Acta Mech 226, 4059–4072 (2015)
DOI 10.1007/s00707-015-1475-3

ORIGINAL PAPER

Lu Yuan · Zhi-Xiang Gu · Zheng-Nan Yin · Heng Xiao

New compressible hyper-elastic models for rubberlike materials

Received: 26 May 2015 / Revised: 31 August 2015 / Published online: 3 October 2015

Abstract Multi-axial hyper-elastic models for large strain rubberlike elasticity are established in a broad sense free of the commonly assumed constraint of incompressibility. Results are derived directly from uniaxial stress–strain relations by means of certain explicit procedures. Novelties in four respects are incorporated in the new models: (i) constitutive parameters of direct physical meanings may be introduced to represent features of rubberlike elasticity; (ii) the usual incompressibility constraint is no longer assumed and relevant issues may be rendered irrelevant; (iii) the incompressibility case may be derived as a natural limit; and (iv) accurate agreement with benchmark data for several deformation modes may be achieved, including uniaxial extension, equi-biaxial extension as well as plane-strain extension and others.

1 Introduction

Rubberlike materials are highly elastic and give rise to large recoverable deformations under relatively low stress levels. Because of their unique properties, such as high flexibility, abrasion resistance, insulation, and waterproof, rubberlike materials are widely used in everyday life and in various fields of engineering and technology. Constitutive models for rubberlike elasticity establish relationships between large elastic deformations and stress responses for rubberlike materials and play a central role in reasonable design and efficient applications of rubber products. The determination of such models has been a central topic both in the theory of finite elasticity and in related fields.

However, large strain rubberlike elasticity exhibits undue complexities arising from strong nonlinearities. It appears difficult to treat such complexities in a general case of large deformation. Usually, results are derived in an approximate or idealized sense of assuming the incompressibility constraint. In the past decades, two approaches, namely statistical approach and phenomenological approach, have been used to establish hyper-elastic constitutive models for rubberlike materials at large incompressible deformations. The former approach derives elastic potentials via averaging procedures based on certain approximations and idealizations concerning the network structures of long chainlike macromolecules. Representative samples of recent results may be found in Arruda and Boyce [2], Wu and van der Giessen [38], Fried [14], Zuniga and Beatty [54], Drozdov and Gottlieb [11], Ogden et al. [32], Zuniga [53], Beatty [4] and many others. On the other side, the latter approach is based on continuum mechanics and directly presents various nonlinear forms of the elastic potential in terms of either strain invariants or principal stretches, with a number of adjustable parameters to be determined. Representative results in this respect may be found in Ogden [27–29,31], Gent [15] and Beatty [3] for recent developments.

L. Yuan · Z.-X. Gu · Z.-N. Yin · H. Xiao (✉)
State Key Laboratory of Advanced Special Steels, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Yanchang Road 149, Shanghai 200072, China
E-mail: chencheng2xh@yahoo.de

Irrespective of the fact that incompressible hyper-elastic models provide good approximations in many cases, issues and anomalies may emerge in association with the incompressibility constraint. In particular, this may be the case in numerical implementations. In fact, particular treatments should be introduced to ensure the incompressibility condition for the purpose of bypassing possible related issues. However, such issues may be rendered irrelevant with models for general compressible deformations without assuming the incompressibility condition. This consideration in a broad sense is just in accord with realistic material behavior. It is noted that the incompressibility is not the reality but merely an approximation of realistic material behavior. Indeed, realistic rubberlike materials display various effects of compressibility, as shown in experimental studies earlier in, e.g., [16] and recently in, e.g., [1,9] and many others. Constitutive models for rubberlike elasticity with compressibility behavior were investigated earlier in, e.g., [7,10,12,13,30] and recently in [6,12,18–20,25,26] and many others.

In a most recent study [42], a new, explicit approach has been proposed to establish multi-axial hyper-elastic models for general compressible deformations. In this contribution, this new approach will be further developed, and new models will be proposed. Results will be derived directly from uniaxial stress–strain relations via certain explicit procedures based on Hencky's logarithmic strain. In contrast to usual models, novelties in four respects will be incorporated in the new models proposed, namely (i) constitutive parameters of direct physical meanings may be introduced to represent features of rubberlike elasticity; (ii) the usual incompressibility constraint is no longer assumed and relevant issues may be bypassed; (iii) the incompressibility case may be derived as a natural limit; and (iv) accurate agreement with benchmark data for several deformation modes may be achieved, including uniaxial extension, equi-biaxial extension as well as plane-strain extension and others.

The main content will be organized as follows. In Sect. 2, large strain hyper-elastic constitutive relations for isotropic, compressible materials will be formulated based on Hencky's logarithmic strain, and, moreover, certain well-chosen Hencky invariants will be introduced. In Sect. 3, Poisson's ratio at infinitesimal strain will be extended to large strain, and a new quantity from this extension will be introduced to characterize the compressibility behavior, and, moreover, single-variable potentials will be derived from uniaxial stress–strain relations with two strain limits. In Sect. 4, compressible multi-axial elastic potentials will be obtained directly from these single-variable potentials by means of multi-axial bridging and matching procedures. In Sect. 5, three usually treated deformation modes are studied, including uniaxial, equi-biaxial and plane-strain extension, and analytic results with compressibility effects are derived from the proposed modes. In Sect. 6, the model predictions for the just mentioned deformation modes are compared with well-known benchmark data from Treloar [37] and Jones and Treloar [24]. Finally, some remarks will be given in Sect. 7.

2 Compressible elastic potentials based on Hencky strain

Consider a material body undergoing finite deformations. Let $\boldsymbol{F}$ be the deformation gradient and $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ the left Cauchy-Green tensor. The three eigenvalues and the three corresponding orthonormal eigenvectors of $\boldsymbol{B}$ are designated by λ_r^2 and $\boldsymbol{n}_r$, $r = 1, 2, 3$. Usually, the λ_r and $\boldsymbol{n}_r$ are referred to as the principal stretches and the principal axes.

As indicated in monographs in finite elasticity (see, e.g., Hill [17], Ogden [31], Saccomandi [35]), hyper-elastic constitutive equations may be formulated in terms of various finite strain measures. From a general mathematical standpoint, different finite strain measures may be used to formulate elastic stress–strain relations. However, that may not be the case for the purpose of determining specific forms of stress–strain relations from test data. In the subsequent study, a formulation with Hencky's logarithmic strain will prove essential. Earlier, Hill [17] demonstrated certain inherent advantages of Hencky strain over other strain measures in a unified study of constitutive inequalities. It has found various applications in treating finite deformation problems. Its far-reaching roles in consistent formulation of Eulerian finite elastoplasticity have been uncovered in, e.g., Bruhns et al. [8] and Xiao et al. [44–49]. A recent review for use of Hencky strain in material modeling at finite deformations may be found in Xiao [40].

In most recent developments [21,23,39,41–43,50–52], new logarithmic invariants have been introduced and found essential in establishing direct, explicit approaches for constructing new elastic potentials. A short account of hyper-elastic relations based on Hencky strain is presented below.

Hencky's logarithmic strain of Eulerian type, denoted $\boldsymbol{h}$, is of the following principal axis form:

$$\boldsymbol{h} = \frac{1}{2}\boldsymbol{B} = \sum_{\alpha=1}^{3} (\ln \lambda_\alpha)\, \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha . \tag{1}$$

The deformation Jacobian $J = \det \boldsymbol{F}$ specifies the volumetric ratio. The logarithm of J is just the trace of Hencky strain, namely

$$\mathrm{tr}\boldsymbol{h} = \ln J. \tag{2}$$

As indicated earlier, usually the incompressibility constraint $J = 1$, namely

$$\mathrm{tr}\boldsymbol{h} = 0 \tag{3}$$

should be imposed as a restrictive condition in an approximate sense. In the subsequent development, this restriction will not be assumed, and general compressible deformations will be taken into consideration. With an elastic potential in terms of Hencky strain,

$$W = W(\boldsymbol{h}), \tag{4}$$

a direct potential relation may be derived as follows (see, e.g., Fitzjerald [13] and Hill [17]; see also Xiao and Chen [49]):

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{h}} \tag{5}$$

for a compressible, isotropic hyper-elastic material. Here, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress.

Instead of Hencky's logarithmic strain, other finite strain measures, such as Green strain and Almansi strain, may be used in treating finite deformation behaviors, as indicated before. Here, advantages of Hencky's strain measure over other strain measures arise out of its unique properties. First, it allows for a direct additive separation of the volumetric and the deviatoric deformation. Second, it leads to a direct potential relation for isotropic, compressible hyper-elasticity, as indicated by Eq. (5), whereas that may not be the case with any other strain measure. By exploiting these unique properties, new, explicit approaches may be established to obtain forms of the hyper-elastic potential that accurately, automatically match test data, as has been and will be shown in most recent studies and in the subsequent development.

The objectivity principle and the material symmetry principle in continuum mechanics jointly require that, for a compressible, isotropic hyper-elastic material, the potential W as shown by Eq. (4) should be an isotropic scalar function of Hencky strain and therefore expressible as a function of an irreducible set of three invariants of Hencky strain. There are infinitely many such sets of invariants. One of them is formed by the first, second, third basic Hencky invariants below:

$$i_q = \mathrm{tr}\boldsymbol{h}^q = (\ln\lambda_1)^q + (\ln\lambda_2)^q + (\ln\lambda_3)^q, \quad q = 1, 2, 3. \tag{6}$$

Note here that the first, i.e., i_1, is just the trace of $\boldsymbol{h}$. Every Hencky invariant may be given in terms of the above three basic Hencky invariants. In particular, that is the case for the two basic invariants of the deviatoric Hencky strain $\tilde{\boldsymbol{h}}$, viz.,

$$j_2 = \mathrm{tr}\tilde{\boldsymbol{h}}^2 = i_2 - \frac{1}{3}i_1^2, \quad j_3 = \mathrm{tr}\tilde{\boldsymbol{h}}^3 = i_3 - i_1 i_2 + \frac{1}{3}i_1^3 \tag{7}$$

where

$$\tilde{\boldsymbol{h}} = \boldsymbol{h} - \frac{1}{3}(\mathrm{tr}\boldsymbol{h})\boldsymbol{I}. \tag{8}$$

Here and henceforward, $\boldsymbol{I}$ is the second-order identity tensor. The elastic potential W, cf. Eq. (4), for a compressible, isotropic hyper-elastic material may be reduced to a function of the three invariants i_1, j_2 and j_3. In general, it is expressible as a function of any other three irreducible Hencky invariants, as indicated before. In the succeeding study, use of three well-chosen Hencky invariants will prove to be essential, which are given by the first basic Hencky invariant i_1 and the following two:

$$\varphi = \sqrt{j_2}\,, \tag{9}$$

$$\gamma = \sqrt{6}\frac{j_3}{\sqrt{J_2^3}}. \tag{10}$$

The invariant φ is just the magnitude of the deviatoric Hencky strain. The invariant γ specifies the ratios of the three principal values of the deviatoric Hencky strain and ranges from -1 to 1, namely

$$-1 \leq \gamma \leq 1. \tag{11}$$

Hencky strain $\boldsymbol{h}$ is determined by the three invariants i_1, φ and γ in a coordinate-free sense. Details may be found in [43]. With theses three Hencky invariants the elastic potential Eq. (4) is now reduced to a three-variable function as shown below:

$$W = W(i_1, \varphi, \gamma) \tag{12}$$

for a compressible, isotropic hyper-elastic material. Whenever a form of the above three-variable function is given, from Eqs. (5) and (12) we then derive the multi-axial stress–strain relation as follows:

$$J\boldsymbol{\sigma} = \frac{\partial W}{\partial i_1}\boldsymbol{I} + \frac{\partial W}{\partial \varphi}\hat{\boldsymbol{h}} + \frac{\sqrt{6}}{\varphi^4}\frac{\partial W}{\partial \gamma}\breve{\boldsymbol{h}} \tag{13}$$

with

$$\hat{\boldsymbol{h}} = \frac{\tilde{\boldsymbol{h}}}{\varphi}, \tag{14}$$

$$\breve{\boldsymbol{h}} = 3\varphi^3\hat{\boldsymbol{h}}^2 - 3j_3\hat{\boldsymbol{h}} - \varphi^3\boldsymbol{I}. \tag{15}$$

Expressing the Cauchy stress $\boldsymbol{\sigma}$ in terms of the principal axes $\boldsymbol{n}_r$, namely

$$\boldsymbol{\sigma} = \sum_{\alpha=1}^{3} \sigma_\alpha \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha \tag{16}$$

with the three principal stresses σ_α and using Eq. (13), we derive the following relations:

$$J(\sigma_1 - \sigma_3) = (h_1 - h_3)(A - Bh_2(h_2 - h_1)(h_2 - h_3)), \tag{17}$$

$$J(\sigma_2 - \sigma_3) = (h_2 - h_3)(A - Bh_1(h_1 - h_2)(h_1 - h_3)), \tag{18}$$

$$J(\sigma_1 - \sigma_2) = (h_1 - h_2)(A - Bh_3(h_3 - h_1)(h_3 - h_2)) \tag{19}$$

with

$$A = \frac{1}{\varphi}\frac{\partial W}{\partial \varphi}, \quad B = \frac{3\sqrt{6}}{\varphi^5}\frac{\partial W}{\partial \gamma}. \tag{20}$$

Here and henceforward, the h_α are used to designate the three principal Hencky strains, viz.,

$$h_\alpha = \ln \lambda_\alpha, \quad \alpha = 1, 2, 3. \tag{21}$$

From Eqs. (12) to (13) it follows that the central issue in modeling rubberlike elasticity for general compressible large deformations is to determine suitable forms of the elastic potential as a three-variable function [cf. Eq. (12)], so that these forms can match given data sets as accurately as possible. In the succeeding sections, we are going to develop the explicit approach suggested in a latest study [42] and obtain new multi-axial elastic potentials from uniaxial stress–strain relations. The main steps are as follows: (i) by virtue of Hencky strain the Poisson ratio at infinitesimal strain will be extended to characterize the compressibility behavior at finite strain; (ii) single-variable potentials will be derived from uniaxial stress–strain relations; and (iii) from these uniaxial potentials, multi-axial potentials will be obtained by means of certain explicit procedures based on well-chosen Hencky invariants and others. Details will be given below.

3 Strain and stress in the uniaxial case

In this section, we are going to present a study of the uniaxial deformation mode. Results given will serve as a starting point for subsequent development.

Consider a uniform cylindrical bar subjected to loading in the axial direction, and then the usual uniaxial deformation mode is generated in this bar. In this simple case, the stress is determined by the true axial stress, denoted σ, while the deformation is determined by the axial stretch and the lateral stretch, denoted λ and $\bar{\lambda}$, respectively.

The material behavior in the uniaxial case may be described by specifying two relations, namely

$$\begin{cases} \sigma = f(\lambda)\,, \\ \bar{\lambda} = P(\lambda). \end{cases} \tag{22}$$

Generally, forms of the above functions rely on the material property and should be determined from test data.

3.1 Poisson's ratio for compressibility

We first consider the relation Eq. (23) between the uniaxial stretch λ and the lateral stretch $\bar{\lambda}$, in which the compressibility effect is incorporated. This relation is determined from test data. Given this relation, we introduce the following quantity:

$$\nu = -\frac{\ln \bar{\lambda}}{\ln \lambda} = -\frac{\ln P(\lambda)}{\ln \lambda}. \tag{23}$$

Hence,

$$\bar{\lambda} = \lambda^{-\nu}. \tag{24}$$

Note in the above that the Hencky strains are used for both the axial and lateral strains. This definition of Poisson ratio is inspired by Beatty and Stalnaker [5] (see also Murphy [25], and Scott [35]), in which different definitions associated with other cases are given. Physically, this ratio characterizes the compressibility property of the material at issue. Its physical meaning may be seen from two particular cases, namely the case of infinitesimal deformations and the case of incompressible deformation. In fact, the ν is exactly Poisson's ratio at infinitesimal strain, and, moreover,

$$\nu = 0.5\,, \tag{25}$$

$$\bar{\lambda} = \lambda^{-0.5}\,, \tag{26}$$

for large incompressible deformations.

The compressibility property of a rubberlike material is represented by Poisson's ratio $\nu = \nu(h)$ with the axial Hencky strain $h = \ln \lambda$. It will play a central role in constructing multi-axial elastic potentials in the sequel. Usually, rubberlike materials exhibit slight compressibility, and Poisson's ratio ν defined by Eq. (24) may be taken to be a constant close to 0.5 with sufficient accuracy.

3.2 Uniaxial stress–strain relations with strain limits

We next take the uniaxial stress–strain relation [cf. Eq. (22)] into account. For rubberlike materials, experimental data reveal that the so-called strain-stiffening effect will emerge, namely the stress will sharply grow up as the stretch approaches a certain limit under either extension or compression, as schematically shown in Fig. 1. Such strain-stiffening effects were observed; refer to, e.g., the early data by Treloar [36] and the recent data by Arruda and Boyce [2] for details, as well as further studies in latest contributions by Puglisi and Saccomandi [34] and De Tommasi et al. [36].

We are in a position to present a uniaxial stress–strain relation that can characterize the strain-stiffening effects at issue and, in the meantime, can accurately match uniaxial data. Most recently [21,41–43,50–52],

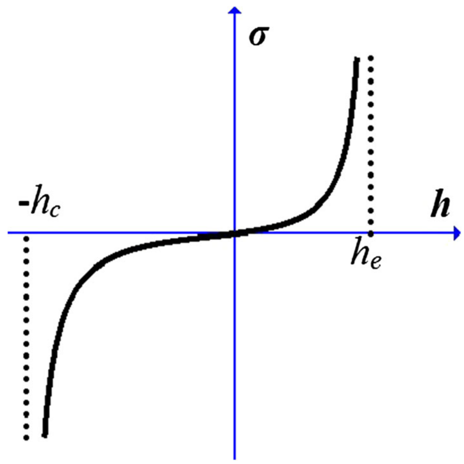

Fig. 1 Uniaxial stress–strain curve with strain limits

it has been found that a simple form of rational functions with the two poles h_e and $-h_c$ may well serve the purpose. Here, such a stress–strain relation is extended to a general compressible case and of the form:

$$\tau = J\sigma = g(h) = \frac{\alpha E h}{\left(1 - \frac{h}{h_e}\right)\left(1 + \frac{h}{h_c}\right)} + (1-\alpha)Eh. \tag{27}$$

In the above, $\tau = J\sigma$ and $h = \ln\lambda$ are the axial Kirchhoff stress and the axial Hencky strain; $h_e > 0$ and $h_c > 0$ are two limits of h at extension and compression, referred to as the extension limit and the compression limit, respectively; E is Young's modulus at infinitesimal strain (i.e., the slope at the origin in Fig. 1), and $\alpha > 0$ prescribes the growth degree of the stress as either of the two limits is approached, and therefore is named the stiffening index.

In summary, it follows from Eqs. (24) to (28) that the five quantities of direct physical meanings, including Poisson's ratio ν, Young's modulus E, the extension limit h_e, the compression limit h_c, as well as the stiffening index α, together characterize features of large strain rubberlike elasticity for general compressible deformations. These constitutive quantities will be incorporated in the multi-axial models that will be established in the next section.

The uniaxial stress–strain relation Eq. (28) is more general than the counterpart given in [41–43]. In fact, the former includes the latter as the special case when the stiffening index α is set to be 2.

It should be pointed out that other forms of rational functions in treating strain-stiffening effects were also found useful from different standpoints; refer to, e.g., Horgan and Saccomandi [20,21].

3.3 Uniaxial elastic potentials

Let w denote the elastic potential at uniaxial strain. Then, the time rate of w is given by

$$\dot{w} = J\,\sigma\,\dot{h}. \tag{28}$$

As a result, we have

$$w = w(h) = \int_0^h J\,\sigma\,dh = \int_0^h g(h)\,dh. \tag{29}$$

It may be clear that the above equation determines a single-variable potential $w = w(h)$ with the axial Kirchhoff stress $\tau = g(h)$ given by Eq. (28). Substituting Eq. (28) into (30) and working out the integration, we obtain the following explicit expression for the potential:

$$w = \frac{1-\alpha}{2}E\,h^2 - \frac{\alpha h_e h_c}{h_e + h_c}\left(h_e \ln\left(1 - \frac{h}{h_e}\right) + h_c \ln\left(1 + \frac{h}{h_c}\right)\right). \tag{30}$$

This gives a unified expression for both cases of extension with $h > 0$ and compression with $h < 0$.

4 Multi-axial elastic potentials

In this section, we are going to develop the explicit approach suggested in a most recent study [42] and obtain new compressible multi-axial elastic potentials directly from the uniaxial elastic potential given in the last section, as well as certain relevant constitutive quantities.

4.1 New explicit approach

In [42], a direct, explicit approach has been proposed to obtain multi-axial elastic potentials $W = W(\boldsymbol{h}) = W(i_1, \varphi, \gamma)$ [cf. Eqs. (4) and (12)] fulfilling the following three requirements:

(i) The uniaxial potential should be exactly given;
(ii) the uniaxial stress–strain relation should be exactly derived; and
(iii) the relation between the lateral and axial stretches should also be derived.

The above requirements mean that, in the uniaxial strain case, the sought multi-axial potential should exactly supply the uniaxial potential, the uniaxial stress–strain relation and the uniaxial Poisson relation as given by Eqs. (30), (28) and (25), respectively. Since the uniaxial case is prescribed by $\gamma = \pm 1$, precisely the following conditions should be satisfied:

$$W|_{\gamma \pm 1} = w(h) = \int_0^h g(h)\, dh\,, \tag{31}$$

$$\left.\frac{\partial W}{\partial \varphi}\right|_{\gamma=\pm 1} = g(h) \tag{32}$$

where the function $\tau = g(h)$ is given by Eq. (28) or may be of any given form. Moreover, the condition (iii) in the foregoing requires that the Poisson relation Eq. (25), namely

$$\overline{h} = -\nu h \tag{33}$$

should be derived by setting

$$\gamma = 1,\ \ \boldsymbol{h} = h\boldsymbol{e} \otimes \boldsymbol{e} + \overline{h}(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e})\,, \tag{34.1,2}$$

with the axial and lateral Hencky strains h and $\overline{h}$. Here, $\boldsymbol{e}$ is a unit vector in the axial direction.

In [42], results are derived from the foregoing requirements. However, the requirements for the derivatives $\partial W/\partial \gamma$ at $\gamma = \pm 1$ are not taken into consideration. This may be understandable, since only one nonvanishing principal stress needs to be treated for the uniaxial mode. That may not be the case for any other deformation mode with at least two nonvanishing principal stresses. From Eqs. (17) to (20) it follows that the three principal stresses should be related to the three derivatives, $\partial W/\partial i_1$, $\partial W/\partial \varphi$ and $\partial W/\partial \gamma$. It is expected that incorporation of these three derivatives will lead to new results that may be more flexible in fitting data for deformation modes other than the uniaxial mode, e.g., plane-strain extension with two nonvanishing principal stresses.

The above understanding leads to a development by further incorporating suitable values of the derivative $\partial W/\partial \gamma$ at $\gamma = \pm 1$. Namely, in addition to the foregoing three requirements [cf., Eqs. (32)–(34)], the following further requirements are introduced:

$$\left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=1} = P_1(h), \quad \left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=-1} = P_{-1}(h)\,, \tag{35}$$

$$\left.\frac{\partial W}{\partial i_1}\right|_{\gamma=\pm 1} = \frac{1}{3} g(h) \tag{36}$$

where the two functions $P_{\pm 1}(h)$ will be given later on. Note that the last condition above is introduced in a sense consistent with Eq. (13).

4.2 Multi-axial potentials via bridging procedure

Now we are in a position to construct a multi-axial potential meeting the requirements prescribed by Eqs. (32)–(37). This will be done by developing the bridging and matching procedures suggested in [42].

Toward our goal, the first step is to extend the single-variable potential $w = w(h)$ to general compressible multi-axial deformations. The idea is to find out two bridging invariants such that they reduce to the axial Hencky strain h in the uniaxial strain case. According to [42], these two invariants may be given by $\pm\pi$ with

$$\pi = \frac{\varphi}{b} \tag{37}$$

where

$$b = \frac{\sqrt{6}}{3}(1+\nu) \tag{38}$$

with Poisson's ratio ν given by Eq. (24). By using Eqs. (34) and (9), (6) and (7), it may readily be verified that, for the uniaxial strain case, together the two invariants $\pm\pi$ indeed provide the axial Hencky strain h.

Now, simply replacing the axial Hencky strain h with the two bridging invariants $\pm\pi$, separately, we may extend the uniaxial potential $w = w(h)$ [cf. Eq. (31)] limited to the uniaxial strain to general compressible multi-axial deformations. This leads to two multi-axial potentials given by $w(\pm\pi)$.

4.3 Unified potential via matching procedure

The next step is to construct a multi-axial potential meeting all the requirements indicated before. Utilizing the two multi-axial potentials $w(\pm\pi)$ and regarding the conditions Eqs. (32), (33) and (36) as interpolating conditions at $\gamma = \pm 1$ as well as considering the conditions Eqs. (34) and (37), we eventually obtain the sought multi-axial potential as follows:

$$W = \frac{1-2\nu}{3}w(q) + \frac{1+\nu}{6}\big(((2-\gamma)w(\pi) + (\gamma-1)Y_1)(1+\gamma)^2 \tag{39}$$

$$+ ((2+\gamma)w(-\pi) + (\gamma+1)Y_{-1})(1-\gamma)^2\big) \tag{40}$$

where π and b are given by Eqs. (38)–(39) and

$$q = \frac{i_1}{1-2\nu}\,, \tag{41}$$

$$Y_1 = \xi\varphi^3\,, \tag{42}$$

$$Y_{-1} = \eta\varphi^3. \tag{43}$$

It should be pointed out that the two functions $Y_{\pm 1}$ given by Eqs. (42), (43) are related to the values of the derivative $\partial W/\partial\gamma$ at $\gamma = \pm$. Here, Eqs. (42), (43) merely supply the simplest form (the cases either linear or quadratic in φ may be excluded).

In the next section, it will be shown that the multi-axial potential given by Eq. (40) indeed fulfills all the conditions prescribed by Eqs. (32)–(37).

Some remarks are now opportune. Since the conditions given by Eqs. (32), (33) are met, the multi-axial potential Eq. (40) can exactly reproduce the uniaxial stress–strain relation of any given form. As a result, whenever the former accurately matches given uniaxial data, accordingly the latter can automatically match the same data. In the next two sections, the predictions of the multi-axial potential proposed will be studied for three usually treated deformation modes, such as uniaxial extension, equi-biaxial extension and plane-strain extension, and results will be compared with test data in literature.

5 Compressible responses for three modes

In this section, analytic results for three usually treated deformation modes will be derived from the proposed model for the purpose of highlighting the compressibility effects. In the next section these results will be used for the purpose of comparison with test data.

5.1 Model prediction for uniaxial extension

We are going to demonstrate that all the requirements given by Eqs. (32)–(37) may be satisfied by the proposed multi-axial potential Eq. (40). The Hencky strain tensor for uniaxial deformation mode is given by Eq. (34.2), while the Cauchy stress tensor in this case is of the simple form

$$\boldsymbol{\sigma} = \sigma\, \boldsymbol{e} \otimes \boldsymbol{e}. \tag{44}$$

As indicated earlier, $\boldsymbol{e}$ is a unit vector in the axial direction, and σ is the true axial stress. By using Eq. (34), we have [cf., Eqs. (7)–(15) (15), (38), (39) and (41)]

$$\tilde{\boldsymbol{h}} = \frac{1}{3}(h - \overline{h})(\boldsymbol{I} - 3\boldsymbol{e} \otimes \boldsymbol{e})\,, \tag{45}$$

$$i_1 = h + 2\overline{h},\ \ j_2 = \frac{2}{3}(h - \overline{h})^2,\ \ j_3 = \frac{2}{9}(h - \overline{h})^3\,, \tag{46}$$

$$\varphi = \frac{\sqrt{6}}{3}|h - \overline{h}|,\ \ \gamma = \pm 1,\ \ \pi = |h - \overline{h}|/b,\ \ q = (h + 2\overline{h})/(1 - 2\nu). \tag{47}$$

Moreover, we have [cf., Eqs. (30) and (40), (41)]

$$\left.\frac{\partial W}{\partial \varphi}\right|_{\gamma=\pm 1} = w'(\pm\pi)\,,\ \ \left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=\pm 1} = \frac{2}{3}(1+\nu)Y_{\pm 1}\,,$$
$$\left.\frac{\partial W}{\partial i_1}\right|_{\gamma=\pm 1} = \frac{1}{3}w'(q) = \frac{1}{3}g(q).$$

From Eq. (13) we deduce

$$J \operatorname{tr}\boldsymbol{\sigma} = 3\frac{\partial W}{\partial i_1}.$$

Then, from the last two expressions and Eq. (44) we infer [cf. Eq. (28)]

$$g(h) = g(q),$$

namely

$$h = q = (h + 2\overline{h})/(1 - 2\nu),\ \text{ i.e., } \overline{h} = -\nu\,,$$

which exactly yields the Poisson relation Eq. (25). Thus, Eqs. (32), (33) and Eqs. (36), (37) may be derived.

Now the nominal axial stress F is given by

$$F = E\frac{\ln\lambda}{\lambda}\left(\frac{\alpha}{\left(1 - \frac{\ln\lambda}{h_e}\right)\left(1 + \frac{\ln\lambda}{h_c}\right)} + 1 - \alpha\right) \tag{48}$$

where λ is the axial stretch, and the volumetric ratio J by

$$J = \lambda^{1-2\nu}. \tag{49}$$

5.2 Model prediction for equi-biaxial extension

Let λ and F be the stretch and the nominal normal stress in the two loading directions, respectively, and let β be the stretch in the direction free of loading. The Hencky strain tensor and the Cauchy stress tensor are of the forms:

$$\boldsymbol{h} = (\ln\lambda)(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}) + ((\ln\beta)\boldsymbol{e} \otimes \boldsymbol{e},\quad \boldsymbol{\sigma} = \sigma(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}). \tag{50}$$

Following the same procedures as in the last subsection, we obtain

$$F = -\frac{E}{2\nu^2}\frac{\ln\lambda}{\lambda}\left(\frac{\alpha}{\left(1+\frac{\lambda}{\nu h_e}\right)\left(1-\frac{\ln\lambda}{\nu h_c}\right)} + 1 - \alpha\right), \tag{51}$$

$$J = \lambda^{-(1-2\nu)/\nu}, \tag{52}$$

for the nominal normal stress in the two loading directions and the volumetric ratio, respectively.

5.3 Model prediction for plane-strain extension

As in the above, let λ and $sigma_1$ be the stretch and the true normal stress in the loading direction, and let σ_2 be the stretch in the free direction, and, moreover, let 2 be the true normal stress in the un-deformed direction. Then, the Hencky strain tensor and the Cauchy stress tensor are of the forms:

$$\boldsymbol{h} = (\ln\lambda)\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + (\ln\beta)\boldsymbol{e}_3\otimes\boldsymbol{e}_3,\ \ \boldsymbol{\sigma} = \sigma_1\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \sigma_2\boldsymbol{e}_2\otimes\boldsymbol{e}_2 \tag{53}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are unit vectors in the loading, un-deformed and free directions, respectively.

From the potential Eq. (40) and the stress–strain relation Eq. (13), each of the three quantities β, σ_1, σ_2 may be derived as a function of the stretch λ. However, in contrast to the incompressibility case with $\beta = \lambda^{-1}$, now the relation $\beta = q(\lambda)$ is very complicated. It does not appear that closed-form results in explicit form are derivable due to the complexity arising from compressibility. As a result, generally numerical procedures should be used. Here, for nearly incompressible deformations with a ν very close to 0.5 we derive analytic results in an approximate sense. Toward this goal, we first present an approximate solution for the stretch β as follows:

$$\beta = \lambda^{-\nu/(1-\nu)}. \tag{54}$$

Then, by setting $\gamma = 0$, approximate results for the two nominal normal stresses F and F_0 and the volumetric ratio J are derived from Eqs. (17) to (20) and (50) and from $J = \lambda\beta$ as follows:

$$J = \lambda^{(1-2\nu)/(1-\nu)}, \tag{55}$$

$$\sigma_1{}' = 4G\alpha h\left(\frac{1-\frac{3h^2}{(1+\nu)^2 h_e h_c}}{\left(1-\frac{3h^2}{(1+\nu)^2 h_e^2}\right)\left(1-\frac{3h^2}{(1+\nu)^2 h_c^2}\right)} + \alpha^{-1} - 1\right) + \sqrt{2}\,(1+\nu)(\xi-\eta)h^2, \tag{56}$$

$$\sigma_2 = \frac{1}{2}\sigma_1 + \frac{\sqrt{6}}{2}(1+\nu)(\xi+\eta)h^2 + C\left(\frac{h_e}{h}\ln\frac{1-\frac{\sqrt{3}h}{(1+\nu)h_e}}{1+\frac{\sqrt{3}h}{(1+\nu)h_e}} + \frac{h_c}{h}\ln\frac{1+\frac{\sqrt{3}h}{(1+\nu)h_c}}{1-\frac{\sqrt{3}h}{(1+\nu)h_c}}\right) \tag{57}$$

where

$$C = \frac{\sqrt{27}}{2}\frac{(1+\nu)^2\alpha G\,h_e\,h_c}{h_e+h_c},\quad h = \ln\lambda.$$

In the above, $2G = E/(1+\nu)$ is the shear modulus. The above results clearly show the effects of compressibility.

6 Numerical examples for model predictions

In presenting numerical examples for the purpose of studying the predictabilities of the proposed model, test data from Treloar [37] and Jones and Treloar [24] will be taken into consideration, separately. These data are concerned with the three deformation modes treated in the last section. In this section, model predictions for these modes will be compared with the data just mentioned.

6.1 Treloar's data

The parameter values are as follows:

$$E = 1.1\,\text{MPa},\ \ \nu = 0.499,\ \ \alpha = 2,\ \ h_e = \ln 8.7,\ \ h_c = \ln 41,\ \ \xi = \eta = 0.$$

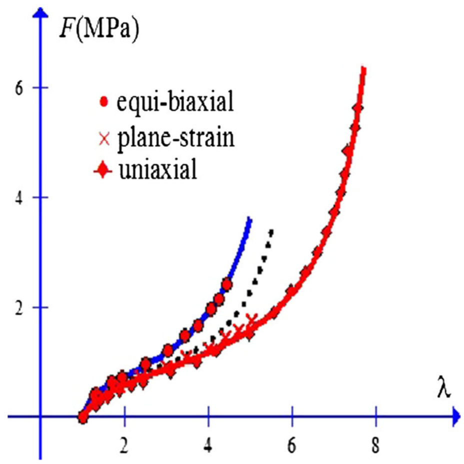

Fig. 2 Comparison of model predictions with Treloar's data in cases of uniaxial, equi-biaxial and plane-strain extension (F for nominal normal stress and for stretch)

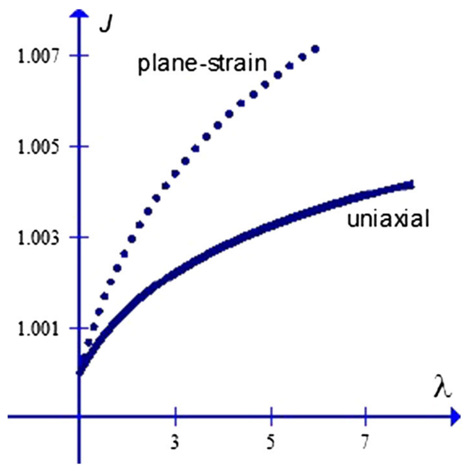

Fig. 3 Volume changes in cases of uniaxial and plane-strain extension

Results are shown in Fig. 2 for the nominal normal stresses in the three cases. Moreover, results for volumetric changes are shown in Figs. 3 and 4. In the latter, test data are unavailable and hence not incorporated. The curves in Figs. 3 and 4 are generated from Eqs. (49), (49) and (51), (52). Result for the stress in the un-deformed direction in the plane-strain case is not involved, since relevant data in this case are also unavailable. It should be pointed out that the two parameters ξ and η play a role in fitting data for the two normal stresses in the loading and un-deformed directions. However, with a lack of data for the stress in the un-deformed direction, here the contributions from $Y_{\pm 1}$ are neglected by setting $\xi = \eta = 0$. This respect will be discussed in the next subsection.

Various models suggested in literature have been used in fitting Treloar's data. In particular, the well-known Gent model was used for this purpose; refer to, e.g., Pucci and Saccomandi [33]. It is noted that the Gent model is for incompressible deformations, while the model proposed here is for general compressible deformations. As such, comparisons of model predictions may not be performed on a common ground. This respect will be treated in a further study.

6.2 Data from Jones and Treloar

In the case of plane-strain extension, there are two nonvanishing principal stresses. However, in Treloar [37], only data for the stress in the loading direction are provided, whereas data for the stress in the un-deformed

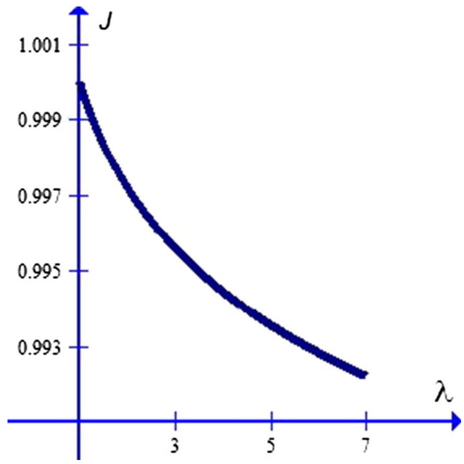

Fig. 4 Volume change in case of equi-biaxial extension

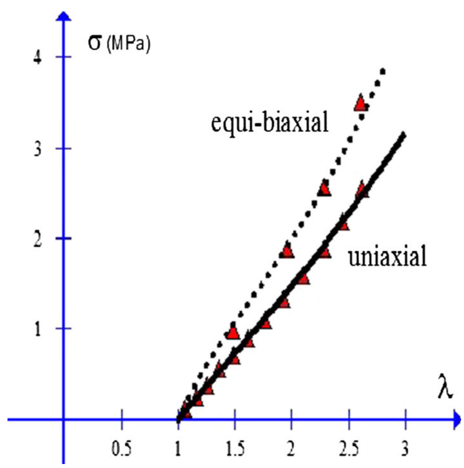

Fig. 5 Comparison of model predictions with Jones and Treloar's data in cases of uniaxial extension and equi-biaxial extension (for true stresses)

direction are not supplied. Data for the stresses just mentioned are available in Jones and Treloar [24] and will be taken into account below.

For Jones and Treloar's data, the following parameter values are found:

$$E = 1.45\,\text{MPa},\ \ \nu = 0.499,\ \ \alpha = 2,\ \ h_e = \ln 11,\ \ h_c = \ln 54,\ \ \xi = \eta = 0.14\,\text{MPa}.$$

Results are shown in Figs. 5 and 6.

In the new model proposed, requirements for the derivative $\partial W/\partial \gamma$ in the uniaxial case are incorporated, as has been done in Sect. 4. The contributions from these requirements are represented by $Y_{\pm 1}$ with two parameters ξ and η [cf., Eqs. (42), (43)]. The effect in this respect is evidenced in Fig. 6, in which the true stress σ_2 in the un-deformed direction is shown for two different cases when $\xi = \eta = 0$ (dotted line) and $\xi = \eta = 0.14\,\text{MPa}$ (solid line). Results with no contributions from $Y_{\pm 1}$ deviate appreciably from test data, whereas results with appropriate contributions from $Y_{\pm 1}$ agree accurately with test data.

7 Concluding remarks

In the previous sections, the explicit approach proposed in a most recent study [22] has been developed toward obtaining a new multi-axial elastic potential for general compressible deformations. The proposed model is free of the commonly assumed constraint of incompressibility and thus bypasses possible issues resulting from this constraint. Numerical examples show good agreement with data from several benchmark tests.

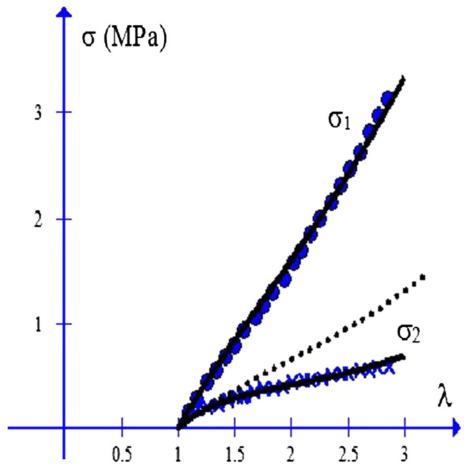

Fig. 6 Comparing model predictions for plane-strain extension with Jones and Treloar's data (1 and 2 for true normal stresses in loading and un-deformed directions, respectively)

In this study, results are derived based merely on a single deformation mode, namely the uniaxial mode. As such, the model may be calibrated simply with uniaxial data. However, it may be expected that more robust models should be based on data for a few suitable deformation modes. Results in this respect have been derived mostly recently in [43] for the case of incompressibility. Further results for general compressible deformations will be presented in ensuing development.

Acknowledgments This study was carried out under the joint support of the start-up fund from the Education Committee of China by Shanghai University (No. S.15-B002-09-032) and the fund for research innovation from Shanghai University (No. S.10-0401-12-001) as well as the fund from Natural Science Foundation of China (Nos. 11372172, 11472164).

References

1. Anderson, M.L., Mott, P.H., Roland, C.M.: The compression of bonded rubber disks. Rubber Chem. Technol. **77**, 293–302 (2004)
2. Arruda, E.M., Boyce, M.C.: A three-dimensional constitutive model for the large stretch behaviour of rubber elastic materials. J. Mech. Phys. Solids **41**, 389–412 (1993)
3. Beatty, M.F.: Topic in finite elasticity: hyper-elasticity of rubber, elastomers, and biological tissues-with examples. Appl. Mech. Rev. **40**, 1699–1733 (1987)
4. Beatty, M.F.: On constitutive models for limited elastic, molecular based materials. Math. Mech. Solids **13**, 375–387 (2008)
5. Beatty, M.F., Stalnaker, D.O.: The Poisson function of finite elasticity. J. Appl. Mech. **53**, 807–813 (1986)
6. Bischoff, E.B., Arruda, E.M., Grush, K.: A new constitutive model for the compressibility of elastomers at finite deformations. Rubber Chem Technol. **74**, 541–559 (2001)
7. Blatz, P.J., Ko, W.L.: Application of finite elasticity theory to the deformation of rubber materials. J. Rheol. **6**, 223–252 (1962)
8. Bruhns, O.T., Xiao, H., Meyers, A.: Self-consistent Eulerian rate type elastoplasticity models based upon the logarithmic stress rate. Int. J. Plast. **15**, 479–520 (1999)
9. Cam, J.B.L., Toussaint, E.: Cyclic volume changes in rubber. Mech. Mater. **41**, 898–901 (2009)
10. Christensen, R.M.: A two material constant, nonlinear elastic stress constitutive equation including the effect of compressibility. Mech. Mater. **7**, 155–162 (1988)
11. Drozdov, A.D., Gottlieb, M.: Ogden-type constitutive equations in finite elasticity of elastomers. Acta Mech. **183**, 231–252 (2006)
12. Ehlers, W., Eipper, G.: The simple tension problem at large volumetric strains computed from finite hyper-elastic material laws. Acta Mech. **130**, 17–27 (1998)
13. Fitzjerald, S.: A tensorial Hencky measure of strain and strain rate for finite deformation. J. Appl. Phys. **51**, 5111–5115 (1980)
14. Fried, E.: An elementary molecular-statistical basis for the Mooney and Rivlin–Saunders theories of rubber elasticity. J. Mech. Phys. Solids **50**, 571–582 (2002)
15. Gent, A.N.: A new constitutive relation for rubber. Rubber Chem. Technol. **69**, 59–61 (1996)
16. Hewitt, F.G., Anthony, R.L.: Measurement of the isothermal volume dilation accompanying the unilateral extension of rubber. J. Appl. Phys. **29**, 1411–1414 (1958)
17. Hill, R.: Constitutive inequalities for isotropic elastic solids under finite strain. Proc. R. Soc. Lond. A **326**, 131–147 (1970)
18. Horgan, C.O., Murphy, J.G.: Compression tests and constitutive models for the slight compressibility of elastic rubberlike materials. Int. J. Eng. Sci. **47**, 1232–1239 (2009)

19. Horgan, C.O., Murphy, J.G.: Constitutive modeling for moderate deformations of slightly compressible rubber. J. Rheol. **53**, 153–168 (2009)
20. Horgan, C.O., Saccomandi, G.: A molecular-statistical basis for the Gent model of rubber elasticity. J. Elast. **68**, 167–176 (2002)
21. Horgan, C.O., Saccomandi, G.: Finite thermo-elasticity with limiting chain extensibility. J. Mech. Phys. Solids **75**, 839–851 (2003)
22. Horgan, C.O., Saccomandi, G.: Constitutive models for compressible nonlinearly elastic materials with limiting chain extensibility. J. Elast. **77**, 123–138 (2004)
23. Jin, T.F., Yu, L.D., Yin, Z.N., Xiao, H.: Bounded elastic potentials for rubberlike materials with strain-stiffening effects. Z. Angew. Math. Mech. (2014). doi:10.1002/zamm.201400109
24. Jones, D.F., Treloar, L.R.G.: The properties of rubber in pure homogeneous strain. J. Phys. D **8**, 1285–1304 (1975)
25. Murphy, J.G.: Strain energy functions for a Poisson power law function in simple torsion of compressible hyper-elastic materials. J. Elast. **60**, 151–164 (2000)
26. Nicholson, D.W., Lin, B.: Theory of thermo-hyperelasticity for near-incompressible elastomers. Acta Mech. **116**, 15–28 (1996)
27. Ogden, R.W.: Large deformation isotropic elasticity-on the correlation of theory and experiment for incompressible rubber-like materials. Proc. R. Soc. Lond. A **326**, 565–584 (1972)
28. Ogden, R.W.: Large deformation isotropic elasticity-on the correlation of theory and experiment for compressible rubber-like materials. Proc. R. Soc. Lond. A **328**, 567–583 (1972)
29. Ogden, R.W.: Volume changes associated with the deformation of rubber-like solids. J. Mech. Phys. Solids **24**, 323–338 (1976)
30. Ogden, R.W.: Nearly isochoric elastic deformations: application to rubberlike solids. J. Mech. Phys. Solids **26**, 37–57 (1978)
31. Ogden, R.W.: Non-Linear Elastic Deformations. Ellis Horwood, Chichester (1984)
32. Ogden, R.W., Saccomandi, G., Sgura, I.: On worm-like chain models within the three-dimensional continuum mechanics framework. Proc. R. Soc. Lond. A **462**, 749–768 (2006)
33. Pucci, E., Saccomandi, G.: A note on the Gent model for rubber-like materials. Rubber Chem. Technol. **75**, 839–851 (2002)
34. Puglisi, G., Saccomandi, G.: The Gent constitutive equation for rubber-like materials: an appraisal of a genial and simple idea. Int. J. Nonlinear Mech. **68**, 17–24 (2015)
35. Saccomandi, G.: Nonlinear elasticity for soft fibrous materials. In: Dorfmann, L., Ogden, R.W. (eds.) *Non-linear Mechanics of Soft Fibrous Materials*. CISM Courses and Lectures No. 559, Springer, Wien
36. De Tommasi, D., Puglisi, G., Saccomandi, G.: Multi-scale mechanics of macromolecular materials with unfolding domains. J. Mech. Phys. Solids **78**, 54–72 (2015)
37. Treloar, L.R.G.: The Physics of Rubber Elasticity. Oxford University Press, Oxford (1975)
38. Wu, P.D., Giessen, E.van der : On improved network models for rubber elasticity and their application to orientation hardening in glassy polymers. J. Mech. Phys. Solids **41**, 427–456 (1993)
39. Wang, X.M., Li, H., Yin, Z.N., Xiao, H.: Multi-axial strain energy functions of rubberlike materials: an explicit approach based on polynomial interpolation. Rubber Chem. Technol. **87**, 168–183 (2014)
40. Xiao, H.: Hencky strain and Hencky model: extending history and ongoing tradition. Multidiscip. Model. Mater. Struct. **1**, 1–52 (2005)
41. Xiao, H.: An explicit, direct approach to obtaining multi-axial elastic potentials that exactly match data of four benchmark tests for rubberlike materials-part 1: incompressible deformations. Acta Mech. **223**, 2039–2063 (2012)
42. Xiao, H.: An explicit, direct approach to obtain multi-axial elastic potentials which accurately match data of four benchmark tests for rubbery materials—part 2: general deformations. Acta Mech. **224**, 479–498 (2013)
43. Xiao, H.: Elastic potentials with best approximation to rubberlike elasticity. Acta Mech. (2014). doi:10.1007/s00707-014-1176-3
44. Xiao, H., Bruhns, O.T., Meyers, A.: Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mech. **124**, 89–105 (1997)
45. Xiao, H., Bruhns, O.T., Meyers, A.: The choice of objective rate s in finite elastoplasticity: general results on the uniqueness of the logarithmic rate. Proc. R. Soc. Lond. A **456**, 1865–1882 (2000)
46. Xiao, H., Bruhns, O.T., Meyers, A.: Explicit dual stress–strain and strain–stress relations of incompressible isotropic hyper-elastic solids via deviatoric Hencky strain and Cauchy stress. Acta Mech. **168**, 21–33 (2004)
47. Xiao, H., Bruhns, O.T., Meyers, A.: Elastoplasticity beyond small deformations. Acta Mech. **182**, 31–111 (2006)
48. Xiao, H., Bruhns, O.T., Meyers, A.: Thermo dynamic laws and consistent Eulerian formulation of finite elastoplasticity with thermal effects. J. Mech. Phys. Solids **55**, 338–365 (2007)
49. Xiao, H., Chen, L.S.: Hencky's logarithmic strain measure and dual stress–strain and strain–stress relations in isotropic finite hyperelasticity. Int. J. Solids Struct. **40**, 1455–1463 (2003)
50. Yu, L.D., Jin, T.F., Yin, Z.N., Xiao, H.: A model for rubberlike elasticity up to failure. Acta Mech. (2014). doi: 10.1007/s00707-014-12626
51. Zhang, Y.Y., Li, H., Wang, X.M., Yin, Z.N., Xiao, H.: Direct determination of multi-axial elastic potentials for incompressible elastomeric solids: An accurate, explicit approach based on rational interpolation. Contin. Mech. Thermodyn. **26**, 207–220 (2013)
52. Zhang, Y.Y., Li, H., Xiao, H.: Further study of rubber-like elasticity: elastic potentials matching biaxial data. Appl. Math. Mech. (Engl. Ed.) **35**, 13–24 (2014)
53. Zuniga, A.E.: A non-Gaussian network model for rubber elasticity. Polymer **47**, 907–914 (2006)
54. Zuniga, A.E., Beatty, M.F.: Constitutive equations for amended non-Gaussian network models of rubber elasticity. Int. J. Eng. Sci. **40**, 2265–2294 (2003)