Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & Q(2) & c-36 & Q(-4) & \cdots \\ 0 & 0 & Q(3) & c-64 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}=0, \tag{16}$$

$$A_{odd}:\quad \det\begin{bmatrix} c-1+P(0) & P(-1) & 0 & 0 & \cdots \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(2) & c-25 & P(-3) & \cdots \\ 0 & 0 & P(3) & c-49 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}=0, \tag{17}$$

$$B_{odd}:\quad \det\begin{bmatrix} c-1-P(0) & P(-1) & 0 & 0 & \cdots \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(2) & c-25 & P(-3) & \cdots \\ 0 & 0 & P(3) & c-49 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}=0, \tag{18}$$

in which

$$Q(m)=\frac{d}{2}+bm-2am^2, \tag{19}$$
$$P(m)=Q\left(m-\frac{1}{2}\right)=\frac{d+b(2m-1)-a(2m-1)^2}{2}. \tag{20}$$

In the next Section, this situation will be explored in more details.

4 Instability boundary curves

In this Section, it is shown how to deduce the transition curves separating stable and unstable regions by using the resulting matrices in Eqs. (15–18). As a first instance, solving Eq. (15) for A-even indices is considered. In order to reach a non-trivial solution, the determinant of the coefficients has to be set to zero, i.e,

$$\det\begin{bmatrix} \delta-\alpha & \frac{\alpha}{2} & 0 & 0 & \cdots \\ \alpha & \delta-\alpha-4(1+\alpha) & \frac{9\alpha}{2} & 0 & \cdots \\ 0 & \frac{9\alpha}{2} & \delta-\alpha-16(1+\alpha) & \frac{25\alpha}{2} & \cdots \\ 0 & 0 & \frac{25\alpha}{2} & \delta-\alpha-36(1+\alpha) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} = 0. \tag{21}$$

Considering a finite-dimension 4×4 expansion of this determinant, the equation determining the boundary of stability in the parameter plane will result as

$$\delta^4 - (60\alpha + 56)\,\delta^3 + \left(784 + 781\alpha^2 + 1736\alpha\right)\delta^2 + \left(-2034 - 2310\alpha^3 - 8836\alpha^2 - 8480\alpha\right)\delta + 2304\alpha + \frac{11025}{8}\alpha^4 + 6580\alpha^3 + 7408\alpha^2 = 0. \tag{22}$$

By solving the above equation, the parametric curve $\delta = \delta(\alpha)$ will arise. According to the fact that the analytical solution is not reachable, a perturbation technique is applied. The expansion is performed with respect to α as the perturbation parameter,

$$\delta = \delta_0 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots + \delta_n\alpha^n. \tag{23}$$

Substituting Eq. (23) into (22) and collecting terms of similar order results in

$$\begin{aligned} &\alpha^0 \to \delta_0(\delta_0-4)(\delta_0-16)(\delta_0-36) = 0 \\ &\alpha^1 \to 1568\delta_0\delta_1 + 4\delta_0^3\delta_1 - 2304\delta_1 - 168\delta_0^2\delta_1 + 2304 - 8480\delta_0 - 60\delta_0^3 + 1736\delta_0^2 = 0 \end{aligned} \tag{24}$$

The first sub-equation for order α^0 has multiple roots 0, 4, 16 and 36. It is worth to mention that these roots are not depending on the considered order of determinant expansion. By selecting the first root ($\delta_0 = 0$) and solving the next equations subsequently, the first transition curve equation in ($\delta - \alpha$) plane results as:

$$\delta = \alpha - \frac{\alpha^2}{8} + \frac{\alpha^3}{8} + O\left(\alpha^4\right). \tag{25}$$

To compare with previous results, $\delta = 1/\beta^2$ is back-substituted, leading to

$$\beta = \sqrt{\frac{8}{8\alpha - \alpha^2(1-\alpha)}}, \tag{26}$$

which is exactly the same expression obtained earlier by the authors via homotopy approach [35] or numerical integration method [19]. It is clear that this result can be enhanced by increasing the order of perturbation expansion and determinant dimension considered. By repeating this trend about other roots and also expanding other determinant equations, a complete map depicting the whole stability regions will be obtained similar to Mathieu's equation diagram, with multiple tongue-like regions of instability. Formally, these regions appear between two consecutive separating curves emerging from a common root in the beam–moving mass parameters plane. The next Section will be devoted to special cases that may be present in the stability diagram of this problem.

5 Coexistence in the beam–moving mass problem

By conducting a more subtle investigation on the structure of matrices (17) and (18), one can perceive that they are identical, therefore possessing the same determinant polynomial which results in a set of coinciding curves in the parameter plane. This can be interpreted as the common occurrence (coexistence) of two independent solutions for the same pair of parameters lying on these curves. As mentioned, the regions of instability are limited to such curves. Coexistence is thus equivalent to the shrinkage of these regions to zero-thickness tongues. Apparently, this internal collapse caused regions of potential instability to disappear. As will be investigated, the persistence of these close-up tongues is sensitive to parameters variation, a fact which makes dangerous to design in the vicinity of these potential "faults."

As far as the present problem is concerned, it is clear that $P(0) = 0$ according to defined parameters in Eq. (10). It is observed that determinants A_{odd} and B_{odd} become equal, resulting in one common equation and hence causing the coexistence to occur. By putting $\varepsilon = 0$ in Eqs. (17, 18) as a first approximation, one obtains that these coexistence curves (tongues of zero thickness) emanate from $c = 1, 9, 25, \ldots$. Using a finite-dimension expansion of determinants Eq. (17) or (18), say of fourth order, leads to

$$c^4 - 84c^3 + \left(1974 - 392\varepsilon^2\right)c^2 + \left(6736\varepsilon^2 - 12916\right)c + 11025 - 10952\varepsilon^2 + 1296\varepsilon^4 = 0. \tag{27}$$

In order to solve the above algebraic equation, the variable c is expanded with respect to ε around emanating points. For example, to find the coexistence curve emanating from $c = 1$, one considers

$$c = 1 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + \cdots. \tag{28}$$

Substituting Eq. (28) in Eq.(27) and gathering terms of similar order of ε, the first coexistence curve becomes

$$c = 1 - \frac{1}{2}\varepsilon^2 - \frac{13}{96}\varepsilon^4 + \cdots. \tag{29}$$

Recalling that $c \triangleq \frac{\delta}{1+\alpha} - \varepsilon$ and $\delta \triangleq 1/\beta^2$, the coexistence curve is expressed in the $\alpha - \beta$ plane as

$$\beta = \left[1 + 2\alpha - \frac{\alpha^2}{2(1+\alpha)} - \frac{13}{96}\frac{\alpha^4}{2(1+\alpha)^3}\right]^{-\frac{1}{2}} \tag{30}$$

which compared accurately to the 2T-periodic solution derived by the IHB method [15].

According to Floquet's theory, it is expected to transit between stable/unstable regions upon crossing over curves corresponding to periodic solutions. But as explained, this curve coexists with another one in the present case, resulting in an unidentifiable instability region. This fact is verified by simulating the system with parameters selected on both sides of this curve, demonstrating a uniform trend of stability as shown in Fig. 1.

Other plausible periodic solutions arise from expanding the determinants A_{even} and B_{even}, Eqs. (15) and (16). These expansions result in different equations and hence correspond to pairs of curves which no more coincide and create tongues of instability with roots emerging from $c = 4, 16, 36, \ldots$. For example, by adapting the aforementioned method, the solutions existing about $c = 4$ can be obtained as

$$c = 4 - \frac{25}{16}\varepsilon^2 + \cdots \tag{31}$$

and

$$c = 4 - \frac{27}{16}\varepsilon^2 + \cdots \tag{32}$$

respectively for A_{even} and B_{even} determinants. The corresponding form of the above curves in the $(\alpha - \beta)$ plane is as follows:

$$\beta = \sqrt{\frac{16(1+\alpha)}{64 + 144\alpha + 55\alpha^2}}, \tag{33}$$

and

$$\beta = \sqrt{\frac{16(1+\alpha)}{64 + 144\alpha + 53\alpha^2}}, \tag{34}$$

which are depicted in Fig. 2 with other tongues and coexistence curves.

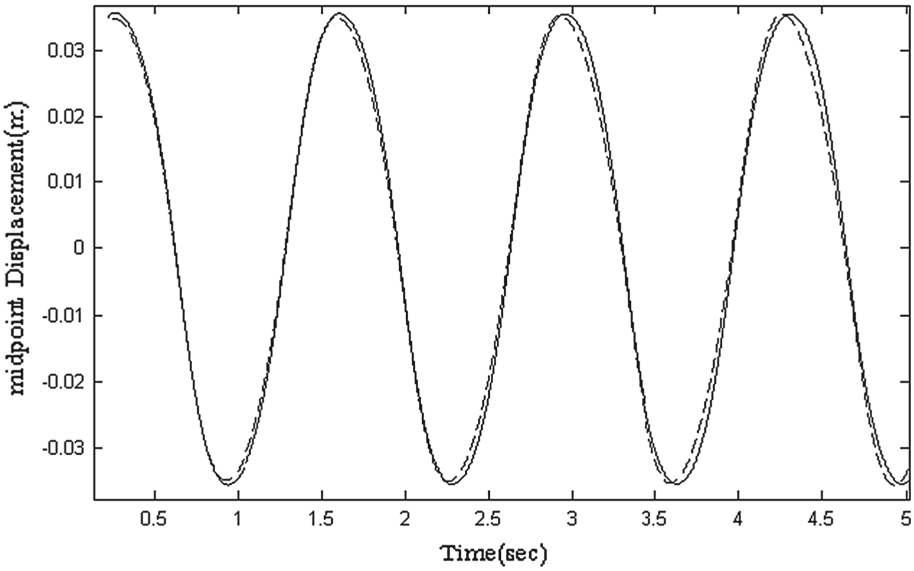

Fig. 1 Comparison of the system response for parameters selected on both sides of the coexistence boundary Eq. (30)

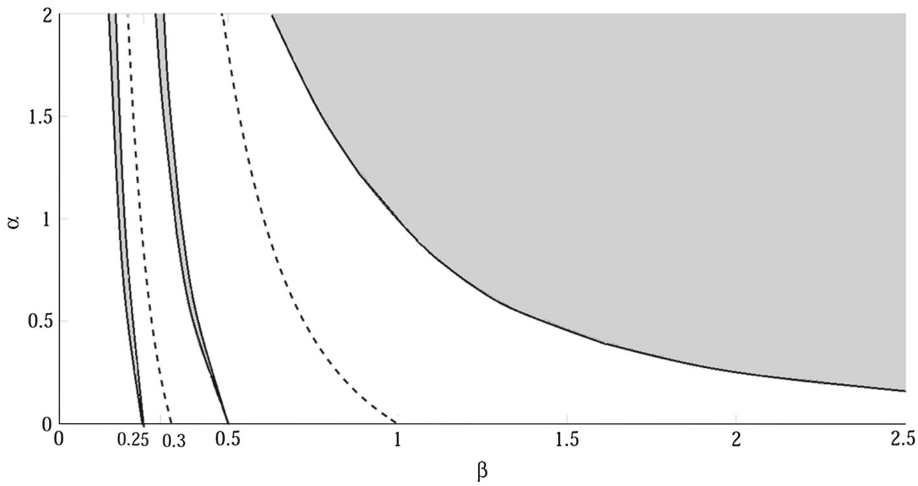

Fig. 2 Complete stability plane of the moving mass and beam interaction problem

The fact that these curves emerge from a common point is easily understood by substituting $\varepsilon = 0$ into the determinants which lead to the vanishing of off-diagonal terms. The determinant expression results as

$$|A_{even}| = c(c-4)(c-16)(c-36)\cdots \tag{35}$$

and

$$|B_{even}| = (c-4)(c-16)(c-36)\cdots \tag{36}$$

which also shows that $c = 0$ is a single transition curve and other curves emanating from $c = 4, 16, 36, \ldots$ form tongues. These curves are represented in Fig. 2 as dashed lines originating from corresponding roots.

6 Opening of coexistence curves

The unique characteristic of the governing Eq. (9) which led to the conditions for the coexistence phenomenon can get challenged by the slightest variation in the model. In fact, it will be shown that the presence of negligible modeling uncertainties or disturbances in the system may result in unpredictable behavior. For instance, one can assume a small oscillating axial force as an arbitrary external loading or appearing due to the reaction of non-ideal supports modeled as horizontal springs. This force which is activated with double the frequency of the moving load passage can be expressed as $(p\cos(2\tau))$ if the span shortening due to the first flexural mode is considered. There is no surprise to note that this term appears alongside the beam stiffness term because of the recognized influence of axial force on flexibility, resulting in the following form for the governing equation:

$$(1+\alpha(1-\cos(2\tau)))\frac{d^2V}{d\tau^2}+2\alpha\sin(2\tau)\frac{dV}{d\tau}+(\delta-\alpha(1-\cos(2\tau))-\lambda\delta\cos(2\tau))V=0 \tag{37}$$

where $\lambda = p/p_{cr}$ and $p_{cr} = \pi^2 EI/l^2$. By performing some algebraic operations, Ince's equation, Eq. (9), reappears with the following coefficients:

$$a \triangleq -\varepsilon, \quad b \triangleq 2\varepsilon, \quad c \triangleq \frac{\delta}{1+\alpha} - \varepsilon, \quad d \triangleq \varepsilon(1+K). \tag{38}$$

In the above definitions, $K = \lambda\delta/\alpha$ represents the axial stiffness reaction of the supports. Substituting these parameters into Eqs. (19) and (20) results in

$$Q(m) = 2\varepsilon m^2 + 2\varepsilon m + \frac{\varepsilon}{2}(1+K), \tag{39}$$

$$P(m) = \frac{\varepsilon(1+K) + 2\varepsilon(2m-1) + \varepsilon(2m-1)^2}{2}. \tag{40}$$

In contrast to the previous ideal case, $P(0) = \varepsilon K/2 \neq 0$ will no more vanish at $m = 0$, which results on this occasion in different expressions for A_{odd} and B_{odd} determinants. Hence the condition for coexistence curves emanating from $c = 1, 9, 25, \ldots$ will no more subsist. This effect will pronounce as the axial stiffness augments, yielding a gradually K-dependent opening of the previously coexisting curve. Hence it can be stated that Eq. (9) with coefficients described in Eq. (38) had buried in it closed-up instability segments which upon being triggered result in unexpected instability regions.

By selecting $K = -1$ and substituting into Eqs. (39–40), it arises that the coefficients $Q(0)$ and $Q(-1)$ both vanish, resulting in the following determinants:

$$A_{even}: \quad \det\begin{bmatrix} c & 0 & 0 & 0 & 0 & \cdots \\ 0 & c-4 & Q(-2) & 0 & 0 & \cdots \\ 0 & Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & 0 & Q(2) & c-36 & Q(-4) & \cdots \\ 0 & 0 & 0 & Q(3) & c-64 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} = 0, \tag{41}$$

$$B_{even}: \quad \det\begin{bmatrix} c-4 & Q(-2) & 0 & 0 & \cdots \\ Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & Q(2) & c-36 & Q(-4) & \cdots \\ 0 & 0 & Q(3) & c-64 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} = 0. \tag{42}$$

In this case, the upper finite part disconnects from the remaining infinite lower portion of the determinant. A comparison between Eqs. (41) and (42) shows that the latter appears as a common factor. Hence, for parameters selected on curves emanating from $c = 4, 16, 36, \ldots$, the conditions for coexistence arise. Consequently, the stability diagram (Fig. 2) will change morphologically with respect to model precision employed, as shown in Fig. 3. The vanishing or appearance of instability regions is noticeable as a critical point to be considered when dealing with such systems. As shown, Fig. 3 is a replot of Fig. 2 with coexistence curves that opened and other zero-thickness instability regions that appeared as dashed lines.

For $K = 1$ (a 180-degree phase difference with the previous loading), no common factor exists between the determinants A_{odd} and B_{odd} or A_{even} and B_{even}. So, the condition for coexistence will no more subsist, and all instability regions will appear as finite tongues. The corresponding stability plane is depicted as shown in Fig. 4.

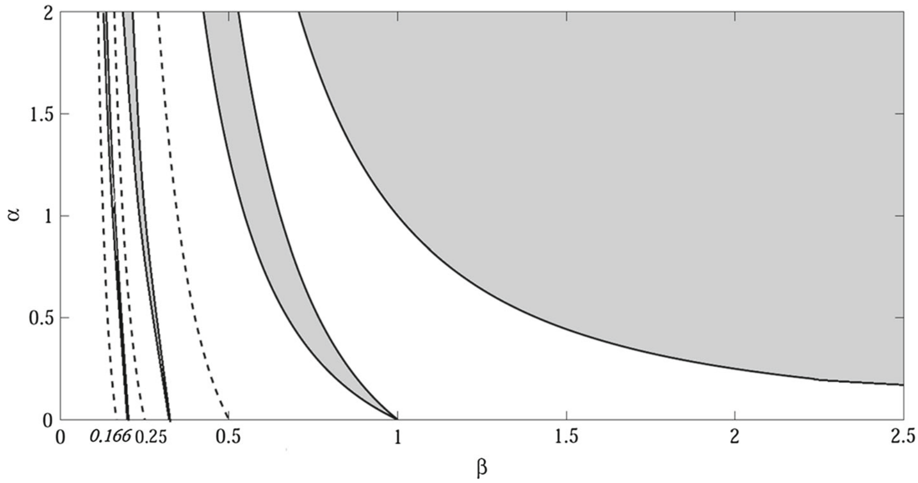

Fig. 3 Modified stability diagram due to in-phase axial force presence

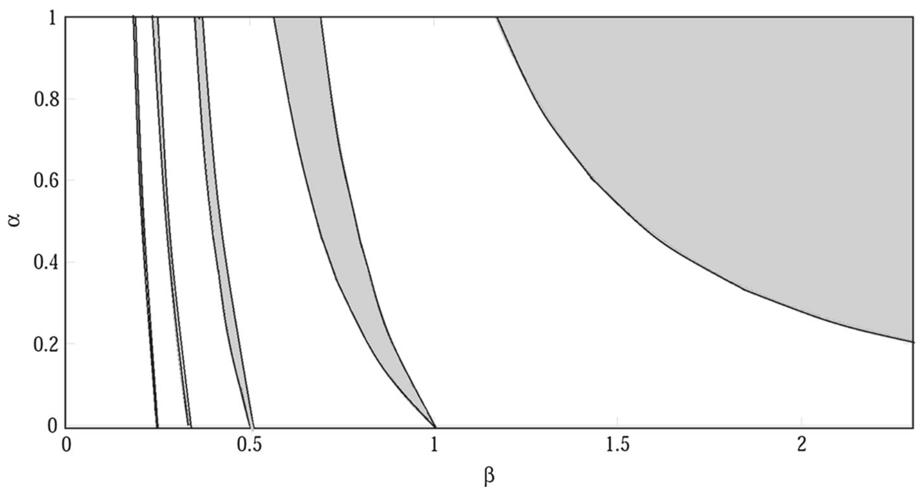

Fig. 4 Modified stability diagram due to out-of-phase axial force presence

7 Simulation results on coexistence occurrence

In purpose of underlying the importance of coexisting solutions, a design whose system parameters are innocently selected in the vicinity of a coexistence curve (dashed line of Fig. 3) is considered. Although selected parameters are apparently situated in the 'middle' of a stable region, the introduction or negligence of certain disturbances effects will lead to the appearance of an instability region at this operating point. Simulations performed under conditions lying on this coexistent curve, before and after the application of the slight deflection-dependent axial force, show an abrupt change of behavior as maintained by the theory (Figs. 5, 6). The fact that system's parameters may be precariously located on a hidden tongue of instability that could open at the first occasion necessitates that the designer be more cautious in the designing process.

8 Conclusions

In this paper, it is shown that the governing equation of a model for the study of moving mass-induced vibrations of a simple beam is reducible to a linear second-order equation with time-periodic coefficients, categorized as a special case of Ince's equation. By using the harmonic balance method, new remarkable stability diagrams are obtained in a more qualitative sense compared to other studies. It is shown that some curves corresponding to periodic solutions for the beam–moving mass system display "coexistence." This situation corresponds to

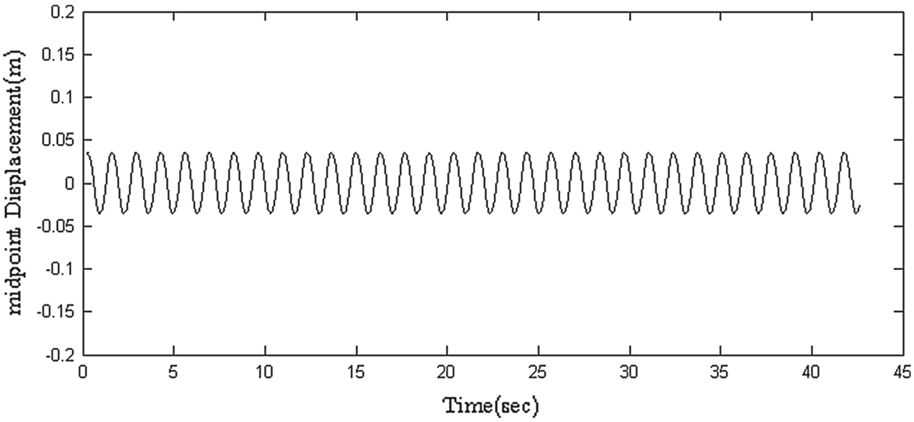

Fig. 5 Simulations performed under conditions lying on the coexistent curve

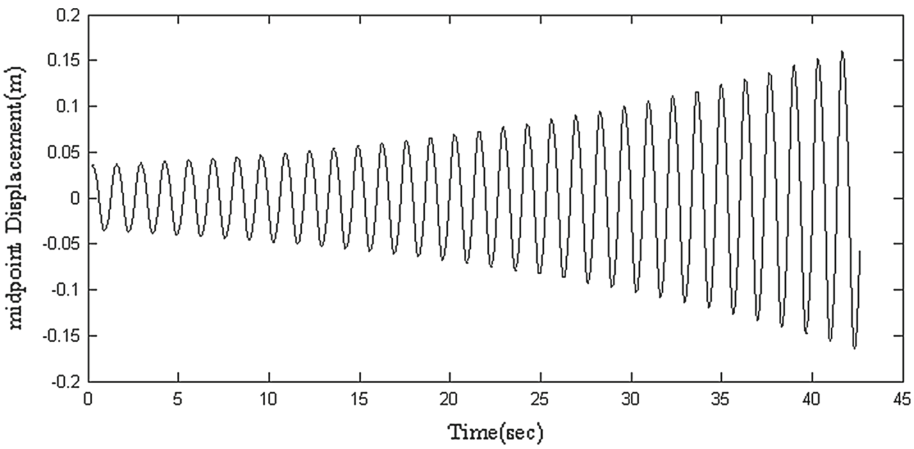

Fig. 6 Simulations performed under conditions lying on the coexistent curve after altering the model

parameters in the stability diagram for which two linearly independent periodic solutions exist simultaneously. They can also be considered as a limiting case of the closure of instability gaps delimited by these curves.

A refined modeling of the original system, such as the reassignment of beam stiffness or readjustment of boundary conditions, may result in the opening-up of the zero-thickness instability region, regarding that conditions leading to coexistence are relatively singular. The designer has to be conscious of the potential incidence of such traps which may emerge in various engineering problems.

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