Acta Mech 227, 549–564 (2016)
DOI 10.1007/s00707-015-1485-1

ORIGINAL PAPER

M. A. Foyouzat · M. Mofid · J. E. Akin

On the dynamic response of beams on elastic foundations with variable modulus

Received: 1 August 2015 / Published online: 13 October 2015

Abstract An exact solution is established pertaining to the dynamic response of an Euler–Bernoulli beam resting on a Winkler foundation with variable subgrade modulus. The solution is performed by employing the infinite power series method. Moreover, using the Frobenius theorem, the proposed method is extended in order to solve the problems wherein the variation of the modulus is not an analytic function. The solution procedure is demonstrated by several illustrative examples, and the correctness of the results has been ascertained by comparison with recognized solutions in the literature. Finally, it is shown that the proposed method of solution is directly applicable to the more general problem of beams on a variable-modulus Pasternak-type foundation.

1 Introduction

Many problems of considerable practical importance can be related to the solution of beams resting on an elastic foundation. The structural analysis of railroad tracks, highway pavements, continuously supported pipelines, and strip foundations is a well-known direct real-world application.

The simplest model to idealize the behavior of the elastic foundation is the one proposed by Winkler [1]. In this model, the relation between the pressure and the deflection of the foundation surface at any point is:

$$q = kw \tag{1}$$

where k is the modulus of subgrade reaction, q is the resting pressure of the foundation, and w is the deflection of the beam. Taking the modulus of subgrade reaction, k, as uniformly distributed below the beam, significantly simplifies the solution of the relevant differential equations. This is the basis of the extensive usage of the constant modulus assumption related to soil models. The well-known text by Hetényi [2] provides a comprehensive treatment of the constant Winkler model for elastic foundations.

However, it is commonly acknowledged that this assumption is far from reality, and a more robust analysis requires the consideration of the inhomogeneity of the subgrade [3–5]. Furthermore, the consideration of the variation of the soil subgrade modulus beneath footings, as stipulated in most design codes (e.g., [6]), is essential in some footing design methods.

M. A. Foyouzat · M. Mofid (✉)
Civil Engineering Department, Sharif University of Technology, Azadi Ave., Tehran, Iran
E-mail: mofid@sharif.edu

M. A. Foyouzat
E-mail: foyozat_mohammadali@mehr.sharif.ir

J. E. Akin
Mechanical Engineering Department, Rice University, Houston, TX, USA
E-mail: akin@rice.edu

The literature contains a wealth of research on the static analysis of beams pertaining to variable Winkler foundations, of which a brief survey is presented here. Franklin and Scott [7] presented a closed-form solution for a linear variation of the foundation modulus, using contour integrals. For a higher-order variation in x (the coordinate along the beam), they presented a partial solution, which is applicable to infinite beams (or piles). Lentini [8] presented a finite difference method to solve the problem when the foundation stiffness varies along x as a power of x. Clastornik et al. [9] presented a solution for finite beams resting on Winkler elastic foundations with a stiffness variation that can be represented as a general polynomial of x.

As far as the dynamic analysis of beams on a variable elastic foundation is concerned, a variety of numerical and approximate methods can be found in the literature. Eisenberger and Clastomik [10,11] have applied a finite element approach to attack the problem for both one- and two-parameter foundations. Moreover, their solution is based primarily on using the exact stiffness, consistent mass, and geometric stiffness matrices. Ding [12] has obtained a general solution in the form of an integral equation. The integrals in the solution have approximately and numerically been calculated by means of the trapezoidal rule.

The current study offers an exact solution to the dynamic response of an Euler–Bernoulli beam supported by a Winkler foundation with variable subgrade modulus. The proposed method is based on the series solution of differential equations, and it is applicable to any type of classical boundary condition (simply supported, clamped, free, elastically supported, etc.). Moreover, it is shown that, using the Frobenius theorem, the methodology can be extended to solve the problems wherein the variation of the modulus is not an analytic function of x. Several examples are presented in order to demonstrate how the work proceeds. The results prove to be in excellent agreement with those available in the literature as well as the finite element results. Finally, the applicability of the proposed solution to beams on a Pasternak-type foundation with a spring layer of variable modulus is set forth.

2 Problem formulation

Consider the problem of an Euler–Bernoulli beam of length s and with constant flexural stiffness EI, resting on a Winkler-type foundation of variable modulus $k(x)$ and subjected to a load $P(x, t)$ (Fig. 1). Assuming that the beam maintains continuous contact with the base, the governing differential equation of this problem can be expressed as

$$EI\frac{\partial^4 w}{\partial x^4} + k(x)w + \rho A\frac{\partial^2 w}{\partial t^2} = P(x, t) \tag{2}$$

where $w(x, t)$ is the deflection of the beam, ρ is the mass density of the material, and A is the cross-sectional area of the beam. In order to seek a solution for Eq. (2), one should begin with the solution of the corresponding homogeneous equation, that is, when $P(x, t) = 0$. Therefore, in what follows, the solution of the homogeneous equation is first elucidated, while the forced vibration response will be treated afterward.

As far as the homogeneous solution is concerned, by applying the method of separation of variables, one can assume a solution as

$$w(x, t) = W(x)T(t) \tag{3}$$

where $W(x)$ is the shape function which describes the modes of the vibration and $T(t)$ is a separable solution of $W(x)$. Introducing the above into Eq. (2) yields

$$\frac{EI}{\rho AW}\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} + \frac{k(x)}{\rho A} = -\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}. \tag{4}$$

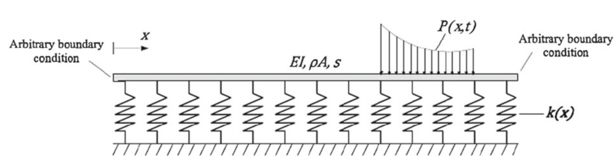

Fig. 1 An Euler–Bernoulli beam supported by a Winkler foundation of variable modulus

Since the left-hand side of this equation is a function of variable x, whereas the right-hand side depends only on time variable t, it can be concluded that either side of Eq. (4) must be equal to a constant. Denoting the aforementioned constant by ω^2, one will get:

$$\frac{d^2T}{dt^2} = -\omega^2 T \tag{5}$$

Solving this for T yields:

$$T = A_1 \cos \omega t + A_2 \sin \omega t \tag{6}$$

The arbitrary constants A_1 and A_2 are determined from specifying the initial conditions. Equation (6) shows that ω is the natural circular frequency of vibration of the beam. Introducing Eq. (6) into Eq. (4), one can reach:

$$EI\frac{d^4W}{dx^4} + \left[k(x) - \omega^2 \rho A\right] W = 0 \tag{7}$$

In the ensuing section, the solution of Eq. (7) is explained further in detail.

3 Method of solution

From the theory of ordinary differential equations, it turns out that point $x = 0$ is an ordinary point for Eq. (7), as long as the function $k(x)$ is analytic in a neighborhood of $x = 0$. The analyticity requires that the Taylor series of $k(x)$ about $x = 0$ exist and converge to $k(x)$. On the other hand, if $k(x)$ is not analytic at $x = 0$, but $x^4 k(x)$ is analytic in a neighborhood of $x = 0$, then point $x = 0$ is called a regular singular point for Eq. (7) (e.g., see [13,14]). It should be noted that the study of the solution can be confined to the neighborhood of $x = 0$, with no loss of generality. In fact, if it is desired to study the solution in the vicinity of a nonzero regular singular point, say $x = x_0 \neq 0$, the equation can be transformed into one for which the singular point is at the origin by a simple change of variable, $x - x_0 = z$.

The solution of Eq. (7) is closely associated with the behavior of function $k(x)$ in the neighborhood of $x = 0$, in the sense that the treatment of the problem differs for the two above-mentioned cases. Accordingly, this paper will first present the solution corresponding to the case when $x = 0$ is an ordinary point for Eq. (7), while the latter case will be treated afterward.

3.1 Solution near an ordinary point

In this case, one can assume a series solution of the form:

$$W(x) = \sum_{m=0}^{\infty} a_m x^m. \tag{8}$$

On substituting the series in Eq. (8) and its derivatives for W in Eq. (7), one can obtain:

$$\sum_{m=0}^{\infty} (m+4)(m+3)(m+2)(m+1)\, a_{m+4} x^m + \frac{k(x) - \omega^2 \rho A}{EI} \sum_{m=0}^{\infty} a_m x^m = 0. \tag{9}$$

In referring to the assumption that $k(x)$ is analytic at $x = 0$, the Taylor series of function $k(x)/EI$ about $x = 0$ exists. This Taylor series could be expressed within the following form:

$$\frac{k(x)}{EI} = \sum_{m=0}^{\infty} Q_m x^m \tag{10}$$

where Q_m's are the coefficients of the Taylor series. Substituting Eq. (10) into Eq. (9) yields:

$$\sum_{m=0}^{\infty} (m+4)(m+3)(m+2)(m+1)\, a_{m+4} x^m + \left(\sum_{m=0}^{\infty} Q_m x^m - \frac{\omega^2 \rho A}{EI}\right) \sum_{m=0}^{\infty} a_m x^m = 0. \tag{11}$$

Making use of the Cauchy product property, the product of the two series can be replaced with one series. In this regard, there is:

$$\left(\sum_{m=0}^{\infty} Q_m x^m\right)\left(\sum_{m=0}^{\infty} a_m x^m\right) = \sum_{m=0}^{\infty} g_m x^m \tag{12}$$

where

$$g_m = \sum_{k=0}^{m} a_k Q_{m-k}. \tag{13}$$

Introducing Eq. (12) to Eq. (11), one will get:

$$\sum_{m=0}^{\infty} (m+4)(m+3)(m+2)(m+1)\, a_{m+4} x^m + \sum_{m=0}^{\infty} g_m x^m - \frac{\omega^2 \rho A}{EI} \sum_{m=0}^{\infty} a_m x^m = 0. \tag{14}$$

Setting the coefficient of each power of x equal to zero and applying Eq. (13) gives:

$$(m+4)(m+3)(m+2)(m+1)\, a_{m+4} - \frac{\omega^2 \rho A}{EI} a_m + \sum_{k=0}^{m} a_k Q_{m-k} = 0. \tag{15}$$

The successive coefficients can be obtained one by one by writing the recurrence relation first for $m = 0$, then for $m = 1$, and so forth. By doing so, the coefficients a_m are evaluated in terms of four coefficients: a_0, a_1, a_2, and a_3. These four unknowns are determined by imposing the four boundary conditions that are required for solving Eq. (2). As far as the free vibration problem is concerned, the existence of a non-trivial solution for this system entails the determinant of its coefficients being equal to zero, whose solution yields the natural frequencies of the problem, ω.

3.2 Solution near a regular singular point

Here, the discussion is restricted primarily to the interval $x > 0$. The interval $x < 0$ can be treated by making the change of variable $x = -z$ and then solving the resulting equation for $z > 0$. According to the Frobenius theorem (e.g., see [13,14]), Eq. (7) has at least one solution that can be represented in the form of

$$W(x) = x^r \sum_{m=0}^{\infty} a_m x^m \tag{16}$$

where exponent r can be any real or complex number and r is chosen so that $a_0 \neq 0$. Moreover, Eq. (7) has solutions other than the solution represented in Eq. (16). The remaining solutions must be obtained in such a way as to make a set of four linearly independent solutions possible. Substituting Eq. (16) and its derivatives into Eq. (7) along with multiplying the result by x^4 gives:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)(m+r-2)(m+r-3)\, a_m x^{m+r} + x^4 \frac{k(x) - \omega^2 \rho A}{EI} \sum_{m=0}^{\infty} a_m x^{m+r} = 0. \tag{17}$$

Moreover, the Taylor series of function $x^4 k(x)/EI$ could be expressed in the following form:

$$\frac{x^4 k(x)}{EI} = \sum_{m=0}^{\infty} L_m x^m \tag{18}$$

where L_m's are the coefficients of the Taylor series. Substituting Eq. (18) into Eq. (17) yields:

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)(m+r-2)(m+r-3)\, a_m x^{m+r} + \left(\sum_{m=0}^{\infty} L_m x^m - \frac{\omega^2 \rho A}{EI} x^4\right) \sum_{m=0}^{\infty} a_m x^{m+r} = 0. \tag{19}$$

Similar to the previous section, use can be made of the Cauchy product property in order to replace the product of the two series with one series, leading to:

$$\left(\sum_{m=0}^{\infty} L_m x^m\right)\left(\sum_{m=0}^{\infty} a_m x^{m+r}\right) = x^r\left(\sum_{m=0}^{\infty} c_m x^m\right) = \sum_{m=0}^{\infty} c_m x^{m+r} \tag{20}$$

where

$$c_m = \sum_{k=0}^{m} a_k L_{m-k} = a_m L_0 + \sum_{k=0}^{m-1} a_k L_{m-k}. \tag{21}$$

For the sake of convenience, the function $F(r)$ will be defined as follows:

$$F(r) = r\,(r-1)\,(r-2)\,(r-3) + L_0. \tag{22}$$

Applying this definition to Eq. (19) gives:

$$\sum_{m=0}^{\infty} a_m \left[F\,(m+r) - L_0\right] x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r} - \frac{\omega^2 \rho A}{EI} \sum_{m=4}^{\infty} a_{m-4} x^{m+r} = 0. \tag{23}$$

Setting the coefficient of each power of x equal to zero gives (note that $a_0 \neq 0$):

$$F(r) = 0, \tag{24.1}$$
$$a_1 F(r+1) + a_0 L_1 = 0, \tag{24.2}$$
$$a_2 F(r+2) + a_0 L_2 + a_1 L_1 = 0, \tag{24.3}$$
$$a_3 F(r+3) + a_0 L_3 + a_1 L_2 + a_2 L_1 = 0, \tag{24.4}$$
$$a_m F(m+r) - \frac{\omega^2 \rho A}{EI} a_{m-4} + \sum_{k=0}^{m-1} a_k L_{m-k} = 0 \quad (m \geq 4)\,. \tag{24.5}$$

In deriving the results above, use is made of Eq. (21).

In view of Eq. (22), Eq. (24.1), which is referred to as the indicial equation, has four roots, namely $r_1,\ r_2,\ r_3$, and r_4. Regarding the Frobenius theory, if $\left(r_i - r_j\right) \notin \mathbb{Z}$ for $i \neq j$ $(i,\ j = 1, \ldots, 4)$, where $\mathbb{Z}$ denotes the set of integers, the four independent solutions of Eq. (7) have the following form:

$$W_i(x) = x^{r_i} \sum_{m=0}^{\infty} a_m^{(i)} x^m \quad (i = 1, \ldots, 4) \tag{25}$$

where superscript (i) in $a_m^{(i)}$ denotes the series coefficients corresponding to the ith solution, $W_i(x)$. Therefore, in this case, the homogeneous solution assumes the form as indicated in Eq. (26):

$$W(x) = C_1 x^{r_1} \sum_{m=0}^{\infty} a_m^{(1)} x^m + C_2 x^{r_2} \sum_{m=0}^{\infty} a_m^{(2)} x^m + C_3 x^{r_3} \sum_{m=0}^{\infty} a_m^{(3)} x^m + C_4 x^{r_4} \sum_{m=0}^{\infty} a_m^{(4)} x^m \tag{26}$$

where C_i's $(i = 1, \ldots, 4)$ are constant coefficients which must be determined by imposing the boundary conditions.

Nevertheless, according to the Frobenius theory, if $\left(r_i - r_j\right)$ is an integer for a couple $i \neq j$, for the solution corresponding to root r_j, i.e., $W_j(r)$, an additional logarithmic term must be added to the series $x^{r_j} \sum_{m=0}^{\infty} a_m^{(j)} x^m$. Otherwise, the two solutions cease to be independent. For $r_i = r_j$, this additional term is $W_i(x)\ln(x)$, while for $\left(r_i - r_j\right)$ being a positive integer, it is a constant multiple of $W_i(x)\ln(x)$, where the constant must be determined by substituting the solution into Eq. (7). However, this constant may turn out to be zero, in which case there is no logarithmic term in the solution.

If x_i is a root of multiplicity m, the other $(m-1)$ solutions are obtained successively by multiplying the preceding solution by $\ln(x)$ and adding the outcome to a power series, whose coefficients are to be determined by substituting the solution into Eq. (7). A discussion on the form of the solutions corresponding to the roots that differ by an integer can be found in the literature (e.g., [13,14]).

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer; hence, there are always two solutions of the form (25). If $r = \lambda + i\mu$, where $\lambda = Re(r)$ and $\mu = Im(r)$, then x^r is defined as:

$$x^{\lambda+i\mu} = x^{\lambda}\left[\cos\left(\mu \ln(x)\right) + i\sin\left(\mu \ln(x)\right)\right].$$

The solution obtained by this method is applicable to any type of boundary condition, that is simply supported, clamped, free, etc.

4 Solution of the forced vibration equation

Up until now, the discussion was devoted to the homogeneous solution of Eq. (2). Basically, the corresponding general solution could be achieved with relative ease, using modal analysis. With regard to Sect. 3, the eigenvalue problem of Eq. (7) has been solved for the natural frequencies and modes; hence, the displacement $w(x,t)$ in Eq. (2) can be given by a linear combination of modes:

$$w(x,t) = \sum_{n=1}^{\infty} W^{(n)}(x) T_n(t) \tag{27}$$

where $W^{(n)}(x)$ is the shape function corresponding to the nth mode and $T_n(t)$ is to be determined. Equation (27) indicates that the response has been expressed as the superposition of the contributions of the individual modes. Substituting Eq. (27) into Eq. (2) gives:

$$\sum_{n=1}^{\infty} \rho A W^{(n)}(x) \frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} + \sum_{n=1}^{\infty}\left[EI\frac{\mathrm{d}^4 W^{(n)}(x)}{\mathrm{d}x^4} + k(x)W^{(n)}(x)\right] T_n(t) = P(x,t). \tag{28}$$

Multiplying each term in Eq. (28) by $W^{(i)}(x)$, integrating it over the length of the beam, and interchanging the order of integration and summation yields:

$$\sum_{n=1}^{\infty} \frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2}\int_0^s \rho A W^{(i)}(x)W^{(n)}(x)\mathrm{d}x + \sum_{n=1}^{\infty} T_n(t)\int_0^s W^{(i)}(x)\left[EI\frac{\mathrm{d}^4 W^{(n)}(x)}{\mathrm{d}x^4} + k(x)W^{(n)}(x)\right]\mathrm{d}x$$
$$= \int_0^s W^{(i)}(x)P(x,t)\mathrm{d}x. \tag{29}$$

By virtue of the orthogonality properties of modes (for proof see “Appendix”), all terms in each of the summations on the left-hand side vanish except the one term for which $i = n$, leaving:

$$\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2}\int_0^s \rho A\left[W^{(n)}(x)\right]^2 \mathrm{d}x + T_n(t)\int_0^s W^{(n)}(x)\left[EI\frac{\mathrm{d}^4 W^{(n)}(x)}{\mathrm{d}x^4} + k(x)W^{(n)}(x)\right]\mathrm{d}x$$
$$= \int_0^s W^{(n)}(x)P\left(x,t\right)\mathrm{d}x. \tag{30}$$

This equation can be rewritten as:

$$M_n\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} + K_n T_n(t) = P_n(t) \tag{31}$$

where

$$M_n = \int_0^s \rho A\left[W^{(n)}(x)\right]^2 \mathrm{d}x, \tag{32}$$

$$K_n = \int_0^s W^{(n)}(x)\left[EI\frac{\mathrm{d}^4 W^{(n)}(x)}{\mathrm{d}x^4} + k(x)W^{(n)}(x)\right]\mathrm{d}x, \tag{33}$$

$$P_n(t) = \int_0^s W^{(n)}(x)P\left(x,t\right)\mathrm{d}x. \tag{34}$$

Moreover, dividing by M_n, Eq. (31) can be rewritten as follows (for proof see "Appendix"):

$$\frac{\mathrm{d}^2 T_n(t)}{\mathrm{d}t^2} + \omega_n^2 T_n(t) = \frac{P_n(t)}{M_n} \tag{35}$$

where ω_n is the natural frequency of vibration corresponding to the nth mode and is known from the homogeneous solution presented in the previous sections.

Thus, an infinite number of equations have been obtained, one for each mode and independent of the equations for all other modes, which can therefore be solved separately. Substituting the solution for $T_n(t)$ in Eq. (27), one will arrive at the general solution of Eq. (2).

5 Illustrative examples

5.1 Example 1

Consider a beam of length s, as shown in Fig. 2, with uniform flexural stiffness EI and uniform cross-sectional area A, resting on a foundation with modulus $k(x) = k_0 \left[1 - \beta \left(x^2/s^2\right)\right]$, where β is a constant and $\beta < 1$. Both ends of the beam are simply supported. The objective is to derive an expression for the deflection response of the beam due to the load distribution shown in Fig. 2. The load is uniformly distributed from $x = s/3$ to $x = s/2$, varying as $P(x, t) = p_0 \sin(\Omega t)$ within this interval. Moreover, the beam is initially at rest, that is $w(x, 0) = 0$ and $\partial w(x, 0)/\partial t = 0$.

To begin with, one should determine the natural vibration frequencies and modes of the system. Since $k(x)$ is a polynomial, it is analytic everywhere, so that $x = 0$ is an ordinary point for Eq. (7). The Taylor series of $k(x)/EI$ about $x = 0$ is given by:

$$\frac{k(x)}{EI} = \frac{k_0}{EI} - \frac{\beta}{s^2}\frac{k_0}{EI}x^2. \tag{36}$$

Hence, regarding Eq. (10), it turns out that $Q_0 = k_0/EI$, $Q_1 = 0$, $Q_2 = -\beta k_0/\left(EIs^2\right)$, and $Q_m = 0$ for $m > 2$. Assuming a series solution of the form presented in Eq. (8) and making use of the recurrence relation obtained in Eq. (15), one will get:

$$\begin{aligned}
&24a_4 - \frac{\omega^2 \rho A}{EI}a_0 + \frac{k_0}{EI}a_0 = 0,\\
&120a_5 - \frac{\omega^2 \rho A}{EI}a_1 + \frac{k_0}{EI}a_1 = 0,\\
&(m+4)(m+3)(m+2)(m+1)\,a_{m+4} - \frac{\omega^2 \rho A}{EI}a_m + \frac{k_0}{EI}a_m - \frac{\beta}{s^2}\frac{k_0}{EI}a_{m-2} = 0 \quad (m > 1),
\end{aligned}$$

or

$$\begin{aligned}
a_4 &= \frac{\left(\omega^2 \rho A - k_0\right)}{24EI}a_0,\ a_5 = \frac{\left(\omega^2 \rho A - k_0\right)}{120EI}a_1,\\
a_6 &= \frac{1}{360EI}\left[\left(\omega^2 \rho A - k_0\right)a_2 + \frac{\beta}{s^2}k_0 a_0\right],
\end{aligned}$$

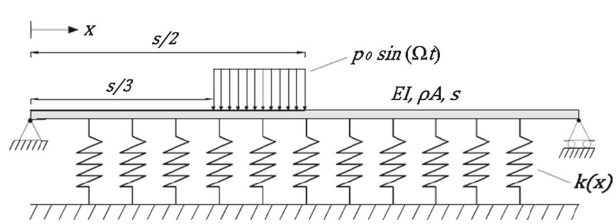

Fig. 2 Example 1: simply supported beam on variable Winkler elastic foundation

$$a_7 = \frac{1}{840EI}\left[\left(\omega^2\rho A - k_0\right)a_3 + \frac{\beta}{s^2}k_0 a_1\right],$$

$$a_8 = \frac{1}{1680EI}\left[\left(\omega^2\rho A - k_0\right)a_4 + \frac{\beta}{s^2}k_0 a_2\right] = \frac{1}{1680EI}\left[\frac{\left(\omega^2\rho A - k_0\right)^2}{24EI}a_0 + \frac{\beta}{s^2}k_0 a_2\right],$$

$$a_9 = \frac{\left(\omega^2\rho A - k_0\right)a_5 + \frac{\beta}{s^2}k_0 a_3}{3024EI} = \frac{1}{3024EI}\left[\frac{\left(\omega^2\rho A - k_0\right)^2}{120EI}a_1 + \frac{\beta}{s^2}k_0 a_3\right],$$

and so forth. The above results show that each coefficient a_m can be expressed in terms of the four coefficients: a_0, a_1, a_2, and a_3. Therefore, substituting a_m's into Eq. (8), the general solution is obtained as follows:

$$\begin{aligned} W(x) = a_0&\left[1 + \frac{\left(\omega^2\rho A - k_0\right)}{24EI}x^4 + \frac{\beta k_0}{360EIs^2}x^6 + \frac{\left(\omega^2\rho A - k_0\right)^2}{40320\,(EI)^2}x^8 + \cdots\right] \\ &+ a_1\left[x + \frac{\left(\omega^2\rho A - k_0\right)}{120EI}x^5 + \frac{\beta k_0}{840EIs^2}x^7 + \frac{\left(\omega^2\rho A - k_0\right)^2}{362880\,(EI)^2}x^9 + \cdots\right] \\ &+ a_2\left[x^2 + \frac{\left(\omega^2\rho A - k_0\right)}{360EI}x^6 + \frac{\beta k_0}{1680EIs^2}x^8 + \cdots\right] \\ &+ a_3\left[x^3 + \frac{\left(\omega^2\rho A - k_0\right)}{840EI}x^7 + \frac{\beta k_0}{3024EIs^2}x^9 + \cdots\right] \end{aligned} \tag{37}$$

or more compactly:

$$W(x) = a_0 W_0(x) + a_1 W_1(x) + a_2 W_2(x) + a_3 W_3(x) \tag{38}$$

where

$$W_0(x) = 1 + \frac{\left(\omega^2\rho A - k_0\right)}{24EI}x^4 + \frac{\beta k_0}{360EIs^2}x^6 + \frac{\left(\omega^2\rho A - k_0\right)^2}{40320\,(EI)^2}x^8 + \cdots, \tag{39}$$

$$W_1(x) = x + \frac{\left(\omega^2\rho A - k_0\right)}{120EI}x^5 + \frac{\beta k_0}{840EIs^2}x^7 + \frac{\left(\omega^2\rho A - k_0\right)^2}{362880\,(EI)^2}x^9 + \cdots, \tag{40}$$

$$W_2(x) = x^2 + \frac{\left(\omega^2\rho A - k_0\right)}{360EI}x^6 + \frac{\beta k_0}{1680EIs^2}x^8 + \cdots, \tag{41}$$

$$W_3(x) = x^3 + \frac{\left(\omega^2\rho A - k_0\right)}{840EI}x^7 + \frac{\beta k_0}{3024EIs^2}x^9 + \cdots \tag{42}$$

Moreover, the simply supported ends impose the following boundary conditions on the general solution:

$$W(0) = 0,\ W''(0) = 0,\ W(s) = 0,\ W''(s) = 0 \tag{43}$$

where the prime is used to indicate a differentiation with respect to x. Furthermore, Eq. (43) is a system of four equations with four unknowns, namely a_0, a_1, a_2, and a_3. The existence of a non-trivial solution for this system entails the determinant of its coefficients being equal to zero; therefore, the frequency equation becomes as follows:

$$\det\begin{bmatrix} W_0(0) & W_1(0) & W_2(0) & W_3(0) \\ W_0''(0) & W_1''(0) & W_2''(0) & W_3''(0) \\ W_0(s) & W_1(s) & W_2(s) & W_3(s) \\ W_0''(s) & W_1''(s) & W_2''(s) & W_3''(s) \end{bmatrix} = 0. \tag{44}$$

Table 1 Values of the frequency parameter for the first five modes in example 1 ($K_0 = 500$)

Reference	β	λ_1	λ_2	λ_3	λ_4	λ_5
Present study	0.2	4.8838	6.7096	9.5613	12.6248	15.7380
	0.6	4.7535	6.6570	9.5426	12.6166	15.7337
Ding [12]	0.2	4.884	6.710	9.562	12.624	15.737
	0.6	4.753	6.657	9.542	12.616	15.733
Finite element method	0.2	4.884	6.711	9.561	12.617	15.749

Considering Eqs. (39)–(42), the above determinant simplifies to

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ W_0(s) & W_1(s) & W_2(s) & W_3(s) \\ W_0''(s) & W_1''(s) & W_2''(s) & W_3''(s) \end{bmatrix} = 0 \tag{45}$$

which reduces to

$$W_1(s)W_3''(s) - W_1''(s)W_3(s) = 0. \tag{46}$$

This equation solves to give the natural frequencies of vibration of the beam.

Ding [12], by applying a numerical method, has solved this problem for various values of β and $K_0 = k_0 s^4/EI$. In Table 1, the values of the non-dimensional frequency parameter, $\lambda = s\sqrt[4]{\omega^2\rho A/EI}$, are shown for $K_0 = 500$ and $\beta = 0.2$ along with 0.6 for the first five modes, and the results are compared with those of Ding [12]. To this end, as it can be observed, excellent agreement is achieved between the two sets of results.

The general solution to the imposed load can now be readily obtained, using the procedure in Sect. 4. The loading function is given by:

$$P(x,t) = \begin{cases} p_0 \sin(\Omega t) & s/3 \le x \le s/2 \\ 0 & 0 \le x < s/3 \quad \text{and} \quad s/2 < x \le s \end{cases}, \tag{47}$$

Applying Eq. (34), one will get:

$$P_n(t) = p_0^{(n)} \sin(\Omega t) \tag{48}$$

where

$$p_0^{(n)} = \int_{s/3}^{s/2} p_0 W^{(n)}(x)\mathrm{d}x \tag{49}$$

and

$$W^{(n)}(x) = a_0 W_0^{(n)}(x) + a_1 W_1^{(n)}(x) + a_2 W_2^{(n)}(x) + a_3 W_3^{(n)}(x) \tag{50}$$

where $W_i^{(n)}(x)$'s ($i = 0, 1, 2, 3$) are obtained from Eqs. (39)–(42) by substituting $\omega = \omega_n$. Imposing the boundary conditions in Eq. (43) on $W^{(n)}(x)$ gives:

$$\begin{aligned} a_0 &= a_2 = 0, \\ a_3 &= -\frac{W_1^{(n)}(s)}{W_3^{(n)}(s)} a_1. \end{aligned} \tag{51}$$

The value of a_1 can be arbitrarily chosen as being equal to unity. Therefore, the mode shapes could be represented in the following form:

$$W^{(n)}(x) = W_1^{(n)}(x) - \frac{W_1^{(n)}(s)}{W_3^{(n)}(s)} W_3^{(n)}(x). \tag{52}$$

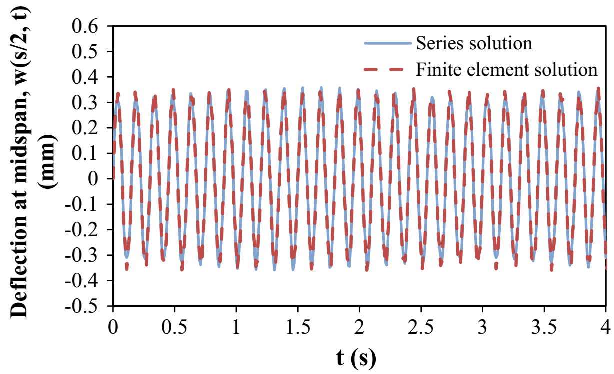

Fig. 3 Example 1: Deflection at midspan as a function of time

Apart from this, considering Eq. (48) as well as the at-rest initial conditions, the solution of Eq. (35) will be:

$$T_n(t) = \frac{p_0^{(n)}}{M_n} \frac{1}{\omega_n^2 - \Omega^2} \left[\sin(\Omega t) - \frac{\Omega}{\omega_n} \sin(\omega_n t) \right]. \tag{53}$$

On substituting Eqs. (52) and (53) into Eq. (27), one will arrive at the expression sought for the deflection response of the beam due to the applied loading:

$$w(x,t) = \sum_{n=1}^{\infty} \frac{p_0^{(n)}}{M_n} \frac{1}{\omega_n^2 - \Omega^2} \left[W_1^{(n)}(x) - \frac{W_1^{(n)}(s)}{W_3^{(n)}(s)} W_3^{(n)}(x) \right] \left[\sin(\Omega t) - \frac{\Omega}{\omega_n} \sin(\omega_n t) \right]. \tag{54}$$

Figure 3 represents the deflection at midspan ($x = s/2$) as a function of time for the following numerical values:

$$s = 4.0\,\text{m},\ EI = 1.8 \times 10^6\,\text{N}\,\text{m}^2,\ \rho A = 15.7\,\text{kg/m},$$
$$K_0 = 500,\ \beta = 0.2,$$
$$p_0 = 4000\,\text{N},\ \Omega = 400\,\text{rpm}.$$

The response is plotted up to $t = 4.0$ s. It is noteworthy to mention that the solution converges by retaining only the first seven modes, and the effect of higher modes becomes less pronounced.

For comparison, the foregoing problem is solved once again by applying a finite element analysis. The dashed curve in Fig. 3 demonstrates the deflection at midspan resulted from the finite element analysis. Basically, as it is discernible in the figure, the solid curve, which was obtained by the series solution, agrees closely with the finite element result insofar as the two sets of results are indistinguishable. Furthermore, the maximum absolute deflection from the exact series solution is $w_{\max} = 0.35867$ mm and from the finite element solution is $w_{\max} = 0.35878$ mm, which are very close to each other.

The finite element solution is also employed in order to obtain the natural vibration frequencies of the beam, and the non-dimensional results are reported in Table 1. Making comparison with the values obtained from the series solution, the agreement between the two sets of results is clear.

5.2 Example 2

The properties of the beam to be investigated herein are shown in Fig. 4. The variation of the foundation modulus along the beam is $k(x) = k_0 s/(x + 0.5s)$. Moreover, both ends of the beam are supposed to be supported by linear translational and rotational point springs, of which the stiffness coefficients are as shown in Fig. 4. The objective is to derive the frequency equation corresponding to the free vibration of the system.

It is clear that function $k(x)$ is not analytic at point $x = -0.5s$. Making use of the change of variable $z = x + 0.5s$, the singular point for Eq. (7) is transformed into point $z = 0$. Introducing this change of variable into Eq. (7) yields:

$$EI \frac{\mathrm{d}^4 W}{\mathrm{d}z^4} + \left[\kappa(z) - \omega^2 \rho A\right] W = 0 \tag{55}$$

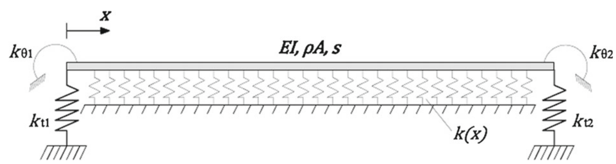

Fig. 4 Example 2: Beam on variable Winkler elastic foundation supported by linear translational and rotational springs at both ends

where $\kappa(z) = k_0 s/z$. It should be noted that if one decides to directly write the Taylor series of $k(x)$ about $x = 0$ and apply a similar treatment as accomplished for example 1, the resulting series solutions will not be convergent at all the points lying in the interval $[0, s]$. In principle, it can be shown that, for this example, the radius of convergence of the series solutions obtained in this particular way is equal to the distance from $x = 0$ to the nearest singular point of Eq. (7), which is $x = -0.5s$. Thus, the radius of convergence will be equal to $0.5s$. Therefore, the solution will be correct only in the interval $[0, 0.5s]$ and does not cover the remaining interval $[0.5s, s]$.

The above discussion clarifies the importance of taking into consideration the point at which the equation is not analytic, i.e., $x = -0.5s$ or, equivalently, $z = 0$. This will guarantee the convergence of the solutions within the whole domain of the problem. Moreover, since $z^4\kappa(z)$ is analytic at the origin, point $z = 0$ is a regular singular point for Eq. (55). The Taylor series of $z^4\kappa(z)/EI$ about $z = 0$ is:

$$\frac{z^4\kappa(z)}{EI} = \frac{k_0 s}{EI} z^3. \tag{56}$$

Therefore, by considering Eq. (18) and replacing x with z, it turns out that $L_3 = k_0 s/EI$, and $L_m = 0$ for $m \neq 3$. Substituting $L_0 = 0$ into Eq. (22)—the indicial equation—gives:

$$F(r) = r(r-1)(r-2)(r-3) = 0. \tag{57}$$

The roots of this equation are $r_1 = 3$, $r_2 = 2$, $r_3 = 1$, and $r_4 = 0$. Accordingly, the complementary solutions take the following forms:

$$W_1 = z^3 \sum_{m=0}^{\infty} a_m z^m, \tag{58}$$

$$W_2 = \alpha W_1 \ln(z) + z^2 \sum_{m=0}^{\infty} b_m z^m, \tag{59}$$

$$W_3 = \beta W_2 \ln(z) + \gamma W_1 \ln(z) + z \sum_{m=0}^{\infty} c_m z^m, \tag{60}$$

$$W_4 = \mu W_3 \ln(z) + \xi W_2 \ln(z) + \theta W_1 \ln(z) + \sum_{m=0}^{\infty} d_m z^m \tag{61}$$

where α, β, γ, μ, ξ, and θ are unknown constants. The unknown coefficients in Eqs. (58) through (61) can be readily determined by substituting these solutions into Eq. (55). For W_1, an alternate way is to apply the recursive formulae expressed by Eqs. (24.1–5). On substituting $r = r_1 = 3$ into these equations, one will get

$$\begin{aligned} a_1 &= a_2 = 0,\ a_3 = \frac{-k_0 s}{360EI} a_0, \\ a_m &= \frac{\omega^2 \rho A a_{m-4} - k_0 s a_{m-3}}{EI\ (m+3)(m+2)(m+1)m}\ (m \geq 4). \end{aligned} \tag{62}$$

a_0 can be arbitrarily chosen as being equal to 1 (note that, as referred in the foregoing section, a_0 is a nonzero coefficient; thus, the arbitrary value assigned to a_0 must be nonzero). For W_2, in addition to the series coefficients, there is an unknown coefficient, α, that must be determined. Substitution of W_2 into Eq. (55) gives:

$$\alpha z^4 \ln(z)\left[\frac{\mathrm{d}^4 W_1}{\mathrm{d}z^4}+\frac{\kappa(z)-\omega^2\rho A}{EI}W_1\right]+4\alpha z^3\frac{\mathrm{d}^3 W_1}{\mathrm{d}z^3}-6\alpha z^2\frac{\mathrm{d}^2 W_1}{\mathrm{d}z^2}+8\alpha z\frac{\mathrm{d}W_1}{\mathrm{d}z}-6\alpha W_1$$
$$+z^4\frac{\kappa(z)-\omega^2\rho A}{EI}\sum_{m=0}^{\infty} b_m z^{m+2}+\sum_{m=0}^{\infty} b_m m(m-1)(m+1)(m+2)z^{m+2}=0. \tag{63}$$

The expression within the brackets is equal to zero, since W_1 is a solution for Eq. (55). Using Eq. (58) and substituting the Taylor series of $z^4\kappa(z)/EI$, the above equation becomes:

$$2\alpha\sum_{m=1}^{\infty} a_{m-1}(2m+1)\left(m^2+m-1\right)z^{m+2}+\frac{k_0 s}{EI}\sum_{m=3}^{\infty} b_{m-3}z^{m+2}-\frac{\omega^2\rho A}{EI}\sum_{m=4}^{\infty} b_{m-4}z^{m+2}$$
$$+\sum_{m=0}^{\infty} b_m m(m-1)(m+1)(m+2)z^{m+2}=0. \tag{64}$$

Rearranging the terms in the ascending order of powers of z yields

$$(6\alpha a_0)z^3+(50\alpha a_1+24b_2)z^4+\left(154\alpha a_2+\frac{k_0 s}{EI}b_0+120b_3\right)z^5$$
$$+\sum_{m=4}^{\infty}\Bigg[2\alpha a_{m-1}(2m+1)\left(m^2+m-1\right)+\frac{k_0 s}{EI}b_{m-3}-\frac{\omega^2\rho A}{EI}b_{m-4}$$
$$+b_m m(m-1)(m+1)(m+2)\Bigg]z^{m+2}=0. \tag{65}$$

In order for this equation to be satisfied for all z, the coefficient of each power of z must vanish independently; for that reason:

$$\alpha=b_2=0,\quad b_3=\frac{-k_0 s}{120EI}b_0,$$
$$b_m=\frac{\omega^2\rho A b_{m-4}-k_0 s b_{m-3}}{EI\, m(m-1)(m+1)(m+2)}\ (m\ge 4). \tag{66}$$

b_0 and b_1 are arbitrary and b_0 must be a nonzero value. Let $b_0=1$ and $b_1=0$. Since coefficient α is turned out to be zero, W_2 does not contain any logarithmic terms.

By substituting W_3 and then W_4 into Eq. (55), c_m's and d_m's can be obtained in a similar manner, the results of which will be

$$\beta=\gamma=0,\quad c_3=\frac{-k_0 s}{24EI},$$
$$c_m=\frac{\omega^2\rho A c_{m-4}-k_0 s c_{m-3}}{EI\, m(m-2)(m-1)(m+1)}\ (m\ge 4) \tag{67}$$

and

$$\mu=\xi=0,\quad \theta=\frac{-k_0 s}{6EI},$$
$$d_m=\frac{3\omega^2\rho A d_{m-4}-3k_0 s d_{m-3}+k_0 s a_{m-3}(2m-3)\left(m^2-3m+1\right)}{3\,EI\, m(m-3)(m-2)(m-1)}\ (m\ge 4) \tag{68}$$

where the values that are assigned to c_0, c_1, c_2, d_0, d_1, d_2, and d_3 were arbitrarily chosen (with regard to the fact that c_0 and d_0 must be nonzero). In substituting the recursive relations acquired above for a_m, b_m, c_m, and d_m in Eqs. (58)–(61), the general solution will be acquired: $W(z)=\sum_{i=1}^{4} C_i W_i$.

For the unknown coefficients C_1, C_2, C_3, and C_4 to be determined, the general solution must satisfy the boundary conditions. The elastically supported ends (at $z=0.5s$ and $z=1.5s$) impose the following boundary conditions on the general solution:

$$k_{t1}W(0.5s)+EIW'''(0.5s)=0,\quad k_{\theta 1}W'(0.5s)-EIW''(0.5s)=0,$$
$$k_{t2}W(1.5s)+EIW'''(1.5s)=0,\quad k_{\theta 2}W'(1.5s)-EIW''(1.5s)=0 \tag{69}$$

where the prime is used to indicate a differentiation with respect to z. Furthermore, Eq. (69) is a system of four equations with four unknowns, namely C_1, C_2, C_3, and C_4. The existence of a non-trivial solution for this system entails the determinant of its coefficients being equal to zero; therefore, the frequency equation becomes as follows:

$$\det\begin{bmatrix} k_{t1}W_1(0.5s)+EIW_1'''(0.5s) & k_{t1}W_2(0.5s)+EIW_2'''(0.5s) & k_{t1}W_3(0.5s)+EIW_3'''(0.5s) & k_{t1}W_4(0.5s)+EIW_4'''(0.5s) \\ k_{\theta 1}W_1'(0.5s)-EIW_1''(0.5s) & k_{\theta 1}W_2'(0.5s)-EIW_2''(0.5s) & k_{\theta 1}W_3'(0.5s)-EIW_3''(0.5s) & k_{\theta 1}W_4'(0.5s)-EIW_4''(0.5s) \\ k_{t2}W_1(1.5s)-EIW_1'''(1.5s) & k_{t2}W_2(1.5s)-EIW_2'''(1.5s) & k_{t2}W_3(1.5s)-EIW_3'''(1.5s) & k_{t2}W_4(1.5s)-EIW_4'''(1.5s) \\ k_{\theta 2}W_1'(1.5s)+EIW_1''(1.5s) & k_{\theta 2}W_2'(1.5s)+EIW_2''(1.5s) & k_{\theta 2}W_3'(1.5s)+EIW_3''(1.5s) & k_{\theta 2}W_4'(1.5s)+EIW_4''(1.5s) \end{bmatrix}=0. \tag{70}$$

The roots of this equation are the natural frequencies of vibration of the system.

6 Extension to Pasternak-type foundation

The linear elastic Winker model explained by Eq. (1) may not be accurate in a number of practical situations. Accordingly, several alternate foundation models have been suggested to achieve some degree of accuracy. The Pasternak foundation model that represents an extension of the Winkler foundation accounting for the effect of in-plane shear may be accurate in certain applications [15]. According to this model, the response of the foundation is given by Eq. (71):

$$q = kw - G\frac{\partial^2 w}{\partial x^2} \tag{71}$$

where both k and G are foundation constants. Consider the problem of an Euler–Bernoulli beam with constant flexural stiffness EI resting on a Pasternak-type elastic foundation, involving a spring layer of variable modulus $k(x)$ and a shear layer of constant modulus G. Moreover, the beam is subjected to a load $P(x,t)$. Assuming that the beam maintains continuous contact with the base, the governing differential equation of this problem can be expressed as

$$EI\frac{\partial^4 w}{\partial x^4} - G\frac{\partial^2 w}{\partial x^2} + k(x)w + \rho A\frac{\partial^2 w}{\partial t^2} = P(x,t) \tag{72}$$

where $w(x,t)$ is the deflection of the beam, ρ is the mass density of the material, and A is the cross-sectional area of the beam. In order to solve Eq. (72), similar to the procedure explained for Eq. (2), one should begin with the solution of the corresponding homogeneous equation, that is, when $P(x,t)=0$. To this aim, one can assume a solution as

$$w(x,t) = W(x)T(t). \tag{73}$$

By pursuing a similar procedure as discussed for Eq. (2), it turns out that $T(t)$ satisfies Eq. (6), and $W(x)$ is the solution of Eq. (74):

$$EI\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - G\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + \left[k(x) - \omega^2\rho A\right]W = 0 \tag{74}$$

which is reduced to Eq. (7) when G is set equal to zero. If $k(x)$ is analytic, the solution of the above equation will be as indicated in Eq. (8). The coefficients a_m are obtained in a manner similar to that discussed in Sect. 3.1, where they appear to be satisfying the following recurrence relation:

$$(m+4)(m+3)(m+2)(m+1)\,a_{m+4} - \frac{G}{EI}(m+1)(m+2)\,a_{m+2} - \frac{\omega^2\rho A}{EI}a_m + \sum_{k=0}^{m} a_k Q_{m-k} = 0, \tag{75}$$

which is reduced to Eq. (15) when G is set equal to zero. On the other hand, if $k(x)$ is not analytic, Eq. (74) has at least one solution in the form represented in Eq. (16). The indicial equation corresponding to Eq. (74) can be derived in a similar manner as performed for Eq. (7), leading to

$$F(r) = 0, \tag{76.1}$$

$$a_1 F(r+1) + a_0 L_1 = 0, \tag{76.2}$$

$$a_2 F(r+2) - \frac{G}{EI} r(r-1) a_0 + a_0 L_2 + a_1 L_1 = 0, \tag{76.3}$$

$$a_3 F(r+3) - \frac{G}{EI} r(r+1) a_1 + a_0 L_3 + a_1 L_2 + a_2 L_1 = 0, \tag{76.4}$$

$$a_m F(m+r) - \frac{G}{EI}(m+r-2)(m+r-3) a_{m-2} - \frac{\omega^2 \rho A}{EI} a_{m-4} + \sum_{k=0}^{m-1} a_k L_{m-k} = 0 \ (m > 3) \tag{76.5}$$

where all the parameters in Eq. (76) are previously defined in Sect. 3.2. Referring to Sect. 3.2, Eq. (76.1) is the indicial equation of the governing differential equation (74), where $F(r)$ is defined in Eq. (22). Solving the indicial equation, the general solution of Eq. (74) can be obtained in a similar particular way as discussed for Eq. (7) in the foregoing sections.

Having solved the homogeneous equation, the general solution of Eq. (72) can be derived by applying the same procedure as discussed for Eq. (2) in Sect. 4.

7 Summary and conclusions

In the present study, an exact solution was set forth for the vibration of an Euler–Bernoulli beam supported by a variable-modulus Winkler foundation. The method of solution was based on the power series solution of ordinary differential equations. Two different cases were examined to treat the problem: the case wherein the variation of the modulus is an analytic function of x, and the case in which the variation is not analytic, but the singularity is of a regular type. The latter case was tackled by applying the Frobenius theorem. Moreover, it was shown that the proposed method could be applied for any type of classical boundary condition (simply supported, clamped, free, etc.) with no restriction. At the same time, several illustrative examples were presented to show the procedure in more detail. The results, where possible, were verified upon comparison with available values in the literature as well as the finite element solution, and excellent agreement was obtained. Finally, it was demonstrated that the proposed method could be readily extended to solve the more general problem of the dynamic response of beams resting on a Pasternak-type foundation with a spring layer of variable modulus.

While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence, power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if one can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, one is likely to need a power series or some equivalent expression if they want to evaluate those functions numerically or to plot their graphs.

Appendix

In order to prove the orthogonality property of the modes, one can substitute the nth mode shape, $W^{(n)}(x)$ into Eq. (7), giving:

$$EI \frac{\mathrm{d}^4 W^{(n)}}{\mathrm{d}x^4} + k(x) W^{(n)}(x) = \omega_n^2 \rho A W^{(n)}(x). \tag{77}$$

Multiplying both sides by $W^{(i)}(x)$ and integrating from 0 to s results in:

$$EI \int_0^s W^{(i)}(x) \frac{\mathrm{d}^4 W^{(n)}}{\mathrm{d}x^4} \mathrm{d}x + \int_0^s k(x) W^{(i)}(x) W^{(n)}(x) \mathrm{d}x = \rho A \omega_n^2 \int_0^s W^{(i)}(x) W^{(n)}(x) \mathrm{d}x. \tag{78}$$

The first integral on the left-hand side of this equation is integrated by parts; applying this procedure twice leads to:

$$\int_0^s W^{(i)}(x)\frac{\mathrm{d}^4W^{(n)}}{\mathrm{d}x^4}\,\mathrm{d}x = \left[W^{(i)}\frac{\mathrm{d}^3W^{(n)}}{\mathrm{d}x^3}\right]_0^s - \left[\frac{\mathrm{d}W^{(i)}}{\mathrm{d}x}\frac{\mathrm{d}^2W^{(n)}}{\mathrm{d}x^2}\right]_0^s + \int_0^s \frac{\mathrm{d}^2W^{(i)}}{\mathrm{d}x^2}\frac{\mathrm{d}^2W^{(n)}}{\mathrm{d}x^2}\,\mathrm{d}x. \tag{79}$$

It is easy to see that the quantities enclosed in the brackets are zero at $x = 0$ and $x = s$ if the ends of the beam are free, simply supported, clamped, or sliding. For example, this is true at a simply supported end, because $W = 0$ and the bending moment is zero (i.e., $W'' = 0$). In general, if the ends of the beam are elastically supported, as in Example 2, the quantities in the brackets resolve into the ones having the same order of differentiation with respect to $W^{(i)}$ and $W^{(n)}$.

For example, in regard to Fig. 4, one could write $W'''(0) = -k_{t1}W(0)/EI$, $W'''(s) = k_{t2}W(s)/EI$, $W''(0) = k_{\theta 1}W'(0)/EI$, and $W''(s) = -k_{\theta 2}W'(s)/EI$. Therefore, Eq. (79) becomes:

$$\int_0^s W^{(i)}(x)\frac{\mathrm{d}^4W^{(n)}}{\mathrm{d}x^4}\,\mathrm{d}x = \frac{k_{t2}}{EI}W^{(i)}(s)W^{(n)}(s) + \frac{k_{t1}}{EI}W^{(i)}(0)\,W^{(n)}(0) + \frac{k_{\theta 2}}{EI}\frac{\mathrm{d}W^{(i)}(s)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(s)}{\mathrm{d}x}$$
$$+\frac{k_{\theta 1}}{EI}\frac{\mathrm{d}W^{(i)}(0)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(0)}{\mathrm{d}x} + \int_0^s \frac{\mathrm{d}^2W^{(i)}}{\mathrm{d}x^2}\frac{\mathrm{d}^2W^{(n)}}{\mathrm{d}x^2}\,\mathrm{d}x. \tag{80}$$

Substituting this equation into Eq. (78) yields:

$$k_{t2}W^{(i)}(s)W^{(n)}(s) + k_{t1}W^{(i)}(0)\,W^{(n)}(0) + k_{\theta 2}\frac{\mathrm{d}W^{(i)}(s)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(s)}{\mathrm{d}x} + k_{\theta 1}\frac{\mathrm{d}W^{(i)}(0)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(0)}{\mathrm{d}x}$$
$$+EI\int_0^s \frac{\mathrm{d}^2W^{(i)}}{\mathrm{d}x^2}\frac{\mathrm{d}^2W^{(n)}}{\mathrm{d}x^2}\,\mathrm{d}x + \int_0^s k(x)W^{(i)}(x)W^{(n)}(x)\mathrm{d}x = \rho A\omega_n^2\int_0^s W^{(i)}(x)W^{(n)}(x)\mathrm{d}x. \tag{81}$$

Similarly, starting with Eq. (77) written for the ith mode, multiplying both sides by $W^{(n)}(x)$, integrating from 0 to s, and using integration by parts twice leads to:

$$k_{t2}W^{(i)}(s)W^{(n)}(s) + k_{t1}W^{(i)}(0)\,W^{(n)}(0) + k_{\theta 2}\frac{\mathrm{d}W^{(i)}(s)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(s)}{\mathrm{d}x} + k_{\theta 1}\frac{\mathrm{d}W^{(i)}(0)}{\mathrm{d}x}\frac{\mathrm{d}W^{(n)}(0)}{\mathrm{d}x}$$
$$+EI\int_0^s \frac{\mathrm{d}^2W^{(i)}}{\mathrm{d}x^2}\frac{\mathrm{d}^2W^{(n)}}{\mathrm{d}x^2}\,\mathrm{d}x + \int_0^s k(x)W^{(i)}(x)W^{(n)}(x)\mathrm{d}x = \rho A\omega_i^2\int_0^s W^{(i)}(x)W^{(n)}(x)\mathrm{d}x. \tag{82}$$

Subtracting Eq. (81) from Eq. (82) gives (note that $\rho A \neq 0$):

$$\left(\omega_i^2 - \omega_n^2\right)\int_0^s W^{(i)}(x)W^{(n)}(x)\mathrm{d}x = 0. \tag{83}$$

Therefore, if $\omega_i \neq \omega_n$,

$$\int_0^s W^{(i)}(x)W^{(n)}(x)\mathrm{d}x = 0, \tag{84}$$

and this substituted into Eq. (78) leads to

$$\int_0^s EIW^{(i)}(x)\frac{\mathrm{d}^4W^{(n)}}{\mathrm{d}x^4}\,\mathrm{d}x + \int_0^s k(x)W^{(i)}(x)W^{(n)}(x)\mathrm{d}x = 0. \tag{85}$$

Equations (84) and (85) are the orthogonality relations for the natural vibration modes. Moreover, if one performs a summation on Eq. (78) over all modes, the result will be:

$$\sum_{i=1}^{\infty}\left\{\int_0^s EIW^{(i)}(x)\frac{\mathrm{d}^4W^{(n)}}{\mathrm{d}x^4}\,\mathrm{d}x + \int_0^s k(x)W^{(i)}(x)W^{(n)}(x)\mathrm{d}x\right\} = \rho A\omega_n^2\sum_{i=1}^{\infty}\int_0^s W^{(i)}(x)W^{(n)}(x)\mathrm{d}x. \tag{86}$$

Applying the orthogonality property, this equation reduces to:

$$\int_0^s W^{(n)}(x)\left[EI\frac{\mathrm{d}^4W^{(n)}(x)}{\mathrm{d}x^4} + k(x)W^{(n)}(x)\right]\mathrm{d}x = \omega_n^2\int_0^s \rho A\left[W^{(n)}(x)\right]^2\mathrm{d}x. \tag{87}$$

In referring to the definitions presented in Eqs. (32) and (33), Eq. (87) may be written as:

$$K_n = \omega_n^2 M_n. \tag{88}$$

This equation verifies the derivation of Eq. (35) from Eq. (31).

References

1. Winkler, E.: Die Lehre von der Elasticitaet und Festigkeit. Prague H. Dominicus **141**, 182–184 (1867)
2. Hetényi, M.: Beams on Elastic Foundation: Theory with Applications in the Fields of Civil and Mechanical Engineering. The University of Michigan Press, Ann Arbor (1946)
3. Dey, A., Basudhar, P.K., Chandra, S.: Distribution of subgrade modulus beneath beams on reinforced elastic foundations. Indian Geotech. J. **41**, 54–63 (2011)
4. Klepikov, S.N.: Calculation of beams on an elastic foundation with a variable modulus of subgrade reaction. Soil Mech. Found. Eng. **2**, 296–299 (1965)
5. Luo, Z., Fu, Z.: Analysis of elastic slab on non-uniform support foundation using a weighted-residual method. In: ICCTP: Critical Issues in Transportation Systems Planning, Development, and Management, pp. 1–7. ASCE, Reston, VA (2009)
6. ASCE Standard, ASCE/SEI 41-13, Seismic Evaluation and Retrofit of Existing Buildings, American Society of Civil Engineers, Reston, VA (2013)
7. Franklin, J.N., Scott, R.F.: Beam equation with variable foundation coefficient. Eng. Mech. Div. ASCE **105**, 811–827 (1979)
8. Lentini, M.: Numerical solution of the beam equation with nonuniform foundation coefficient. J. Appl. Mech. ASME **46**, 90–904 (1979)
9. Clastomik, J., Eisenberger, M., Yankelevsky, D.Z., Adin, M.A.: Beams on variable Winkler elastic foundation. J. Appl. Mech. ASME **53**, 925–928 (1986)
10. Eisenberger, M., Clastornik, J.: Vibrations and buckling of a beam on variable Winkler elastic foundation. J. Sound Vib. **115**, 233–241 (1987)
11. Eisenberger, M., Clastornik, J.: Beams on variable two-parameter elastic foundation. J. Eng. Mech. ASCE **113**, 1454–1466 (1987)
12. Ding, Z.: A general solution to vibrations of beams on variable Winkler elastic foundation. Comput. Struct. **47**, 83–90 (1993)
13. Bender, C.M., Orszag, A.: Advanced Mathematical Methods for Scientists and Engineers I. Mc-Graw Hill, New York (1978)
14. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)
15. Pasternak, P.L.: On a new method of analysis of an elastic foundation by means of two foundation constants. Gasudarstvennoe Izdatelstvo Literaturi po Stroitelstvui Arkhitekture, Moscow (1954) (**in Russian**)