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ORIGINAL PAPER

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Physical variational principle in dissipative media

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Abstract The physical variational principle (PVP) is a physical principle which is implied in the thermodynamic laws. For a conservative system, the PVP is implied in the first thermodynamic law and gives the motion equation. But for a dissipative system, PVP is implied in the extended Gibbs equation, which is the result of the first and second thermodynamic laws. The precision of the PVP in a dissipative system is in the same order of the Gibbs equation. The dissipative work and its converted internal irreversible heat are simultaneously included in the PVP to get the governing equation and the boundary condition of the dissipated variables. The "generalized motion equations" including governing equations of the mechanical momentum and thermoelastic, thermal viscoelastic, thermal elastoplastic, linear thermoelastic diffusive and linear electromagnetic thermoelastic materials etc. can be derived by the PVP of dissipative media in this paper. The conservative system is the special case of the dissipative system. Other than the mathematical variational principle, which is obtained by a known governing equation, the PVP is used to deduce the governing equation. The PVPs including the hyperbolic temperature wave equation with a finite phase speed are also discussed shortly.

1 Introduction

As a branch of mathematics, the variational method is much studied and has strict mathematical theory and is extensively used in physics and engineering [1,2]. The mathematical variational principle is got by a known governing equation and is applied as a mathematical method. However, the physical variational principle (PVP), which is a result of thermodynamics, is studied rarely. In the theory of elasticity, the potential energy principle points out that in all allowable displacements the true displacement makes the total potential energy minimum, where the total potential energy is equal to the strain energy minus the external work, but it is not always connected with the first thermodynamic law. In fact, the potential energy principle is a special case of the PVP.

In the thermodynamic process of a closed system, the first law is

$$\mathrm{d}U = \mathrm{d}W + \mathrm{d}Q \tag{1}$$

where U is the internal energy of the system, W is the work done by the external force, Q is the heat absorbed from the external heat source. The author pointed out that [3–11] from the first thermodynamic law we can assume a PVP for the conservative system:

$$\delta U = \delta W + \delta Q, \text{ or } \quad \delta \Pi = \delta U - \delta W - \delta Q = 0 \tag{2}$$

where δ is a variation symbol. Equation (2) shows that in all stable processes for all allowable displacements, Π takes extremum, and its rationality will be proved by that the correct governing equations can be derived from

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this principle for various materials and processes. The postulation of Eq. (2) is rational and can be considered that it is implied by the first thermodynamic law. The virtual work rate principle in elasticity can be considered as a special case of Eq. (2). If we reverse the derivation procedure from the various correct governing equations, we can get the unified PVP. However, it is not needed because we consider that the PVP is a universal principle in continuum mechanics. The energy principle discusses the real process varied with time, but the PVP compares the different paths in a same time. It is obvious that Eq. (2) is held for the true displacements due to the first thermodynamic law. Therefore Eq. (2) cannot be held for all other allowable displacements except at a branch point on the moving path, but the branch point should be discussed by stability or other theories. When the external force is not related to the deformation of a material, the explicit expression of the constitutive equation is not needed in PVP expressed by Eq. (2), so it is universal for all materials. However when the external force is related to the deformation of a material, the explicit expression of the energy or the constitutive equation is needed for PVP, such as in electrostrictive and magnetostrictive media where the electromagnetic force is related to the structure and deformation of a material. For the electrostrictive material without dissipation, the PVP is shown in literatures [3–11]. In this case, the PVP is for a given kind of material and not universal for all materials. In this paper, we do not discuss the case where the external force is related to the deformation, but for the dissipative media it will be discussed elsewhere.

Equation (2) is successfully applied to conservative systems. From Eq. (2), the momentum equation can be derived for all the conservative systems. Some variational principles for the dissipative system had been discussed in previous papers [12–14], but these variational principles were deduced from known governing equations. In this paper, we show that the physical foundation of the PVP for the dissipative media, such as media with heat conduction, electric current, diffusion and chemical reaction, is the extended Gibbs equation and gives a general theory of the PVP in dissipative media. From the PVP in dissipative media, all the governing equations and the boundary conditions of the reversible and irreversible variables, such as the displacement, temperature, electric potential, chemical potential, can be obtained. These governing equations may be called the "general motion equations". It is noted that the constitutive equations of the reversible variables and evolution equations between irreversible variables cannot directly be got from the PVP, and these relations should be obtained by experiments and thermodynamics.

In order to emphasize the physical feature, in this paper we only discuss the small deformation. The finite deformation case can be discussed straightforwardly, but its mathematical formulas are more complex. The PVP including the inertial entropy concept are also discussed in this paper shortly.

2 The PVP in thermoelastic media

2.1 Extended Gibbs equation

For the dissipative media, we prefer the following extended Gibbs equation to the usual one:

$$\begin{aligned}
&\mathrm{d}\int_V u\mathrm{d}V = \mathrm{d}Q^{(t)} + \mathrm{d}W^{(r)}, \quad \mathrm{d}Q^{(t)} = \mathrm{d}Q + \mathrm{d}Q^{(i)}, \quad \mathrm{d}W^{(r)} = \mathrm{d}W - \mathrm{d}h,\\
&\mathrm{d}\int_V g\mathrm{d}V = \mathrm{d}\tilde{Q}^{(t)} + \mathrm{d}\tilde{W}^{(r)}, \quad \mathrm{d}\tilde{Q}^{(t)} = \mathrm{d}\tilde{Q} + \mathrm{d}\tilde{Q}^{(i)}, \quad \mathrm{d}\tilde{W}^{(r)} = \mathrm{d}W - \mathrm{d}\tilde{h}
\end{aligned} \tag{3}$$

where $\mathrm{d}Q^{(t)} = \int_V T\mathrm{d}s$ is the total heat rate absorbed by the medium, which includes the heat rate $\mathrm{d}Q$ supplied by the environment and the dissipative heat rate $\mathrm{d}Q^{(i)} = \int_V T\mathrm{d}s^{(i)}$ produced in the internal irreversible process; $\mathrm{d}\tilde{Q}^{(t)} = -\int_V s\mathrm{d}T$ is the total complement heat rate, which includes the complement heat rate $\mathrm{d}\tilde{Q}$ and the complement dissipative heat rate $\mathrm{d}\tilde{Q}^{(i)} = -\int_V s^{(i)}\mathrm{d}T$; $\mathrm{d}W^{(r)}$ is the reversible part of the external work rate, which is equal to the external work rate $\mathrm{d}W$ minus the irreversible work rate $\mathrm{d}h$. $\mathrm{d}Q^{(i)}$ is converted from $\mathrm{d}h$ and they are the same in value, but different in the energy form. A similar explanation is for $\mathrm{d}\tilde{W}^{(r)}$, $\mathrm{d}\tilde{Q}^{(i)}$ and $\mathrm{d}\tilde{h}$. Similar to Eqs. (1) and (2), from Eq. (3) we can propose the PVP for dissipative media as:

$$\begin{aligned}
&\delta\Pi = \delta\int_V u\mathrm{d}V - \delta Q^{(t)} - \delta W^{(r)} = 0, \quad \delta Q^{(t)} = \delta Q + \delta Q^{(i)}, \quad \delta W^{(r)} = \delta W - \delta h,\\
&\delta\tilde{\Pi} = \delta\int_V g\mathrm{d}V - \delta\tilde{Q}^{(t)} - \delta\tilde{W}^{(r)} = 0, \quad \delta\tilde{Q}^{(t)} = \delta\tilde{Q} + \delta\tilde{Q}^{(i)}, \quad \delta\tilde{W}^{(r)} = \delta W - \delta\tilde{h}.
\end{aligned} \tag{4}$$

Equation (4) is the footstone of the PVP for the dissipative media. Equation (4) may also be called the "generalized virtual work rate principle". For a conservative system $\delta Q^{(i)} = \delta \tilde{Q}^{(i)} = \delta h = \delta \tilde{h} = 0$, and in this case the extended Gibbs equation is reduced to the exact energy equation.

It is noted that though the extended Gibbs equation and the usual Gibbs equation are the same substantially, the former is clearer. The extended Gibbs equation expresses explicitly the total heat as the sum of the external and internal irreversible heats, and the dissipative work is explicitly subtracted from the external work. Using the usual Gibbs equation, the meaning of each term is often confused in it. If we use the usual Gibbs equation in the above sense correctly, it is also appropriate.

Because the internal irreversible heat is enclosed in the total heat, from the variational equation we can get the temperature equation including the internal irreversible heat.

Usually, the governing equations in thermoelasticity are in the form of stress and temperature, so using the free energy is more convenient. In the later sections, we mainly discuss $\delta \tilde{\Pi}$.

Equation (4) is a rational proposition, we can get a correct governing equation by using this method. Contrarily, if we reverse the derivation procedure from the various correct governing equations we can get the same Eq. (4). If the current governing equation is not fully correct, we can not get the PVP from this incorrect governing equation, so the PVP can also be used to check the current governing equation. We can say that the conservation of mass, balance of energy and PVP (or the principle of energy stationary value) are three fundamental natural rules. One can also say that the PVP is implied in the thermodynamic laws. Because the extended Gibbs equation is in the precision of the first order, the equation describing the dissipative process in the generalized motion equation is in the precision of first order also, but the equation describing the nondissipative process is fully exact.

2.2 The entropy in a thermoelastic system

All the irreversible work converts to dissipative heat, so the heat has the central role in irreversible thermodynamics. Due to this reason, the discussion of PVP in the dissipative media will begin from the thermoelastic system. In the classical irreversible thermodynamics [15–17] for a thermoelastic system, the entropy density s is divided into the reversible part $s^{(r)}$ and the irreversible part $s^{(i)}$, i.e.,

$$\begin{aligned} &T\dot{s} = \dot{r} - q_{i,i}, \quad \dot{s} = \dot{s}^{(r)} + \dot{s}^{(i)}, \\ &T\dot{s}^{(r)} = \dot{r} - T\dot{\eta}_{i,i} = \dot{r} - q_{i,i} + \dot{\eta}_i T_{,i}, \quad \dot{\boldsymbol{\eta}} = \boldsymbol{q}/T, \quad \boldsymbol{\eta} = \int_0^t (\mathbf{q}/T)\,\mathrm{d}\tau, \\ &T\sigma = T\dot{s}^{(i)} = T\dot{s} - T\dot{s}^{(r)} = -T_{,i}\dot{\eta}_i = -\left(T_{,i}/T\right)(T\dot{\eta}_i) \geq 0 \end{aligned} \tag{5}$$

where r is the internal heat source strength, T is the temperature, $\boldsymbol{q}$ is the heat flow vector per interface area supplied by the environment, $\boldsymbol{\eta}$ is the entropy displacement vector, $\dot{\boldsymbol{\eta}}$ is the entropy flow vector. In Eq. (5), a dot over a variable means the differentiation with respect to time of that variable, i.e., $()^{\cdot} = \mathrm{d}()/\mathrm{d}t$, such as $\dot{\boldsymbol{D}} = \mathrm{d}\boldsymbol{D}/\mathrm{d}t$; a comma followed by index i in a subscript indicates partial differentiation with respect to x_i, such as $u_{i,j} = \partial u_i/\partial x_j$, and the summation rule applies for repeated indices.

Using the theory of the thermodynamics of irreversible processes (TIP), from $T\sigma$ in Eq. (5) [15–17] we can get the linear evolution equation, which is just Fourier's law:

$$\begin{aligned} &T\dot{\eta}_i = q_i = -\lambda'_{il}\left(T_{,l}/T\right) = -\lambda_{il}T_{,l}, \\ &\lambda'_{ij} = \lambda'_{ji}, \quad \lambda'_{ij} = T\lambda_{ij}; \quad \eta_j = -\int_0^t \lambda'_{ij}T^{-2}T_{,i}\mathrm{d}\tau \end{aligned} \tag{6.1}$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}'/T$ is the usual heat conductive coefficient. We can also consider that $T\sigma$ in Eq. (5) gives a pair of irreversible variables, but the evolution equation between them can be in a more general form:

$$T\dot{\eta}_i = q_i = F(T_{,l}/T). \tag{6.2}$$

The explicit expression of $F(T_{,l}/T)$ in Eq. (6.2) should be obtained by experiments and rational theories. How to select a state variable is a very important problem [18]. If a state variable is selected well, the governing equation may be simple and it is easy to solve. Though the state variables are selected by the experience and some theories, such as the entropy production rate, a fundamental principle cannot be violable, i.e., the product of the general force and displacement should have the energy dimension.

In the following sections, we mainly discuss the linear approximate evolution equation.

2.3 PVP by using free energy

Under assumptions that the mechanical displacement $\boldsymbol{u}$ and the temperature $\vartheta = T - T_0$, where T_0 is the reference temperature of the environment, satisfy their own boundary conditions, $u_i = u_i^*$, $\vartheta = \vartheta^*$ on a_u and a_T, respectively. Based on Eq. (4), the PVP in the form of the free energy for the thermoelasticity under the small deformation can be expressed as:

$$
\begin{aligned}
&\delta\tilde{\Pi} = \int_V \delta g \mathrm{d}V - \delta\tilde{Q}^{(t)} - \delta\tilde{W}^{(r)} = 0, \\
&\delta\tilde{Q}^{(t)} = -\int_V \delta\vartheta \int_0^t (\dot{r}/T)\mathrm{d}\tau \mathrm{d}V - \int_V \delta\vartheta \int_0^t \dot{s}^{(i)}\mathrm{d}\tau \mathrm{d}V + \int_{a_q} \delta\vartheta \int_0^t \dot{\eta}^* \mathrm{d}\tau \mathrm{d}a, \\
&\delta\tilde{W}^{(r)} = \int_V (f_k - \rho\ddot{u}_k)\delta u_k \mathrm{d}V + \int_{a_\sigma} p_k^* \delta u_k \mathrm{d}a - \int_V \delta\tilde{h}\mathrm{d}V, \\
&\delta g = \sigma_{ij}\delta\varepsilon_{ij} - s\delta\vartheta; \quad \delta\tilde{h} = -T_{,i}\delta\eta_i + \delta\left(\eta_i T_{,i}\right) = \eta_j \delta\vartheta_{,j}
\end{aligned}
\tag{7}
$$

where σ_{ij} is the stress tensor, $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ is the strain tensor; f_k, p_k^* and $\dot{\eta}^* = \dot{\eta}_i^* n_i$ are the given mechanical body force, surface traction and surface entropy flow rate, respectively. In Eq. (7), the variation $\delta\vartheta$ is at current time t and does not depend on the past time τ. It is noted that in the free energy the independent variables are $\boldsymbol{\varepsilon}$ and ϑ, so we use the complement heat rate $s\delta\vartheta$ and the irreversible complement work rate $\eta_i \delta\vartheta_{,i}$.

In Eq. (7), we have

$$
\begin{aligned}
&\int_V \delta g \mathrm{d}V = \int_a \sigma_{ij} n_j \delta u_i \mathrm{d}a - \int_V \sigma_{ij,j}\delta u_i \mathrm{d}V - \int_V s\delta\vartheta \mathrm{d}V, \\
&\int_V \delta\tilde{h}\mathrm{d}V = \int_V \eta_j \delta\vartheta_{,j}\mathrm{d}V = \int_V \delta\vartheta_{,j} \int_0^t \dot{\eta}_j \mathrm{d}\tau \mathrm{d}V.
\end{aligned}
$$

Finishing the variational calculation, we get

$$
\begin{aligned}
\delta\tilde{\Pi} = &\int_{a_\sigma} (\sigma_{ij} n_j - p_i^*)\delta u_i \mathrm{d}a - \int_V (\sigma_{ij,j} + f_i - \rho\ddot{u}_i)\delta u_i \mathrm{d}V \\
&+ \int_{a_q} \left(\int_0^t \dot{\eta}_i \mathrm{d}\tau n_j - \eta^*\right)\delta\vartheta \mathrm{d}a - \int_V \left[s + \int_0^t \left(-T^{-1}\dot{r} + \dot{\eta}_{i,i} - \dot{s}^{(i)}\right)\mathrm{d}\tau\right]\delta\vartheta \mathrm{d}V = 0.
\end{aligned}
\tag{8}
$$

By using the relation $T\dot{\eta}_{j,j} - T\dot{s}^{(i)} = T\dot{\eta}_{j,j} + \dot{\eta}_i T_{,i} = (T\dot{\eta}_i)_{,i} = q_{i,i}$ and the arbitrariness of δu_i and $\delta\vartheta$, from Eq. (8) we get

$$
\begin{aligned}
&\sigma_{ij,j} + f_i = \rho\ddot{u}_i; \quad T\dot{s} = \dot{r} - q_{i,i}, \quad \text{in } V; \\
&\sigma_{kl} n_l = p_k^*, \quad \text{on } a_\sigma; \quad \dot{\eta}_i n_i = \dot{\eta}^*, \quad \text{or } q_n = q_n^*, \quad \text{on } a_q.
\end{aligned}
\tag{9}
$$

The governing equations of thermoelasticity will consist of Eq. (9) and the following displacement and temperature boundary conditions which are the prior condition of the PVP:

$$
\mathbf{u} = \mathbf{u}^*, \quad \text{on } a_u; \quad \vartheta = \vartheta^* (or\ T = T^*), \quad \text{on } a_T.
\tag{10}
$$

For a dynamical problem, the initial conditions should also be added. From the above discussion, it is seen that the governing equations for the thermoelastic media derived by the PVP are natural and simple. It is also noted that in the thermoelastic problem the elastic process is nondissipative, so the momentum equation in Eq. (9) is fully exact, but the heat equation is in the precision of first order because the heat process is a dissipative process. In the literature [14], Benjeddou and Andrianarison gave a similar variational principle for the thermoelastic media:

$$
\begin{aligned}
&\left\{\int_V \left[\sigma_{ij}\delta\varepsilon_{ij} + (\rho\ddot{u}_i - f_i)\,\delta u_i\right]\mathrm{d}V - \int_{a_\sigma} T_i \delta u_i \mathrm{d}a\right\} \\
&\quad + \left\{\int_V \left[q_i \delta\vartheta_{,i} - (\vartheta_0 \dot{s} - \dot{r})\,\delta\vartheta\right]\mathrm{d}V - \int_{a_q} q_i^* n_i \delta\vartheta \mathrm{d}a\right\} = 0.
\end{aligned}
$$

This formula is similar to Eq. (8), but as in all present literatures, their variational formula is derived from the known governing equations by a purely mathematical method. In our theory, the governing equation Eq. (9) is derived by PVP, Eq. (7). In other words, the mathematical variational formulas are deduced from the known governing equations, but the PVP is used to deduce the governing equations.

If we assume the explicit expression of g in the form

$$\begin{aligned} &g(\varepsilon_{kl}, \vartheta) = u - \vartheta s = (1/2) C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \alpha_{ij}\varepsilon_{ij}\vartheta - (1/2T_0) C\vartheta^2 \\ &\quad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad \alpha_{ij} = \alpha_{ji} \end{aligned} \tag{11}$$

where $\boldsymbol{C}$, $\boldsymbol{\alpha}$ and C are the elastic coefficient, stress–temperature coefficient and the specific heat at constant deformation, respectively, the constitutive and evolution equations corresponding to Eq. (11) are, respectively:

$$\begin{aligned} &\sigma_{ij} = \partial g/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - \alpha_{ij}\vartheta, s = -\partial g/\partial\vartheta = \alpha_{ij}\varepsilon_{ij} + C\vartheta/T_0, \\ &\eta_i = \partial\tilde{h}/\partial\vartheta_{,i}, \quad T\dot{\eta}_i = q_i = -\lambda_{ij}\vartheta_{,j}. \end{aligned} \tag{12}$$

The explicit motion and temperature equations are:

$$\begin{aligned} &\left(C_{ijkl}\varepsilon_{kl} - \alpha_{ij}\vartheta\right)_{,j} + f_i = \rho\ddot{u}_i, \\ &T\left(\alpha_{ij}\varepsilon_{ij} + C\vartheta/T_0\right)^{\cdot} = \dot{r} + \left(\lambda_{ij}\vartheta_{,j}\right)_{,i} \quad \text{in } V. \end{aligned} \tag{13}$$

2.4 PVP by using internal energy

Under the assumption that the mechanical displacement $\boldsymbol{u}$ and the entropy density s satisfy their own boundary conditions, $u_i = u_i^*$, $s = s^*$ on a_u and a_s, respectively. Based on Eq. (4), the PVP in the form of the internal energy for the thermoelasticity under the small deformation can be expressed as:

$$\begin{aligned} &\delta\Pi = \int_V \delta u \mathrm{d}V - \delta Q^{(t)} - \delta W^{(r)} = 0, \\ &\delta Q^{(t)} = \int_V T\delta(\dot{r}/T)\mathrm{d}V + \int_V T\delta s^{(i)}\mathrm{d}V - \int_{a_q} T\delta\eta^*\mathrm{d}a, \\ &\delta W^{(r)} = \int_V (f_k - \rho\ddot{u}_k)\delta u_k \mathrm{d}V + \int_{a_\sigma} p_k^*\delta u_k \mathrm{d}a - \int_V \delta h \mathrm{d}V, \\ &\delta u = \sigma_{ij}\delta\varepsilon_{ij} + T\delta s; \quad \delta h = -T_{,i}\delta\eta_i. \end{aligned} \tag{14}$$

In Eq. (14), we have

$$\int_V \delta u \mathrm{d}V = \int_a \sigma_{ij} n_j \delta u_i \mathrm{d}a - \int_V \sigma_{ij,j}\delta u_i \mathrm{d}V + \int_V T\delta s \mathrm{d}V; \quad \int_V \delta h \mathrm{d}V = -\int_V T_{,i}\delta\eta_i \mathrm{d}V.$$

Finishing the variational calculation, we get

$$\begin{aligned} \delta\Pi = &\int_{a_\sigma} \left(\sigma_{ij} n_j - p_k^*\right)\delta u_i \mathrm{d}a - \int_V \left(\sigma_{ij,j} + f_i - \rho\ddot{u}_i\right)\delta u_i \mathrm{d}V \\ &+ \int_V T\delta\left[s - (\dot{r}/T) - s^{(i)} + \eta_{i,i}\right]\mathrm{d}V - \int_{a_q} T\delta\left(n_i\eta_i - \eta^*\right)\mathrm{d}a = 0. \end{aligned} \tag{15}$$

By using the arbitrariness of δu_i, we get the motion equation from Eq. (15). Due to $T \neq 0$, from Eq. (15) $s - (\dot{r}/T) - s^{(i)} + \eta_{i,i}$ in V and $n_i\eta_i - \eta^*$ in a_q must be zero, which is the entropy equation and its boundary condition, but this method is not consistent with the usual variational method. It may be of interest to investigate this problem.

3 The PVP in thermal viscoelastic media

In a linear thermal viscoelastic material, the irreversible heat consists of two parts: One is produced by the heat conduction, and the other is produced by the viscous effect.

In a linear thermal viscoelastic material, the entropy equation (5) is changed to [17]

$$\begin{aligned}
&T\dot{s}=\dot{r}+\sigma_{ij}^{(i)}\dot{\varepsilon}_{ji}-q_{i,i};\quad \dot{s}=\dot{s}^{(r)}+\dot{s}^{(i)};\quad T\dot{s}^{(r)}=\dot{r}-T\dot{\eta}_{i,i},\\
&T\sigma=T\dot{s}^{(i)}=T\left(\dot{s}-\dot{s}^{(r)}\right)=T\dot{s}-\dot{r}+T\dot{\eta}_{i,i}=\sigma_{ij}^{(i)}\dot{\varepsilon}_{ji}-T_{,i}\dot{\eta}_i\geq 0,\\
&\delta\tilde{h}=\sigma_{ij}^{(i)}\delta\varepsilon_{ji}+\eta_j\delta\vartheta_{,j}=\sigma_{ij}^{(i)}\delta\varepsilon_{ji}+\left(\int_0^t\dot{\eta}_j\mathrm{d}\tau\right)\delta\vartheta_{,j}
\end{aligned}\tag{16}$$

where $\sigma_{ij}^{(i)}$ is the irreversible part of the stress σ_{ij}. It is usually considered that the average stress $\sigma_{mm}^{(i)}/3$ is reversible, only the deviator part $\sigma'^{(i)}_{ij}=\sigma_{ij}^{(i)}-\sigma_{mm}^{(i)}\delta_{ij}/3$ of $\sigma_{ij}^{(i)}$ is irreversible, so in this case the viscous dissipative work rate $\sigma_{ij}^{(i)}\dot{\varepsilon}_{ji}$ in Eq. (16) will be given instead by $\sigma'^{(i)}_{ij}\dot{\varepsilon}'_{ij}$, $\dot{\varepsilon}'_{ij}=\dot{\varepsilon}_{ij}-\dot{\varepsilon}_{mm}\delta_{ij}/3$, which will be converted to irreversible heat. From Eq. (16), we get the linear evolution equation by the theory of TIP as

$$\sigma_{kl}^{(i)}=\beta_{ijkl}\dot{\varepsilon}_{ij}-\kappa_{ikl}\vartheta_{,i},\quad T\dot{\eta}_i=-\lambda_{ij}\vartheta_{,j}-\kappa_{ikl}\dot{\varepsilon}_{kl}\tag{17}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ are the coefficient tensors. In an isotropic material, the Curie principle shows that there is no coupling between tensors with different rank, so in Eq. (17) $\boldsymbol{\kappa}=\boldsymbol{0}$.

The PVP is still expressed by Eq. (7), but $\dot{s}^{(i)}$ and $\tilde{h}$ are changed to Eqs. (16), i.e.,

$$\begin{aligned}
&\delta\tilde{\Pi}=\int_V\delta g\mathrm{d}V-\delta\tilde{Q}^{(t)}-\delta\tilde{W}^{(r)}=0,\\
&\delta\tilde{Q}^{(t)}=-\int_V\delta\vartheta\int_0^t(\dot{r}/T)\mathrm{d}\tau\mathrm{d}V-\int_V\delta\vartheta\int_0^t\dot{s}^{(i)}\mathrm{d}\tau\mathrm{d}V+\int_{a_q}\delta\vartheta\int_0^t\dot{\eta}^*\mathrm{d}\tau\mathrm{d}a,\\
&\delta\tilde{W}^{(r)}=\int_V(f_k-\rho\ddot{u}_k)\delta u_k\mathrm{d}V+\int_{a_\sigma}p_k^*\delta u_k\mathrm{d}a-\int_V\delta\tilde{h}\mathrm{d}V,\\
&\delta g=\sigma_{ij}^{(r)}\delta\varepsilon_{ij}-s\delta\vartheta;\quad \delta\tilde{h}=\sigma_{ij}^{(i)}\delta\varepsilon_{ji}+\eta_j\delta\vartheta_{,j},\\
&\sigma_{ij}=\sigma_{ij}^{(r)}+\sigma_{ij}^{(i)},\quad \sigma_{ij}^{(r)}=\partial g/\partial\varepsilon_{ij}
\end{aligned}\tag{18}$$

where $\sigma_{ij}^{(r)}$ is the reversible part of the stress σ_{ij}. Finishing the variational calculation, we get

$$\begin{aligned}
\delta\tilde{\Pi}=&\int_{a_\sigma}(\sigma_{ij}n_j-p_i^*)\delta u_i\mathrm{d}a-\int_V(\sigma_{ij,j}+f_i-\rho\ddot{u}_i)\delta u_i\mathrm{d}V\\
&-\int_{a_q}\left[\int_0^t\dot{\eta}_in_j\mathrm{d}\tau-\eta^*\right]\delta\vartheta\mathrm{d}a-\int_V\left\{s+\int_0^t\left[-T^{-1}\dot{r}+\dot{\eta}_{j,j}-\dot{s}^{(i)}\right]\mathrm{d}\tau\right\}\delta\vartheta\mathrm{d}V=0.
\end{aligned}\tag{19}$$

From Eq. (19), we get

$$\begin{aligned}
&\sigma_{ij,j}+f_i=\rho\ddot{u}_i;\quad T\dot{s}=\dot{r}-q_{i,i}+\sigma_{ij}^{(i)}\dot{\varepsilon}_{ji}\quad\text{in the medium},\\
&\sigma_{kl}n_l=p_k^*,\quad on\quad a_\sigma;\quad \dot{\eta}_i=-T^{-1}\lambda_{ij}\vartheta_{,i},\quad \dot{\eta}_in_i=\dot{\eta}^*,\quad\text{or}\quad q_n=q_n^*,\quad\text{on}\quad a_q
\end{aligned}\tag{20}$$

For an isotropic incompressible viscous fluid, $\boldsymbol{\kappa}=\boldsymbol{0}$, let $\sigma_{ij}^{(r)}=-p\delta_{ij}$, $\sigma'^{(i)}_{ij}=2\mu\dot{\varepsilon}_{ij}$, $\dot{\varepsilon}_{ij}=\dot{\varepsilon}'_{ij}$ due to $\varepsilon_{mm}=0$, where p and μ are the pressure and viscosity coefficient; we get the theory of the thermal viscous Newtonian fluid, i.e.,

$$-p_{,i}+\mu\dot{u}_{i,jj}+f_i=\rho\ddot{u}_i;\quad T\dot{s}=\dot{r}-q_{i,i}+\mu\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ji}\quad\text{in }V.$$

A similar work in the thermal viscous fluid can also be seen in Bejan's work [19].

If we assume the explicit expression of g is still Eq. (11), we get

$$\begin{aligned}&\sigma_{ij}^{(r)}=\partial g/\partial\varepsilon_{ij}=C_{ijkl}\varepsilon_{kl}-\alpha_{ij}\vartheta,\quad s=-\partial g/\partial\vartheta=\alpha_{ij}\varepsilon_{ij}+C\vartheta/T_0,\\&\sigma_{ij}=\sigma_{ij}^{(r)}+\sigma_{ij}^{(i)}=C_{ijkl}\varepsilon_{kl}-\alpha_{ij}\vartheta+\beta_{ijkl}\dot{\varepsilon}_{kl}-\kappa_{ijk}\vartheta_{,k}.\end{aligned}\tag{21}$$

The explicit motion and temperature equations are:

$$\begin{aligned}&\left(C_{ijkl}\varepsilon_{kl}-\alpha_{ij}\vartheta+\beta_{ijkl}\dot{\varepsilon}_{ij}-\kappa_{ikl}\vartheta_{,i}\right)_{,j}+f_i=\rho\ddot{u}_i,\\&T\left(\alpha_{ij}\varepsilon_{ij}+C\vartheta/T_0\right)^{\cdot}=\dot{r}+\left(\lambda_{ij}\vartheta_{,j}+\kappa_{klj}\dot{\varepsilon}_{lj}\right)_{,i}+\left(\beta_{ijkl}\dot{\varepsilon}_{ij}-\kappa_{ikl}\vartheta_{,i}\right)\dot{\varepsilon}_{ji}.\end{aligned}\tag{22}$$

For a more complicated thermo-viscoelastic integral constitutive equation, we can discuss in the same way.

4 The PVP in a thermal elastoplastic material under small deformation

In the usual elastoplastical theory [17,20], according to the type of the constitutive equation, we should distinguish two regions: in region V_1 including the elastic region, elastoplastic region at the unloading or neutral state, the constitutive equation is the elastic one; and in the elastoplastic loading region V_2, the constitutive equation is the elastoplastic one. The loading and unloading regions are determined by the loading criterion [17,20]. In V_2, part of the plastic deformation can transform to heat. The entropy equation Eq. (3) is changed to

$$\begin{aligned}&T\dot{s}=\dot{r}+\sigma_{ij}\dot{\varepsilon}_{ij}^{(p)}-q_{i,i};\quad\dot{s}=\dot{s}^{(r)}+\dot{s}^{(i)};\quad T\dot{s}^{(r)}=\dot{r}-T\dot{\eta}_{i,i},\\&T\sigma=T\left(\dot{s}-\dot{s}^{(r)}\right)=T\dot{s}-\dot{r}+T\dot{\eta}_{i,i}=\sigma_{ij}\dot{\varepsilon}_{ij}^{(p)}-T_{,i}\dot{\eta}_i\geq 0;\quad\text{in }V_2,\\&T\dot{s}=\dot{r}-q_{i,i};\quad\dot{s}=\dot{s}^{(r)}+\dot{s}^{(i)};\quad T\dot{s}^{(r)}=\dot{r}-T\dot{\eta}_{i,i},\\&T\sigma=T\left(\dot{s}-\dot{s}^{(r)}\right)=T\dot{s}-\dot{r}+T\dot{\eta}_{i,i}=-T_{,i}\dot{\eta}_i\geq 0;\quad\text{in }V_2\end{aligned}\tag{23}$$

where $\varepsilon_{ij}=\varepsilon_{ij}^{(e)}+\varepsilon_{ij}^{(p)}$, $\varepsilon_{ij}^{(e)}$ and $\varepsilon_{ij}^{(p)}$ are the elastic and plastic strain, respectively. For simplicity, it is assumed that the plastic work is fully changed to irreversible heat. From Eq. (23), it is known that $\sigma_{ij}, \dot{\varepsilon}_{ij}^{(p)}$ and $T_{,i}, \dot{\eta}_i$ are two pairs of irreversible variables definitely, but their evolution equation cannot be obtained by the classical theory of TIP; it should be determined by experiments and rational theories [17,20].

On the interface of V_1 and V_2, the connective conditions should be satisfied, i.e.,

$$\sigma_{ij}^{(1)}=\sigma_{ij}^{(2)},\ u_i^{(1)}=u_i^{(2)},\ T^{(1)}=T^{(2)}\quad\text{on interface }\ a_{\text{int}}\tag{24}$$

where $\sigma_{ij}^{(1)}, u_i^{(1)}, T^{(1)}$ and $\sigma_{ij}^{(2)}, u_i^{(2)}, T_i^{(2)}$ are the stress, displacement and temperature in V_1 and V_2, respectively. Because the constitutive equations in V_1 and V_2 are different, the PVP should be divided into two parts. Under the conditions that the boundary and interface conditions of $\boldsymbol{u}$ and T on a_u, a_T and a_{int} are satisfied prior, the PVP for the thermal elastoplastic field can be expressed as

$$\begin{aligned}&\delta\tilde{\Pi}=\sum\nolimits_{j=1}^{2}\left(\int_{V_j}\delta g^{(j)}\mathrm{d}V-\delta\tilde{Q}^{(t)(j)}-\delta\tilde{W}^{(r)(j)}\right)=0,\\&\delta\tilde{Q}^{(t)(j)}=-\int_{V_j}\delta\vartheta\int_0^t(\dot{r}/T)\mathrm{d}\tau\mathrm{d}V-\int_{V_j}\delta\vartheta\int_0^t\dot{s}^{(i)}\mathrm{d}\tau\mathrm{d}V+\int_{a_{qj}}\delta\vartheta\int_0^t\dot{\eta}^*\mathrm{d}\tau\mathrm{d}a,\\&\delta\tilde{W}^{(r)(j)}=\int_{V_j}(f_k-\rho\ddot{u}_k)\delta u_k\mathrm{d}V+\int_{a_{\sigma j}}p_k^*\delta u_k\mathrm{d}a-\int_{V_j}\delta\tilde{h}_j\mathrm{d}V,\\&\delta g^{(j)}=\sigma_{kl}\delta\varepsilon_{kl}^{(j)}-s^{(k)}\delta\vartheta^{(k)},\quad V=V_1+V_2,\quad a_q=a_{q1}+a_{q2},\quad a_\sigma=a_{\sigma1}+a_{\sigma2},\\&\delta\tilde{h}_1=0,\quad\delta\tilde{h}_2=\sigma_{ij}\delta\varepsilon_{ij}^{p}.\end{aligned}\tag{25}$$

Because Eq. (24) is assumed, the integral on the interface is eliminated, so

$$\sum_{j=1}^{2}\int_{a_{\sigma j}+a_{\sigma}^{\text{int}}} p_k^*\delta u_k \mathrm{d}a+\int_{a_{qj}+a_q^{\text{int}}}\delta\vartheta\int_0^t\dot{\eta}^*\mathrm{d}\tau\mathrm{d}a=\int_{a_\sigma} p_k^*\delta u_k \mathrm{d}a+\int_{a_q}\delta\vartheta\int_0^t\dot{\eta}^*\mathrm{d}\tau\mathrm{d}a.$$

The above result has been used in Eq. (25). Finishing the variational calculation, finally we get

$$\begin{aligned}&\sigma_{ij,j}+f_i=\rho\ddot{u}_i,\quad \text{in } V;\quad \sigma_{kl}n_l=p_k^*,\quad \text{on } a_\sigma,\\&T\dot{s}=\dot{r}-q_{i,i}+\sigma_{ij}\dot{\varepsilon}_{ij}^{(p)}\quad \text{in } V_2;\quad T\dot{s}=\dot{r}-q_{i,i}\quad \text{in } V_1;\quad q_n=q_n^*,\quad \text{on } a_q.\end{aligned}\tag{26}$$

For an isotropic hardening material with associated flow rule under small deformation, the constitutive equations are discussed in many textbooks, such as in [17,20],

$$\begin{aligned}&\dot{\boldsymbol{\sigma}}=2G\left(\dot{\boldsymbol{\varepsilon}}+\frac{\nu}{1-2\nu}\dot{\varepsilon}_{mm}\boldsymbol{I}_2\right)-(\boldsymbol{\alpha}T)^{\cdot},\quad \text{in } V_1;\quad \dot{\boldsymbol{\sigma}}=2G\left(\dot{\tilde{\boldsymbol{\varepsilon}}}+\frac{\nu}{1-2\nu}\dot{\tilde{\varepsilon}}_{mm}\boldsymbol{I}_2-3\dot{\lambda}\boldsymbol{\sigma}'\right),\quad \text{in } V_2,\\&\dot{\lambda}=\frac{\boldsymbol{\sigma}':\dot{\tilde{\boldsymbol{\varepsilon}}}-\sqrt{J_2/3}k\dot{T}/G}{6J_2\left(1+H/3G\right)},\quad \dot{\tilde{\boldsymbol{\varepsilon}}}=\dot{\boldsymbol{\varepsilon}}-(\alpha_{mm}T)^{\cdot}\boldsymbol{I}-\dot{\boldsymbol{A}}^{-1}\boldsymbol{\sigma},\quad H=\frac{\partial\sigma_s}{\partial\varepsilon},\quad k=\frac{\partial\sigma_s}{\partial T},\\&\sigma_r^2=\frac{3}{2}\boldsymbol{\sigma}':\dot{\boldsymbol{\sigma}}'=3J_2,\quad \boldsymbol{\sigma}'=\boldsymbol{\sigma}-\sigma_{mm}\boldsymbol{I}_2,\quad A^{-1}=\frac{1}{2G}\left(\boldsymbol{I}_4-\frac{\nu}{1+\nu}\boldsymbol{I}_2\otimes\boldsymbol{I}_2\right)\end{aligned}\tag{27}$$

where σ_s is the yielding stress, H is the tangential modulus in a uniaxial tension, G is the shear modulus, ν is Poisson's ratio, and $\boldsymbol{I}_2$ and $\boldsymbol{I}_4$ are the unit tensors of second order and fourth order, respectively.

Yang et al. [21] give a variational principle in a dissipative material by using the thermodynamics with the internal variable theory under finite deformation. TIP (The thermodynamics of irreversible processes) and the internal variable theory are two important theories in irreversible thermodynamics. As we know, in mechanics there are two types of governing equations: One is the universal for all materials, and the other is special for a special material. In TIP, the Gibbs equation is a universal principle in the first-order precision, and the difficult evolution equation is put in the entropy production rate equation. PVP is based on the extended Gibbs equation, so it is also a universal principle in the first-order precision, if the external force is free of the deformation. If a variational principle includes the constitutive and evolution equations, the variational principle cannot be a universal physical principle, but a variational principle of a kind of material, though it is valuable in engineering. We can also use some mathematical methods [1,17] to get the generalized physical variational principle including the constitutive and evolution equations from PVP.

5 The PVP in a linear thermoelastic diffusive material

5.1 Fick's laws [22]

$$\text{First law}\quad \boldsymbol{J}=-D'\nabla c,\quad \boldsymbol{J}=\rho\boldsymbol{w}=\rho(\boldsymbol{v}'-\boldsymbol{v}),\tag{28.1}$$

$$\text{Modified Fick's first law}\quad \boldsymbol{J}=-D\nabla\mu,\tag{28.2}$$

$$\text{Second law}\quad \dot{c}=\partial c/\partial t=-\nabla\cdot\boldsymbol{J}=-J_{i,i}\tag{28.3}$$

where $\dot{c}=-J_{i,i}$ is the mass conservation equation, ρ is the density; c is the concentration (kg/m^3); μ is the chemical potential, $\boldsymbol{J}$ is the diffusion mass flux of the diffusing material $(\mathrm{kg}/(\mathrm{m}^2\cdot\mathrm{s}))$, $\boldsymbol{w}=\boldsymbol{v}'-\boldsymbol{v}$ is the diffuse velocity, $\boldsymbol{v}$ is the centroid velocity of an element (consisting of matrix and diffusing material), and $\boldsymbol{v}'$ is the velocity of the diffusing material. When we use $\boldsymbol{J}=-D\nabla\mu$ in Eq. (28.2), Eq. (28.2) can apply to downhill and uphill diffusions, but $\boldsymbol{J}=-D'\nabla c$ in Eq. (28.1) only appropriate for the downhill diffusion.

5.2 Momentum and energy equations for a linear thermoelastic diffusive material

The momentum equation, which can be found in all textbooks of continuum mechanics, is [17,23]

$$\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\rho\dot{\boldsymbol{v}}. \tag{29}$$

The thermoelastic diffusive problem is an open system, so the energy conservation equation is:

$$\mathrm{d}U=\mathrm{d}W+\mathrm{d}Q+\mathrm{d}\Phi,\quad \mathrm{d}\Phi=-\int_a \mu\boldsymbol{J}\cdot\boldsymbol{n}\mathrm{d}a \tag{30}$$

where $\mathrm{d}\Phi$ is the energy rate introduced by the mass conversion through the boundary.

Applying the Gauss theorem and the momentum equation, from Eq. (30) we get the local energy conservation equation:

$$\dot{u}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}+\dot{r}-\nabla\cdot\boldsymbol{q}-\nabla\cdot(\mu\boldsymbol{J}). \tag{31}$$

In some literatures, e.g., [24], there is no term $\nabla\cdot(\mu\boldsymbol{J})$ in the local energy conservation equation. In this paper, we shall use the energy equation Eq. (31) to give the appropriate PVP and derive the governing equations.

5.3 The Gibbs equation and the entropy production rate

The Gibbs equation is [15]:

$$\dot{u}=\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}+T\dot{s}+\mu\dot{c}. \tag{32}$$

Substituting the energy equation (31) into the Gibbs equation (32), we get the entropy equation:

$$\begin{aligned}
&T\dot{s}=\dot{r}-\nabla\cdot\boldsymbol{q}-\nabla\cdot(\mu\boldsymbol{J})-\mu\dot{c}=\dot{r}-\nabla\cdot\boldsymbol{q}-\nabla\mu\cdot\boldsymbol{J},\\
&T\dot{s}=T\left(\dot{s}^{(r)}+\sigma\right),\quad \dot{s}^{(r)}=\dot{r}/T-\nabla\cdot(\boldsymbol{q}/T),\\
&T\sigma=T\left(\dot{s}-\dot{s}^{(r)}\right)=-\boldsymbol{q}\cdot(\nabla T/T)-\nabla\mu\cdot\boldsymbol{J}=-T_{,i}\dot{\eta}_i-\mu_{,i}\dot{\xi}_i;\quad \boldsymbol{J}=\dot{\boldsymbol{\xi}}.
\end{aligned} \tag{33}$$

Equation (33) shows that a part of the external heat is stored in the system, and the other part is dissipated by the heat conduction. However, the diffuse process is fully dissipative. The dissipative heat $T\sigma$ has the dimension of energy. So we prefer $T\sigma$ to σ. According to TIP, $(\boldsymbol{q},\nabla\boldsymbol{T}/T)$ and $(\nabla\mu,\boldsymbol{J})$ are two pairs of conjugate variables. So we have the following evolution equations:

$$\begin{aligned}
&T\dot{\boldsymbol{\eta}}=\boldsymbol{q}=-\hat{\boldsymbol{\lambda}}\cdot(\nabla T/T)-\boldsymbol{L}\cdot\nabla\mu,\quad \dot{\boldsymbol{\xi}}=\boldsymbol{J}=-\boldsymbol{L}\cdot(\nabla T/T)-\boldsymbol{D}\cdot\nabla\mu,\\
&T\dot{\eta}_i=-\lambda_{ij}\vartheta_{,j}-L_{ij}\mu_{,j},\quad \dot{\xi}_i=-L_{ij}\left(\vartheta_{,j}/T\right)-D_{ij}\,\mu_{,j},\\
&\hat{\lambda}_{ij}=\hat{\lambda}_{ji},L_{ij}=L_{ji},D_{ij}=D_{ji},\lambda_{ij}=\hat{\lambda}_{ij}/T
\end{aligned} \tag{34}$$

where $\hat{\boldsymbol{\lambda}}=\boldsymbol{\lambda}T,\boldsymbol{L},\boldsymbol{D}$ are material constants. Equations (33) and (34) modify the entropy and evolution equations adopted in some previous literatures.

5.4 PVP in thermoelastic diffusive theory

Under the assumption that the mechanical displacement $\boldsymbol{u}$, the temperature ϑ and the chemical potential μ satisfy their own boundary conditions, $\boldsymbol{u}=\boldsymbol{u}^*$, $\vartheta=\vartheta^*$ and $\mu=\mu^*$ on a_u, a_T and a_μ, respectively. When the variation of temperature is not large, the PVP for the thermoelastic diffusive problem is

$$\begin{aligned}
\delta\tilde{\Pi} &= \int_V \delta g \mathrm{d}V - \delta\tilde{Q}^{(t)} - \delta\tilde{\Phi} - \delta\tilde{W}^{(r)} = 0,\\
\delta\tilde{Q}^{(t)} &= -\int_V \delta\vartheta \int_0^t \dot{r}T^{-1}\mathrm{d}\tau\mathrm{d}V - \int_V \delta\vartheta \int_0^t \dot{s}^{(i)}\mathrm{d}\tau\mathrm{d}V + \int_{a_q} \delta\vartheta \int_0^t \dot{\eta}^*\mathrm{d}\tau\mathrm{d}a,\\
\delta\tilde{\Phi} &= \int_{a_\Phi} \delta\mu \int_0^t \dot{\xi}^*\mathrm{d}\tau\mathrm{d}a,\\
\delta\tilde{W}^{(r)} &= \int_V (f_k - \rho\ddot{u}_k)\delta u_k \mathrm{d}V + \int_{a_\sigma} p_k^*\delta u_k \mathrm{d}a - \int_V \delta\tilde{h}\mathrm{d}V,\\
\delta g &= \sigma_{ij}\delta\varepsilon_{ij} - s\delta\vartheta - c\delta\mu,\\
\delta\tilde{h} &= -T_{,i}\delta\eta_i - \mu_{,i}\delta\xi_i + \delta\left(T_{,i}\eta_i\right) + \delta(\mu_{,i}\xi_i) \;=\; \eta_j\delta\vartheta_{,j} + \xi_j\delta\mu_{,j}.
\end{aligned} \tag{35}$$

In Eqs. (35), f_k, p_k^* $\dot{\eta}^* = \dot{\eta}_i^* n_i$ and $\dot{\xi}^* = \dot{\xi}_i^* n_i$ are given values, and $\delta\tilde{\Phi}$ is the complement diffusion energy. Using the following relations:

$$\begin{aligned}
&\int_V \delta g \mathrm{d}V = \int_a \sigma_{ij}n_j\delta u_i \mathrm{d}a - \int_V \sigma_{ij,j}\delta u_i \mathrm{d}V - \int_V s\delta\vartheta\mathrm{d}V - \int_V c\delta\mu\mathrm{d}V,\\
&\int_V \delta\tilde{h}\mathrm{d}V = \int_V \left(\eta_j\delta\vartheta_{,j} + \xi_j\delta\mu_{,j}\right)\mathrm{d}V = \int_a \left(\eta_j n_j\delta\vartheta + \xi_j n_j\delta\mu\right)\mathrm{d}a - \int_V \left(\eta_{j,j}\delta\vartheta + \xi_{j,j}\delta\mu\right)\mathrm{d}V
\end{aligned} \tag{36}$$

finishing the variational calculation, we get

$$\begin{aligned}
\delta\Pi = &\int_{a_\sigma} (\sigma_{ij}n_j - p_i^*)\delta u_i \mathrm{d}a - \int_V (\sigma_{ij,j} + f_i - \rho\ddot{u}_i)\delta u_i \mathrm{d}V - \int_V s\delta\vartheta\mathrm{d}V - \int_V c\delta\mu\mathrm{d}V\\
&+ \int_V \delta\vartheta \int_0^t \dot{r}T^{-1}\mathrm{d}\tau\mathrm{d}V + \int_V \delta\vartheta \int_0^t \dot{s}^{(i)}\mathrm{d}\tau\mathrm{d}V - \int_{a_q} \delta\vartheta \int_0^t \dot{\eta}^*\mathrm{d}\tau\mathrm{d}a - \int_{a_\Phi} \delta\mu \int_0^t \dot{\xi}^*\mathrm{d}\tau\mathrm{d}a\\
&+ \int_a \left(\eta_j n_j\delta\vartheta + \xi_j n_j\delta\mu\right)\mathrm{d}a - \int_V \left(\eta_{j,j}\delta\vartheta + \xi_{j,j}\delta\mu\right)\mathrm{d}V\\
= &\int_{a_\sigma} (\sigma_{ij}n_j - p_i^*)\delta u_i \mathrm{d}a - \int_V (\sigma_{ij,j} + f_i - \rho\ddot{u}_i)\delta u_i \mathrm{d}V\\
&- \int_V \left\{s + \int_0^t \left(-T^{-1}\dot{r} + \dot{\eta}_{j,j} - \dot{s}^{(i)}\right)\mathrm{d}\tau\right\}\delta\vartheta\mathrm{d}V\\
&- \int_V \left(c + \dot{\xi}_{j,j}\right)\delta\mu c\mathrm{d}V + \int_{a_q} \left(\eta_j n_j - \eta^*\right)\delta\vartheta\mathrm{d}a + \int_{a_\Phi} \left(\xi_j n_j - \xi^*\right)\delta\mu\mathrm{d}a = 0.
\end{aligned} \tag{37}$$

By using

$$-T^{-1}\dot{r} + \dot{\eta}_{j,j} - \dot{s}^{(i)} = -T^{-1}\left(\dot{r} - T\dot{\eta}_{j,j} - T_{,i}\dot{\eta}_i - \mu_{,i}\dot{\xi}_i\right) = -T^{-1}\left[\dot{r} - q_{j,j} - \mu_{,i}\dot{\xi}_i\right]$$

and the arbitrariness of $\delta\boldsymbol{u}$, $\delta\vartheta$ and $\delta\mu$, from Eq. (37) we get

$$\sigma_{kl,l} + f_k = \rho\ddot{u}_k, \text{ in} V; \quad \sigma_{kl}n_l = p_k^*, \text{ on } a_\sigma \tag{38}$$

and

$$\begin{aligned}
&\int_V \int_0^t \dot{s}\mathrm{d}\tau\mathrm{d}V = \int_V \int_0^t T^{-1}\left(\dot{r} - q_{j,j} - \mu_{,i}\dot{\xi}_i\right)\mathrm{d}\tau\mathrm{d}V,\\
&\int_V \int_0^t \dot{c}\mathrm{d}\tau\mathrm{d}V = -\int_V \int_0^t \dot{\xi}_{j,j}\mathrm{d}\tau\mathrm{d}V \quad \text{in } V,\\
&\dot{\eta}_j n_j = \dot{\eta}^*, \quad \text{or } q_n = q_n^* \quad \text{on } a_q,\\
&\dot{\xi}_j n_j = \dot{\xi}^*, \quad \text{or } J_n = J_n^* \quad \text{on } a_\Phi.
\end{aligned} \tag{39}$$

The first two formulas in Eq. (39) can be rewritten as

$$\begin{aligned}&T\dot{s} = \dot{r} - q_{j,j} - \mu_{,i}\dot{\xi}_i; \quad \dot{c} = -\dot{\xi}_{j,j}; \quad \text{or}\\&T\dot{s} + \mu\dot{c} = \dot{r} - q_{j,j} - \mu_{,i}\dot{\xi}_i - \mu\dot{\xi}_{j,j} = \dot{r} - q_{j,j} - \left(\mu J_i\right)_{,i}; \quad \text{in } V.\end{aligned} \tag{40}$$

The last equation in Eq. (40) is just the same as that in Eq. (33).

The above variational principle requests prior that the $\mathbf{u}$, ϑ and μ satisfy their own boundary conditions, so in the governing equations the following equations should also be added:

$$\mathbf{u} = \mathbf{u}^*, \quad \text{on } a_u; \quad \vartheta = \vartheta^*, \quad \text{on } a_T; \quad \mu = \mu^*, \quad \text{on } a_\mu. \tag{41}$$

Equations (38)–(41) are the governing equations of the generalized thermal diffusion theory.

5.5 Governing equations of the temperature and chemical potential

In irreversible thermodynamics, the Gibbs free energy g can be selected as

$$g\left(\varepsilon_{kl}, \vartheta, \mu\right) = \frac{1}{2}C_{ijkl}\varepsilon_{ji}\varepsilon_{lk} - \alpha_{ij}\varepsilon_{ij}\vartheta - \frac{1}{2T_0}C\vartheta^2 - \frac{1}{2}b\mu^2 - b_{ij}\varepsilon_{ij}\mu - a\mu\vartheta \tag{42}$$

where a, b, b_{ij} are also material constants. The constitutive equations are:

$$\begin{aligned}&\sigma_{ij} = \partial g/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - \alpha_{ij}\vartheta - b_{ij}\mu,\\&s = -\partial g/\partial\vartheta = \alpha_{ij}\varepsilon_{ij} + C\vartheta/T_0 + a\mu, c = -\partial g/\partial\mu = b\mu + b_{ij}\varepsilon_{ij} + a\vartheta.\end{aligned} \tag{43}$$

Substituting Eqs. (34) and (43) into Eq. (33) and Second Fick's law Eq. (28.3), we get the temperature and chemical potential equations:

$$\begin{aligned}&T\left(\alpha_{ij}\varepsilon_{ij} + C\vartheta/T_0 + a\mu\right)^{\cdot} = \dot{r} + \left(\lambda_{ij}\vartheta_{,j} + L_{ij}\mu_{,j}\right)_{,i} + \mu_{,i}\left(L_{ij}T^{-1}\vartheta_{,j} + D_{ij}\ \mu_{,j}\right),\\&\left(b\mu + b_{ij}\varepsilon_{ij} + a\vartheta\right)^{\cdot} = \left(L_{ij}T^{-1}\vartheta_{,j} + D_{ij}\ \mu_{,j}\right)_{,i}; \quad \text{in } V.\end{aligned} \tag{44}$$

If there is no coupling in ϑ and μ, i.e., let $L_{ij} = 0$ and $a = 0$, then for $\dot{r} = 0$ and constant coefficients, the linearized form of Eq. (44) is

$$\begin{aligned}&T\left(\alpha_{ij}\dot{u}_{i,j} + C\dot{\vartheta}/T_0 + a\dot{\mu}\right) = \lambda_{ij}\vartheta_{,j},\\&b\dot{\mu} + b_{ij}\dot{u}_{i,j} + a\dot{\vartheta} = D_{ij}\mu_{,ji}; \quad \text{in } V.\end{aligned} \tag{45}$$

In the literature formulas analogous to Eq. (45) can be found.

If there is no diffusion, the thermoelastic diffusive problem is reduced to the thermoelastic problem as shown in Sect. 2. The PVP and governing equation of an isothermal diffusive problem can also be obtained from the thermoelastic diffusive problem by omitting r, T and $\boldsymbol{q}$.

6 The PVP in linear electromagnetic thermoelastic media with current

6.1 The Maxwell equations

The electromagnetic theory has been discussed in many literatures [25–28]. The Maxwell equation is:

$$\nabla\times\boldsymbol{H} = -\dot{\boldsymbol{D}} + \boldsymbol{I}, \quad \nabla\times\boldsymbol{E} = -\dot{\boldsymbol{B}}, \quad \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{D} = \rho_e; \quad \text{in } \boldsymbol{V} \tag{46.1}$$

where $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}$ and $\boldsymbol{B}$ are the electric field intensity, electric displacement or electric flux density, magnetic field intensity and magnetic induction or the magnetic flux density, respectively; ρ_e is the free electric charge density; $\boldsymbol{I}$ is the total electric current density. Some time $\boldsymbol{I}$ is divided into $\boldsymbol{I}^{(\mathrm{e})}$ and $\boldsymbol{I}^{(\gamma)}$ corresponding to the given external exciting current density and the induction or eddy current density, respectively.

If we introduce an electric scalar potential φ and a magnetic vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad \text{or } B_i = \varpi_{ikl}A_{l,k}; \quad \boldsymbol{E} = -\nabla\varphi - \dot{\boldsymbol{A}}, \tag{46.2}$$

the equations $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{J} = -\dot{\rho}_{\mathrm{e}}$ will be satisfied automatically. In Eq. (46.2), $\boldsymbol{\varpi}$ is the permutation notation. The boundary conditions are

$$\begin{aligned}
&\boldsymbol{D} \cdot \boldsymbol{n} = -\sigma^*; \quad \text{on } a_D; \quad \varphi = \varphi^*; \quad \text{on } a_\varphi,\\
&\boldsymbol{A} = \boldsymbol{A}^*; \quad \text{on } a_A; \quad \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{I}^* \quad \text{on } a_H
\end{aligned} \tag{47}$$

where σ^*, $\boldsymbol{I}^*$, φ^* and $\boldsymbol{A}^*$ are given values.

6.2 The PVP in a linear electromagnetic thermoelastic field with current

As in section 2, by using TIP [15–17] the entropy equation can be written as

$$\begin{aligned}
&T\dot{s} = \dot{r} + E_i I_i - q_{i,i}; \quad \dot{s} = \dot{s}^{(r)} + \dot{s}^{(i)}; \quad T\dot{s}^{(r)} = \dot{r} - T\dot{\eta}_{i,i},\\
&T\sigma = T\dot{s}^{(i)} = T\left(\dot{s} - \dot{s}^{(r)}\right) = E_i I_i - T_{,i}\dot{\eta}_i \geq 0
\end{aligned} \tag{48}$$

where $\boldsymbol{E} \cdot \boldsymbol{I}$ is the dissipative heat produced by the current. From Eq. (48), we get the evolution equations as:

$$\boldsymbol{I} = \boldsymbol{\gamma} \cdot \boldsymbol{E} + \boldsymbol{\beta} \cdot \nabla T/T, \quad T\dot{\boldsymbol{\eta}} = \boldsymbol{q} = -\boldsymbol{\lambda} \cdot \nabla T + \boldsymbol{\beta} \cdot \boldsymbol{E} \tag{49}$$

where $\boldsymbol{\gamma}$ is the electric conductivity, $\boldsymbol{\beta}$ is the current–temperature coefficient.

Under the conditions that on the boundary a the boundary conditions are satisfied prior, and the time conditions at the initial time $t = 0$ and the terminal time t_f satisfy

$$\delta\boldsymbol{A}(0) = \delta\boldsymbol{A}(t_f) = \boldsymbol{0}, \tag{50}$$

the PVP for the linear electromagnetic thermoelastic field can be expressed as

$$\begin{aligned}
&\delta\tilde{\Pi} = \int_0^{t_f}\left(\int_V \delta g \mathrm{d}V - \delta\tilde{Q}^{(t)} - \delta\tilde{W}^{(r)}\right)\mathrm{d}t = 0,\\
&\delta\tilde{Q}^{(t)} = -\int_V \delta\vartheta \int_0^t \dot{r}T^{-1}\mathrm{d}\tau\mathrm{d}V - \int_V \delta\vartheta \int_0^t \dot{s}^{(i)}\mathrm{d}\tau\mathrm{d}V + \int_{a_q} \delta\vartheta \int_0^t \dot{\eta}^*\mathrm{d}\tau\mathrm{d}a,\\
&\delta\tilde{W}^{(r)} = \int_V (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \delta\boldsymbol{u}\mathrm{d}V - \int_V \rho_e\delta\varphi\mathrm{d}V - \int_V \delta\tilde{h}\mathrm{d}V - \int_{a_D} \sigma^*\delta\varphi\mathrm{d}a + \int_{a_\sigma} p^* \cdot \delta\boldsymbol{u}\mathrm{d}a + \int_{a_H} \boldsymbol{I}^* \cdot \delta\boldsymbol{A}\mathrm{d}a,\\
&\delta\tilde{h} = -\boldsymbol{I} \cdot \delta\boldsymbol{A} + \left(\int_0^t \dot{\eta}_j\mathrm{d}\tau\right)\delta\vartheta_{,j},\\
&\delta g = \sigma_{ij}\delta\varepsilon_{ji} - D_k\delta E_k + H_k\delta B_k - s\delta\vartheta
\end{aligned} \tag{51}$$

where $-\boldsymbol{I} \cdot \delta\boldsymbol{A}$ is the dissipative work density rate of the electromagnetic field with current. Equation (51) is similar to the Hamilton variational principle in dynamics. Certainly, we can reduce Eq. (51) to a variational principle without the time integral, just as in the previous sections.

By using Eq. (50), we have

$$\begin{aligned}
\int_0^{t_f}\mathrm{d}t\int_V D_i\delta\dot{A}_i\mathrm{d}V &= \int_0^{t_f}\mathrm{d}t\int_V \partial(D_i\delta A_i)/\partial t\mathrm{d}V - \int_0^{t_f}\mathrm{d}t\int_V \partial D_i/\partial t\delta A_i\mathrm{d}V\\
&= \left(\int_V D_i\delta A_i\mathrm{d}V\right)\Bigg|_0^{t_f} - \int_0^{t_f}\mathrm{d}t\int_V \partial D_i/\partial t\delta A_i\mathrm{d}V = -\int_0^{t_f}\mathrm{d}t\int_V \dot{D}_i\delta A_i\mathrm{d}V.
\end{aligned} \tag{52}$$

By using Eq. (52), we have

$$\begin{aligned}
\int_V \delta g\mathrm{d}V &= \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V + \int_V D_i\delta\varphi_{,i}\mathrm{d}V + \int_V D_i\delta\dot{A}_i\mathrm{d}V + \int_V H_i\delta B_i\mathrm{d}V - \int_V s\delta\vartheta\mathrm{d}V\\
&= \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V + \int_V D_i\delta\varphi_{,i}\mathrm{d}V - \int_V \dot{D}_i\delta A_i\mathrm{d}V + \int_V H_i\delta B_i\mathrm{d}V - \int_V s\delta\vartheta\mathrm{d}V.
\end{aligned} \tag{53}$$

Applying Eq. (52), the first relation in Eq. (51) can be reduced to

$$\int_V \delta g \mathrm{d}V - \delta \tilde{Q}^{(t)} - \delta \tilde{W}^{(r)} = 0. \tag{54}$$

Substituting Eq. (53) into Eq. (54), we get

$$\begin{aligned}
\delta \Pi =& \int_V \sigma_{ij} \delta \varepsilon_{ij} \mathrm{d}V + \int_V D_i \delta \varphi_{,i} \mathrm{d}V - \int_V \dot{D}_i \delta A_i \mathrm{d}V + \int_V H_i \delta B_i \mathrm{d}V - \int_V s \delta \vartheta \mathrm{d}V - \int_{a_\sigma} p_i^* \delta u_i \mathrm{d}a \\
&+ \int_V \delta \vartheta \int_0^t \dot{r} T^{-1} \mathrm{d}\tau \mathrm{d}V + \int_V \delta \vartheta \int_0^t \dot{s}^{(i)} \mathrm{d}\tau \mathrm{d}V - \int_{a_q} \delta \vartheta \int_0^t \dot{\eta}^* \mathrm{d}\tau \mathrm{d}a - \int_V (f_i - \rho \ddot{u}_i) \delta u_i \mathrm{d}V \\
&+ \int_V \rho_e \delta \varphi \mathrm{d}V + \int_a \sigma^* \delta \varphi \mathrm{d}a - \int_V I_i \delta A_i \mathrm{d}V - \int_{a_H} I_l^* \delta A_l \mathrm{d}V + \int_V \left(\int_0^t \dot{\eta}_j \mathrm{d}\tau \right) \delta \vartheta_{,j} \mathrm{d}V \\
=& \int_a \left(\sigma_{ij} \right) n_j \delta u_i \mathrm{d}a - \int_V \left[\left(\sigma_{ij} \right)_{,j} + f_i - \rho \ddot{u}_i \right] \delta u_i \mathrm{d}V + \int_V D_i \left[(\delta \varphi)_{,i} - \varphi_{,p} \delta u_{p,i} \right] \mathrm{d}V - \int_{a_\sigma} p_i^* \delta u_i \mathrm{d}a \\
&+ \int_V H_i \left(\varpi_{ikl} \delta A_l \right)_{,k} \mathrm{d}V + \int_V \rho_e \delta \varphi \mathrm{d}V - \int_V \left(I_i + \dot{D}_i \right) \delta A_i \mathrm{d}V + \int_a \sigma^* \delta \varphi \mathrm{d}a - \int_{a_H} I_l^* \delta A_l \mathrm{d}V \\
&+ \int_V \delta \vartheta \left(-s + \int_0^t \dot{r} T^{-1} \mathrm{d}\tau + \int_0^t \dot{s}^{(i)} \mathrm{d}\tau \right) \mathrm{d}V + \int_a \delta \vartheta \eta_j n_j - \int_V \eta_{j,j} \delta \vartheta \mathrm{d}V - \int_{a_q} \delta \vartheta \int_0^t \dot{\eta}^* \mathrm{d}\tau \mathrm{d}a \\
=& \int_a \left(\sigma_{ij} n_j - p_i^* \right) \delta u_i \mathrm{d}a - \int_V \left(\sigma_{ij,j} + f_i - \rho \ddot{u}_i \right) \delta u_i \mathrm{d}V - \int_V \left(D_{i,i} - \rho_e \right) \delta \varphi \mathrm{d}V \\
&- \int_V \left(\varpi_{ikl} H_{i,k} + \dot{D}_l + I_l \right) \delta A_l \mathrm{d}V + \int_a \left(D_k n_k + \sigma^* \right) \delta \varphi \mathrm{d}a + \int_{a_H} \left(\varpi_{mli} H_i n_m - I_l^* \right) \delta A_l \mathrm{d}V \\
&- \int_V \delta \vartheta \left[s - \int_0^t T^{-1} \left(-\dot{r} - T \dot{s}^{(i)} + T \dot{\eta}_{j,j} \right) \mathrm{d}\tau \right] \mathrm{d}V + \int_{a_q} \delta \vartheta \int_0^t \left(\dot{\eta}_j n_j - \dot{\eta}^* \right) \mathrm{d}\tau \mathrm{d}a = 0.
\end{aligned} \tag{55}$$

Due to the arbitrariness of $\delta \boldsymbol{u}$, $\delta \boldsymbol{A}$, $\delta \varphi$, etc, we get

$$\begin{aligned}
& \sigma_{ji,j} + f_i - \rho \ddot{u}_i = 0; \quad D_{i,i} - \rho_e = 0; \quad \varpi_{mli} H_{i,m} + \dot{D}_l + I_l = 0; \\
& T \dot{s} = \dot{r} + T \dot{s}^{(i)} - T \dot{\eta}_{j,j} = 0, \quad \text{or } T \dot{s} = \dot{r} - q_{i,i} + E_i I_i; \quad \text{in } V, \\
& \sigma_{ji} n_j = p_i^*, \quad \text{on } a_\sigma; \quad D_k n_k + \sigma^* = 0, \quad \text{on } a_D, \\
& \varpi_{mli} H_i n_m - I_l^* = 0, \quad \text{on } a_H; \quad \dot{\eta}_j n_j = \dot{\eta}^*; \quad \text{on } a_q
\end{aligned} \tag{56}$$

where $\varpi_{mli} H_{i,m} = -\nabla \times \boldsymbol{H}$ and $\varpi_{mli} H_i n_m = \boldsymbol{H} \times \boldsymbol{n}$. $\boldsymbol{I}$ and $\boldsymbol{q}$ are expressed in Eq. (49). For the case without current, some variational principles in the thermal electric material were also seen in literature, like in [12,29].

Equation (56) and the prior conditions on the interface and boundary of the PVP and the initial condition form the governing equations for the linear electromagnetic thermoelastic analysis under the general electromagnetic field with current.

In engineering power systems, the frequency of the electromagnetic variables is usually less than 10MHz, so comparing with electric current density $\boldsymbol{I}$, $\dot{\boldsymbol{D}}$ can be neglected. The above theory can still be used, but the PVP is without $\dot{\boldsymbol{D}}$.

6.3 The governing equation of temperature

According to the Continuum thermodynamics, the electric Gibbs function g in the linear electromagnetic thermoelastic media can be assumed as

$$\begin{aligned}
g =& (1/2) C_{ijkl} \varepsilon_{ji} \varepsilon_{lk} - (1/2) \in_{kl} E_k E_l - e_{kij}^{\mathrm{e}} E_k \varepsilon_{ij} + e_{kij}^{\mathrm{m}} B_k \varepsilon_{ij} \\
&+ (1/2) \mu_{ij}^{-1} B_i B_j - \alpha_{ij} \varepsilon_{ij} \vartheta - \tau_i^{\mathrm{e}} E_i \vartheta + \tau_i^{\mathrm{m}} B_i \vartheta - (1/2T_0) C \vartheta^2, \\
C_{ijkl} =& C_{jikl} = C_{ijlk} = C_{klij}, \quad e_{kij}^{\mathrm{e}} = e_{kji}^{\mathrm{e}}, \quad \in_{kl} = \in_{lk}, \quad e_{kij}^{\mathrm{m}} = e_{kji}^{\mathrm{m}}, \quad \mu_{kl} = \mu_{lk}, \quad \alpha_{ij} = \alpha_{ji}
\end{aligned} \tag{57}$$

where $\boldsymbol{C}$, $\boldsymbol{e}^{\mathrm{e}}$, $\in$, $\boldsymbol{e}^{\mathrm{m}}$, $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, $\boldsymbol{\tau}^{\mathrm{e}}$, $\boldsymbol{\tau}^{\mathrm{m}}$ and C are the elastic, piezoelectric, permittivity, piezomagnetic, permeability, thermo-mechanical, thermo-electric, thermo-magnetic coefficients, and the heat capacity, respectively. The consecutive equation corresponding to Eq. (57) is

$$\begin{aligned}\sigma_{ji} &= \partial g/\partial\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} - e^{\mathrm{e}}_{kij}E_k + e^{\mathrm{m}}_{kij}B_k - \alpha_{ij}\vartheta,\\ D_i &= -\partial g/\partial E_i = \in_{ij} E_j + e^{\mathrm{e}}_{ikl}\varepsilon_{kl} + \tau^{\mathrm{e}}_i\vartheta,\\ H_i &= \partial g/\partial B_i = \mu^{-1}_{ij}B_j + e^{\mathrm{m}}_{ikl}\varepsilon_{kl} + \tau^{\mathrm{m}}_i\vartheta,\\ s &= -\partial g/\partial\vartheta = \alpha_{ij}\varepsilon_{ij} + \tau^{\mathrm{e}}_i E_i - \tau^{\mathrm{m}}_i B_i + C\vartheta/T_0.\end{aligned} \tag{58}$$

By using Eqs. (48), (56), (58), the temperature equation can be expressed as

$$T\left(\alpha_{ij}\varepsilon_{ij} + \tau^{\mathrm{e}}_i E_i - \tau^{\mathrm{m}}_i B_i + C\vartheta/T_0\right)^{\cdot} = \dot{r} + \left(\lambda_{ij}T_{,j} + \beta_{ij}E_j\right)_{,i} + E_i\left(\gamma_{ij}E_j + \beta_{ij}T_{,j}/T\right). \tag{59}$$

7 The PVP including the hyperbolic temperature wave equation

Due to the paradox of infinite thermal wave velocity and the Landau second sound speed observed by experiments in liquid helium and in solids at low temperatures, the hyperbolic temperature wave equation instead of the parabolic thermal equation is developed. Kuang pointed out that [30,31] the variation of temperature with time should be supplied by extra heat from the environment, which is formally similar to the inertial work produced by the inertial force in the mechanical dynamics, i.e., when the temperature is varied with time the first law of thermodynamics for a closed system is changed to [30,31]

$$\dot{U} = \dot{W} + \dot{Q} - \dot{Q}^{(a)};\ \dot{Q}^{(a)} = \int_V T\dot{s}^{(a)}\mathrm{d}V \tag{60}$$

where $Q^{(a)}$ is the inertial heat, and $s^{(a)}$ is the inertial entropy density. The usual local entropy equation (5) is changed to

$$\begin{aligned}&T\left(\dot{s} + \dot{s}^{(a)}\right) = \dot{r} - q_{i,i},\ \dot{s} = \dot{s}^{(r)} + \dot{s}^{(i)},\\ &T\left(\dot{s}^{(r)} + \dot{s}^{(a)}\right) = \dot{r} - T\dot{\eta}_{i,i};\quad \dot{s}^{(a)} = \rho_{s0}C\ddot{T}/T,\\ &T\sigma = T\left(\dot{s} - \dot{s}^{(r)}\right) = T\left[\left(\dot{s} + \dot{s}^{(a)}\right) - \left(\dot{s}^{(r)} + \dot{s}^{(a)}\right)\right] = -\dot{\eta}_i T_{,i}\end{aligned} \tag{61}$$

where ρ_{s0} is a coefficient, and C is the specific heat per volume. From Eq. (61), it is seen that if instead of $\dot{s}$ in usual entropy equation there is $\dot{s}+\dot{s}^{(a)}$, we get the entropy equation of inertial entropy theory. If we use $\dot{s}+\dot{s}^{(a)}$ instead of $\dot{s}$ in the PVP of the dissipative media established in the previous sections, the hyperbolic temperature wave equation will be instead of the parabolic temperature equation, and the other governing equations are kept. Therefore by using the inertial entropy theory, the temperature wave equation in the dissipative media can easily be obtained.

For the thermoelastic media in Eq. (13), the temperature governing equation is changed to

$$T\left(\dot{s} + \dot{s}^{(a)}\right) = \dot{r} - q_{i,i},\quad \text{or}\quad T\left(\alpha_{ij}\dot{\varepsilon}_{ij} + C\dot{\vartheta}/T_0\right) + \rho_{s0}C\ddot{\vartheta} = \dot{r} + \left(\lambda_{ij}\vartheta_{,j}\right)_{,i}\quad \text{in}V. \tag{62}$$

For the thermal viscoelastic media in Eq. (22), the temperature governing equation is changed to

$$\begin{aligned}&T\left(\dot{s} + \dot{s}^{(a)}\right) = \dot{r} - q_{i,i} + \sigma^{(i)}_{ij}\dot{\varepsilon}_{ji},\quad \text{or}\\ &T\left(\alpha_{ij}\dot{\varepsilon}_{ij} + C\dot{\vartheta}/T_0\right) + \rho_{s0}C\ddot{\vartheta} = \dot{r} + \left(\lambda_{ij}\vartheta_{,j} + \kappa_{klj}\dot{\varepsilon}_{lj}\right)_{,i} + \left(\beta_{ijkl}\dot{\varepsilon}_{ij} - \kappa_{ikl}\vartheta_{,i}\right)\dot{\varepsilon}_{ji}.\end{aligned} \tag{63}$$

The similar case is also true for the thermal elastoplastic media.

For the thermoelastic diffusive media, a generalized inertial entropy theory is introduced, where analogous to the inertial entropy concept an inertial concentration concept is introduced in the thermo-diffusion problem

[30,31]. In the generalized inertial entropy theory, the entropy equation of the classical thermal diffusion theory, Eq. (33), is changed to:

$$\begin{aligned} &T\left(\dot{s}+\dot{s}^{(a)}\right)+\mu\left(\dot{c}+\dot{c}^{(a)}\right)=\dot{r}-\nabla\cdot\boldsymbol{q}, \\ &\dot{s}^{(a)}=\rho_s\ddot{T}=\rho_{s0}(C/T)\ddot{T},\ \dot{c}^{(a)}=\rho_\mu\ddot{\mu};\ \dot{c}=-\nabla\cdot\boldsymbol{J}, \\ &T\sigma=\rho T\left(\dot{s}-\dot{s}^{(r)}\right)=-T_{,i}\dot{\eta}_i-\mu_{,i}\dot{\xi}_i;\quad \boldsymbol{J}=\dot{\boldsymbol{\xi}} \end{aligned} \tag{64}$$

where ρ_μ is the inertial concentration coefficient. Instead of Eq. (44), the temperature and chemical potential governing equations are, respectively, changed to

$$\begin{aligned} &T\left(\alpha_{ij}\dot{\varepsilon}_{ij}+C\dot{\vartheta}/T_0+a\dot{\mu}\right)+\rho_{s0}C\ddot{\vartheta}=\dot{r}+\left(\lambda_{ij}\vartheta_{,j}+L_{ij}\mu_{,j}\right)_{,i}+\mu_{,i}\left(L_{ij}T^{-1}\vartheta_{,j}+D_{ij}\ \mu_{,j}\right), \\ &b\dot{\mu}+b_{ij}\dot{\varepsilon}_{ij}+a\dot{\vartheta}+\rho_\mu\ddot{\vartheta}=\left(L_{ij}T^{-1}\vartheta_{,j}+D_{ij}\ \mu_{,j}\right)_{,i}. \end{aligned} \tag{65}$$

If considering the temperature inertia for the electromagnetic thermoelastic media with current, the temperature wave equation in Eqs. (48) and (59) is changed to

$$\begin{aligned} &T\left(\dot{s}+\dot{s}^{(a)}\right)=\dot{r}-q_{i,i}+E_iI_i;\quad \text{or} \\ &T\left(\alpha_{ij}\varepsilon_{ij}+\tau_i^{\mathrm{e}}E_i-\tau_i^{\mathrm{m}}B_i+C\vartheta/T_0\right)^{\cdot}+\rho_{s0}C\ddot{\vartheta}=\dot{r}+E_i\left(\gamma_{ij}E_j+\beta_{ij}T_{,j}/T\right)-\left(\lambda_{ij}T_{,j}+\beta_{ij}E_j\right)_{,i}. \end{aligned} \tag{66}$$

From the above discussion, it is found that when we study the temperature wave equation the inertial entropy theory is the easiest theory compared with other theories [32,33].

8 Conclusions

In this paper, we give an extended Gibbs equation for dissipative media, and on this basis the PVP for dissipative media, which is a universal principle, is established. The dissipation work and its converted internal irreversible heat are simultaneously included in the PVP. From this PVP, the governing equation, including thermodynamic and irreversible variables, and the boundary conditions of the dissipative media can be obtained. The conservative system is the special case of the dissipative system. The governing equations of the thermoelastic, thermal viscoelastic, thermal elastoplastic, linear thermoelastic diffusive and linear electromagnetic thermoelastic media are derived by the PVP in this paper or the PVP for dissipative media gives the "generalized motion equations". The PVP in dissipative media can be applied to more complex multi-field problems. Applying the inertial entropy theory, the PVP is easily extended to the case where the temperature propagates with finite wave speed. Other than the mathematical variational principle, where the variational principle is derived from known governing equations, in the PVP the governing equations are obtained by it. The PVP can also be used to check the current governing equation. When we accept PVP, the universal principles in continuum mechanics are the mass conservation, the energy conservation and the PVP (or the principle of energy stationary value) laws. In the engineering applications, these governing equations derived by PVP can be simplified according to the practical situation. The constitutive and evolution equations and the material constants should be determined by experiments and other appropriate theories.

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