Acta Mech 227, 3213–3237 (2016)
DOI 10.1007/s00707-015-1534-9

ORIGINAL PAPER

M. H. B. M. Shariff

Anisotropic separable free energy functions for elastic and non-elastic solids

Received: 1 December 2014 / Revised: 14 June 2015 / Published online: 14 January 2016

Abstract In incompressible isotropic elasticity, the Valanis and Landel strain energy function has certain attractive features from both the mathematical and physical view points. This separable form of strain energy has been widely and successfully used in predicting isotropic elastic deformations. We prove that the Valanis–Landel *hypothesis* is part of a general form of the isotropic strain energy function. The Valanis–Landel form is extended to take anisotropy into account and used to construct constitutive equations for anisotropic problems including stress-softening Mullins materials. The anisotropic separable forms are expressed in terms of spectral invariants that have clear physical meanings. The elegance and attractive features of the extended form are demonstrated, and its simplicity in analysing anisotropic and stress-softening materials is expressed. The extended anisotropic separable form is able to predict, and compares well with, numerous experimental data available in the literature for different types of materials, such as soft tissues, magneto-sensitive materials and (stress-softening) Mullins materials. The simplicity in handling some constitutive inequalities is demonstrated. The work here sets an alternative direction in formulating anisotropic solids in the sense that it does not explicitly use the standard classical invariants (or their variants) in the governing equations.

1 Introduction

The Valanis and Landel [40] separable strain energy function has certain attractive features, and this form of strain energy function has been widely and successfully used in predicting isotropic elastic deformations [23,26,30]. It is simple in form in the sense that it contains only a general single-variable function of a principal stretch with a clear physical meaning, and this facilitates the seeking of specific forms of the single-variable function via experimental data [26,30]. Inspired by the principal stretch successes and the simple form of the Valanis–Landel function, the author [4,31,36,37] recently developed, similar but somewhat different, separable forms to model anisotropic solids. In this paper, we report on the efficacy of such separable forms to model anisotropic problems. We use new and existing results to show that the proposed anisotropic separable forms have the potential to model anisotropic solids. The main aim of this paper is to set a platform for future modelling in a setting different from the classical invariant setting, where most of anisotropic models in the current literature are based on.

Spectral (principal axis) invariants are required to formulate a separable constitutive equation. These invariants have clear physical meanings, and hence they can be more attractive when looking for expressions for the total energy function via experimental data; they also can be more attractive in seeking to design a rational programme of experiments for anisotropic solids. In addition to this, the classical invariants (and most of their variants) can be explicitly expressed in terms of spectral invariants, and hence if the constitutive equation is

M. H. B. M. Shariff (✉)
Department of Applied Mathematics and Science, Khalifa University of Science, Technology and Research,
Sharjah, UAE
E-mail: shariff@kustar.ac.ae

initially written in terms of the classical invariants, the relevant formulations can be easily formulated in terms of both classical and spectral invariants. However, if on the onset the constitutive equation is written in terms of spectral invariants, it is generally impossible to convert it explicitly in terms of classical invariants, and we cannot write the corresponding constitutive equations, explicitly, in terms of the classical invariants. Hence, in general, the separable forms proposed here cannot be written explicitly in terms of classical invariants, and in view of this, spectral formulations are used here to deal with the proposed spectral separable forms.

In Sect. 2, we prove that the Valanis–Landel separable form is not a hypothesis but part of a general spectral form. The spectral stress components are given in Sect. 3. Section 4 is concerned with single preferred direction problems. In this section, new separable forms for transversely isotropic soft tissue and rubber-like materials are proposed, and existing separable results on magneto-isotropic materials are discussed. Also, in Sect. 4, a new Ogden-type series constitutive equation is proposed. Section 5 deals with two preferred direction problems, where existing results for orthotropic passive myocardium are given and a new separable constitutive equation for a transversely isotropic magneto-elastic body is proposed. The efficacy of separable constitutive equation for inelastic Mullins materials is reported in Sect. 6. Some remarks are given in Sect. 7, and the conclusion is given in Sect. 8.

2 The Valanis–Landel function

In this communication, all subscripts i, j and k take the values 1, 2 and 3, unless stated otherwise, and we denote the strain energy for isotropic and anisotropic solids as W_e. Let $W_e = W(\lambda_1, \lambda_2, \lambda_3)$ be the strain energy function of an isotropic elastic solid, where λ_i is an eigenvalue (principal stretch) of the right stretch tensor $\boldsymbol{U}$. Isotropy requires the symmetry property

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3) = W(\lambda_3, \lambda_1, \lambda_2). \tag{1}$$

This symmetry requirement is difficult to manage in searching for an explicit expression of W_e, and to overcome this difficulty, Valanis and Landel [40] postulate the *separable* form

$$W(\lambda_1, \lambda_2, \lambda_3) = r_v(\lambda_1) + r_v(\lambda_2) + r_v(\lambda_3). \tag{2}$$

Initially, there is no experimental reason to postulate (2); only convenience and experience may suggest (2). Indeed, Valanis and Landel [40] stated that:

This postulated form is not fortuitous but is a natural generalization of more particular forms that already exist in the literature.

In addition to the above, Sacommandi [29] stated that:

Therefore for Ogden the Valanis–Landel hypothesis follows from the postulate of shape invariance, but there is no physical or theoretical reason for this postulate, this only a metaphoric interpretation of a special class of experimental data.

Several workers [26,30] used the form (2) to obtain specific strain energy functions, and they have shown that, for moderate strains, these functions have good agreement with experimental data of several different types of rubber-like materials [30].

In this section, we, however, prove that the form (2) is part of a general form of $W(\lambda_1, \lambda_2, \lambda_3)$ and not just a hypothesis as suggested in the literature. In order to prove a general functional form for an incompressible isotropic material, we consider the polynomial expansion

$$W_e = \sum_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma}(\lambda_1^{\alpha}\lambda_2^{\beta}\lambda_3^{\gamma} - 1), \tag{3}$$

where the terms $C_{\alpha,\beta,\gamma}$ are constants and, α, β, and γ are non-negative integers. We do not intend to use the above polynomial form as a constitutive model or as an "Nth"-order approximation; we only use it to obtain a general functional form for W_e.

For an incompressible material, $\lambda_1\lambda_2\lambda_3 = 1$, and we can write (3) in the form

$$\begin{aligned} W_e = &\sum_{\gamma+r,\gamma+s,\gamma} C_{\gamma+r,\gamma+s,\gamma}(\lambda_1^r\lambda_2^s - 1) + \sum_{\beta+r,\beta,\beta+s} C_{\beta+r,\beta,\beta+s}(\lambda_1^r\lambda_3^s - 1) \\ &+ \sum_{\alpha,\alpha+r,\alpha+s} C_{\alpha,\alpha+r,\alpha+s}(\lambda_2^r\lambda_3^s - 1), \end{aligned} \tag{4}$$

where r and s are non-negative integers. The above expression can be rewritten as

$$W_e = \sum_{r,s} C^{(1)}_{r,s}(\lambda_1^r\lambda_2^s - 1) + \sum_{r,s} C^{(2)}_{r,s}(\lambda_1^r\lambda_3^s - 1) + \sum_{r,s} C^{(3)}_{r,s}(\lambda_2^r\lambda_3^s - 1). \tag{5}$$

To obtain the symmetry given in Eq. (1), certain conditions have to be imposed on the coefficients $C^{(i)}_{r,s}$. Before we do this, we write the expansion given in Eq. (5) in the form

$$\begin{aligned} W_e = &\sum_{r=0} C^{(1)}_{r,0}(\lambda_1^r - 1) + \sum_{s=1} C^{(1)}_{0,s}(\lambda_2^s - 1) + \sum_{r=0} C^{(2)}_{r,0}(\lambda_1^r - 1) + \sum_{s=1} C^{(2)}_{0,s}(\lambda_3^s - 1) \\ &+ \sum_{r=0} C^{(3)}_{r,0}(\lambda_2^r - 1) + \sum_{s=1} C^{(3)}_{0,s}(\lambda_3^s - 1) \\ &+ \sum_{r,s\neq 0} C^{(1)}_{r,s}(\lambda_1^r\lambda_2^s - 1) + \sum_{r,s\neq 0} C^{(2)}_{r,s}(\lambda_1^r\lambda_3^s - 1) + \sum_{r,s\neq 0} C^{(3)}_{r,s}(\lambda_2^r\lambda_3^s - 1). \end{aligned} \tag{6}$$

To satisfy the symmetry given in (1), $C^{(i)}_{r,s}$ must take certain forms (as shown below), and since $\lambda_i^0 - 1 = 0$, we can rewrite the above equation in the form

$$\begin{aligned} W_e = &\sum_{r=0} D_r(\lambda_1^r - 1) + \sum_{r=0} E_r(\lambda_1^r - 1) + \sum_{r=0} D_r(\lambda_2^r - 1) + \sum_{r=0} E_r(\lambda_2^r - 1) \\ &\sum_{r=0} D_r(\lambda_3^r - 1) + \sum_{r=0} E_r(\lambda_3^r - 1) \\ &+ \sum_{r,s\neq 0} c_{r,s}(\lambda_1^r\lambda_2^s - 1) + \sum_{r,s\neq 0} c_{r,s}(\lambda_1^r\lambda_3^s - 1) + \sum_{r,s\neq 0} c_{r,s}(\lambda_2^r\lambda_3^s - 1), \end{aligned} \tag{7}$$

where $c_{r,s} = c_{s,r}$. From the above equation and in view of Weierstrass approximation theorem, we can write the strain energy function in the separable form

$$W_e = r_v(\lambda_1) + r_v(\lambda_2) + r_v(\lambda_3) + g(\lambda_1, \lambda_2) + g(\lambda_1, \lambda_3) + g(\lambda_2, \lambda_3), \tag{8}$$

where

$$\begin{aligned} r_v(x) &= \sum_{r=0} (D_r + E_r)(x^r - 1), \\ g(x, y) &= \sum_{r,s\neq 0} c_{r,s}(x^r y^s - 1) = g(y, x) \neq f_s(x) + f_s(y), \end{aligned} \tag{9}$$

where f_s is an arbitrary function. It is clear from (8) that the Valanis and Landel [40] hypothesis is part of a general form of W_e.

In a similar way, the form (8) can also be obtained via the series

$$W_e = \sum_{r,s} A_{r,s}(I_1 - 3)^r(I_2 - 3)^s \tag{10}$$

by substituting $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$ and using the incompressibility condition $\lambda_1\lambda_2\lambda_3 = 1$.

In a biaxial deformation [30], we have

$$\begin{aligned} \sigma_1 - \sigma_2 = &\lambda_1 r_v(\lambda_1) - \lambda_2 r_v(\lambda_2) + \lambda_1\left(\frac{\partial g}{\partial\lambda_1}(\lambda_1, \lambda_2) + \frac{\partial g}{\partial\lambda_1}(\lambda_1, \lambda_3)\right) \\ &- \lambda_2\left(\frac{\partial g}{\partial\lambda_2}(\lambda_1, \lambda_2) + \frac{\partial g}{\partial\lambda_2}(\lambda_2, \lambda_3)\right), \quad \sigma_3 = 0, \end{aligned} \tag{11}$$

where σ_i is a spectral components of the Cauchy stress. It is found in Jones and Treloar [20] experiment data that $\sigma_1 - \sigma_2$ stress versus λ_1 curves have the same shape for different values of λ_2. The same shape curves shifted vertically for different values of λ_2. Hence, in view of (11) and the fact that $\sigma_1 - \sigma_2 = 0$ for $\lambda_1 = \lambda_2 = 1$, we must have $g = 0$. This concludes that the general form (8) takes the Valanis and Landel form for rubber-like materials which display this shape invariance behaviour. We note that Ogden [27] uses a different approach to obtain the Valanis–Landel form for the shape invariant class of rubber-like materials [30].

3 Spectral stress components

The modelling of an anisotropic strain energy function, written in terms of spectral invariants such as the principal stretches, requires spectral formulations which have recently been developed [31,32,34]. An anisotropic elastic strain energy function W_e can be written in the form

$$W_e = \hat{W}(\boldsymbol{C}) = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \tag{12}$$

with the symmetry property

$$\tilde{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = \tilde{W}(\lambda_2, \lambda_1, \lambda_3, \boldsymbol{e}_2, \boldsymbol{e}_1, \boldsymbol{e}_3) = \tilde{W}(\lambda_3, \lambda_2, \lambda_1, \boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_1), \tag{13}$$

where $\boldsymbol{e}_{1-3}$ are the eigenvectors of the right stretch tensor $\boldsymbol{U}$ and $\boldsymbol{C} = \boldsymbol{U}^2$ is the right Cauchy Green tensor. In view of the non-unique values of $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ when $\lambda_i = \lambda_j$, a unique value $\tilde{W}$ should be independent of $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ when $\lambda_i = \lambda_j$ and a unique value $\tilde{W}$ should be independent of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ when $\lambda_1 = \lambda_2 = \lambda_3$. We call this independent property together with the symmetrical property (13), the *P-property*. All the free energy functions proposed in this paper are required to satisfy the *P-property*.

Spectral formulations require the components of $\dfrac{\partial W_e}{\partial \boldsymbol{C}}$ relative the basis $\{\boldsymbol{e}_{1-3}\}$. Following the work of Shariff [31], we have

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial \tilde{W}}{\partial \lambda_i}, \quad i \text{ not summed}, \tag{14}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{\dfrac{\partial \tilde{W}}{\partial \boldsymbol{e}_i}\cdot \boldsymbol{e}_j - \dfrac{\partial \tilde{W}}{\partial \boldsymbol{e}_j}\cdot \boldsymbol{e}_i}{2(\lambda_i^2 - \lambda_j^2)}, \quad i \neq j. \tag{15}$$

It is assumed that $\tilde{W}$ has sufficient regularity to ensure that, as the value of λ_i approaches λ_j, (15) has a limit. The Cauchy stress $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = 2\boldsymbol{F}\frac{\partial W_e}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p\boldsymbol{I}, \tag{16}$$

where p is the Lagrange multiplier associated with the incompressible constraint $\lambda_1\lambda_2\lambda_3 = 1$ and $\boldsymbol{F}$ is the deformation gradient tensor. The Eulerian spectral Cauchy stress components $\hat{\tau}_{ij}$ take the form

$$\hat{\tau}_{ii} = \lambda_i \frac{\partial \tilde{W}}{\partial \lambda_i} - p, \quad i \quad \text{not summed}, \tag{17}$$

$$\hat{\tau}_{ij} = \frac{\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}\left(\frac{\partial \tilde{W}}{\partial \boldsymbol{e}_i}\cdot \boldsymbol{e}_j - \frac{\partial \tilde{W}}{\partial \boldsymbol{e}_j}\cdot \boldsymbol{e}_i\right), \quad i \neq j. \tag{18}$$

4 Anisotropy due to the preferred direction *a*

4.1 Transversely isotropic elastic solid

The classical invariants

$$I_1 = tr(\boldsymbol{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \frac{I_1^2 - \text{tr}(\boldsymbol{C}^2)}{2} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{19}$$

$$I_4 = \boldsymbol{a}\cdot\boldsymbol{C}\boldsymbol{a} = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \quad I_5 = \boldsymbol{a}\cdot\boldsymbol{C}^2\boldsymbol{a} = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \tag{20}$$

are commonly used arguments for the strain energy function W_e of an incompressible transversely isotropic elastic solid. It is clear that I_4 and I_5 satisfy the *P-property* as described below:

For $\lambda_1 = \lambda_2 = \lambda$ (say), we have

$$I_4 = (\zeta_1 + \zeta_2)\lambda^2 + \zeta_3\lambda_3^2 = (1-\zeta_3)\lambda^2 + \zeta_3\lambda_3^2, \quad I_5 = (\zeta_1 + \zeta_2)\lambda^4 + \zeta_3\lambda_3^4 = (1-\zeta_3)\lambda^4 + \zeta_3\lambda_3^4; \tag{21}$$

hence, both I_4 and I_5 are independent of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Similarly, when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, we have

$$I_4 = 3\lambda^2, \quad I_5 = 3\lambda^4, \tag{22}$$

both are independent of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. For a transversely isotropic solid, the set of spectral invariants $T = \{\lambda_1, \lambda_1, \lambda_3, \zeta_1, \zeta_2, \zeta_3\}$, where $\zeta_i = (\boldsymbol{a} \cdot \boldsymbol{e}_i)^2$, $\zeta_3 = 1 - \zeta_2 - \zeta_3$ was proposed by Shariff [32] to characterize the strain energy function of transversely isotropic solids. The elements of T have immediate physical interpretation; the physical meaning of λ_i is obvious, and it is clear that ζ_i is the square of the cosine of the angle between the principal direction $\boldsymbol{e}_i$ and the preferred direction $\boldsymbol{a}$. We now propose an anisotropic separable functional form

$$\Phi = \sum_{i=1}^{3} \zeta_i \phi(\lambda_i) \tag{23}$$

as a candidate for the construction of W_e. It is clear that Φ satisfies the *P-property*, and we consider (23) as an extension of Valanis–Landel form to transversely isotropic elasticity due to its *separable* nature. We note that the invariants I_4 and I_5 are of the form (23) and we can create infinitely many types of invariants using (23); however, we shall not dwell on this issue in this paper. In this section, we use the form (23) to propose the separable strain energy function

$$W_e = W_T(\lambda_{1-3}, \zeta_{1-3}) = \sum_{i=1}^{3} [\mu_T(\lambda_i) r_1(\lambda_i) + 2(\mu_L(I_4) - \mu_T(\lambda_i))\zeta_i r_2(\lambda_i)] + \frac{\beta}{2}(I_4)\left(\sum_{i=1}^{3} \zeta_i r_3(\lambda_i)\right)^2, \tag{24}$$

where $\mu_T(\lambda_i)$, $\mu_L(I_4)$ and $\beta(I_4)$ are ground-state constants. These discrete ground-state constant functions are more general in the sense that μ_T may not have the same constant value for both $\lambda_i \geq 1$ and $\lambda_i < 1$, and μ_L and β constant values may not be the same for both $I_4 \geq 1$ and $I_4 < 1$. To be consistent with the classical linear theory of incompressible transversely isotropic elasticity, appropriate for infinitesimal deformations, we must have the relations

$$r_1(1) = r_1'(1) = r_2(1) = r_2'(1) = r_3(1) = 0, \quad r_1''(1) = 2, \quad r_2''(1) = 2, \quad r_3'(1) = 1. \tag{25}$$

It is clear that W_e in (24) satisfies the *P-property*. We note that in formulating W_e in the spectral form (24), not only we have the flexibility in defining the ground-state constants, we also have the flexibility in constructing the functions r_1, r_2 and r_3, since their arguments are principal stretches with a clear physical meaning. For example, there is no reason why the functional forms of $r_1(\lambda_i)$, $r_2(\lambda_i)$ and $r_3(\lambda_i)$ should be the same for both $\lambda_i \geq 1$ and $\lambda_i < 1$. This useful concept of different functional forms for different ranges of λ_i is generally alien in classical invariant formulation, where the concept can only be applied to a function (part of the classical invariant strain energy function) which depends on I_4 only; it has no physical meaning when applied to functions that depend on I_1 or I_2 or I_5 (or any combination of them).

The spectral components (14) and (15) take the forms

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial W_T}{\partial \lambda_i}, \quad (i \quad \text{not summed}), \tag{26}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{\dfrac{\partial W_T}{\partial \zeta_i} - \dfrac{\partial W_T}{\partial \zeta_j}}{(\lambda_i^2 - \lambda_j^2)} \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j, \quad i \neq j, \tag{27}$$

where $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$(dyadic product). It is explicit in (26) and (27) that the second Piola–Kirchhoff stress

$$\boldsymbol{T}^{(2)} = 2\frac{\partial W_e}{\partial \boldsymbol{C}} - p\boldsymbol{C}^{-1} \tag{28}$$

is coaxial with $\boldsymbol{U}$ when the preferred direction $\boldsymbol{a}$ is parallel to one of the principal directions. This explicitness may not be obtained if the strain energy function is expressed in terms of the classical invariants (19) and (20) (or possibly most types of invariants found in the literature). The Eulerian principal components of the Cauchy stress are

$$\hat{\tau}_{ii} = \lambda_i \frac{\partial W_T}{\partial \lambda_i} - p, \quad (i \text{ not summed}), \tag{29}$$

$$\hat{\tau}_{ij} = 2\frac{\lambda_i \lambda_j}{\lambda_i^2 - \lambda_j^2}\left(\frac{\partial W_T}{\partial \zeta_i} - \frac{\partial W_T}{\partial \zeta_j}\right)\boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j, \quad i \neq j. \tag{30}$$

4.1.1 Infinitesimal strain energy function

Strain energy function for infinitesimal deformations can be a useful tool in facilitating the construction of a nonlinear strain energy function. For example, it facilitates the process of selecting the appropriate invariants for the nonlinear strain energy function (see the Appendix) and allows us to easily put constraints on the ground-state constants. Due to the separable nature of (24), it can be easily linearized using (25) to obtain the strain energy for an infinitesimal deformation (which is separable in nature), i.e.

$$W_e = \sum_{i=1}^{3} \mu_T e_i^2 + 2(\mu_L - \mu_T)\sum_{i=1}^{3} \zeta_i e_i^2 + \frac{\beta}{2}\sum_{i,j=1}^{3} \zeta_i \zeta_j e_i e_j, \tag{31}$$

where e_i is the principal strain value of the infinitesimal strain tensor $\boldsymbol{E}$ and $\operatorname{tr}(\boldsymbol{E}) = e_1 + e_2 + e_3 = 0$. The ground-state constants in (31) may depend on e_i and I_4 as described in Sect. 4.1.

To ensure physically reasonable responses, restrictions are imposed on the infinitesimal strain energy function which in turn restrict the values of the material constants. If we let $\boldsymbol{a} \equiv [1, 0, 0]^T$, we have, after taking into account the incompressible constraint $e_{11} + e_{22} + e_{33} = 0$,

$$W_e = \frac{1}{2}[(\beta + 4\mu_L)e_{11}^2 + 4\mu_T e_{11}e_{22} + 4\mu_T e_{22}^2 + 4\mu_T e_{32}^2 + 4\mu_L e_{31}^2 + 4\mu_L e_{12}^2], \tag{32}$$

where e_{ij} is the Cartesian component of $\boldsymbol{E}$. Since e_{11}, e_{22}, e_{12}, e_{31} and e_{32} are independent, necessary and sufficient conditions for (32) to be positive definite are:

$$\mu_T > 0, \quad \mu_L > 0, \quad \beta + 4\mu_L - \mu_T > 0. \tag{33}$$

For simplicity, in Sects. 4.1.2, 4.1.3 and 4.3, we assume that the ground state constants are independent of strain. However, in the near future, we will use discrete strain-dependent ground state constants to compare our theory with experiments.

4.1.2 Modelling of fibre-reinforced rubber-like materials

For rubber-like materials, we consider, as a first approximation, the specific forms [30]

$$r(x) = r_1(x) = r_2(x) = x\ln(x) - x + 1 + d_0\left(-e^{1-x} + \frac{x^2 - 4x + 5}{2}\right) + d_1\left(e^{x-1} - \frac{x^2+1}{2}\right) \tag{34}$$

and

$$r_3(x) = \ln(x). \tag{35}$$

Our theory is compared with Ciarletta et al. [6] uniaxial experiment which depicts the first Piola–Kirchoff stress versus strain. The uniaxial stretch is in the $\boldsymbol{e}_1$ direction. The nonzero axial first Piola–Kirchoff stress component is

$$P_{11} = (\mu_T + 2\mu_1)r'(\lambda_1) + \beta r_3(\lambda_1)r_3'(\lambda_1) - \mu_T \frac{\lambda_3 r'(\lambda_3)}{\lambda_1} \tag{36}$$

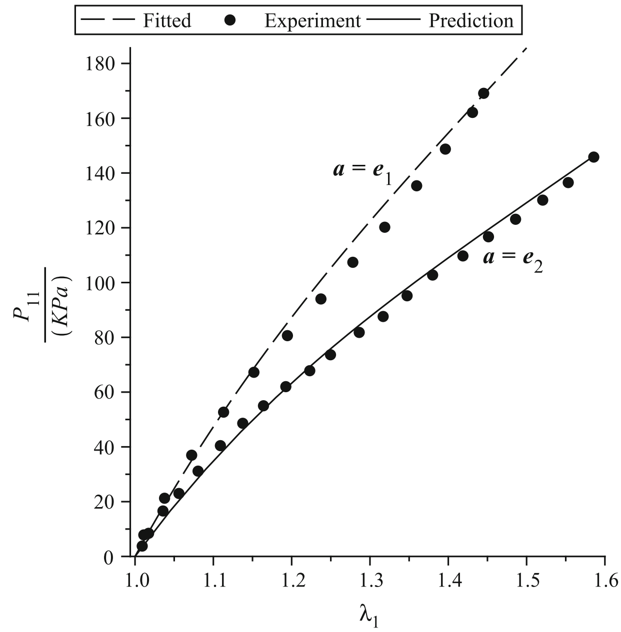

Fig. 1 First Piola–Kirchoff stress versus stretch. Ciarletta et al. [6] uniaxial experiment. Data are obtained from Al-Kinani et al. [1]. $\mu_T = 120$, $\mu_L = 160$, $\beta = 0$, $\mathrm{d}_0 = -3$, $d_1 = 2$

$(\mu_1 = \mu_L - \mu_T)$ for the case when $\boldsymbol{a} = \boldsymbol{e}_1$ and in this case $\lambda_3 = \frac{1}{\sqrt{\lambda_1}}$. In Fig. 1 we visually curve fit the $\boldsymbol{a} = \boldsymbol{e}_1$ data since we know that $\lambda_3 = \frac{1}{\sqrt{\lambda_1}}$. However, we cannot curve fit for the case $\boldsymbol{a} = \boldsymbol{e}_2$, since we do not know the values of λ_3. In this case we have to predict the experimental data using the stress

$$P_{11} = \mu_T \left(r'(\lambda_1) - \frac{\lambda_3 r'(\lambda_3)}{\lambda_1} \right), \tag{37}$$

where λ_3 is obtained in terms of λ_1 from solving the first Piola–Kirchoff component equations $P_{22} = P_{33} = 0$ and $\lambda_1\lambda_3\lambda_3 = 1$. It is clear from Fig. 1 that we are able to fit and predict very well using the above specific forms.

Restrictions on $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$: The restriction on the values of the parameters d_0 and d_1 is governed by the restriction on the function r. We do this by considering a special set of admissible ground-state constant values, where $\mu_T > 0$ and the rest have zero values. This set of values corresponds to the strain energy of an isotropic material. Using Hill's [16] inequality, it is shown in Shariff [30] that, to ensure physically reasonable responses for incompressible isotropic materials, we require the condition $h'(x) > 0$, for $x > 0$, where $h(x) = xr'(x)$; in this paper, we use this necessary condition to restrict the values of d_0 and d_1 for the proposed anisotropic model. The admissible ranges for d_0 and d_1 are not straightforward to obtain. However, for given values of d_0 and d_1 we can easily (and non rigorously) verify whether $h'(x) > 0$ by plotting $h'(x)$ for practical values of $x > 0$. The concepts of polyconvexity [18], convexity and stability [18] can also be used to restrict the values of our material constants, and we hope to do this in the near future. However, we note that stability in an infinitesimal deformation (relative to a stress-free ground-state configuration) is achieved if the classical ground-state constants have the restricted values.

4.1.3 Modelling of soft tissue

In soft tissues, the initial large extension is generally achieved at relatively low levels of stress with subsequent stiffening at higher levels of extension. This behaviour is due to the recruitment of collagen fibres as they become uncrimped and reach their natural lengths [17,28]. The inverse error function $erf^{-1}(x)$ seems a good candidate to describe the above-mentioned soft tissue stress-strain behaviour since it has low initial gradients followed by high gradients at higher values of x. In view of this, for simplicity, we propose the functional forms

$$r_1(x) = \int_1^x \frac{4}{\alpha_1\sqrt{\pi}} erf^{-1}(\alpha_1 \ln(y))\,\mathrm{d}y, \quad r_2(x) = \int_1^x \frac{4}{\alpha_2\sqrt{\pi}} erf^{-1}(\alpha_2 \ln(y))\,\mathrm{d}y \tag{38}$$

$$r_3(x) = \frac{2}{\alpha_3\sqrt{\pi}} erf^{-1}(\alpha_3 \ln(x)), \tag{39}$$

where $\alpha_{1-3} \neq 0$ are dimensionless material parameters. It is possible that the values of α_{1-3} are not the same for both $x \geq 1$ and $x < 1$; however, in this communication we assume they are the same for both $x \geq 1$ and $x < 1$. In addition to this, we only consider a less general strain energy function by letting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_0$; there is no physical or theoretical reason for this postulate, and it merely reduces the number of material constants. Hence, in this section, we let

$$r(x) = r_1(x) = r_2(x) = \int_1^x \frac{4}{\alpha_0\sqrt{\pi}} erf^{-1}(\alpha_0 \ln(y))\,\mathrm{d}y, \quad s(x) = r_3(x) = \frac{2}{\alpha_0\sqrt{\pi}} erf^{-1}(\alpha_0 \ln(x)). \tag{40}$$

In this paper, we will use the above specific forms to compare our theory with several different types of soft tissue experiments.

The restriction on α_0 is governed by the restriction on the function r_1, similar to that described in Sect. 4.1.2.

Weinberg and Kaazempur-Mofrad [41] biaxial experiment

In this section, we show the efficacy of the special constitutive form using the mitral valve tissue biaxial data of Weinberg and Kaazempur-Mofrad [41]. We emphasize that care must be taken in interpreting the results from a curve-fitting exercise. For example, the ground-state constant values will not be accurately obtained if there are insufficient data at low strains or due to inappropriate low strain data.

The relevant stress components take the form

$$S_f = (\mu_T + 2\mu_1)\frac{r'(\lambda_1)}{\lambda_1} + \beta\frac{s(\lambda_1)s'(\lambda_1)}{\lambda_1} - \mu_T\frac{\lambda_3 r'(\lambda_3)}{\lambda_1^2}, \quad S_t = \mu_T\left(\frac{r'(\lambda_2)}{\lambda_2} - \frac{\lambda_3 r'(\lambda_3)}{\lambda_2^2}\right), \tag{41}$$

where S_t and S_f are the components of the second Piola–Kirchhoff stress in the cross-fibre (perpendicular to fibre) and fibre directions, respectively. In Figs. 2, 3, 6, and 7, we visually fit our theory to the equibiaxial data of mitral valve anterior and posterior leaflets, respectively. We then use the material constant values obtained from these fittings to predict the $\frac{\lambda_1-1}{\lambda_2-1} = \frac{2}{1}$ biaxial data in Figs. 4, 5, 8 and 9. We note that for the 2:1 biaxial data $\lambda_1 = 2\lambda_2 - 1$ or $\lambda_2 = \frac{\lambda_1+1}{2}$, hence $\lambda_2 > 1$ when $\lambda_1 > 1$ and vice-versa, which indicates that the fibres are always in tension during this type of deformation. It is clear from these figures that our theory compares well with Weinberg and Kaazempur-Mofrad [41] biaxial data.

Chui et al. [5] uniaxial experiment

In this part, we compare our theory with the uniaxial Chui et al. [5] experiment on porcine liver. Here, we plot the nominal stresses $T_f = \lambda_1 S_f$ and $T_t = \lambda_2 S_t$ against λ_1 and λ_2, respectively. For the uniaxial deformation in the fibre direction, $\lambda_3 = \lambda_2 = \frac{1}{\sqrt{\lambda_1}}$, $T_t = 0$ and the T_f takes the form

$$T_f = (\mu_T + 2\mu_1)r'(\lambda_1) + \beta s(\lambda_1)s'(\lambda_1) - \mu_T\frac{\frac{1}{\sqrt{\lambda_1}}r'\left(\frac{1}{\sqrt{\lambda_1}}\right)}{\lambda_1}. \tag{42}$$

Fitting in Fig. 10 is done visually, and we use $\mu_T = 200$, $\beta = 0$ and $\mu_1 = 400$. In the case when the uniaxial deformation is in the 2-direction (perpendicular to the fibre direction), the axial stress

$$T_t = \mu_T\left(r'(\lambda_2) - \frac{\lambda_3 r'(\lambda_3)}{\lambda_2}\right) \tag{43}$$

is plotted against λ_2. In this case $\lambda_3 \neq \lambda_1$, in general, and λ_3 for (43) are obtained from solving the stress-free condition

$$T_f = (\mu_T + 2\mu_1)r'\left(\frac{1}{\lambda_2\lambda_3}\right) + \beta s\left(\frac{1}{\lambda_2\lambda_3}\right)s'\left(\frac{1}{\lambda_2\lambda_3}\right) - \mu_T\lambda_2\lambda_3^2 r'(\lambda_3) = 0 \tag{44}$$

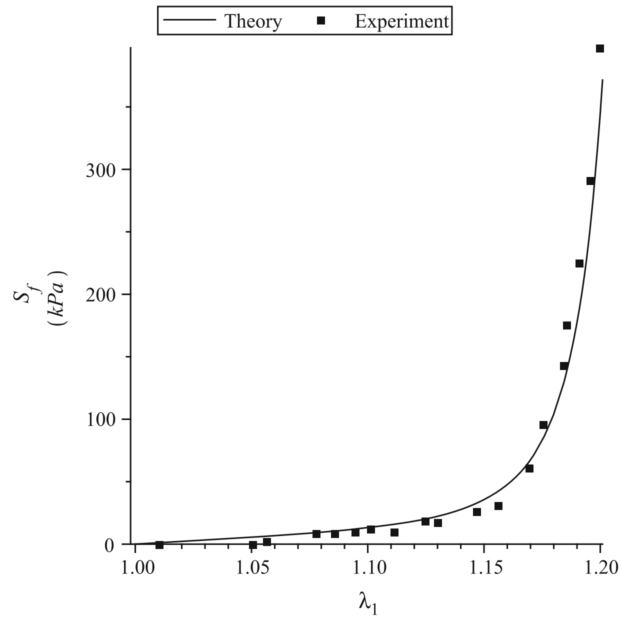

Fig. 2 Fitting Weinberg and Kaazempur-Mofrad [41] mitral valve anterior leaflet equibiaxial experiment. $\lambda_1 = \lambda_2$. $\mu_T = 5\,\text{kPa}$, $\mu_1 = 10\,\text{kPa}$, $\beta = 20\,\text{kPa}$, $\alpha_0 = 6$

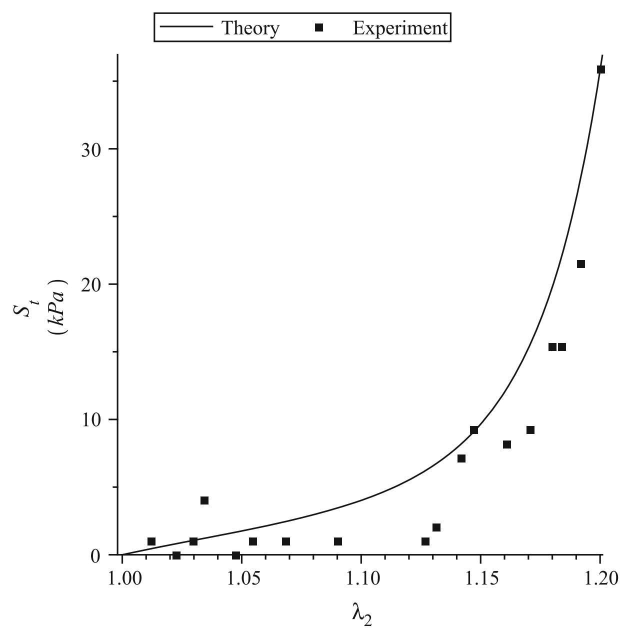

Fig. 3 Fitting Weinberg and Kaazempur-Mofrad [41] mitral valve anterior leaflet equibiaxial experiment. $\lambda_1 = \lambda_2$. $\mu_T = 5\,\text{kPa}$, $\alpha_0 = 6$

for a given value of λ_2. Since the values of λ_1 (or λ_3) are not given in Chui et al.'s [5] experiment, we cannot curve fit the T_t data; hence, we can only predict this data. It is clear from Fig. 11 that our theory predict the data quite well.

Weiss et al. [42] simple shear experiment

Very few suitable simple shear data for soft tissue can be found in the literature; one of them is the simple shear data on ligament tissue given in Weiss et al. [41]. The shear stress used to fit the experimental data is given by [32]

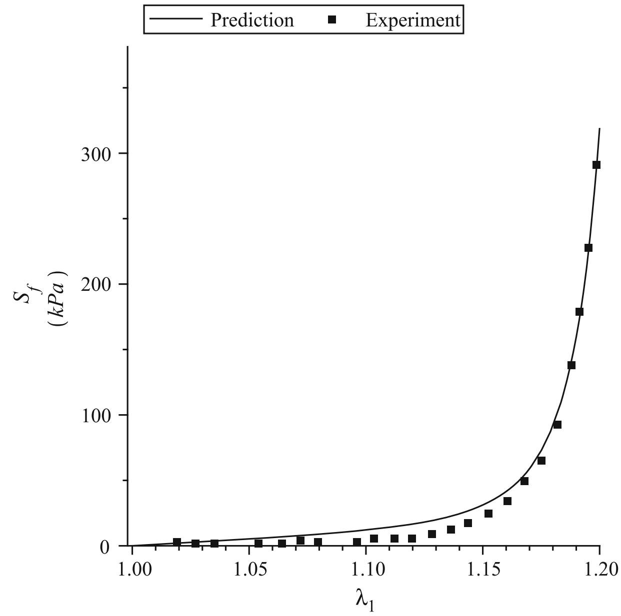

Fig. 4 Predicting Weinberg and Kaazempur-Mofrad [41] mitral valve anterior leaflet 2:1 biaxial experiment. $\lambda_2 = \frac{\lambda_1+1}{2}$. $\mu_T = 5\,\text{kPa}$, $\mu_1 = 10\,\text{kPa}$, $\beta = 20\,\text{kPa}$, $\alpha_0 = 6$

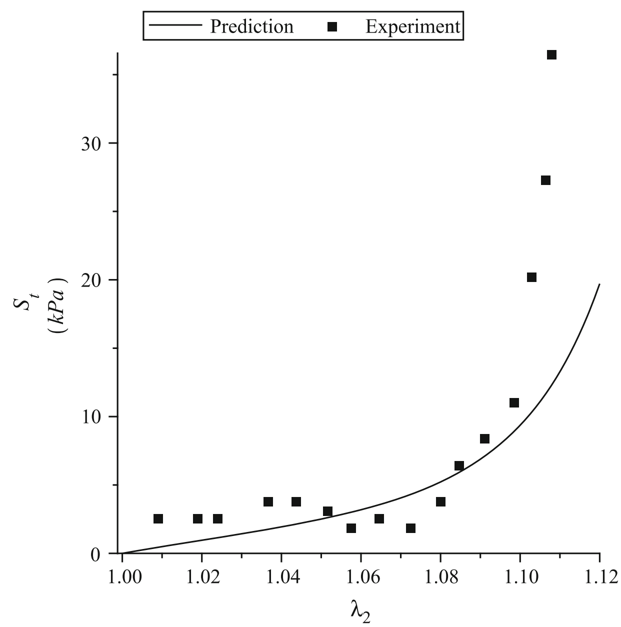

Fig. 5 Predicting Weinberg and Kaazempur-Mofrad [41] mitral valve anterior leaflet 2:1 biaxial experiment. $\lambda_1 = 2\lambda_2 - 1$. $\mu_T = 5\,\text{kPa}$, $\alpha_0 = 6$

$$\sigma_{12} = 2\left[l_1(\gamma s^2 + cs) + l_2(\gamma c^2 - cs) + l_4 \gamma cs\right], \tag{45}$$

where

$$l_\alpha = \frac{1}{\lambda_\alpha}\left(\mu_T r_1'(\lambda_\alpha) + 2\mu_1 \zeta_\alpha r_2'(\lambda_\alpha) + \beta\left[\sum_{i=1}^{3} \zeta_i r_3(\lambda_i)\right]\zeta_\alpha r_3'(\lambda_\alpha)\right), \quad \alpha = 1, 2, \tag{46}$$

$$l_4 = \frac{\boldsymbol{e}_1 \cdot \boldsymbol{A}\boldsymbol{e}_2}{\lambda_1^2 - \lambda_2^2}\left(2\mu_1[r_2(\lambda_1) - r_2(\lambda_2)] + \beta\left[\sum_{i=1}^{3} \zeta_i r_3(\lambda_i)\right][r_3(\lambda_1) - r_3(\lambda_2)]\right), \tag{47}$$

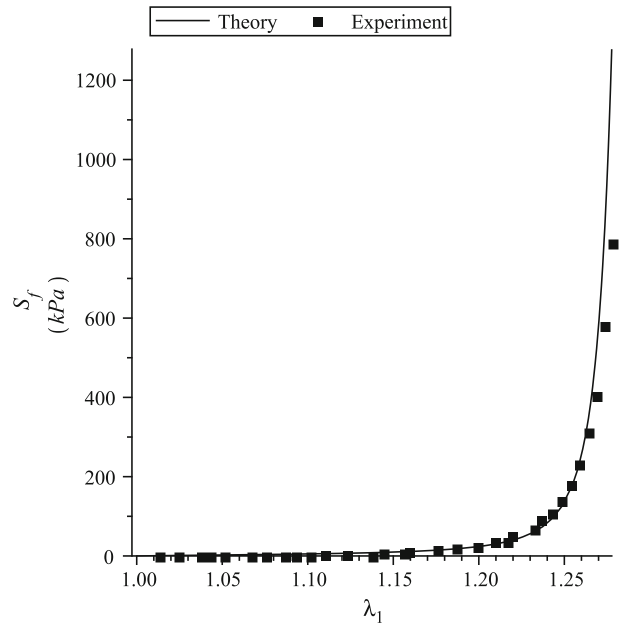

Fig. 6 Fitting Weinberg and Kaazempur-Mofrad [41] mitral valve posterior leaflet equibiaxial experiment. $\lambda_1 = \lambda_2$. $\mu_T = 6$ kPa, $\mu_1 = 1$ kPa, $\beta = 2$ kPa, $\alpha_0 = 5.1$

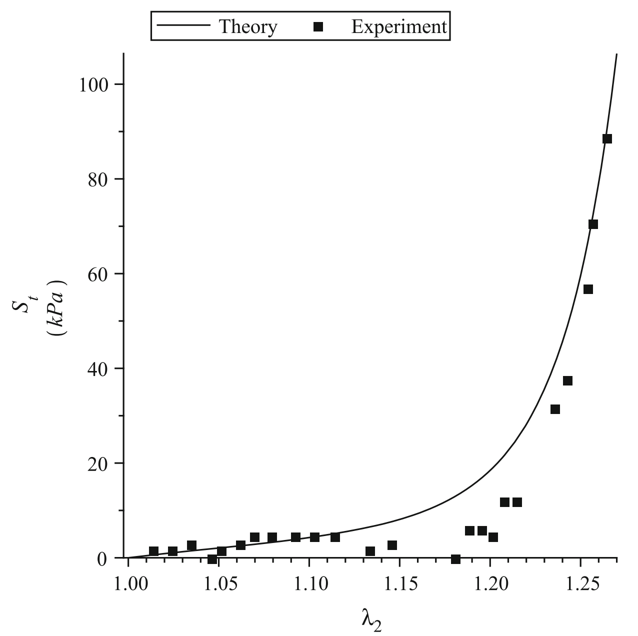

Fig. 7 Fitting Weinberg and Kaazempur-Mofrad [41] mitral valve posterior leaflet equibiaxial experiment. $\lambda_1 = \lambda_2$. $\mu_T = 6$ kPa, $\alpha_0 = 5.1$

$$c = \frac{1}{\sqrt{1+\lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}, \tag{48}$$

$$\lambda_1 = \frac{\gamma + \sqrt{\gamma^2+4}}{2} \geq 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2+4} - \gamma}{2} \leq 1, \quad \lambda_3 = 1, \tag{49}$$

and γ is the amount of shear. It is clear in Fig. 12 that our proposed crude constitutive function (visual fit) fits the experimental data well. In Fig. 13 we plot the values of I_4 for the preferred direction $\boldsymbol{a}$ taking the values

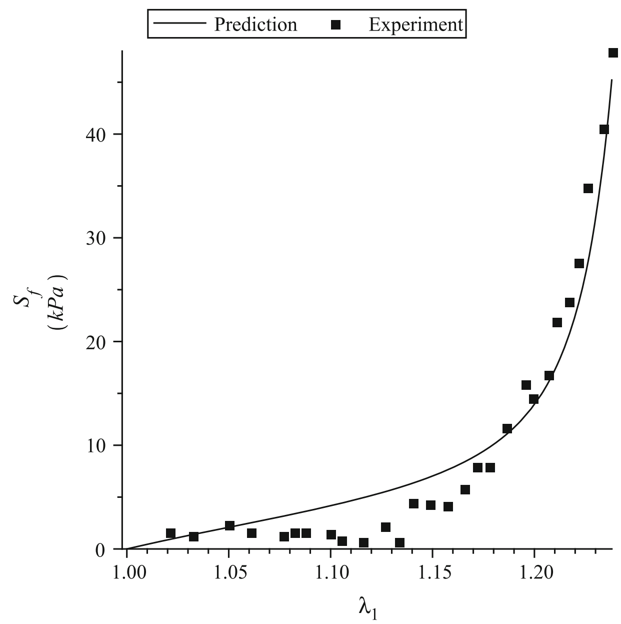

Fig. 8 Predicting Weinberg and Kaazempur-Mofrad [41] mitral valve posterior leaflet 2:1 biaxial experiment. $\lambda_2 = \frac{\lambda_1+1}{2}$. $\mu_T = 6\,\text{kPa}$, $\mu_1 = 1\,\text{kPa}$, $\beta = 2\,\text{kPa}$, $\alpha_0 = 5.1$

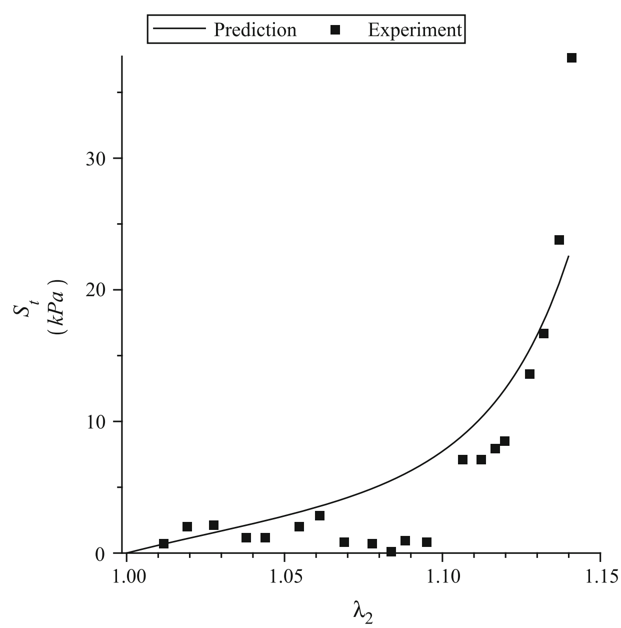

Fig. 9 Predicting Weinberg and Kaazempur-Mofrad [41] mitral valve posterior leaflet 2:1 biaxial experiment. $\lambda_1 = 2\lambda_2 - 1$. $\mu_T = 6\,\text{kPa}$, $\alpha_0 = 5.1$

$[1, 0, 0]$, $[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]$ and $[0, 1, 0]$. Using these values of $\boldsymbol{a}$, we plot σ_{12} versus γ in Fig. 14. From Fig. 14, we see that the shear stress behave as expected, i.e. at a fixed value of γ, the shear stress increases monotonically as I_4 (the fibre stretch) increases.

4.2 Ogden-type series for incompressible transversely isotropic solids

The Ogden series [26] has been widely and successfully used in predicting isotropic elastic deformations. Some of its attractive features are its generality and mathematical simplicity. However, the Ogden series has

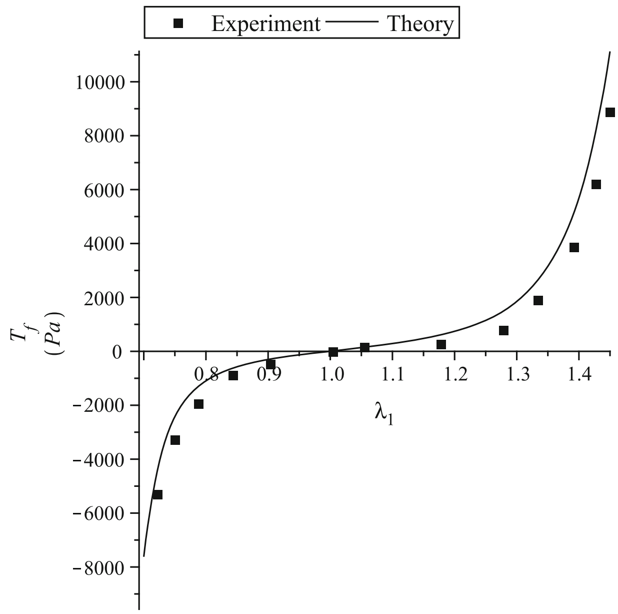

Fig. 10 Fitting Chui et al. [5] porcine liver uniaxial deformation in the fibre direction. $\mu_T = 200\,\text{Pa}$, $\mu_1 = 400\,\text{Pa}$, $\beta = 0\,\text{Pa}$, $\alpha_0 = 5.7$

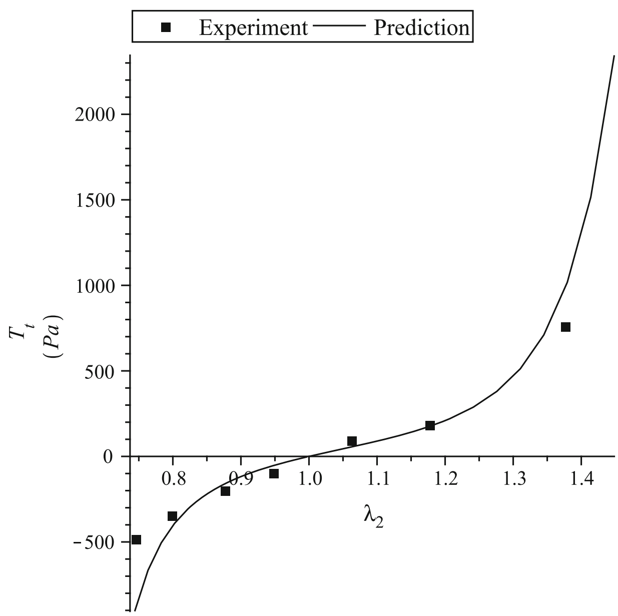

Fig. 11 Predicting Chui et al. [5] porcine liver uniaxial deformation in transverse direction. $\mu_T = 200\,\text{Pa}$, $\mu_1 = 400\,\text{Pa}$, $\beta = 0\,\text{Pa}$, $\alpha_0 = 5.7$

not been utilized to characterize the strain energy function for incompressible transversely isotropic elastic solids. This is due to the fact that a concise spectral formulation for anisotropic materials was not developed until the recent past. In this section, we use (24) to construct a modified Ogden strain energy function, where

$$r_1(\lambda) = \sum_r \frac{\tilde{a}_r}{\tilde{b}_r} \int_1^{\lambda} \frac{x^{\tilde{b}_r} - 1}{x}\,\mathrm{d}x, \quad r_2(\lambda) = \sum_r \frac{\tilde{c}_r}{\tilde{d}_r} \int_1^{\lambda} \frac{x^{\tilde{d}_r} - 1}{x}\,\mathrm{d}x, \tag{50}$$

$$r_3(\lambda) = \sum_r \frac{\tilde{\nu}_r(\lambda^{\tilde{\alpha}_r} - 1)}{\tilde{\alpha}_r}. \tag{51}$$

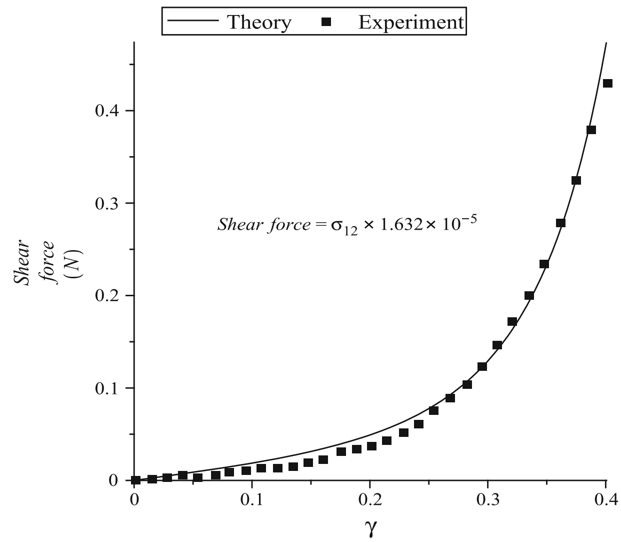

Fig. 12 Fitting Weiss et al. [42] ligament simple shear experimental data. $\mu_T = 350\,\mathrm{N/m^2}$, $\mu_1 = 8000\,\mathrm{N/m^2}$ $\beta = 0\,\mathrm{N/m^2}$, $\alpha_0 = 9.8$

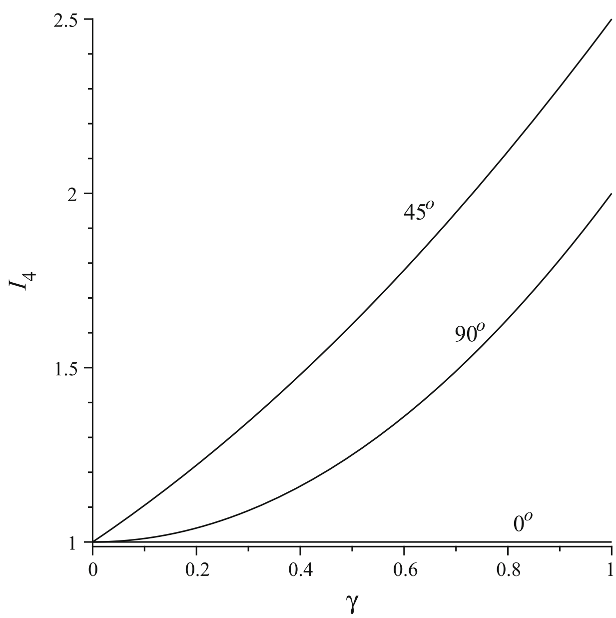

Fig. 13 Plot of I_4 versus γ for different fibre angles in the anticlockwise sense relative to the X_1 axis

The functions in (50) and (51) are Ogden-type series and have the properties

$$r_1''(1) = \sum_r \tilde{a}_r = 2, \quad r_2''(1) = \sum_r \tilde{c}_r = 2, \quad r_3'(1) = \sum_r \tilde{\nu}_r = 1. \tag{52}$$

The Eulerian spectral components for the Cauchy stress are:

$$\hat{\tau}_{ii} = \mu_T \sum_r \frac{\tilde{a}_r}{\tilde{b}_r}(\lambda_i^{\tilde{b}_r} - 1) + 2(\mu_L - \mu_T)\zeta_i \sum_r \frac{\tilde{c}_r}{\tilde{d}_r}(\lambda_i^{\tilde{d}_r} - 1) + \beta\left[\sum_{k=1}^{3}\zeta_k \sum_r \frac{\tilde{\nu}_r(\lambda_k^{\tilde{\alpha}_r} - 1)}{\tilde{\alpha}_r}\right]\zeta_i \sum_r \tilde{\nu}_r \lambda_i^{\tilde{\alpha}_r} - p,$$
$$i \quad \text{not summed.} \tag{53}$$

$$\hat{\tau}_{ij} = \frac{2\lambda_i\lambda_j}{\lambda_i^2 - \lambda_j^2}\left(2(\mu_L - \mu_T)(r_2(\lambda_i) - r_2(\lambda_j)) + \beta(r_3(\lambda_i) - r_3(\lambda_j))\left[\sum_{k=1}^{3}\zeta_k r_3(\lambda_k)\right]\right)\boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j. \tag{54}$$

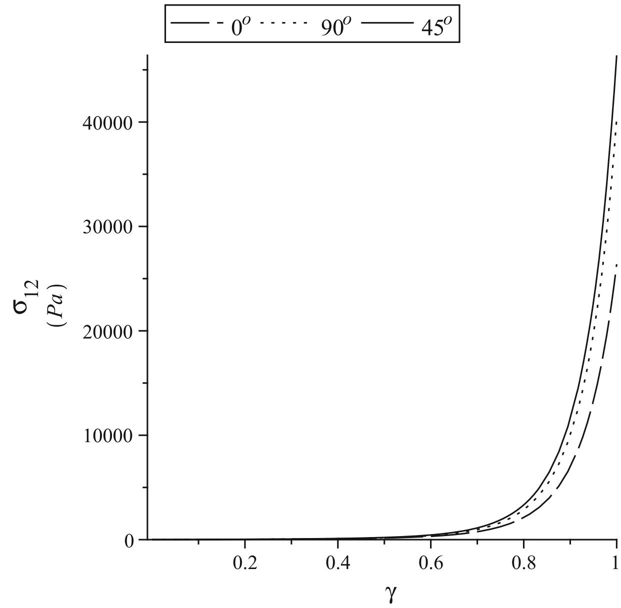

Fig. 14 Shear stress versus γ for different fibre angles in the anticlockwise sense relative to the X_1 axis. $\mu_T = 1$ Pa, $\mu_1 = 100$ Pa, $\beta = 0$ Pa, $\alpha_0 = 6$

It is important to note that the above Ogden-type series cannot be constructed using the standard classical invariants I_1, I_2, I_4, I_5. Since, representing a function via an infinite series can be considered a general function, it is expected that the Ogden-type series function is a able to model, for moderate strains, a wide range of different types of incompressible transversely isotropic elastic solids; however, due to the scope of this paper, we will not discuss the efficacy of the proposed modified Ogden series strain energy function in this paper. This will be carried out in the near future.

4.3 Modelling nonlinear magneto-elastic deformations

Magneto-sensitive (MS) elastomers correspond to a class of rubber-like material filled with magneto-active particles, which can react to the presence of magnetic fields. Following the work of Bustamante and Shariff [4], the free energy function Ω_M for magneto-sensitive elastomers can be expressed as

$$\Omega_M = \Omega(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, H), \tag{55}$$

where Ω satisfy the P-*property*,

$$\boldsymbol{a} = \frac{\boldsymbol{H}_l}{H}, \quad H \neq 0, \quad H = |\, \boldsymbol{H}_l \,|, \tag{56}$$

$\boldsymbol{H}_l$ is the Lagrangian counterpart in the reference configuration of the magnetic field $\boldsymbol{H}$. The *total* Cauchy-like stress $\boldsymbol{\tau}$ is related to Ω_M via [4]

$$\boldsymbol{\tau} = 2\boldsymbol{F}\frac{\partial \Omega_M}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p\boldsymbol{I}. \tag{57}$$

In view of (14) and (15), the principal Eulerian components of the *total* Cauchy-like stress are:

$$\tau_{ii} = \lambda_i \frac{\partial \Omega}{\partial \lambda_i} - p, \quad i \text{ not summed}, \tag{58}$$

$$\tau_{ij} = 2\lambda_i\lambda_j \frac{\dfrac{\partial \Omega}{\partial \zeta_i} - \dfrac{\partial \Omega}{\partial \zeta_j}}{\left(\lambda_i^2 - \lambda_j^2\right)} \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j, \quad i \neq j. \tag{59}$$

For simplicity, Bustamante and Shariff [4] proposed the particular separable form

$$\Omega_M = \sum_{i=1}^{3}(r_4(\lambda_i) + \zeta_i r_5(\lambda_i, H)), \tag{60}$$

where

$$r_4(\lambda_i) = \mu \ln(\lambda_i)^2 + \hat{c}_0 \left| \frac{\lambda_i^2}{2} - 2\lambda_i + \ln(\lambda_i) + 1.5 \right|, \tag{61}$$

$$r_5(\lambda_i, H) = \hat{c}_1 \frac{H^2}{\lambda_i^2 + \lambda_i} - \mu_0 \frac{H^2}{2\lambda_i^2} + \mu_0 H^2 e^{\left(-\frac{1}{H^2}\right)} \frac{1}{2\lambda_i^2}, \tag{62}$$

μ is the ground-state shear modulus of the isotropic body, $\mu_0 = 1.2566 \times 10^{-3}$ kN/kA^2 is the magnetic permeability in vacuo, and $\hat{c}_0$ and $\hat{c}_1$ are material constants. In Bustamante and Shariff [4], we show that our theory compares well with experimental data of Bellan and Bossis [2] and agrees with Kankanala and Triantafyllidis [21] results.

5 Two preferred direction anisotropy

5.1 Orthotropic solid

In this case the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal. The classical invariants commonly used as arguments for an orthotropic strain energy function are I_1, I_2, I_4, I_5 given in Eqs. (21) and (22) and

$$I_6 = \boldsymbol{b} \cdot \boldsymbol{C}\boldsymbol{b} = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \quad I_7 = \boldsymbol{b} \cdot \boldsymbol{C}^2\boldsymbol{b} = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \tag{63}$$

where $\xi_i = (\boldsymbol{b} \cdot \boldsymbol{e}_i)^2$. It is clear from (63) that I_6 and I_7 satisfy the *P-property*. Shariff [34] expressed the strain energy in terms of spectral invariants, i.e.

$$W_e = W_O(\lambda_{1-3}, \zeta_{1-3}, \xi_{1-3}). \tag{64}$$

Shariff [35] has shown that only six of the invariants in (64) are independent. The required spectral components are:

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial W_O}{\partial \lambda_i}, \quad \text{i not summed}, \tag{65}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{1}{\lambda_i^2 - \lambda_j^2}\left(\left(\frac{\partial W_O}{\partial \zeta_i} - \frac{\partial W_O}{\partial \zeta_j}\right)\boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j + \left(\frac{\partial W_O}{\partial \xi_i} - \frac{\partial W_O}{\partial \xi_j}\right)\boldsymbol{e}_i \cdot \boldsymbol{B}\boldsymbol{e}_j\right), \tag{66}$$

where $\boldsymbol{B} = \boldsymbol{b} \otimes \boldsymbol{b}$. In order to propose a separable form for W_e, we introduce (as in (23)) a separable function

$$\hat{\Phi} = \sum_{i=1}^{3} \xi_i \hat{\phi}(\lambda_i) \tag{67}$$

which satisfies the *P-property*. We use the separable forms (67) and (23) to construct the strain energy

$$W_e = \sum_{i=1}^{3}\left(\mu r_1(\lambda_i) + \hat{\mu}_1\zeta_i r_2(\lambda_i) + \mu_2\xi_i r_6(\lambda_i)\right) + \frac{\beta_1}{2}\left(\sum_{i=1}^{3}\zeta_i r_3(\lambda_i)\right)^2 + \frac{\beta_2}{2}\left(\sum_{i=1}^{3}\xi_i r_7(\lambda_i)\right)^2 \tag{68}$$

with the properties

$$r_6(1) = r_6'(1) = r_7(1) = 0, \quad r_6''(1) = 2, \quad r_7'(1) = 1, \tag{69}$$

where μ, $\hat{\mu}_1$, μ_2, β_1, and β_2 are the classical ground-state elastic constants. In this Section, unlike Sect. 4.1 (for transversely isotropic solids), for simplicity, we only consider $\hat{\mu}_1$,β_1 and μ_2, β_2 to be discrete functions of I_4 and I_6, respectively; μ is assumed to be independent of λ_i.

5.1.1 Modelling passive myocardium

Passive myocardium tissue can be considered as an orthotropic material [17]. Locally within the architecture of the myocardium, three mutually orthogonal directions can be identified, forming planes with distinct material responses. In this section, we consider the left ventricular myocardium which is non-homogeneous, thick-walled, nonlinearly elastic and incompressible material. To reduce the number of material constants, Shariff [36] proposed the simple forms

$$r_1(x) = s_p^2(x), \quad r_2(x) = s_p(x)^2, \quad r_6(x) = s_p(x)^2, \quad r_3 = s_p(x), \quad r_7 = s_p(x), \tag{70}$$

where

$$s_p(x) = \frac{2}{\bar{\alpha}_0\sqrt{\pi}} erf^{-1}(\bar{\alpha}_0 \ln(x)) + \bar{\alpha}_1(e^{1-x} + x - 2), \tag{71}$$

to model the mechanical behaviour of passive myocardium tissue. Using a similar analysis as in Sect. 4.1.1, the ground-state constants must satisfy the conditions [36],

$$c_3^{(m)} > 0, c_4^{(m)} > 0, c_5^{(m)} > 0, c_1^{(m)} + 2\mu > 0, (c_1^{(m)} + 2\mu)(c_2^{(m)} + 2\mu) > (\beta_3 + 2\mu)^2, \tag{72}$$

where $c_1^{(m)} = \beta_1 + 2\mu + 4\hat{\mu}_1$, $c_2^{(m)} = \beta_2 + 2\mu + 4\mu_2$, $c_3^{(m)} = \mu + \mu_2$, $c_4^{(m)} = \mu + \hat{\mu}_1$, $c_5^{(m)} = \mu + \hat{\mu}_1 + \mu_2$. In Shariff [34], the necessary condition $2xs_p(x)s_p'(x) > 0$ is used to impose constraints on the material constants $\bar{\alpha}_0$ and $\bar{\alpha}_1$. It was shown in [36] that the constitutive equation using the functional form (71) compares well with the simple shear experimental data of Dokos et al. [10] and the biaxial data of Yin et al. [43].

5.2 Modelling nonlinear transversely magneto-elastic deformations

During the curing process, when the magneto-active particles are added to the rubber-like matrix material, it is possible to apply an external magnetic field, which produces a relative alignment of the magneto-active particles and remains locked inside forming chains when the body solidifies. This class of magneto-active elastomers is called transversely isotropic MS elastomer [3], and it has been shown that the magnetostriction effect in such materials is much stronger in comparison with the case of isotropic MS elastomers, therefore making them more interesting from the point of view of the possible applications of such materials. In the case of transversely MS elastomers, when an external magnetic field is applied, the material behaves as a solid with two families of fibres, where one preferred direction $\boldsymbol{b}$ is given by the magneto-active particle chains, whereas the additional preferred direction $\boldsymbol{a}$ is induced by the magnetic forces. In general $\boldsymbol{a}$ is not orthogonal to $\boldsymbol{b}$, and the classical invariants required to describe the free energy function Ω_M for a transversely MS elastomer are $I_1, I_2, I_4, I_5, I_6, I_7$ and

$$I_8 = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{a}\cdot\boldsymbol{C}\boldsymbol{b} = \sum_{i=1}^{3}\lambda_i^2\chi_i, \quad I_9 = (\boldsymbol{a}\cdot\boldsymbol{b})^2, \quad I_{10} = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{a}\cdot\boldsymbol{C}^2\boldsymbol{b} = \sum_{i=1}^{3}\lambda_i^4\chi_i, \tag{73}$$

where $\chi_i = (\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\boldsymbol{e}_i)(\boldsymbol{b}\cdot\boldsymbol{e}_i)$. Note that

$$I_9 = \chi_1 + \chi_2 + \chi_3. \tag{74}$$

It is clear from (73) and (74) that I_8, I_9 and I_{10} satisfy the *P-property*. In this section we express the free energy Ω_M in terms of the spectral invariants [39], i.e.

$$\Omega_M = \Omega_T(\lambda_{1-3}, \zeta_{1-3}, \xi_{1-3}, \chi_{1-3}, I_9, H). \tag{75}$$

Shariff and Bustamante [38] shows that only seven of 13 invariants $\lambda_{1-3}, \zeta_{1-3}, \xi_{1-3}, \chi_{1-3}, I_9$ are independent. The required spectral components for the derivative $\dfrac{\partial\Omega_M}{\partial\boldsymbol{C}}$ are:

$$\left(\frac{\partial\Omega_M}{\partial\boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial\Omega_T}{\partial\lambda_i} \quad (i \quad \text{not summed}) \tag{76}$$

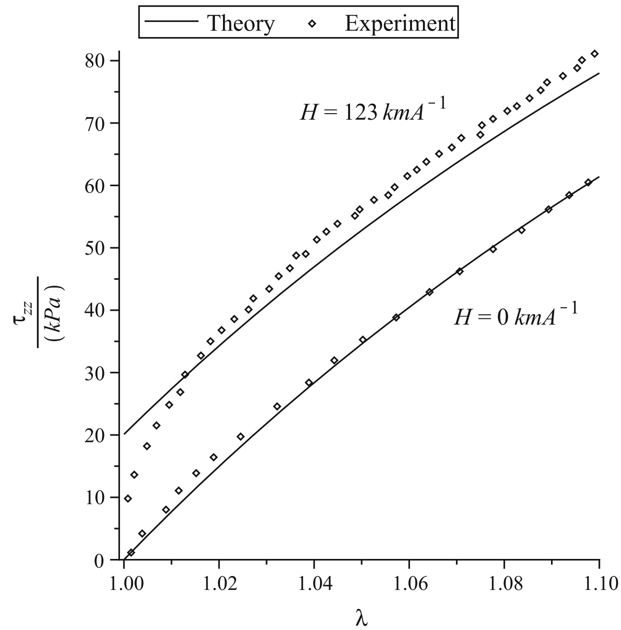

Fig. 15 Bellan and Bossis [2] uniaxial experiment. $\mu = 110$, $a_1 = -0.7$, $a_2 = 480$

and the shear components

$$\left(\frac{\partial \Omega_M}{\partial \boldsymbol{C}}\right)_{ij} = \frac{1}{(\lambda_i^2 - \lambda_j^2)} \left\{ \left(\frac{\partial \Omega_T}{\partial \zeta_i} - \frac{\partial \Omega_T}{\partial \zeta_j}\right) \boldsymbol{e}_i \cdot \boldsymbol{A} \boldsymbol{e}_j + \left(\frac{\partial \Omega_T}{\partial \xi_i} - \frac{\partial \Omega_T}{\partial \xi_j}\right) \boldsymbol{e}_i \cdot \boldsymbol{B} \boldsymbol{e}_j + \left(\frac{\partial \Omega_T}{\partial \chi_i} - \frac{\partial \Omega_T}{\partial \chi_j}\right) (\boldsymbol{e}_i \cdot \boldsymbol{AB} \boldsymbol{e}_j + \boldsymbol{e}_j \cdot \boldsymbol{AB} \boldsymbol{e}_i) \right\}, \quad i \neq j. \tag{77}$$

For this MS material we propose a simple separable form

$$\Omega_M = \sum_{i=1}^{3} r_8(\lambda_i) + \zeta_i r_9(\lambda_i, H) + \xi_i r_{10}(\lambda_i) + \chi_i r_{11}(\lambda_i, H). \tag{78}$$

It is clear from the above that Ω_M satisfies the *P-property*. To be more specific in our example, we choose the forms

$$r_8(\lambda) = \mu \ln(\lambda)^2, \quad r_9(\lambda, H) = a_1 \frac{\mu_0 H^2}{2\lambda^2} - \frac{\mu_0 H^2}{2\lambda^2},$$
$$r_{10}(\lambda) = a_2 \int_1^{\lambda} \frac{\ln(x)}{x^5}\, \mathrm{d}x, \quad r_{11}(\lambda, H) = 0, \tag{79}$$

where μ is the ground-state shear modulus for the corresponding isotropic body, a_1 and a_2 are constants. With the above specific forms, our theory compares well with the uniaxial tension experimental data of Bellan and Bossis [2], where the stress–strain behaviour is depicted in Fig. 15 for different values of H. Figure 15 indicates that to maintain a zero strain under an external magnetic field, a tensile stress is required to overcome the attractive interparticle forces. The tensile stress in the z-direction is given by

$$\tau_{zz} = \lambda s_T(\lambda) - \lambda^{-\frac{1}{2}} s_T\left(\lambda^{-\frac{1}{2}}\right) - a_1 \frac{\mu_o H^2}{\lambda^2} + \frac{\mu_o H^2}{\lambda^2} + a_2 \frac{\ln(\lambda)}{\lambda^4}, \tag{80}$$

where

$$s_T(x) = 2\mu \ln(x). \tag{81}$$

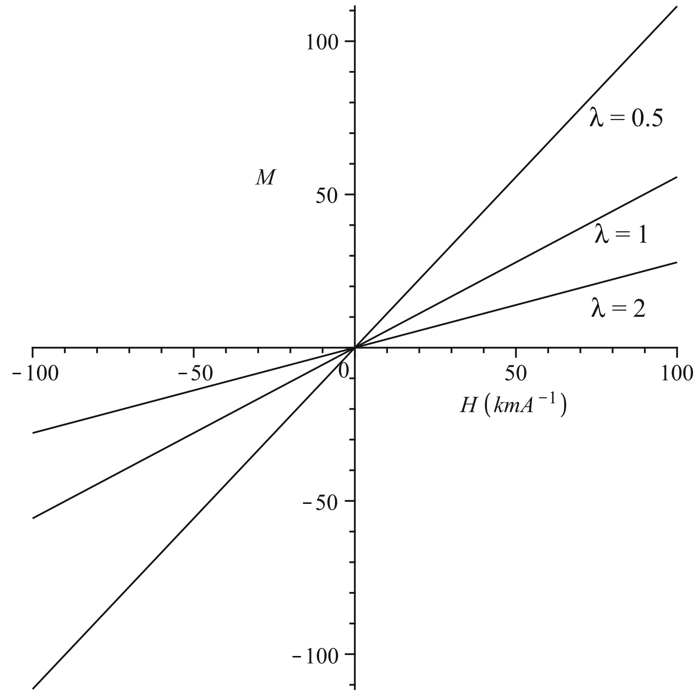

Fig. 16 Magnetization M versus magnetic field H_0 for the cylindrical body. For extension ($\lambda > 1$) the particle distance increases thus lowering the specimen's magnetization compared to $\lambda = 1$. The opposite is true for compression ($\lambda < 1$) due to shorter distances among the magnetic particles. $\mu = 110$, $a_1 = -0.7$, $a_2 = 480$

It is evident from (80) that the influence of H_0 on the tensile stress τ_{zz} diminishes as λ increases. This is due to that the average interparticle distance increases and the interparticle forces are weaker for the same imposed H_0; hence, the influence of H_0 on the tensile stress τ_{zz} diminishes [21].

Let $\boldsymbol{H}$ and $\tilde{\boldsymbol{B}}$ denote the magnetic field and the magnetic induction, respectively, in the current configuration. In the absence of electric interactions and time effects, the magnetic field and the magnetic induction have to satisfy the simplified form of the Maxwell equations

$$\operatorname{div} \tilde{\boldsymbol{B}} = 0, \quad \operatorname{curl} \boldsymbol{H} = \boldsymbol{0}. \tag{82}$$

It is possible to define the following Lagrangian counterparts in the reference configuration of the magnetic field and the magnetic induction $\boldsymbol{H}_l$ and $\boldsymbol{B}_l$ [3]

$$\boldsymbol{H}_l = \boldsymbol{F}^{\mathrm{T}} \boldsymbol{H}, \quad \boldsymbol{B}_l = J \boldsymbol{F}^{-1} \tilde{\boldsymbol{B}}. \tag{83}$$

In vacuum the magnetic field and the magnetic induction are related by the equation

$$\tilde{\boldsymbol{B}} = \mu_0 \boldsymbol{H}, \tag{84}$$

where μ_0 is the magnetic permeability in vacuo. For a condensed matter, an additional field is required, which is the magnetization field $\boldsymbol{M}$ and it is related to $\tilde{\boldsymbol{B}}$ and $\boldsymbol{H}$ through

$$\tilde{\boldsymbol{B}} = \mu_0 [\boldsymbol{H} + \boldsymbol{M}]. \tag{85}$$

$\boldsymbol{B}_l$ is related to $\boldsymbol{H}_l$ via [3]

$$\boldsymbol{B}_l = -\frac{\partial \Omega_T}{\partial \boldsymbol{H}_l}. \tag{86}$$

In Fig. 16, we depict the behaviour of the magnetization $\boldsymbol{M}$ with respect to $\boldsymbol{H}_l$ for several uniaxial strain values, and it shows that for extension ($\lambda > 1$) the particle distance increases thus lowering the specimen's magnetization compared to $\lambda = 1$. The opposite is true for compression ($\lambda < 1$) due to shorter distances among the magnetic particles [21]. In view of the above illustrations, we note that a construction of a more sophisticated specific constitutive equation is trivial, and we will do this in the future when the appropriate experimental data are available.

6 Inelastic Mullins stress-softening materials

When subjected to cyclic loadings, many rubber-like and biological materials exhibit an anisotropic stress-softening phenomenon widely known as the Mullins effect [25]. There is a wide literature on the Mullins effect; readers are referred to the literature [7,11,31,33] for detail description on the anisotropic behaviour of the Mullins effect. Softening-induced anisotropy is demonstrated by performing successive non-proportional loadings (i.e. successive loadings with changing the directions of stretching or the type of loading), and recently, several non-proportional experiments [7–9,13,20,22] were conducted.

In this section, we briefly demonstrate that a separable form of free energy is able to model anisotropic stress softening. Detailed description of this model can be found in Shariff [37]; hence, we just give an outline of this model here.

Shariff [37] proposed the direction-dependent free energy

$$W_f = \sum_{i=1}^{3} \left[\eta(\Upsilon_i, \hat{\alpha}_i) r_f(\Upsilon_i) + \phi(\Upsilon_i, \hat{\alpha}_i) \right] \tag{87}$$

which satisfies the *P-property*, where

$$\Upsilon_1 = \mathrm{tr}(\boldsymbol{UA}) = \lambda_1\zeta_1 + \lambda_2\zeta_2 + \lambda_3\zeta_3, \quad \Upsilon_2 = \mathrm{tr}(\boldsymbol{UB}) = \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3,$$
$$\Upsilon_3 = \lambda_1(1 - \zeta_1 - \xi_1) + \lambda_2(1 - \zeta_2 - \xi_2) + \lambda_3(1 - \zeta_3 - \xi_3) \tag{88}$$

and

$$\phi(\Upsilon_i, \hat{\alpha}_i) = -\int_1^{\Upsilon_i} r_f(y) \frac{d\eta}{dy}(y, \hat{\alpha}_i)\, \mathrm{d}y. \tag{89}$$

The softening function η is introduced in (87) to soften the stress and has the property $0 < \eta \leq 1$ and $\eta(y, y) = 1$. The free energy (87) is direction dependent since the damage parameter $\hat{\alpha}_i$ is direction dependent, i.e.

$$\hat{\alpha}_i = \begin{cases} s_i^{(\max)} & \text{when } \lambda_i > 1, \\ s_i^{(\min)} & \text{when } \lambda_i < 1, \end{cases} \tag{90}$$

where

$$s_i^{(\min)} \leq \lambda_i \leq s_i^{(\max)}, \tag{91}$$
$$s_i^{(\max)} = \max_{0 \leq z \leq t} \sqrt{\boldsymbol{e}_i \cdot \boldsymbol{C}(z)\boldsymbol{e}_i}\,, \quad \text{and} \quad s_i^{(\min)} = \min_{0 \leq z \leq t} \sqrt{\boldsymbol{e}_i \cdot \boldsymbol{C}(z)\boldsymbol{e}_i}; \tag{92}$$

the material is subjected to a deformation history up to the current time t, and z denotes a running time variable. We note that in (90), we do not consider $\lambda_i = 1$ because our model is constructed in such a way that $\hat{\alpha}_i$ does not contribute to stress softening when $\lambda_i = 1$. In the case when $\lambda_i = \lambda_j$ $(i \neq j)$, the directions $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ are not unique. In view of this, we let

$$\hat{\alpha}_i = \hat{\alpha}_j = \begin{cases} \dfrac{1}{\sqrt{s_k^{(\min)}}} & \text{when } \lambda_i = \lambda_j > 1, \\ \dfrac{1}{\sqrt{s_k^{(\max)}}} & \text{when } \lambda_i = \lambda_j < 1, \end{cases} \tag{93}$$

where $i \neq j \neq k$. In the case when all the principal stretches are equal, the principal directions are all non-unique. However, for an incompressible material this can only happen when $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and as mentioned before $\hat{\alpha}_i$ does not contribute to stress softening in this case; hence their values are not given.

In the direction-dependent model, $\boldsymbol{a} = \boldsymbol{e}_1$ and $\boldsymbol{b} = \boldsymbol{e}_2$(always), and hence, we obtain the separable form

$$W_f = \sum_{i=1}^{3} \left(\eta(\lambda_i, \hat{\alpha}_i) r_f(\lambda_i) + \phi(\lambda_i, \hat{\alpha}_i) \right), \tag{94}$$

where $\phi(\lambda_i, \hat{\alpha}_i) = -\int_1^{\lambda_i} r_f(y)\dfrac{d\eta}{dy}(y, \hat{\alpha}_i)\,\mathrm{d}y$. In view of (65) and (66), the second Piola–Kirchhoff stress is always coaxial with the tensor $\boldsymbol{U}$; this simulates the stress behaviour in experiments, especially, in non-proportional simple tension loadings of Machado et al. [22].

Shariff [37] only considers η to have the particular form

$$\eta(y, d) = \hat{\eta}(g(y), g(d)) = e^{b_1(g(y)-g(d))g(y)^{b_2}} - b_3 e^{-b_4 g(y)}(g(d) - g(y)), \tag{95}$$

where g is a damage function. The condition

$$\frac{\partial \hat{\eta}}{\partial g(\hat{\alpha}_i)}(g(\lambda_i), g(\hat{\alpha}_i)) < 0 \tag{96}$$

is imposed so that W_f decreases monotonically as $g(a_i)$ increases.

On the primary loading, $\eta = 1$, the free energy function simply becomes the separable form, i.e.

$$W_f = \sum_{i=1}^{3}[r_f(\lambda_i) + +\phi(\lambda_i, \hat{\alpha}_i)]. \tag{97}$$

Based on the work of Shariff [30] on nonlinear isotropic elasticity, the specific form for r, i.e.

$$r_f(\lambda_i) = \int_1^{\lambda_i} \frac{f(y)}{y}\,\mathrm{d}y \tag{98}$$

is proposed, where $f(1) = 0$, $f(y) > 0$ for $y > 1$ and $f(y) < 0$ for $y < 1$. It is clear that $r(1) = 0$, $r'(1) = 0$, $0 = r(1) \le r(y)$ and $r(y)$ increases (strictly) monotonically away from $y = 1$. Following the work of Shariff [30],

$$f(y) = \sum_{i=1}^{4} d_i^{(m)} \phi_i(y), \tag{99}$$

is proposed, where

$$\phi_1(y) = \frac{2}{3}\ln(y), \quad \phi_2(y) = e^{(1-y)} + y - 2, \quad \phi_3(y) = e^{(y-1)} - y,$$
$$\phi_4(y) = \frac{(y-1)^3}{y^{\bar{k}}}, \tag{100}$$

$d_{1-4}^{(m)}$ and $\bar{k}$ are material constants.

Shariff [37] has shown that the above separable form is able to predict and compares well with experimental data available in the literature for different types of rubber-like materials and different types of experiments.

7 Remarks

We note that the separable constitutive equations for anisotropic problems, proposed in this paper, are just heuristic proposals. They are based on the forms developed, in the past, for anisotropic solids (see for example, Shariff [36] and Bustamante and Shariff [4]). An important property of a separable form is that it satisfies the *P-property*, which is required in spectral constitutive equations. An attractive property is that it is simple in form in the sense that it contains only a single-variable arbitrary function that depends on a principle stretch, which is a mechanically useful invariant when compared to the classical invariants I_1, I_2, I_5 and etc. Some unattractive features of some classical invariants and attractive features of spectral invariants have been discussed, for example, in Shariff [32]. The simple separable forms contain spectral invariants (which have immediate physical interpretation) that can be more attractive when looking for expressions for the free energy function by fitting experimental data, and also in order to design a rational programme of experiments for anisotropic materials. The mechanical behaviour of an anisotropic solid is much easier to analyse via the separable form [34,34]. It is shown in Shariff [32,36] that general spectral free energy functions for incompressible transversely isotropic and orthotropic solids are of the form

$$W_e = \sum_{i=1}^{3} r_T(\lambda_i, \zeta_i) + \hat{g}(\lambda_1, \lambda_2, \zeta_1, \zeta_2) + \hat{g}(\lambda_1, \lambda_3, \zeta_1, \zeta_3) + \hat{g}(\lambda_2, \lambda_3, \zeta_2, \zeta_3) \tag{101}$$

and

$$W_e = \sum_{i=1}^{3} r_O(\lambda_i, \zeta_i, \xi_i) + \bar{g}(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) + \bar{g}(\lambda_1, \lambda_3, \zeta_1, \zeta_3, \xi_1, \xi_3) + \bar{g}(\lambda_2, \lambda_3, \zeta_2, \zeta_3, \xi_2, \xi_3), \tag{102}$$

respectively. The functions $\hat{g}$ and $\bar{g}$ have the symmetries $\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j) = \hat{g}(\lambda_j, \lambda_i, \zeta_j, \zeta_i)$ and $\bar{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = \bar{g}(\lambda_j, \lambda_i, \zeta_j, \zeta_i, \xi_j, \xi_i)$, $i \neq j$, respectively. The spectral forms (101) and (102) must satisfy the *P-property*, and this property is difficult to implement. However, the separable forms, proposed in this paper, ensure that the *P-property* is satisfied. The relations between the separable and the general (101) and (102) forms are given below:

For a transversely isotropic solid, in view of Eqs. (101) and (24), we have

$$r_T(\lambda_i, \zeta_i) = \mu_T r_1(\lambda_i) + 2(\mu_L - \mu_T)\zeta_i r_2(\lambda_i) + \frac{\beta}{2}\zeta_i^2 r_3^2(\lambda_i), \quad \hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j) = \beta\zeta_i\zeta_j r_3(\lambda_i) r_3(\lambda_j). \tag{103}$$

In the case of an orthotropic solid, we have, in view of Eqs. (102) and (68),

$$\begin{aligned} r_O(\lambda_i, \zeta_i, \xi_i) &= \mu r_1(\lambda_i) + \hat{\mu}_1\zeta_i r_2(\lambda_i) + \mu_2\xi_i r_6(\lambda_i) + \frac{\beta_1}{2}\zeta_i^2 r_3^2(\lambda_i) + \frac{\beta_2}{2}\xi_i^2 r_7^2(\lambda_i), \\ \bar{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j\xi_i, \xi_j) &= \beta_1\zeta_i\zeta_j r_3(\lambda_i) r_3(\lambda_j) + \beta_2\xi_i\xi_j r_7(\lambda_i) r_7(\lambda_j). \end{aligned} \tag{104}$$

8 Conclusion

In this communication, we have shown, mathematically, that the Valanis–Landel hypothesis is part of a general strain energy function of an isotropic elastic solid. An anisotropic extension of the separable Valanis–Landel form is proposed to facilitate the construction of separable anisotropic constitutive equations, and we show that particular separable forms are capable of modelling anisotropic elastic, MS and non-elastic Mullins solids. Construction and analysis of more sophisticated specific constitutive equations, such as the Ogden-type series, via the proposed separable forms will be carried out in the near future. We hope that the proposed separable forms will set a platform for future modelling of anisotropic problems using spectral invariants. The spectral invariants used in this work can be more attractive than the standard classical invariants presented in the literature because a clearer physical meaning can be attached to each one of them, and because when solving boundary value problems, the different expressions for the stresses in terms of the deformation and the magnetic induction are simpler than when considering the standard classical invariant theory. The spectral invariants also imparts experimental advantage over classical invariants presented in the literature, e.g. a simple triaxial test can vary a single invariant while keeping the remaining invariants fixed [32].

Appendix

In this appendix, we show very simply for a transversely isotropic solid, via spectral analysis, the effect of not selecting the full set of invariants on the ground-state constants. We demonstrate this effect via some models given in the literature. Before we do this, we consider the second derivative of $W_P(\lambda_1, \lambda_1, \zeta_1, \zeta_3) = \tilde{W}(\lambda_1, \lambda_1, \zeta_1, \zeta_2 = 1 - \zeta_1 - \zeta_3)$ with respect to λ_1 at reference configuration, i.e.

$$\frac{\partial^2 W_P}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_3) = 4\mu_T + 4\mu_1(\zeta_1 + \zeta_3) + \beta(\zeta_1 - \zeta_3)^2. \tag{A1}$$

The terms δ_i , $i = 1, 2, \ldots$ given below are material constants.

Ciarletta et al. [6] model

We can write Ciarletta et al. [6] model, for rubber-like materials, in terms of spectral invariants, i.e.

$$W_C(\lambda_1, \lambda_1, \zeta_1, \zeta_3) = \delta_1 I_1(\boldsymbol{G}) + \delta_2 I_2(\boldsymbol{G}) + \delta_3 I_4(\boldsymbol{G}), \tag{A2}$$

where

$$I_1(\boldsymbol{G}) = tr(\boldsymbol{G}) = \sum_{i=1}^{3} q_i, \quad I_2(\boldsymbol{G}) = \sum_{i=1}^{3} q_i^2, \quad I_4(\boldsymbol{G}) = \sum_{i=1}^{3} \zeta_i q_i, \quad q_i = \frac{\left(\lambda_i - \dfrac{1}{\lambda_i}\right)^2}{4}. \tag{A3}$$

The corresponding ground-state second derivative for this model is

$$\frac{\partial^2 W_C}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_3) = 4\delta_1 + 2\delta_3(\zeta_1 + \zeta_3). \tag{A4}$$

Since ζ_1 and ζ_3 are arbitrary in the reference state, comparing (A1) and (A4), we have

$$\mu_T = \delta_1, \quad \mu_1 = \frac{\delta_3}{2}, \quad \beta = 0. \tag{A5}$$

Humphrey et al. [15] model

In view of (19) and (20), Humphrey et al. [15] strain energy function can be written as

$$W_H(\lambda_1, \lambda_1, \zeta_1, \zeta_3) = \delta_1(\sqrt{I_4} - 1)^2 + \delta_2((\sqrt{I_4} - 1)^3 + \delta_3(I_1 - 3) + \delta_4(I_1 - 3)(\sqrt{I_4} - 1) + \delta_5(I_1 - 3)^2. \tag{A6}$$

The second derivative for this model is

$$\frac{\partial^2 W_H}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_3) = 8\delta_3 + 2\delta_1(\zeta_1 - \zeta_3)^2. \tag{A7}$$

Hence, we have

$$\mu_T = \frac{\delta_3}{2}, \quad \mu_1 = 0, \quad \beta = 2\delta_1. \tag{A8}$$

May-Newmann and Yin [24]

$$W_M(\lambda_1, \lambda_1, \zeta_1, \zeta_3) = \frac{\delta_1}{2}\left(e^{\delta_2(I_1-3)^2 + \delta_3(\sqrt{I_4}-1))^4} - 1\right), \tag{A9}$$

$$\frac{\partial^2 W_H}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_3) = 0, \tag{A10}$$

$$\mu_T = 0, \quad \mu_1 = 0, \quad \beta = 0. \tag{A11}$$

Humphrey and Yin [14]

$$W_{HY}(\lambda_1, \lambda_1, \zeta_1, \zeta_3) = \delta_1(e^{\delta_2(I_1-3)^2} - 1) + \delta_3(e^{\sqrt{I_4}(\sqrt{I_4}-1)^2} - 1), \tag{A12}$$

$$\frac{\partial^2 W_{HY}}{\partial \lambda_1^2}(1, 1, \zeta_1, \zeta_3) = 8\delta_1\delta_2 + 2\delta_3(\zeta_1 - \zeta_3)^2, \tag{A13}$$

$$\mu_T = 2\delta_1\delta_2, \quad \mu_1 = 0, \quad \beta = 2\delta_3. \tag{A14}$$

Ciarletta et al. [6] model assumed $\beta = 0$ on the onset. The rest of the models have assumed, on the onset, the relation $\mu_L = \mu_T$. In the case of the May-Newmann and Yin [24] model, the conditions $\mu_L = \mu_T = 0$ and $\beta = 0$ are also assumed on the onset. These assumptions, the author believes, are not rigorously derived since there is no strong experimental evidence to support these assumptions (see, for example, Feng et al. [12]).

References

1. Al-Kinani, R., Hartmann, S., Netz, T.: Transversal isotropy based on a multiplicative decomposition of the deformation gradient within p-version finite elements. Z. Angew. Math. Mech. (2014). doi:10.1002/zamm.201300155
2. Bellan, C., Bossis, G.: Field dependence of viscoelastic properties of MR elastomers. Int. J. Mod. Phys. B. **16**, 2447–2453 (2002)
3. Bustamante, R.: Transversely isotropic nonlinear magneto-active elastomers. Acta Mech. **210**, 183–214 (2010)
4. Bustamante, R., Shariff, M.H.B.M.: A principal axis formulation for nonlinear magnetoelastic deformations: isotropic bodies. Eur. J. Mech. A Solids **50**, 17–27 (2015)
5. Chui, C., Kobayashi, E., Chen, X., Hisada, T., Sakuma, I.: Transversely isotropic properties of porcine liver tissue: experiments and constitutive modeling. Med. Bio. Eng. Comput. **45**, 99–106 (2007)
6. Ciarletta, P., Izzo, I., Micera, S., Tendick, F.: Stiffening by fiber reinforcement in soft materials: a hyperelastic theory at large strains and its application. J. Mech. Behav. Biomed. Mater. **4**, 1359–1368 (2011)
7. Dargazany, R., Itskov, M.: A network evolution model for the anisotropic mullins effect in carbon black filled rubbers. Int. J. Solids Struct. **46**, 2967–2977 (2009)
8. Diani, J., Brieu, M., Gilormini, P.: Observation and modeling of the anisotropic visco-hyperelastic behavior of a rubberlike material. Int. J. Solids Struct. **43**(10), 3044–3056 (2006)
9. Diani, J., Brieu, M., Vacherand, M.: A damage directional constitutive model for Mullins effect with permanent set and induced anisotropy. Eur. J. Mech. A Solids **25**, 483–496 (2006)
10. Dokos, S., Smaill, B.H., Young, A.A., LeGrice, I.J.: Shear properties of passive ventricular myocardium. Am. J. Physiol. Heart Circ. Physiol. **283**, H2650–H2659 (2002)
11. Dorfmann, A., Pancheri, F.: A constitutive model for the Mullins effect with changes in material symmetry. Int. J. Nonlinear Mech. **47**(8), 874–887 (2012)
12. Feng, Y., Okamoto, R.J., Namani, R., Genin, G.M., Bayly, P.V.: Measurements of mechanical anisotropy in brain tissue and implications for transversely isotropic material models of white matter. J. Mech. Behav. Biomed. Mater. **23**, 117–132 (2013)
13. Hanson, D.E., Hawley, M., Houlton, R., Chitanvis, K., Rae, P., Orler, B.E., Wrobleski, D.A.: Stress softening experiments in silica-filled polydimethylsiloxane provide insight into a mechanism for the Mullins effect. Polymer **46**(24), 10989–10995 (2005)
14. Humphrey, J., Yin, F.: Biomechanical experiments on excised myocardium: Theoretical considerations. J. Biomech. **22**(4), 377–383 (1989)
15. Humphrey, J., Strumpf, R., Yin, F.: Determination of a constitutive relation for passive myocardium: I. A new functional form. J. Biomech. Eng. **112**(3), 340–346 (1990)
16. Hill, R.: Constitutive inequalities for isotropic elastic solids under finite strain. Proc. R. Soc. Lond. A **314**, 457–472 (1970)
17. Holzapfel, G.A., Ogden, R.W.: Constitutive modeling of passive myocardium: a structurally based framework of material characterization. Philos. Trans. R. Soc. A **367**, 3445–3475 (2009)
18. Itskov, M., Aksel, N.A.: Class of orthotropic and transversely isotropic hyperelastic constitutive models based on polyconvex strain energy function. Int J Solids Stuct 41, 3833–3848 (2004)
19. Itskov, M., Halberstroh, E., Ehret, A.E., Vhringer, M.C.: Experimental observation of the deformation induced anisotropy of the Mullins effect in rubber. Kaut. Gummi. Kunstst. 93–96 (2006)
20. Jones, D.F., Treloar, L.R.G.: The properties of rubber in pure homogeneous strain. J. Phys. D Appl. Phys. **8**(11), 1285–1304 (1975)
21. Kankanala, S.V., Triantafyllidis, N.: On finitely strained magnetorheological elastomers. J. Mech. Phys. Solids **52**, 2869–2908 (2004)
22. Machado, G., Chagnon, G., Favier, D.: Induced anisotropy by the Mullins effect in filled silicone rubber. Mech. Mater. **50**, 70–80 (2012)
23. Marckmann, G., Verron, E.: Comparison of hyperelastic models for rubber-like materials. Rubber Chem. Technol. **79**, 835–858 (2006)
24. May-Newman, K., Yin, F.: A constitutive law for mitral valve tissue. J. Biomech. Eng. **120**(1), 38–47 (1998)
25. Mullins, L., Tobin, N.R.: Theoretical model for the elastic behaviour of filler reinforced vulcanized rubbers. Rubber Chem. Technol. **30**, 551–571 (1957)
26. Ogden, R.W.: Large deformation isotropic elasticity: on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. Lond. A **326**, 565–584 (1972)
27. Ogden, R.W.: Non-linear Elastic Deformations. Dover, New York (1997)
28. Ogden, R.W.: Nonlinear elasticity, anisotropy and residual stresses in soft tissue. In: Holzapfel G.A., Ogden R.W. (eds) Biomechanics of soft tissue in cardiovascular systems. CISM courses and lectures vol 441, pp. 65–10. Springer (2003)
29. Saccomandi, G.: Phenomenological theory of rubber-like elasticity. In: Saccomandi, G., Ogden, R.W. (2004) Thermomechanics of Rubber-Like Elasticity: CISM Lectures Notes 452, pp. 91–134. Springer Wien, NewYork (2004)
30. Shariff, M.H.B.M.: Strain energy function for filled and unfilled rubberlike material. Rubber Chem. Technol. **73**, 1–21 (2000)
31. Shariff, M.H.B.M.: An anisotropic model of the Mullins effect. J. Eng. Math. **56**, 415–435 (2006)
32. Shariff, M.H.B.M.: Nonlinear transversely isotropic elastic solids: an alternative representation. Q. J. Mech. Appl. Math. **61**, 129–149 (2008)
33. Shariff, M.H.B.M.: Anisotropic stress-softening model for compressible solids. Z. Angew. Math. Phys. (ZAMP) **60**, 1112–1134 (2009)
34. Shariff, M.H.B.M.: Physical invariants for nonlinear orthotropic solids. Int. J. Solids Struct. **48**, 1906–1914 (2011)
35. Shariff, M.H.B.M.: Nonlinear orthotropic elasticity: only six invariants are independent. J. Elast. **110**, 237–241 (2013a)
36. Shariff, M.H.B.M.: Physical invariant strain energy function for passive myocardium. Biomech. Model Mechanobiol. **12**, 215–223 (2013b)
37. Shariff, M.H.B.M.: Direction dependent orthotropic model for Mullins materials. Int. J. Solids Struct. **51**, 4357–4372 (2014)

38. Shariff, M.H.B.M., Bustamante, R.: On the independence of strain invariants of two preferred direction nonlinear elasticity. Int. J. Eng. Sci. **97**,18–25 (2015)
39. Shariff, M.H.B.M., Bustamante, R., Hossain, M., Steinmann, P.: A novel spectral formulation for transversely isotropic magneto-elasticity Math. Mech. Solids (2015). doi:10.1177/1081286515618999
40. Valanis, K.C., Landel, R.F.: The strain-energy function of hyperelastic material in terms of the extension ratios. J. Appl. Phys. **38**, 2997–3002 (1967)
41. Weinberg, E.J., Kaazeempur-Mofrad, M.R.: A large-strain finite element formulation for biological tissues with application to mitral valve leaflet tissue mechanics. J. Biomech. **39**, 1557–1561 (2006)
42. Weiss, J.A., Gardiner, J.C., Bonifasi-Lista, C.: Ligament material behavior is nonlinear, viscoelastic and rate-independent under shear loading. J. Biomech. 35, 943–950 (2002)
43. Yin, F.C.P., Strumpf, R.K, Chew, P.H., Zeger, S.L.: Quantification of the mechanical properties of noncontracting canine myocardium under simultaneous biaxial loading. J. Biomech. **20**, 577–589 (1987)