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ORIGINAL PAPER

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Dynamic Finite Element analysis of fractionally damped structural systems in the time domain

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Abstract Visco-elastic material models with fractional characteristics have been used for several decades. This paper provides a simple methodology for Finite-Element-based dynamic analysis of structural systems with viscosity characterized by fractional derivatives of the strains. In particular, a re-formulation of the well-known Newmark method taking into account fractional derivatives discretized via the Grünwald–Letnikov summation allows the analysis of structural systems using standard Finite Element technology.

Keywords Fractional calculus · Viscoelasticity · Finite Elements · Newmark method · Fractional damping

1 Introduction

With increasing advanced manufacturing processes, visco-elastic materials are very attractive for mitigation of vibrations, provided that you have advanced studies for capturing the realistic behavior of such materials. Experimental verification of the visco-elastic behavior is limited to some well-known low-order models like the Maxwell or Kelvin–Voigt models. However, both models are not sufficient to model the visco-elastic behavior of real materials, since only the Maxwell type can capture the relaxation tests and the Kelvin–Voigt the creep tests, respectively. Very recently, it has been stressed that the most suitable model for capturing visco-elastic behavior is the springpot, characterized by a fractional constitutive law. A detailed description how to arrive at a fractional visco-elastic formulation from experimental data is given in [7]. Once asserted that this is the proper model for understanding the visco-elastic behavior, the structural response was evaluated considering this internal bond and fundamental analyses of beams with fractional visco-elastic properties were carried out in [17]. Time domain analysis of fractionally damped oscillators has been investigated earlier [14]. Numerical solution procedures are discussed in [11]. Now the challenge is that such a model can be utilized in the engineering field practically. To this aim, since the Finite Element method is commonly used in engineering, the algorithm solution has been suitably modified inserting the terms representing the discretized fractional derivative. In this context, a Finite Element formulation using the central difference method is described in [20] and a boundary element formulation is presented in [12]. The Newmark method was applied to a single-degree-of-freedom system in [21]. A comparison of various explicit and implicit integration schemes is carried out in [5]. Although focused on frequency domain analysis, the condensation technique described in

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[5] can also be relevant for time domain analysis. Additional considerations regarding the numerical treatment of nonlinearities in fractional material behavior can be found in the recent work [13]. In order to arrive at a generally applicable methodology, it is essential that existing Finite Element formulations can be used without modification for fractional visco-elastic analysis. This paper provides such a non-intrusive approach by combining a time-discrete formulation of the fractional derivative with an implicit discrete time integration procedure. It is demonstrated that this approach does not require a treatment of visco-elasticity at material point level, but rather at the level of global system matrices. The validity of this approach is demonstrated by comparison to existing analytical solutions for special structural models based on the Euler–Bernoulli and Timoshenko beam theories, and by applying it to a realistic structural model subject to earthquake excitation.

2 Elements of fractional calculus

2.1 Formal definitions

Among the various possible definitions of fractional derivatives, the so-called *Caputo* derivative [19] is considered to be the most suitable for describing visco-elastic material behavior. The formal definition is

$$D_t^\alpha f(t) = \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{f^{(r)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-r}}, \quad r-1 < \alpha < r, \tag{1}$$

in which r is an integer number and $f^{(r)}(\tau)$ is the r-th derivative of the function f w.r.t. τ. This definition requires $t \geq 0$ and is therefore suitable for initial value problems which usually arise in structural dynamics. In fact, the typical initial value problem is defined by "at-rest" initial conditions (i.e., the displacements and the velocities are zero initially).

2.2 Grünwald–Letnikov summation

The fractional derivative $D^\alpha f(t)$ of a smooth function $f(t)$ with $f(t) = 0$ for $t \leq 0$ can be expressed in terms of the limit

$$D_t^\alpha f(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^\alpha} \sum_{k=0}^{\infty} g_k f(t - k\Delta t) \tag{2}$$

in which the coefficients g_k can be easily computed from the recursion formula (e.g., [19])

$$g_k = 1 - \frac{\alpha+1}{k} g_{k-1}; \quad g_0 = 1. \tag{3}$$

Note that for $0 \leq \alpha \leq 1$, all coefficients g_k with $k > 0$ will be non-positive and of (slowly) decreasing magnitude. Eq. (2) implies that a fractional derivative contains memory from the past, i.e., it is not a local quantity such as integer order derivatives. For practical computations, the limiting process in Eq. (2) is omitted and the summation stops at a finite upper limit m. Hereby a sufficiently small value for Δt is chosen, such that we obtain the approximation

$$D_t^\alpha f(t) \approx \frac{1}{\Delta t^\alpha} \sum_{k=0}^{m} g_k f(t - k\Delta t). \tag{4}$$

The summation limit m is implicitly defined by reaching the condition $t - m\Delta t \leq 0$. For convenience, new coefficients q_k are introduced by

$$q_k = \frac{1}{\Delta t^\alpha} g_k \tag{5}$$

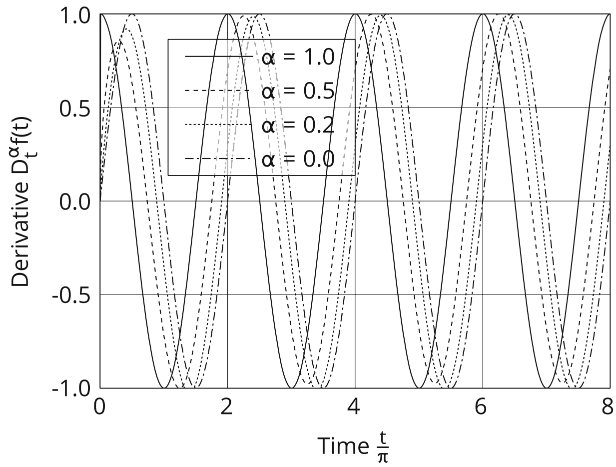

Fig. 1 Grünwald–Letnikov fractional derivatives of the function $f(t) = \sin t$

such that

$$D_t^\alpha f(t) \approx \sum_{k=0}^{m} q_k f(t - k\Delta t). \tag{6}$$

It should be mentioned that a very detailed discussion on numerical methods for computing fractional derivatives can be found in [8]. As a demonstration example, consider the function $f(t) = \sin t$. In Fig. 1 the fractional derivatives of this function are shown for different numerical values of α. Herein the time step Δt was chosen such that number m is at most $m = 1000$ for $t = 8\pi$. It can be seen that apart from some initial transient phenomena, the fractional derivatives essentially introduce a phase shift into the trigonometric function. This phase shift depends on the magnitude of α, and it accounts for damping effects in structural vibration. Note that for the case $\alpha = 1$ (i.e., the regular first derivative) the numerical result using the Grünwald–Letnikov expansion matches the exact result $\cos t$ exactly apart from the initial jump from 0 to 1. So together with the usual "at-rest" initial conditions, Eq. (6) is a suitable discrete representation of Caputo's derivative.

3 Visco-elasticity

Visco-elastic behavior is fully consistent with the reality of almost all materials since they exhibit a mixture of the two simple behaviors: purely elastic and purely viscous. A visco-elastic material does not maintain a constant strain under constant stress, but it undergoes a strain slowly varying with time, that is, it *creeps*; and if deformed at constant strain, the stress required to hold it diminishes gradually with time, that is, it *relaxes*.

In particular, the uniaxial, isothermal stress–strain equation for a linear visco-elastic material is ruled by the Boltzman superposition integral

$$\sigma(t) = \int_0^t E\left(t - \bar{t}\right) \frac{\mathrm{d}\varepsilon\left(\bar{t}\right)}{\mathrm{d}\bar{t}} \mathrm{d}\bar{t} \tag{7}$$

which in this form, is valid for a quiescent system at $t = 0$. The right-hand side of Eq. (7) is a convolution integral in which the relaxation function $E(t)$ plays the role of kernel of the integral. From experimental data, it is always observed that $E(t)$is a decaying function and the shape of such decaying function depends on the material. In [4,9,16], the simplest and most used model for capturing the relaxation function $E(t)$ is the Maxwell model (a spring in series with a dashpot, Fig. 2) whose constitutive law, the relation between the stress $\sigma(t)$ and the strain $\varepsilon(t)$, is ruled by a differential equation of integer order of the type

$$\dot{\sigma}(t) + \frac{E}{c}\sigma(t) = E\dot{\varepsilon}(t), \tag{8}$$

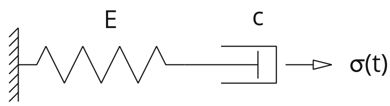

Fig. 2 Maxwell model

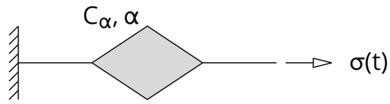

Fig. 3 Springpot element

where E is Young's modulus and c the viscosity coefficient.

This type of constitutive law leads to an exponential function decay as a relaxation function $E(t)$ [3]

$$E(t) = E\exp\left(-\frac{E}{c}t\right). \tag{9}$$

However, at the beginning of the twentieth century, Nutting [15] and Gemant [10] observed that, for visco-elastic materials such as rubber, bitumen, polymers, concrete etc., the experimental data coming from the relaxation test were well fitted by a power law decay, that is

$$E(t) \propto (t)^{-\alpha} \qquad 0 < \alpha < 1. \tag{10}$$

Then, selecting the coefficient of proportionality in Eq. (10) in the form $C_\alpha/\Gamma(1-\alpha)$, where $\Gamma(\cdot)$ is the Gamma function, C_α and α are coefficients characterizing the material at hand, that is

$$E(t) = \frac{C_\alpha}{\Gamma(1-\alpha)}t^{-\alpha}, \tag{11}$$

and introducing this α power law function of $E(t)$ into the Boltzman superposition principle Eq. (7) leads to the constitutive law ruled by a differential equation of real order that is just α:

$$\sigma(t) = \frac{C_\alpha}{\Gamma(1-\alpha)}\int_0^t \left(t-\bar{t}\right)^{-\alpha}\frac{\mathrm{d}\varepsilon\left(\bar{t}\right)}{\mathrm{d}\bar{t}}\mathrm{d}\bar{t} = C_\alpha\left(D_t^\alpha\varepsilon\right)(t). \tag{12}$$

In particular comparing Eq. (12) with Eq. (1), it is apparent that the stress $\sigma(t)$ is related to the Caputo's fractional derivative of the strain $\varepsilon(t)$.

Further, it has to be underscored that the constitutive law in Eq. (12) interpolates the purely elastic behavior ($\alpha = 0$) and the purely viscous behavior ($\alpha = 1$) and is termed in literature as springpot element depicted in Fig. 3.

4 Finite Element formulation

4.1 Basic principles

It is well known (see e.g., [1]) that the principle of virtual work leads to a straightforward formulation of the force–displacement relation in linear Finite Elements. In the general case, this principle states that for any virtual element displacement vector $\delta\mathbf{U}_\mathrm{e}$ and compatible element strain field $\delta\boldsymbol{\varepsilon}$ we have

$$\delta W = \mathbf{F}_\mathrm{e}^\mathrm{T}\delta\mathbf{U}e - \int_{V_\mathrm{e}} \boldsymbol{\sigma}^\mathrm{T}\delta\boldsymbol{\varepsilon}\mathrm{d}V = 0. \tag{13}$$

In this equation, the stress and strain tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are re-arranged into vector form. $\mathbf{F}_\mathrm{e}$ is the vector of element nodal forces, and V_e denotes the element volume.

In a displacement-based element formulation, the strains are related to the element displacements by shape functions $\mathbf{B}_\mathrm{e}$ such that

$$\boldsymbol{\varepsilon} = \mathbf{B}_\mathrm{e}\mathbf{U}_\mathrm{e}, \tag{14}$$

from which a compatible virtual strain field is readily found as

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}_\mathrm{e}\delta\mathbf{U}_\mathrm{e}. \tag{15}$$

In the case of a linear-elastic material, the stresses and strains are related by the elasticity tensor $\mathbf{E}$ (again re-arranged into matrix form)

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}, \tag{16}$$

so that from Eq. (13) we get after canceling $\delta\mathbf{U}_\mathrm{e}$

$$\mathbf{F}_{\mathrm{e,el}} = \int_{V_\mathrm{e}} \mathbf{B}_\mathrm{e}^\mathrm{T}\mathbf{E}\mathbf{B}_\mathrm{e}\mathrm{d}V\mathbf{U}_\mathrm{e}. \tag{17}$$

Hence elastic element force vector $\mathbf{F}_{\mathrm{e,el}}$ is related to the element displacement vector $\mathbf{U}_\mathrm{e}$ by the element stiffness matrix $\mathbf{K}_\mathrm{e}$

$$\mathbf{F}_{\mathrm{e,el}} = \mathbf{K}_\mathrm{e}\mathbf{U}_\mathrm{e} \tag{18}$$

and the element stiffness matrix is obtained by integration over the element volume V_e

$$\mathbf{K}_\mathrm{e} = \int_{V_\mathrm{e}} \mathbf{B}_\mathrm{e}^\mathrm{T}\mathbf{E}\mathbf{B}_\mathrm{e}\mathrm{d}V. \tag{19}$$

This relation can be exploited to look into other relations between stresses and strains as well, e.g., for (fractional) visco-elasticity. So if the stress–strain relation is given by a fractional law with a viscosity tensor $\mathbf{C}$

$$\boldsymbol{\sigma} = \mathbf{C}D_t^\alpha\boldsymbol{\varepsilon}, \tag{20}$$

in which the stress tensor written as vector is

$$\boldsymbol{\sigma} = \left[\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \tau_{xy}\ \tau_{xz}\ \tau_{yz}\right]^\mathrm{T} \tag{21}$$

and the strain tensor written as vector is

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}\right]^\mathrm{T}, \tag{22}$$

then the corresponding relation between element forces and fractional derivatives of element displacements is

$$\mathbf{F}_{\mathrm{e,vis}} = \mathbf{C}_\mathrm{e}D_t^\alpha\mathbf{U}_\mathrm{e}. \tag{23}$$

The fractional element viscosity matrix is then constructed in full analogy to the element stiffness matrix, the only difference being that the material stiffness tensor is replaced by the material fractional damping tensor:

$$\mathbf{C}_\mathrm{e} = \int_{V_\mathrm{e}} \mathbf{B}_\mathrm{e}^\mathrm{T}\mathbf{C}\mathbf{B}_\mathrm{e}\mathrm{d}V. \tag{24}$$

For isotropic materials, the elasticity and viscosity tensors are described by at most two parameters. In the following it is assumed for simplicity that Poisson's ratio ν and the corresponding parameter in the viscosity tensor have equal values. This can be formalized as

$$\mathbf{E} = E\mathbf{N}(\nu);\ \ \mathbf{C} = C\mathbf{N}(\nu), \tag{25}$$

in which E denotes the scalar modulus of elasticity, C is the scalar modulus of fractional viscoelasticity, and $\mathbf{N}(\nu)$ is a tensor depending only on Poisson's ratio ν. The explicit form of the tensor $\mathbf{N}(\nu)$ used in this paper is—for 3D visco-elasticity—given by

$$\mathbf{N}(\nu)=\frac{1}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0\\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0\\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2\end{bmatrix}. \tag{26}$$

For 2D visco-elasticity (plain stress) with a stress tensor

$$\sigma=\left[\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}\right] \tag{27}$$

and a strain tensor

$$\varepsilon=\left[\varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy}\right], \tag{28}$$

this reduces to

$$\mathbf{N}(\nu)=\frac{1}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}. \tag{29}$$

For each type of finite element, the appropriate formulation has to be chosen. For the sake of completeness, in the case of 1D elasticity, the tensor $\mathbf{N}$ reduces to a scalar unit value.

The required coding was done within the multi-purpose software package slangTNG [2]. The overall strategy to implement fractional visco-elasticity in a Finite Element context is to assemble the global stiffness matrix $\mathbf{K}_0$ with unit modulus of elasticity and fixed Poisson's ratio ν. In the next step, the actual stiffness matrix is generated by multiplying with the actual modulus of elasticity E, and the fractional component matrix arising from the fractional Kelvin–Voigt material is generated by multiplying $\mathbf{K}_0$ with C [cf. Eq. (25)]

$$\mathbf{K}=E\mathbf{K}_0;\ \ \mathbf{C}=C\mathbf{K}_0. \tag{30}$$

If the structure contains different materials with different visco-elastic characteristics, then the stiffness matrix assembly has to be done separately for all elements within a group having the same materials (e.g., all steel columns and all concrete floor slabs in a building).

The inertia effects are considered in terms of the kinetic energy T which can be expressed by the shape functions such that within one finite element we have the kinetic energy

$$T_\mathrm{e}=\frac{1}{2}\int\limits_{V_\mathrm{e}}\rho\dot{u}^2\mathrm{d}V_\mathrm{e}=\frac{1}{2}\int\limits_{V_\mathrm{e}}\rho\dot{\mathbf{u}}^\mathrm{T}\dot{\mathbf{u}}\mathrm{d}V_\mathrm{e}. \tag{31}$$

The displacement field $\mathbf{u}(x, y, z)$ within the element is expressed as

$$\mathbf{u}=\mathbf{H}_\mathrm{e}\mathbf{U}_\mathrm{e}, \tag{32}$$

in which $\mathbf{H}_\mathrm{e}$ denotes the matrix of displacement shape functions for this element. From this, the velocity field $\dot{\mathbf{u}}$ within the element can be written as

$$\dot{\mathbf{u}}=\mathbf{H}_\mathrm{e}\dot{\mathbf{U}}_\mathrm{e}. \tag{33}$$

Following Lagrange's method, the kinetic energy must be first derived with respect to all components of the generalized velocity vector and then with respect to time, which results in the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_\mathrm{e}}{\partial\dot{U}_\mathrm{e}}=\int\limits_{V_\mathrm{e}}\rho\mathbf{H}^\mathrm{T}\mathbf{H}\ddot{\mathbf{U}}_\mathrm{e}\mathrm{d}V_\mathrm{e}=\mathbf{M}_\mathrm{e}\ddot{\mathbf{U}}_\mathrm{e}. \tag{34}$$

Note that the derivative with respect to a vector means derivation with respect to all components of this vector. The matrix

$$\mathbf{M}_\mathrm{e} = \int_{V_\mathrm{e}} \rho \mathbf{H}^\mathrm{T} \mathbf{H} \ddot{\mathbf{U}}_\mathrm{e} \mathrm{d}V_\mathrm{e} \tag{35}$$

is the *element mass matrix*. Together with the generalized visco-elastic forces as derived above and the externally applied loads $\mathbf{P}_\mathrm{e}(t)$, this leads to the element equations of motion

$$\mathbf{M}_\mathrm{e}\ddot{\mathbf{U}}_\mathrm{e} + \mathbf{C}_\mathrm{e} D_t^\alpha \mathbf{U}_\mathrm{e} + \mathbf{K}_\mathrm{e}\mathbf{U}_\mathrm{e} = \mathbf{P}_\mathrm{e}(t). \tag{36}$$

Standard assembly of the element equations and introducing the appropriate boundary conditions finally leads to the equation of motion for the finite element assembly [1]

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C} D_t^\alpha \mathbf{U} + \mathbf{K}\mathbf{U} = \mathbf{P}(t). \tag{37}$$

4.2 Newmark method

The Newmark method (e.g., [1]) is an implicit direct integration method, i.e., for a known displacement vector $\mathbf{U}(t)$ and velocity vector $\dot{\mathbf{U}}(t)$, and it satisfies the equations of motion (dynamic equilibrium) at time $t + \Delta t$. For the special case of the constant average acceleration method, the kinematic assumptions are

$$\begin{aligned} \dot{\mathbf{U}}(t+\Delta t) &= \dot{\mathbf{U}}(t) + \frac{\Delta t}{2}\left[\ddot{\mathbf{U}}(t) + \ddot{\mathbf{U}}(t+\Delta t)\right], \\ \mathbf{U}(t+\Delta t) &= \mathbf{U}(t) + \Delta t\dot{\mathbf{U}}(t) + \frac{\Delta t^2}{4}\left[\ddot{\mathbf{U}}(t) + \ddot{\mathbf{U}}(t+\Delta t)\right]. \end{aligned} \tag{38}$$

These two equations together with the equation of motion at time $t + \Delta t$

$$\mathbf{M}\ddot{\mathbf{U}}(t+\Delta t) + \mathbf{C} D_t^\alpha \mathbf{U}(t+\Delta t) + \mathbf{K}\mathbf{U}(t+\Delta t) = \mathbf{P}(t+\Delta t) \tag{39}$$

define the integration scheme. Re-arranging these equations and using the Grünwald–Letnikov sum for representing the fractional derivative D_t^α, we obtain the recursion scheme

$$\begin{aligned} \left(\frac{4}{\Delta t^2}\mathbf{M} + q_0\mathbf{C} + \mathbf{K}\right)\mathbf{U}(t+\Delta t) &= \mathbf{P}(t+\Delta t) + \mathbf{M}\left[\frac{4}{\Delta t^2}\mathbf{U}(t) + \frac{4}{\Delta t}\dot{\mathbf{U}}(t) + \ddot{\mathbf{U}}(t)\right] \\ &\quad -\mathbf{C}\sum_{k=1}^{m} q_k \mathbf{U}(t + \Delta_t - k\Delta t). \end{aligned} \tag{40}$$

It should be noted that in the summation up to m in the last term of the previous equation is it implicitly assumed that the displacements are zero for $t < 0$. So effectively the number of terms to be actually considered will increase with increasing time. For updating the acceleration, the second of Eq. (38) is re-arranged into

$$\ddot{\mathbf{U}}(t+\Delta t) = \frac{4}{\Delta t^2}\left[\mathbf{U}(t+\Delta t) - \mathbf{U}(t)\right] - \frac{4}{\Delta t}\dot{\mathbf{U}}(t) - \ddot{\mathbf{U}}(t). \tag{41}$$

For updating the velocity, the first of Eq. (38) can then be directly used.

While strictly speaking the time step for the Grünwald–Letnikov series does not have to match the time step used for the Newmark scheme, it is computationally advantageous to use the same time step Δt (cf. [20]).

5 Numerical examples

5.1 Creep response of a Timoshenko Beam

As a reference example to check the validity of the numerical procedure consider the simple Timoshenko beam previously analyzed in [17]. A sketch of this beam is shown in Fig. 4. For the numerical analysis, the values $L = 2$, $b = 0.2$, $q = 10^6$ are chosen. The material is characterized by fractional viscosity with $C_\alpha = 10^{11}$ and $\alpha = 0.25$. Inertia effects are neglected in this example. The midspan deflection of the beam is computed using 20 finite Timoshenko beam elements (after [18]). The time resolution for the Newmark scheme is $\Delta t = 0.1$.

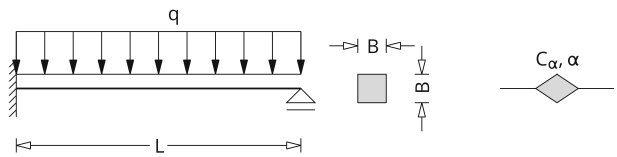

Fig. 4 Simple Timoshenko beam and constitutive model

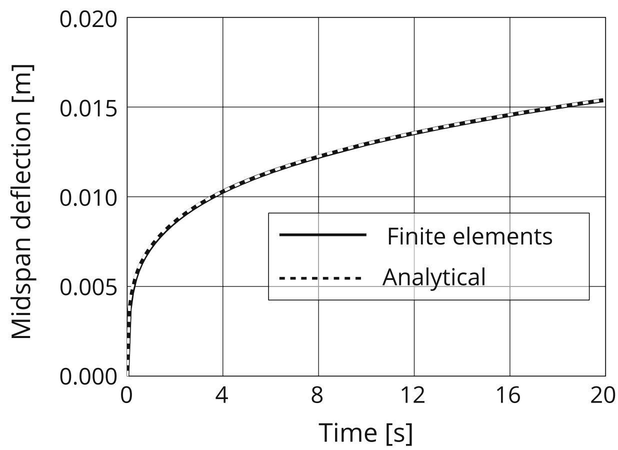

Fig. 5 Midspan deflection of Timoshenko beam. Analytical result taken from [17].

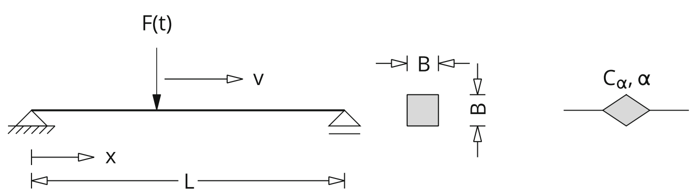

Fig. 6 Euler–Bernoulli beam under moving load

The results are shown in Fig. 5. In this figure, the numerical solution according to [17] is shown as well. The match is very good which indicates that the Newmark procedure as described works extremely well.

As a variation, now consider an Euler–Bernoulli beam under a traveling load $F(t) = 10^5$ (cf. Fig. 6). It is assumed that this load moves with constant velocity $v = 0.2$ from the left support to the right support. The beam has a square cross section of size $b = 0.2$. Its fractional viscosity modulus is $C_\alpha = 10^5$, and the fractional order is chosen as $\alpha = 0.75$ and $\alpha = 0.25$. Again, inertia effects are neglected in this example. For this case, the midspan deflection as a function of time is shown in Fig. 7. The results are compared to analytical solutions obtained by modal superposition according to the procedure reported in Sect. 4, having considered only the first three modes [17]. Again, the agreement between the results of the two methods is excellent.

5.2 Dynamic response of an Euler–Bernoulli beam

A simply supported beam (cf. Fig. 8) of length $L = 5$ with a cross section defined in terms of cross-sectional area A and moment of inertia I under a static concentrated load $F_0 = 10$ kN at position $x_F = 2$ is considered [6]. This load is suddenly applied at time $t = 0$. The ensuing free vibration response is computed assuming fractional visco-elastic behavior of the Kelvin–Voigt type characterized by a modulus of elasticity E and a springpot with a coefficient C_α and an order α. The mass density is ρ. Numerical values are $A = 1$ m^2, $I = 8.226\times10^{-6}$ m^4, $E = 21$ GPa, $c = 965$ MPa, $\rho = 7850$ kg/m^3. The coefficient C_α is computed according to $C_\alpha = E\left(\frac{c}{E}\right)^\alpha$. Figure 9 shows the displacement response at the location $x_D = 2.35$ m.

It can be seen that the Finite Element results agree extremely well with the analytical result based on modal superposition as reported in [6].

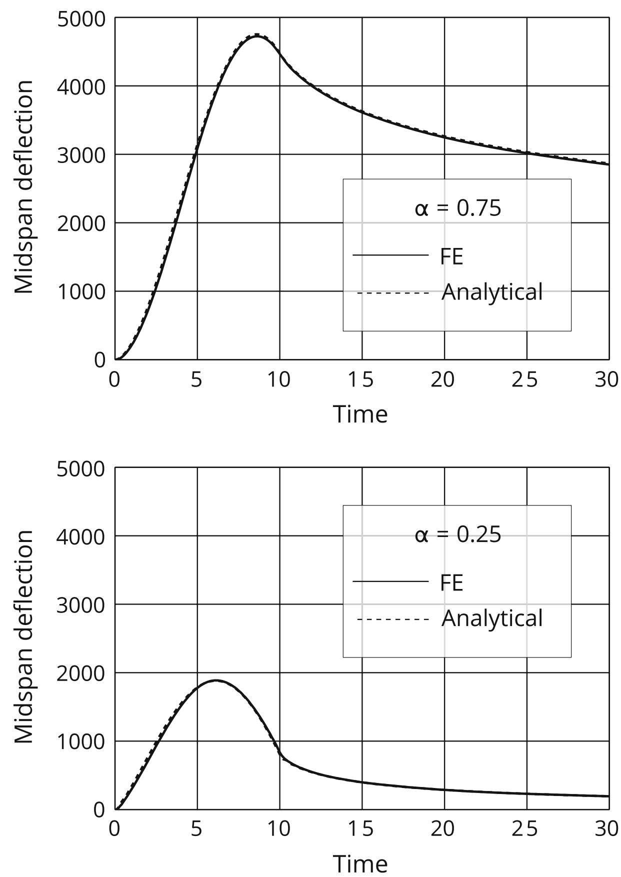

Fig. 7 Midspan deflection of Euler–Bernoulli beam under moving load for two different values of the fractional order α

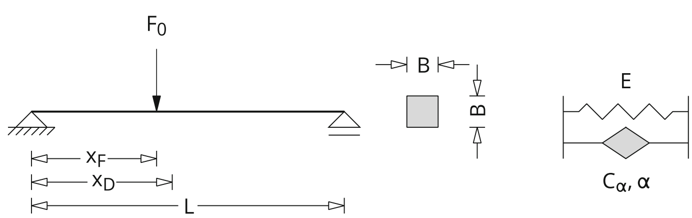

Fig. 8 Simple Euler–Bernoulli beam and constitutive model

5.3 Earthquake response of a multi-storey building

This somewhat more complex example demonstrates the application of the procedure as outlined to a structural model with a larger number of degrees of freedom (378 DOF). A sketch of this 7-storey structure is shown in Fig. 10. The model consists of columns, shear walls and floor slabs. The columns are modeled by beam elements with quadratic cross sections (30 × 30 cm), the shear walls and the floor slabs are modeled by triangular plate elements with a thickness of 20 cm. The material properties are chosen close to reinforces concrete (mass density ρ = 2500 kg/m^3, modulus of elasticity $E = 30$ GPa). All elements are assumed to have the same fractional visco-elasticity with a coefficient $C_\alpha = 1.5$ GPa (s/m)$^\alpha$, and α is varied in the analysis.

The dynamic response to the well-known El Centro (NS) acceleration acting in x-direction is computed. Due to the lack of symmetry of the building plan, considerable torsional motion is introduced. The horizontal displacements u_x and u_y of the top front corner node (cf. Fig. 10) in x- and y-directions are shown in Fig. 11 for different values of the fractional order α.

From a structural engineering perspective, internal forces such as bending moments or shear forces are more relevant for structural design as compared to displacements. Figure 12 compares the bending moments

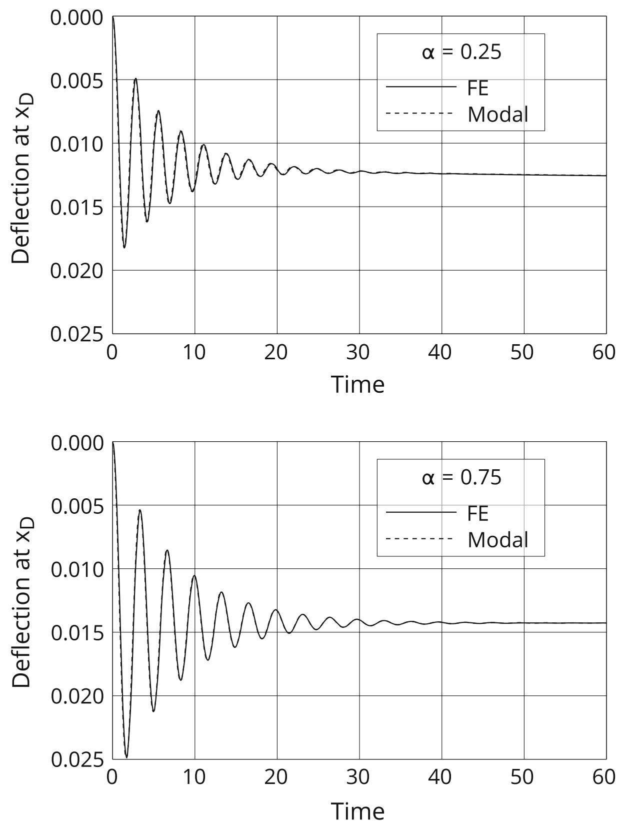

Fig. 9 Dynamic response of beam under suddenly applied concentrated load for different values of α. Modal superposition results taken from [6]

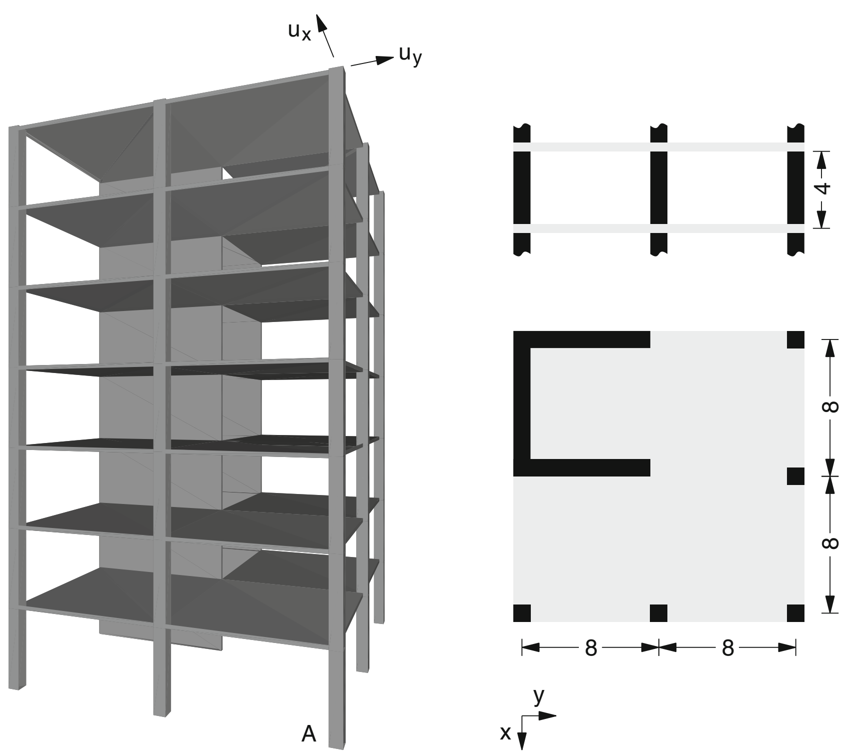

Fig. 10 Frame structure under earthquake loading

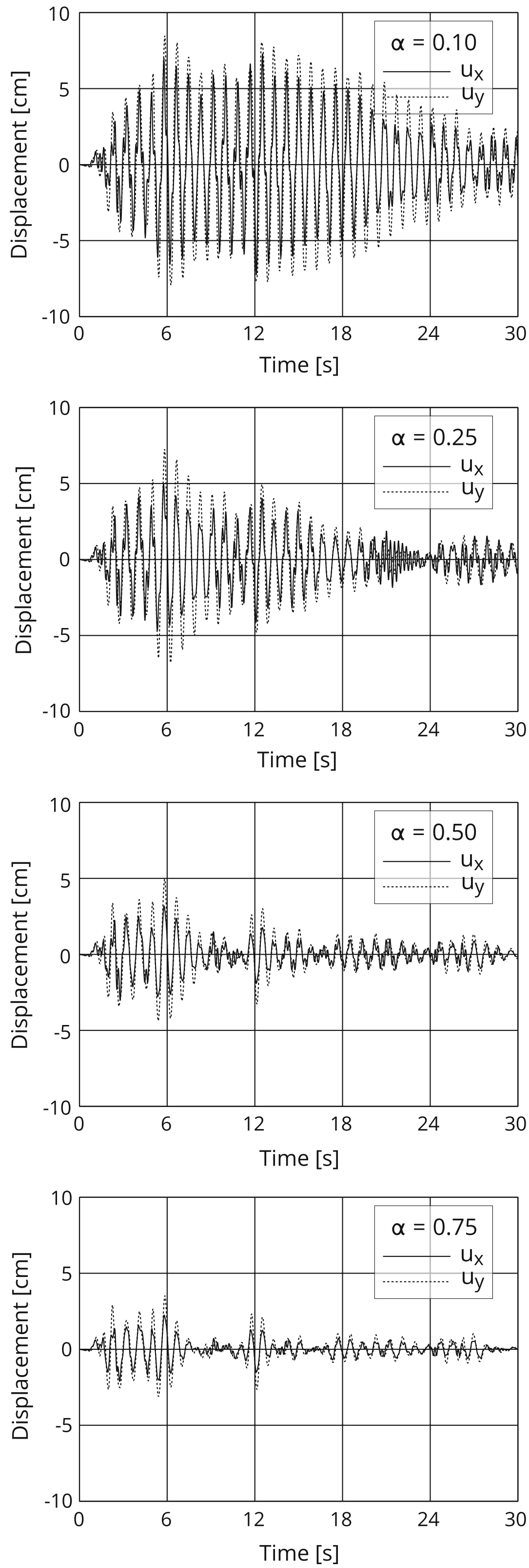

Fig. 11 Dynamic displacement response of frame structure under earthquake loading

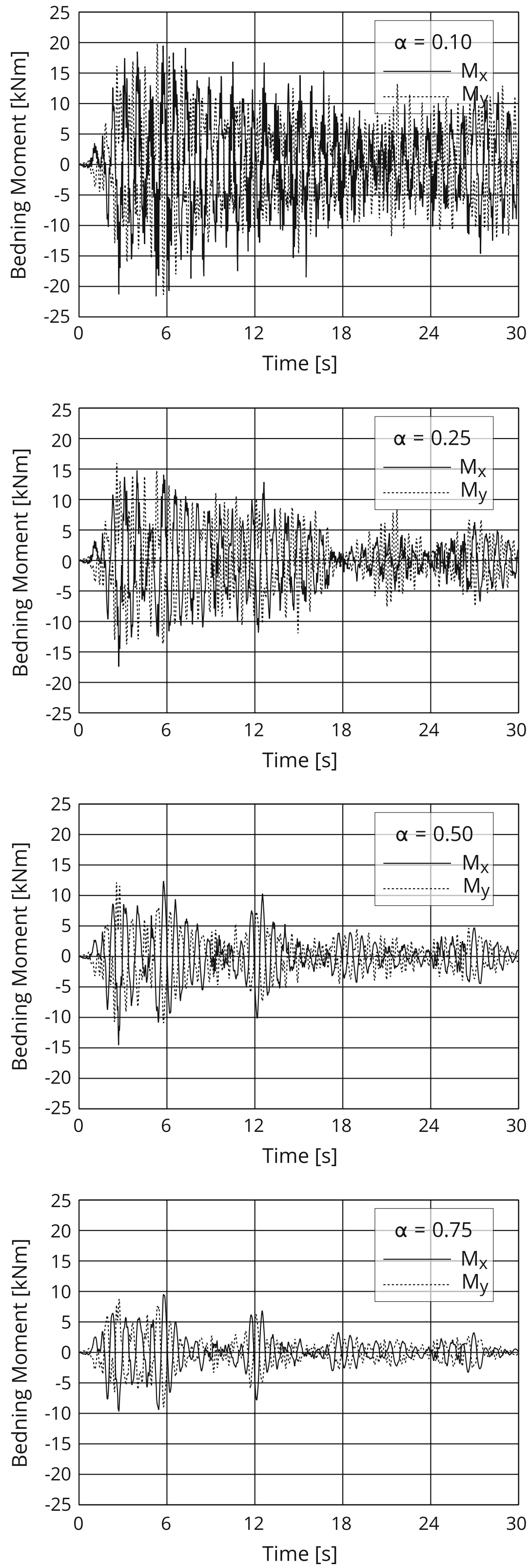

Fig. 12 Dynamic bending moment response of frame structure under earthquake loading

M_x and M_y about the x- and y-axes, respectively, in support point A (cf. Fig. 10) for different values of the fractional order α.

The bending moments exhibit a similarly decreasing trend with increasing α as the displacements. The frequency content, however, is quite different in the sense that substantially more high-frequency components are present in the bending moments. This is not too surprising, as the moments are obtained from the displacements by taking second spatial derivatives.

6 Concluding remarks

A straightforward Finite Element formulation for the numerical treatment of fractional viscoelasticity has been presented. This formulation has the advantage of being non-intrusive with respect to the material formulations used in the FE code. Hence, a largely code-independent algorithm could be developed. This algorithm is based on the standard Newmark implicit time-stepping scheme in conjunction with a discrete version of the Grünwald–Letnikov representation of Caputo's fractional derivative.

The algorithm has been applied to several test cases. The numerical results show excellent agreement to results previously obtained using alternative method based on modal superposition. Due to the memory inherent in the fractional viscoelasticity and the discrete Grünwald–Letnikov formulation, it is necessary to store the past displacement history for all degrees of freedom. This requires some amount of storage which, however, is not significant in relation to storage requirements related to the system matrices, in particular the factorized stiffness matrix.

Since all operations involved are carried out at the level of system matrices and vectors (and not at material point level), it is very easy and straightforward to implement the solution procedure into any general-purpose Finite Element code.

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