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On one new approach to the solving of an elasticity mixed plane problem for the semi-strip

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Abstract It is well known that the main approaches of the analytical solving of the elasticity mixed plane problems for a semi-strip are based on the different representations of the equilibrium equations' solutions: the representations through the harmonic and by harmonic functions, through the stress function, Fadle–Papkovich functions and so on. The main shortcoming of these approaches is connected with the fact that to obtain the expression for the real mechanical characteristics, one should execute additional operations, not always simple ones. The approach that is proposed in this paper allows the direct solution of the equilibrium equations. With the help of the matrix integral transformation method applied directly to the equilibrium equations, the initial boundary problem is reduced to a vector boundary problem in the transformation's domain. The use of matrix differential calculations and Green's matrix function leads to the exact vector solution of the problem. Green's matrix function is constructed in the form of a bilinear representation which simplifies the calculations. The method is demonstrated by the solving of the thermoelastic problem for the semi-strip. The zones and conditions of the strain stress occurrence on the semi-strip's lateral sides, important to engineering applications, are investigated.

Mathematics Subject Classification 74B05 · 14B05

1 Introduction

The plain elasticity problems for a semi-strip have been solved by many authors, and this can be explained by the importance of these problems as the model example for different engineering applications. Some of the investigations, due to the obtained theoretical results, became classics, such as the works of Kolosov, Muskchelishvili, Babeshko, Vorovich [1–3]. The methods proposed in these works are based on the use of the complex variable functions' theory and Koshy-type integrals. A detailed bibliography dedicated to the integral transformation method in such problem-solving is given in [4]. The standard methods which are based on the presentation of the solution in Papkovich–Neuber form or on the presentation of the solution in the form of the superposition of the few harmonic functions are often used during the solving.

The importance of the elasticity problem-solving for a continuous half-strip is due to the necessity of solving more complex problems of stress concentration around the cracks and the inclusions inside it.

In [5], the elastostatic plane problem of an infinite strip having a circular hole and containing two symmetrically located internal cracks perpendicular to the boundary is formulated in terms of triply coupled integral

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equations. The solution of the problem is obtained for various crack geometries and for uniaxial tension applied to the strip away from the crack region. In [6], the three-dimensional (3-D) elastodynamic interaction between a penny-shaped crack and a thin elastic interlayer joining two elastic half-spaces is investigated by an improved boundary integral equation method or boundary element method.

A short review of the different approaches to the solving of the plane elasticity problems for an elastic half-strip is given below.

In [7], the authors considered the problem on the symmetrically loaded semi-strip fixed by the short edge. The solving is reduced to the Fredholm's integral equation of the first kind with regard to the normal stress at the fixed edge. As is noted in the work, this approach complicates the equation's investigation and its numerical solving also. With the help of the stress function, in [8,9] the problem was solved for a semi-strip with free longitudinal sides, when the self-equilibrium loading influences the short edge. The application of the sine transformation to the equation and a stress function reduced the problem of the semi-strip's strain with free longitudinal edges and fixed short edge to an infinite system of linear equations in [10]. The approach where the problem is formulated for a stress function is used also in [11], but here Laplace's integral transformation is used. Often, the stress function is performed as combination of Fourier integrals and series. An analogous approach is used in [12–15] for the problems on a semi-strip. As a result, the authors obtain an infinite system of algebraic equations (the question of its regularity they leave without research).

In [16], the problem for a semi-strip with free lateral edges and a short edge loaded by displacements is considered. It is supposed that the displacements have the form of polynomials. The method of orthogonal polynomials is used to solve the problem. The variation method is used for the analogous problem in [17]. When the lateral edges of the semi-strip are free and the short edge is under the concentrated force, the problem-solving can be based on the energetic method [18]. In [19], the authors construct a special system of biorthogonal functions, with the help of which they solve the problem for a load on the short edge semi-strip. In [20], the mixed elasticity problem is solved with the help of the variation Castigliano method, where the varying parameters are expressed through the solving of an infinite system of linear algebraic equations.

Another big class of problems in the solving of the elasticity problem for a semi-strip is based on the use of Fadle–Papkovich functions. In [21], the solution was constructed for the classical boundary valued problem for a rectangular semi-strip with free lateral edges. The same problem was solved in [22], where the author applies the Borel's transformation in the class of quasi-integer functions of the exponential kind. In [23], also the Fadle–Papkovich functions are used for the representation of the solution in the form of series by these functions' system.

As can be seen from the review, the two main approaches are used—the analytical approach and the numerical analytical one. The choice between them is determined by the boundary conditions at the semi-strip's edge. All these approaches use the representation of the equilibrium equations' solutions through the harmonic, by harmonic and other functions. So to obtain the real mechanical displacements and stress requires an additional, often not very simple, steps.

With the aim of avoiding this shortcoming in this paper the method, which was proposed by Popov [24], is used. According to it, the integral transformations are applied directly to the equilibrium equations and boundary conditions of a problem. In most cases, it leads to a one-dimensional boundary problem in the transformation's domain. The last one is formulated as a vector boundary value problem and solved with the apparatuses of the matrix differential calculations and Green's matrix function. It leads to the solving of a singular integral equation. The orthogonal polynomials' method [25], taking into consideration the real singularities of the unknown function at the ends of the integration interval, is proposed for the equation solving.

2 The statement of a problem

The elastic (G is the shear modulus, μ is a Poison's coefficient) semi-strip, $0 < x < a, 0 < y < \infty$ is loaded at the edge $y = 0, 0 < x < a$,

$$\sigma_y|_{y=0} = -p(x), \quad \tau_{yx}|_{y=0} = 0, \quad 0 < x < a, \tag{1}$$

where $p(x)$ is a known function. At the edges $x = 0, 0 < y < \infty$ and $x = a,\ 0 < y < \infty$, the boundary conditions of the general type are given:

$$U_0[f] = 0, \quad U_1[f] = 0. \tag{2}$$

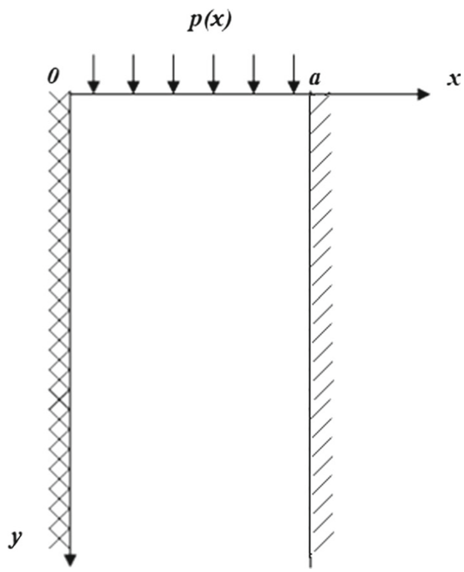

Fig. 1 Geometry and coordinate system of the semi-strip

Here, $U_i\left[f(x)\right] = \alpha_i f(a_i, y) + \beta_i f'(a_i, y), i = 0, 1$, are boundary functionals of general form (they will be detailed later), $f(x, y) = (u(x, y), v(x, y))^T$ is the vector of displacements, and $u(x, y) = u_x(x, y), v(x, y) = u_y(x, y)$ are the displacements which satisfy the Lamé's equations (Fig. 1).

The equations are written in the following form:

$$\begin{cases} \mu_* \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + \mu_0 \frac{\partial^2 v(x,y)}{\partial x \partial y} + X_1(x, y) = 0, \\ \frac{\partial^2 v(x,y)}{\partial x^2} + \mu_* \frac{\partial^2 v(x,y)}{\partial y^2} + \mu_0 \frac{\partial^2 u(x,y)}{\partial x \partial y} + X_2(x, y) = 0, \end{cases} \tag{3}$$

where $\mu_0 = \frac{1}{1-2\mu}$, $\mu_* = \mu_0 + 1$, $X_1(x, y) = X_x(x, y)$, $X_2(x, y) = X_y(x, y)$ are the components of the volume force. There can be, for example, the corresponding derivatives of the temperature that one obtains after solving the thermal conductivity problem for a semi-strip. After the expression of the constants μ_0, μ_* through the Muskchelishvili constant $\kappa = 3 - 4\mu$, one obtains the system (3) in another form:

$$\begin{cases} \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\kappa-1}{\kappa+1} \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{2}{\kappa+1} \frac{\partial^2 v(x,y)}{\partial x \partial y} + \frac{\kappa-1}{\kappa+1} X_1(x, y) = 0, \\ \frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\kappa+1}{\kappa-1} \frac{\partial^2 v(x,y)}{\partial y^2} - \frac{2}{\kappa-1} \frac{\partial^2 u(x,y)}{\partial x \partial y} + X_2(x, y) = 0. \end{cases} \tag{4}$$

The boundary conditions are reformulated with the terms of the displacements

$$\begin{aligned} &2G\mu_0 \left(\mu \frac{\partial u(x,0)}{\partial x} + (1-\mu) \frac{\partial v(x,0)}{\partial y} \right) = -p(x), \\ &\frac{\partial u(x,0)}{\partial y} + \frac{\partial v(x,0)}{\partial x} = 0. \end{aligned} \tag{5}$$

One needs to solve the boundary value problem (2), (4), (5) to estimate the stress state of the semi-strip and to evaluate the absolute values and conditions of the stretching stress occurrence at the lateral side of the strip.

3 The general solving scheme of the problems on the semi-strip stress state estimation

The Fourier's transformation is applied to the system of Lamé's equation and to the boundary conditions by the scheme

$$\begin{bmatrix} u_\beta(x), X_{1\beta}(x) \\ v_\beta(x), X_{2\beta}(x) \end{bmatrix} = \int_0^\infty \begin{bmatrix} u(x, y), X_1(x, y) \\ v(x, y), X_2(x, y) \end{bmatrix} \begin{bmatrix} \cos \beta y \\ \sin \beta y \end{bmatrix} \mathrm{d}y \tag{6}$$

with the inverse formula

$$\begin{bmatrix} u(x,y), X_1(x,y) \\ v(x,y), X_2(x,y) \end{bmatrix} = \frac{2}{\pi}\int_0^\infty \begin{bmatrix} u_\beta(x), X_{1\beta}(x) \\ v_\beta(x), X_{2\beta}(x) \end{bmatrix} \begin{bmatrix} \cos\beta y \\ \sin\beta y \end{bmatrix} \mathrm{d}\beta. \tag{7}$$

After this, the initial equations (4) have the form

$$\begin{cases} \frac{\mathrm{d}^2 u_\beta(x)}{\mathrm{d}x^2} - \frac{\beta^2(\kappa-1)}{\kappa+1}u_\beta(x) + \frac{2\beta}{\kappa+1}\frac{dv_\beta(x)}{\mathrm{d}x} = \frac{3-\kappa}{\kappa+1}\chi'(x) - \frac{\kappa-1}{\kappa+1}X_{1\beta}(x), \\ \frac{\mathrm{d}^2 v_\beta(x)}{\mathrm{d}x^2} - \frac{\beta^2(\kappa+1)}{\kappa-1}v_\beta(x) - \frac{2\beta}{\kappa-1}\frac{du_\beta(x)}{\mathrm{d}x} = -\beta\frac{\kappa+1}{\kappa-1}\chi(x) - X_{2\beta}(x). \end{cases} \tag{8}$$

Here the new unknown function $\chi(x) = v(x,0)$, $\chi'(x) = v'(x,0)$ is introduced. As can be seen from the second boundary condition (5), $u^\bullet(x,0) = -\chi'(x)$, so the second condition is satisfied automatically.

With the aim to reduce the problem to a vector boundary problem, one must introds the vectors and the matrixes $\vec{y}_\beta(x) = \begin{pmatrix} u_\beta(x) \\ v_\beta(x) \end{pmatrix}$, $\vec{f}(x) = \begin{pmatrix} \frac{3-\kappa}{\kappa+1}\chi'(x) - \frac{\kappa-1}{\kappa+1}X_{1\beta}(x) \\ -\beta\frac{\kappa+1}{\kappa-1}\chi(x) - X_{2\beta}(x) \end{pmatrix}$, $P = \begin{pmatrix} \frac{\kappa-1}{\kappa+1} & 0 \\ 0 & \frac{\kappa+1}{\kappa-1} \end{pmatrix}$, $Q = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ -\frac{1}{\kappa-1} & 0 \end{pmatrix}$. Then, the equations in vector form will be written as the vector equation $L_2\vec{y}_\beta(x) = \vec{f}(x)$, where L_2 is a differential operator of second order, $L_2\vec{y}_\beta(x) = I\vec{y}_\beta''(x) + 2\beta Q\vec{y}_\beta'(x) - \beta^2 P\vec{y}_\beta(x)$, and I is an identity matrix. The integral transformations also should be applied to the boundary conditions, with the aim to formulate the boundary functionals in the transformations' domain. As a result, the vector boundary problem is constructed:

$$\begin{gathered} L_2\vec{y}_\beta(x) = \vec{f}(x), \\ U_0\left[\vec{y}_\beta\right] = 0, \quad U_1\left[\vec{y}_\beta\right] = 0. \end{gathered} \tag{9}$$

4 The solution of the vector boundary value problem

The solution of the vector boundary problem will be searched as the superposition of a homogenous vector equation's general solution $\vec{y}_\beta^{\,0}(x)$ and a particular solution of the inhomogeneous one $\vec{y}_\beta^{\,1}(x)$,

$$\vec{y}_\beta(x) = \vec{y}_\beta^{\,0}(x) + \vec{y}_\beta^{\,1}(x). \tag{10}$$

These solutions will be constructed with the help of the matrix differential calculation apparatus. As has been shown earlier [26] for the construction of the homogenous vector equation's solution, one must first construct the solution of a homogenous matrix equation

$$L_2 Y_\beta(x) = 0, \quad 0 < x < a. \tag{11}$$

Here, $Y_\beta(x)$ is the matrix of order 2×2. This matrix $Y_\beta(x)$ should be chosen in a form $Y_\beta(x) = e^{\xi x}I$ and be substituted into the matrix equation (11). As a result, the equality $L_2 e^{\xi x}I = M(\xi)e^{\xi x}$ is obtained, where the matrix $M(\xi)$ has the form

$$M(\xi) = I\xi^2 + 2\beta Q\xi - \beta^2 P = \begin{pmatrix} \xi^2 - \beta^2\frac{\kappa-1}{\kappa+1} & \frac{2\beta\xi}{\kappa+1} \\ -\frac{2\beta\xi}{\kappa-1} & \xi^2 - \beta^2\frac{\kappa+1}{\kappa-1} \end{pmatrix}. \tag{12}$$

According to [27], the solution of the homogenous matrix equation is constructed by a formula

$$Y(x) = \frac{1}{2\pi i}\oint_C e^{\xi x}M^{-1}(\xi)\mathrm{d}\xi, \tag{13}$$

where $M^{-1}(\xi)$ is the inverse matrix to $M(\xi)$. The closed contour C covers all singularity points of the matrix $M^{-1}(\xi)$. For the estimation of these singularity points, we use the fact that the inverse matrix can be expressed as the ratio of a transposed matrix of the algebraic additions and a determinant of the initial matrix. So the determinant of the matrix $M(\xi)$ is found as

$$\det M(\xi) = \xi^4 - 2\beta^2\xi^2 + \beta^4 = (\xi - \beta)^2(\xi - \beta)^2. \tag{14}$$

After transposing the matrix of the algebraic additions, one can write the view of the inverse matrix:

$$M^{-1}(\xi) = \frac{1}{(\xi - \beta)^2(\xi + \beta)^2}\begin{pmatrix} \xi^2 - \beta^2\frac{\kappa+1}{\kappa-1} & -\frac{2\beta\xi}{\kappa+1} \\ \frac{2\beta\xi}{\kappa-1} & \xi^2 - \beta^2\frac{\kappa-1}{\kappa+1} \end{pmatrix}. \tag{15}$$

After the substitution of the constructed inverse matrix (15) into the expression (13), one can find that the expression under the contour integral has two multiple poles $\xi = \beta, \xi = -\beta$. With the help of the residual theorem and after the calculation of the residuals, one obtains the following matrix system of the fundamental matrix solutions:

$$Y_1(x) = \frac{e^{\beta x}}{2}\begin{pmatrix} \frac{\kappa-\beta x}{\beta(\kappa-1)} & -\frac{x}{\kappa+1} \\ \frac{x}{\kappa-1} & \frac{\kappa+\beta x}{\beta(\kappa+1)} \end{pmatrix}, \quad Y_2(x) = \frac{e^{-\beta x}}{2}\begin{pmatrix} -\frac{\kappa+\beta x}{\beta(\kappa-1)} & \frac{x}{\kappa+1} \\ -\frac{x}{\kappa-1} & -\frac{\kappa-\beta x}{\beta(\kappa+1)} \end{pmatrix}. \tag{16}$$

After the substitution of the searched matrixes $Y_1(x)$, $Y_2(x)$ in the formula (10), the solution of the vector equation can be rewritten:

$$\vec{y}_\beta(x) = Y_1(x)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + Y_2(x)\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \vec{y}^{\,1}_\beta(x), \tag{17}$$

where the constants $c_i,\ i = \overline{1,4}$ are founded from the boundary conditions.

5 The construction of the Green's matrix function

For the obtaining of the vector boundary problem's particular solution, one needs to construct the Green's matrix function. It can be done with the help of the matrix integral transformations' method [25]. Let's construct the Green's matrix function $G(x,\xi)$ for the vector boundary problem of the structure

$$\begin{cases} L_2\vec{y}(x) = \vec{f}(x), \\ V_i\left[\vec{y}(x)\right] = 0, \quad i = 0, 1, \end{cases} \tag{18}$$

where V_0, V_1 are the boundary functionals of the following form:

$$\begin{aligned} V_0\left[\vec{y}(x)\right] &= \alpha_n\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\vec{y}(0) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\vec{y}'(0), \\ V_1\left[\vec{y}(x)\right] &= \alpha_n\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\vec{y}(a) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\vec{y}'(a). \end{aligned} \tag{19}$$

The kernel of the integral transformation was taken in the form

$$H(x,\alpha_n) = \begin{pmatrix} \sin\alpha_n x & 0 \\ 0 & \cos\alpha_n x \end{pmatrix}, \quad \alpha_n = \frac{n\pi}{a}, \quad n = 0, 1, 2\ldots \tag{20}$$

The multiplication of Eq. (18) on both sides by the kernel (20) and the following integration by parts on the segment $[0; a]$ lead to the correspondence

$$\Omega_\beta(\alpha_n)\vec{y}_n = \vec{f}_n, \tag{21}$$

where $\vec{y}_n = \int_0^a \vec{y}(x) H(x,\alpha_n)\mathrm{d}x$, $\tilde{Q} = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ \frac{1}{\kappa-1} & 0 \end{pmatrix}$, $\Omega_\beta(\alpha_n) = -I\alpha_n^2 - 2\beta\alpha_n\tilde{Q} - \beta^2 P = \begin{pmatrix} -\alpha_n^2 - \beta^2\frac{\kappa-1}{\kappa+1} & -\frac{2\beta\alpha_n}{\kappa+1} \\ -\frac{2\beta\alpha_n}{\kappa-1} & -\alpha_n^2 - \beta^2\frac{\kappa+1}{\kappa-1} \end{pmatrix}$.

The solution of Eq. (21) has the form

$$\vec{y}_n = \Omega_\beta^{-1}(\alpha_n)\,\vec{f}_n, \tag{22}$$

where

$$\Omega_\beta^{-1}(\alpha_n) = \frac{1}{(\alpha_n^2+\beta^2)^2}\begin{pmatrix} -\frac{\alpha_n^2(\kappa-1)+\beta^2(\kappa+1)}{\kappa-1} & \frac{2\beta\alpha_n}{\kappa+1} \\ \frac{2\beta\alpha_n}{\kappa-1} & -\frac{\alpha_n^2(\kappa+1)+\beta^2(\kappa-1)}{\kappa+1} \end{pmatrix}.$$

The inverse integral transformation is applied to the solution $\vec{y}_n = \begin{pmatrix} y_{n1} \\ y_{n2} \end{pmatrix}$. In component-wise form, it can be written as

$$\begin{aligned} y_1(x) &= \frac{2}{a}\sum_{n=1}^{\infty} y_{n1}\sin(\alpha_n x) = \frac{2}{a}\sum_{n=0}^{\infty}{}' y_{n1}\sin(\alpha_n x), \\ y_2(x) &= \frac{y_{02}}{a} + \frac{2}{a}\sum_{n=1}^{\infty} y_{n2}\cos(\alpha_n x) = \frac{2}{a}\sum_{n=0}^{\infty}{}' y_{n2}\cos(\alpha_n x). \end{aligned} \tag{23}$$

In vector form, one has $\vec{y}(x) = \frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\,\vec{y}_n$; here, dot denotes that the zero term should be multiplied by 1/2. On the other hand—$\vec{y}_n = \int_0^a H(x,\alpha_n)\,\vec{y}(x)\mathrm{d}x$.

After the union of these two results, the formula for the vector calculations is obtained: $\vec{y}(x) = \frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\,\Omega_\beta^{-1}(\alpha_n)\,\vec{f}_n$. Taking into consideration the formula $\vec{f}_n = \int_0^a H(\xi,\alpha_n)\vec{f}(\xi)\,\mathrm{d}\xi$, one can write the correspondence

$$\begin{aligned} \vec{y}(x) &= \frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\,\Omega_\beta^{-1}(\alpha_n)\int_0^a H(\xi,\alpha_n)\vec{f}(\xi)\,\mathrm{d}\xi \\ &= \int_0^a \left[\frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\,\Omega_\beta^{-1}(\alpha_n)\,H(\xi,\alpha_n)\right]\vec{f}(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{24}$$

From the last formulae, it shows that Green's matrix function of the boundary valued problem (16) can be constructed in the form

$$G(x,\xi) = \frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\,\Omega_\beta^{-1}(\alpha_n)\,H(\xi,\alpha_n). \tag{25}$$

The representation (25) is the bilinear expansion for the Green's matrix function.

One can be sure that all properties of Green's function are executed, and in particular that the boundary conditions in (16) are satisfied:

$$V_0[G(x,\xi)] = 0, \quad V_1[G(x,\xi)] = 0. \tag{26}$$

For each component of the Green's matrix, summarizing was executed with regard to the known formulae (1.445(1–2), [28]). The representation of the Green's matrix function's components is shown in Appendix A.

The solution of the inhomogeneous boundary problem is constructed in the form

$$\vec{y}_\beta(x) = Y_1(x)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + Y_2(x)\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \int_0^a G(x,\xi)\,f(\xi)v\xi. \tag{27}$$

The application of the inverse integral transformations' formulae (5) to the expression (17) completes the construction of the displacement field.

The solution of the initial problem would be finally found if one would know the unknown function $\chi(x)$ in the right-hand part $f(x)$ of Eq. (9). With the aim to find it, one needs to satisfy the first boundary condition in (5) that is unsatisfied yet. That leads to a singular integral or integro-differential equation with respect to this function. The solving method of this singular equation is the method of the orthogonal polynomials allowing to take into account the real singularity of the solution at the ends of the integration interval [25].

Let us demonstrate the proposed solution method for the particular thermoelasticity problem for an elastic semi-strip.

6 The solution of the thermoelasticity problem for an elastic semi-strip

Let us solve the boundary valued problem (4), (5) for an elastic semi-strip with the following conditions (2) on the lateral sides: The semi-strip's edge $x = 0, 0 < y < \infty$ is fixed, and the edge $x = a, 0 < y < \infty$ is in a smooth contact condition. Thus, the detailed conditions (2) have the form

$$\begin{aligned} &u(0, y) = 0, \quad v(0, y) = 0, \quad x = 0, 0 < y < \infty, \\ &u(a, y) = 0, \quad \tau_{xy}(a, y) = 0 \quad x = a, 0 < y < \infty, \end{aligned} \tag{28}$$

or, with the help of the boundary functionals, one can write

$$U_0\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u(0, y) \\ v(0, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad U_1\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u(a, y) \\ v'(a, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{29}$$

The temperature $T(x, y)$ is taken as the volume force. It was found from the thermal conductivity problem for a semi-strip

$$\begin{aligned} &\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < \infty, \\ &T(x, 0) = f(x), \quad 0 < x < a, \\ &\frac{\partial T}{\partial x}(0, y) = 0,\ \frac{\partial T}{\partial x}(a, y) = 0, \quad 0 < y < \infty. \end{aligned} \tag{30}$$

The solution of this problem was constructed earlier in the form [29]

$$T(x, y) = \begin{cases} \frac{e^{-\frac{2\pi y}{a}}}{2a}\int_0^a f(\xi)\left[\delta^+(y, x, \xi) + \delta^-(y, x, \xi)\right]\mathrm{d}\xi, & 0 \le x \le a, 0 < y < \infty, \\ \frac{1}{a}\int_0^a f(\xi)\,\mathrm{d}\xi + \sum_{k=1}^{\infty}\frac{2}{a}\int_0^a f(\xi)\cos\frac{\pi k\xi}{a}\mathrm{d}\xi\cos\frac{\pi k x}{a}, & 0 \le x \le a, y = 0, \end{cases}$$

where $\delta^{\pm}(y, x, \xi) = \frac{1}{1+e^{-\frac{2\pi y}{a}} - 2e^{-\frac{\pi y}{a}}\cos\frac{\pi(x\pm\xi)}{a}}$.

The right-hand part of the Lamé's equations will be transformed taking into consideration the view of the volume force and will be written in the transformation domain in the form

$$\begin{aligned} &\frac{\mathrm{d}^2 u_\beta(x)}{\mathrm{d}x^2} - \frac{\beta^2(\kappa-1)}{\kappa+1}u_\beta(x) + \frac{2\beta}{\kappa+1}\frac{dv_\beta(x)}{\mathrm{d}x} = \frac{3-\kappa}{\kappa+1}\chi'(x) + \frac{\tilde{\rho}}{\kappa+1}\frac{\mathrm{d}T_\beta}{vx}(x), \\ &\frac{\mathrm{d}^2 v_\beta(x)}{\mathrm{d}x^2} - \frac{\beta^2(\kappa+1)}{\kappa-1}v_\beta(x) - \frac{2\beta}{\kappa-1}\frac{du_\beta(x)}{\mathrm{d}x} = -\beta\frac{\kappa+1}{\kappa-1}\chi(x) - \frac{\beta\rho}{\kappa-1}T_\beta(x). \end{aligned}$$

Here, $T_\beta(x) = \int_0^\infty T(x, y)\cos\beta y\mathrm{d}y$, $\tilde{\rho} = \rho\frac{\kappa-1}{\kappa+1}$, $\rho = 2\frac{\mu+1}{1-2\mu}\alpha_t$, and α_t is a linear expansion coefficient.

According to the constructed solution (27), the displacements' formulae will be the following:

$$\begin{aligned} u_\beta(x) = {} & Y_1^{11}(x)c_1 + Y_1^{12}(x)c_2 + Y_2^{11}(x)c_3 + Y_2^{12}(x)c_4 \\ & + \frac{3-\kappa}{\kappa+1}\int_0^a G^{11}(x, \xi)\chi'(\xi)\,\mathrm{d}\xi + \tilde{\rho}\int_0^a G^{11}(x, \xi)\frac{\mathrm{d}T_\beta}{\mathrm{d}\xi}(\xi)\,\mathrm{d}\xi \\ & - \beta\frac{\kappa+1}{\kappa-1}\int_0^a G^{12}(x, \xi)\chi(\xi)\,\mathrm{d}\xi - \beta\rho\int_0^a G^{12}(x, \xi)T_\beta(\xi)\,\mathrm{d}\xi, \\ v_\beta(x) = {} & Y_1^{21}(x)c_1 + Y_1^{22}(x)c_2 + Y_2^{21}(x)c_3 + Y_2^{22}(x)c_4 \\ & + \frac{3-\kappa}{\kappa+1}\int_0^a G^{21}(x, \xi)\chi'(\xi)\,\mathrm{d}\xi + \tilde{\rho}\int_0^a G^{21}(x, \xi)\frac{\mathrm{d}T_\beta}{\mathrm{d}\xi}(\xi)\,\mathrm{d}\xi \\ & - \beta\frac{\kappa+1}{\kappa-1}\int_0^a G^{22}(x, \xi)\chi(\xi)\,\mathrm{d}\xi - \beta\rho\int_0^a G^{22}(x, \xi)T_\beta(\xi)\,\mathrm{d}\xi, \end{aligned}$$

where $G^{i,j}(x, \xi)$ is the Green's matrix function element in row i and column j. The integrals with the derivatives of the function $T(\xi)$ and the function $\chi(\xi)$ are calculated by parts. During the calculation, the facts that $G^{11}(x,a) = G^{11}(x,0) = G^{21}(x,a) = G^{21}(x,0) = 0$ are taken into consideration, so the expressions for the displacements are modified:

$$\begin{aligned} u_\beta(x) &= Y_1^{11}(x)\,c_1 + Y_1^{12}(x)\,c_2 + Y_2^{11}(x)\,c_3 + Y_2^{12}(x)\,c_4, \\ v_\beta(x) &= Y_1^{21}(x)\,c_1 + Y_1^{22}(x)\,c_2 + Y_2^{21}(x)\,c_3 + Y_2^{22}(x)\,c_4 \\ &\quad + \frac{3-\kappa}{\kappa+1}\int_0^a G^{21}(x,\xi)\,\chi'(\xi)\,\mathrm{d}\xi - \tilde{\rho}\int_0^a \frac{\partial G^{21}(x,\xi)}{\partial \xi} T_\beta(\xi)\,\mathrm{d}\xi \\ &\quad + \beta\frac{\kappa+1}{\kappa-1}\int_0^a \Im^{22}(x,\xi)\,\chi'(\xi)\,\mathrm{d}\xi - \beta\rho\int_0^a G^{22}(x,\xi)\,T_\beta(\xi)\,\mathrm{d}\xi, \end{aligned}$$

where $\Im^{22}(x,\xi) = \int G^{22}(x,\xi)\,\mathrm{d}\xi$.

The unknown constants $c_i,\ i=\overline{1,4}$, are found from the boundary conditions (28) and are shown in Appendix B. These formulae would be the final ones if the unknown function $\chi'(\xi)$ is known. For its finding with, one must satisfy the boundary condition (5), which is unsatisfied yet. With this aim, the inverse integral transformations' formulae should be applied to the displacements' transformations (7):

$$\begin{aligned} u(x,y) = \int_0^a \Bigg[&\chi'(\xi)\int_0^\infty f_1(x,\xi,\beta)\cos(\beta y)\,\mathrm{d}\beta \\ &+ \int_0^\infty T(\xi,\eta)\int_0^\infty f_2(x,\xi,\beta)\,\frac{(\cos\beta(y+\eta)+\cos\beta(y-\eta))}{2}\mathrm{d}\beta\mathrm{d}\eta\Bigg]\mathrm{d}\xi, \end{aligned} \tag{31}$$

$$\begin{aligned} v(x,y) = \int_0^a \Bigg[&\chi'(\xi)\int_0^\infty g_1(x,\xi,\beta)\sin(\beta y)\,\mathrm{d}\beta\ , \\ &+ \int_0^\infty T(\xi,\eta)\int_0^\infty g_2(x,\xi,\beta)\,\frac{(\sin\beta(y+\eta)+\sin\beta(y-\eta))}{2}\mathrm{d}\beta\mathrm{d}\eta\Bigg]\mathrm{d}\xi, \end{aligned} \tag{32}$$

where $f_i(x,\xi,\beta)\,,\ g_i(x,\xi,\beta),\ i=\overline{1,2}$ are shown in Appendix C.

It should be taken into consideration that integrals in these correspondences are conditionally convergent integrals. So, before differentiating the displacements' expressions, at first one must extract the weakly convergence parts of these integrals. With this aim, the following method is proposed. An conditionally convergent integral $\int_0^\infty a(x)\,\mathrm{d}x$ is divided on two summands $\int_0^\infty a(x)\,\mathrm{d}x = \int_0^A a(x)\,\mathrm{d}x + \int_A^\infty a(x)\,\mathrm{d}x$. In the second summand, the functions under the integral sign is substituted by its asymptotic expression $\tilde{a}(x)$ when $x\to\infty$. Then, the summand $\int_0^A \tilde{a}(x)\,\mathrm{d}x$ is added and subtracted. Instead of initial integral, one obtains the expression $\int_0^\infty a(x)\,\mathrm{d}x = \int_0^\infty \tilde{a}(x)\,\mathrm{d}x + \left(\int_0^A a(x)\,\mathrm{d}x - \int_0^A \tilde{a}(x)\,\mathrm{d}x\right)$. The first integral in this expression is the table integral and is calculated with the formulae (3.941(1), 3.944(11–12)) [28]. After this procedure, one can differentiate the displacements' expression and satisfy the condition (5). It leads to the singular integral equation

$$\int_0^a \chi'(\xi)\left[\frac{1}{\xi-x} + f(\xi,x)\right]\mathrm{d}\xi = r(x) - \int_0^a\int_0^\infty T(\xi,\eta)\,g(\xi,\eta,x)\,\mathrm{d}\eta\mathrm{d}\xi, \quad 0<x<a.$$

Here, $r(x)\,,\ f(\xi,x)\,,\ g(\xi,\eta,x)$ are the known regular functions.

7 The solution of the singular integral equation

The change in the variables $\xi = \frac{\xi^*}{a}, x = \frac{x^*}{a}$ is done for switching to the integration interval [0; 1]. As a result, the integral equation is transformed to the form

$$\int_0^1 \tilde{\chi}(\xi)\left[\frac{1}{\xi - x} + af(a\xi, ax)\right] d\xi = r(ax)$$
$$-a\int_0^1\int_0^\infty T(a\xi,\eta)\, g(a\xi,\eta,ax)\, d\eta d\xi, \quad 0 < x < 1, \tag{33}$$

where $\tilde{\chi}(\xi) = \chi'(a\xi)$. The integral equation is solved approximately by the orthogonal polynomials' method [25]. This method allows taking into consideration the real singularities of the solution at the ends of the integration interval. The order of singularities is found from the known solution for an edge with the opening angle $\pi/2$ [4]. With regard to this, the function $\tilde{\chi}(\xi)$ is expanded into the series of the Jacobi polynomials

$$\tilde{\chi}(\xi) = \sum_{n=0}^{\infty} \tilde{c}_n \xi^\alpha (1-\xi)^\beta P_n^{\alpha,\beta}(1-2\xi), \tag{34}$$

where $P_n^{\alpha,\beta}(x)$ is a Jacobi polynomial, $\alpha = -0.31$, $\beta = 0$ [4]. This expression is substituted into the integral equation (33), and the standard scheme of the orthogonal polynomials' method is realized. The spectral correspondence [30]

$$\frac{1}{\pi}\int_0^1 \frac{y^\alpha (1-y)^\beta P_n^{\alpha,\beta}(1-2y)\, dy}{y - x} = \mathrm{ctg}(\pi\alpha)\, x^\alpha (1-x)^\beta P_n^{\alpha,\beta}(1-2x), \quad 0 < x < 1$$

is used. As a result, the infinite system of linear algebraic equations related to the unknown coefficients $\tilde{c}_i, \; i = 0, 1, 2...$, is obtained:

$$\tilde{c}_m + \sum_{n=0}^{\infty} \tilde{c}_n d_{mn} = f_m, m = 0, 1, 2... \tag{35}$$

The system (35) is solved by the reduction method (its applicability is proved by the scheme proposing in [25]). The substitution of the constants in the formula (34) and then using the formulae (31) completes the construction of the problem's solution.

8 The results of the numerical analyses

The calculations were done for an elastic half-strip ($G = 82.03125 \cdot 10^9$ Pa, $\mu = 0.28$) with the side a = 10. Figures 2, 3 and 4 show the values of the stress σ_x and σ_y on the lateral sides, the edge and inside the semi-strip under the mechanical and temperature influences. The changes in the stress field were investigated depending on the changes in the temperature loading by the given mechanical loading and vice versa—by the changing in the mechanical loading by the given temperature loading influence on a semi-strip.

To keep the accuracy of the calculations below 10^{-4}, it was enough to save 15 equations in the reduced system of the linear algebraic equation (35). The integrals of the type (31) were calculated using the Gauss quadrature formulas with 35 nodes. The accuracy of the problem boundary conditions' executing was investigated for the validation of the numerical calculations. This test showed that the numerical error in satisfying the boundary conditions is 10^{-4}.

From Fig. 2, one can admit that the values of the normal stress σ_y at the lateral side $x = 0$ are substantially higher than those at the lateral side $x = 10$, and the absolute values of the stress increase with the temperature loading increasing. A similar effect of temperature influence is observed during the analysis of the stress

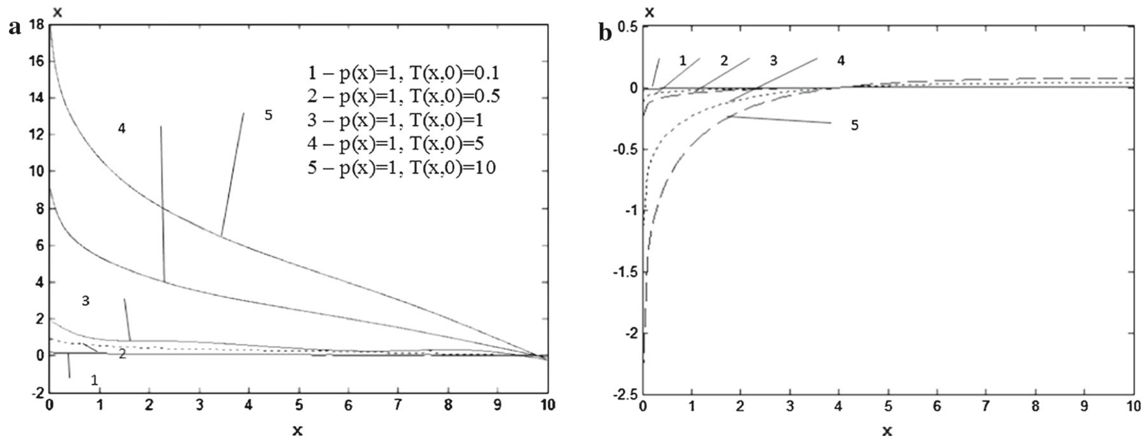

Fig. 2 Stress' distribution of $\sigma_y\ (x, y)$ at the strip's points $\{[x, y] : x = 0, 0 \leq y \leq 10\}$ (**a**) and $\{[x, y] : x = 10, 0 \leq y \leq 10\}$ (**b**)

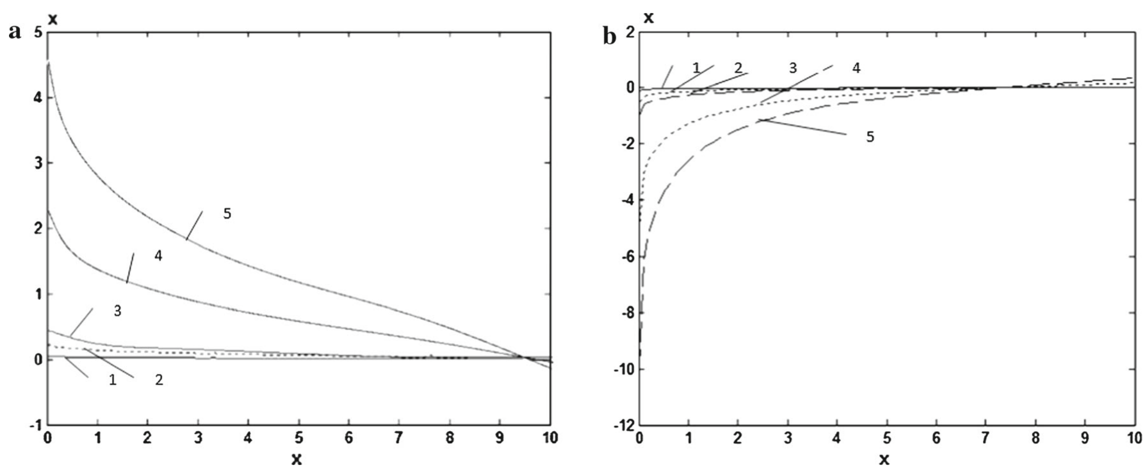

Fig. 3 Stress' distribution of $\sigma_x\ (x, y)$ at the strip's points $\{[x, y] : x = 0, 0 \leq y \leq 10\}$ (**a**) and $\{[x, y] : x = 10, 0 \leq y \leq 10\}$ (**b**)

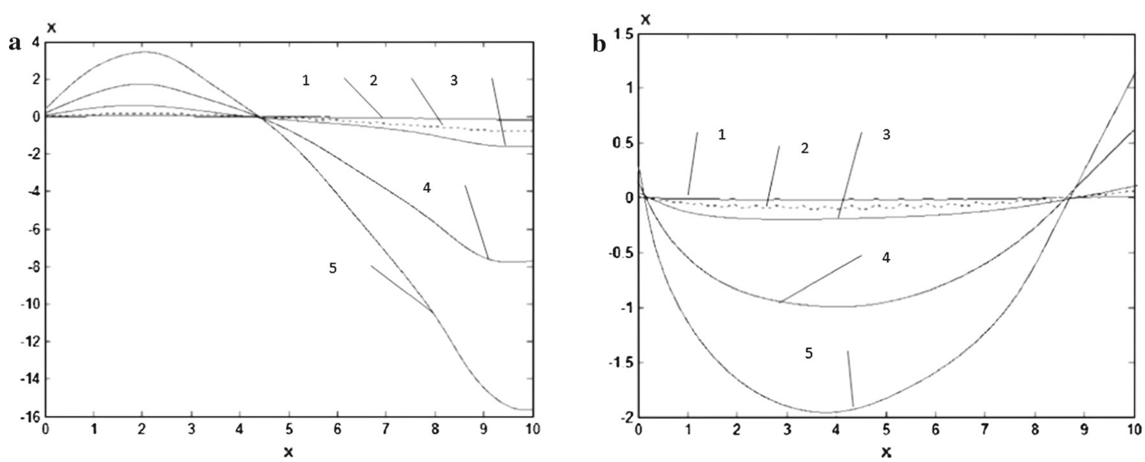

Fig. 4 Stress' distribution of $\sigma_y\ (x, y)$ at the strip's points $\{[x, y] : 0 \leq x \leq 10, y = 1\}$ (**a**) and $\{[x, y] : 0 \leq x \leq 10, y = 4\}$ (**b**)

σ_x (Fig. 3). On the fixed lateral side, the occurrence of the tensile stresses is observed. No tensile stresses are admitted on the lateral side, where the conditions of the slide contact are given. With increasing applied temperature, the absolute values of the stresses on both surfaces also increase.

During the analysis of the stresses within the semi-strip, one can note the decrease in the normal stress' absolute values with increasing distance from the location of the applied loads (Fig. 4). An increase in the applied temperature loading leads to increased stress values within the semi-strip.

After the analysis of the numerical calculations it was estimated that the effect of the temperature change has more influence on the stress state than the change of the mechanical loading. The increase in the temperature results in significantly higher absolute values of the stresses. At the same time, on the fixed edge of the strip the occurrence of tensile stresses is admitted, which increases with the increasing temperature effect in the applied loading.

9 The conclusions

1. The proposed method allows to solve mixed plane elasticity problems for a semi-strip with different conditions on its lateral sides. Due to the method, integral transformations are applied directly to Lamé's equations, and one does not need to use the known different representations of their solutions.
2. The method allows the reduction in the problems for a semi-strip to singular integral equations, which can be solved approximately with the help of the orthogonal polynomials' method, taking into consideration the singularity's order of the unknown function at the ends of the integration interval.
3. For the thermoelasticity problem of a semi-strip, the zones and conditions of the strain stress occurrence on the semi-strip's lateral sides, important to engineering applications, are investigated.

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Appendix A: The Green's matrix function's components

$$
\begin{aligned}
G_{11}(x,\xi) = {} & \frac{\kappa ch(\beta(\xi-a+x)) - \kappa ch(\beta(a-|\xi-x|))}{2\beta sh(a\beta)(\kappa-1)} \\
& - \frac{1}{2(sh(a\beta))^2(\kappa-1)}[sh(a\beta)(\xi sh(\beta(\xi-a+x)) + x sh(\beta(\xi-a+x)) \\
& - a sh(\beta(\xi-a+x)) - (a-|\xi-x|)\, a sh(\beta(a-|\xi-x|))) \\
& - a ch(a\beta)(ch(\beta(\xi-a+x)) - ch(\beta(a-|\xi-x|)))]
\end{aligned}
$$

$$
\begin{aligned}
G_{12}(x,\xi) = {} & \frac{1}{2sh(a\beta)(\kappa+1)}\Bigg[-\frac{a ch(a\beta)}{\left((ch(a\beta))^2-1\right)(\kappa+1)}(sh(\beta(\xi-a+x)) \\
& - \operatorname{sgn}(x-\xi)\, sh(\beta(a-|\xi-x|))) \\
& + (ch(\beta(\xi-a+x))(\xi-a+x) - \operatorname{sgn}(x-\xi)\, ch(\beta(a-|\xi-x|))(a-|\xi-x|))\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
G_{21}(x,\xi) = {} & \frac{1}{2sh(a\beta)(\kappa-1)}\Bigg[-\frac{a ch(a\beta)}{\left((ch(a\beta))^2-1\right)(\kappa-1)}(sh(\beta(\xi-a+x)) \\
& + \operatorname{sgn}(x-\xi)\, sh(\beta(a-|\xi-x|))) \\
& + (ch(\beta(\xi-a+x))(\xi-a+x) + \operatorname{sgn}(x-\xi)\, ch(\beta(a-|\xi-x|))(a-|\xi-x|))\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
G_{22}(x,\xi) = {} & -\frac{ch(\beta(\xi-a+x)) + ch(\beta(a-|\xi-x|))}{2\beta sh(a\beta)} \\
& + \frac{1}{2\beta(sh(a\beta))^2(\kappa+1)}[sh(a\beta)(ch(\beta(\xi-a+x)) + ch(\beta(a-|\xi-x|)) \\
& + \beta((a-x-\xi)\, sh(\beta(\xi-a+x)) - (a-|\xi-x|)\, sh(\beta(a-|\xi-x|))) \\
& + a\beta ch(a\beta)(ch(\beta(\xi-a+x)) + ch(\beta(a-|\xi-x|)))]
\end{aligned}
$$

Appendix B: The constants c_i, $i = \overline{1,4}$

$$
\begin{aligned}
c_1 = &\left\{2\beta\left[\left(a\kappa e^{-a\beta} - a\kappa e^{a\beta}\right) T_4 + \left(\kappa + 2a\beta + \kappa e^{-2a\beta} + \kappa^2 e^{-2a\beta} + \kappa^2 + 2a\beta\kappa\right) T_1\right.\right.\\
&+ \left(a\beta e^{-a\beta} - \kappa e^{-a\beta} - \kappa^2 e^{a\beta} - \kappa^2 e^{-a\beta} - a\beta\kappa e^{a\beta} - \kappa e^{a\beta}\right) T_3\\
&\left.\left.+\ (-2a\beta - 2a\beta\kappa)\, T_2\right](\kappa - 1)\right\} / \left\{\kappa\,(1+\kappa)\left(4a\beta + \kappa e^{-2a\beta} - \kappa e^{2a\beta}\right)\right\};\\
c_2 = &\left\{2T_4\left(\kappa^2 e^{-a\beta} - \kappa^2 e^{a\beta} + a\beta\kappa e^{a\beta} + a\beta\kappa e^{-a\beta}\right)\right.\\
&+ 2T_2\left(2a\beta^2 - \beta\kappa - \beta\kappa^2 + \beta\kappa^2 e^{-2a\beta} + \beta\kappa e^{-2a\beta} + 2a\beta^2\kappa\right)\\
&+ 2T_3\left(\beta\kappa e^{a\beta} - \beta\kappa e^{-a\beta} + a\beta^2\kappa e^{a\beta} + a\beta^2\kappa e^{-a\beta}\right)\\
&\left.- 2T_1\left(2a\beta^2 + 2a\beta^2\kappa\right)\right\} / \left\{\kappa\left(4a\beta + \kappa e^{-2a\beta} - \kappa e^{2a\beta}\right)\right\};\\
c_3 = &\left\{2\beta\left[\left(a\kappa e^{a\beta} - a\kappa e^{-a\beta}\right) T_4 + \left(\kappa e^{a\beta} + \kappa e^{-a\beta} + \kappa^2 e^{-a\beta} + a\beta\kappa e^{a\beta} - a\beta\kappa e^{-a\beta}\right) T_3\right.\right.\\
&+ \left(2a\beta - \kappa e^{2a\beta} - \kappa^2 e^{2a\beta} - \kappa^2 + 2a\beta\kappa\right) T_1\\
&\left.\left.+\ (2a\beta + 2a\beta\kappa)\, T_2\right](\kappa - 1)\right\} / \left\{\kappa\,(1+\kappa)\left(4a\beta + \kappa e^{-2a\beta} - \kappa e^{2a\beta}\right)\right\};\\
c_4 = &\left\{2T_4\left(\kappa^2 e^{-a\beta} - \kappa^2 e^{a\beta} + a\beta\kappa e^{a\beta} + a\beta\kappa e^{-a\beta}\right)\right.\\
&- 2T_2\left(2a\beta^2 + \beta\kappa + \beta\kappa^2 - \beta\kappa^2 e^{2a\beta} - \beta\kappa e^{2a\beta} + 2a\beta^2\kappa\right)\\
&+ 2T_3\left(\beta\kappa e^{a\beta} - \beta\kappa e^{-a\beta} + a\beta^2\kappa e^{a\beta} + a\beta^2\kappa e^{-a\beta}\right)\\
&\left.- 2T_1\left(2a\beta^2 + 2a\beta^2\kappa\right)\right\} / \left\{\kappa\left(4a\beta + \kappa e^{-2a\beta} - \kappa e^{2a\beta}\right)\right\}
\end{aligned}
$$

Here, $T_1 = -u_\beta^1(0)$, $T_2 = -v_\beta^1(0)$, $T_3 = -u_\beta^1(a)$, $T_4 = -v_\beta^1{}'(a)$.

Appendix C: The formulae for the $f_i(x,\xi,\beta)$, $g_i(x,\xi,\beta)$, $i = \overline{1,2}$ functions from the displacements' final formulae

$$
\begin{aligned}
&f_1(x,\xi,\beta)\\
&= -\frac{2}{\pi\kappa\left(\kappa e^{-4a\beta} - \kappa + 4a\beta e^{-2a\beta}\right)}\left\{2\beta\left[\frac{1}{(\kappa+1)\left(e^{-2a\beta}-1\right)}\left[\kappa\,(a-\xi)\left(x\left(e^{\beta(\xi-6a+x)} - e^{-\beta(\xi+x)}\right.\right.\right.\right.\right.\\
&\left.- e^{-\beta(2a-\xi+x)} + e^{-\beta(4a+\xi-x)}\right) - (2a-x)\left(-e^{-\beta(2a+\xi+x)} + e^{\beta(\xi-4a+x)} - e^{-\beta(2a+\xi-x)}\right.\\
&\left.\left.\left.+ e^{-\beta(4a-\xi+x)}\right)\right)\right] - \frac{1}{(\kappa+1)\left(e^{-2a\beta}-1\right)^2}\left[a\kappa\left(x\left(e^{\beta(\xi-6a+x)} + e^{-\beta(\xi+x)} - e^{-\beta(2a-\xi+x)} - e^{-\beta(4a+\xi-x)}\right)\right.\right.\\
&\left.\left.\left.+ (2a-x)\left(e^{-\beta(2a+\xi+x)} - e^{-\beta(4a-\xi+x)} + e^{\beta(\xi-4a+x)} - e^{-\beta(2a+\xi-x)}\right)\right)\left(e^{-2a\beta}+1\right)\right]\right]\\
&- \frac{\kappa}{e^{-2a\beta}-1}\left[x\left(e^{\beta(\xi-6a+x)} + e^{-\beta(\xi+x)} - e^{-\beta(2a-\xi+x)} - e^{-\beta(4a+\xi-x)}\right) + (2a-x)\left(e^{-\beta(2a+\xi+x)}\right.\right.\\
&\left.\left.\left.- e^{-\beta(4a-\xi+x)} + e^{\beta(\xi-4a+x)} - e^{-\beta(2a+\xi-x)}\right)\right]\right\}\\
&f_2(x,\xi,\beta)\\
&= -\frac{\beta\rho\,(\kappa-1)}{(\kappa+1)\left(e^{-2a\beta}-1\right)\pi\left(\kappa e^{-4a\beta} - \kappa + 4a\beta e^{-2a\beta}\right)}\left[x\left(e^{\beta(\xi-6a+x)} - e^{-\beta(\xi+x)} - e^{-\beta(2a-\xi+x)}\right.\right.\\
&\left.\left.+ e^{-\beta(4a+\xi-x)}\right) + \left(e^{\beta(\xi-4a+x)} - e^{-\beta(2a+\xi+x)} + e^{-\beta(2a+\xi-x)} - e^{-\beta(4a-\xi+x)}\right)(2a-x)\right]\\
&g_1(x,\xi,\beta)\\
&= \frac{2}{\pi}\left\{\frac{1}{\kappa\left(\kappa e^{-4a\beta} - \kappa + 4a\beta e^{-2a\beta}\right)}\left\{\kappa\left[\frac{1}{e^{-2a\beta}-1}\left[2a\left(e^{-\beta(2a+\xi+x)} - e^{\beta(\xi-4a+x)} + e^{-\beta(2a+\xi-x)}\right.\right.\right.\right.\right.\\
&\left.- e^{-\beta(4a-\xi+x)}\right) - x\left(e^{-\beta(2a+\xi+x)} - e^{\beta(\xi-4a+x)} + e^{\beta(\xi-6a+x)} - e^{-\beta(\xi+x)} + e^{-\beta(2a+\xi-x)} + e^{-\beta(2a-\xi+x)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-e^{-\beta(4a+\xi-x)}-e^{-\beta(4a-\xi+x)}\Big)+\frac{2\kappa\,(a-\xi)}{\kappa+1}\Big(e^{-\beta(2a+\xi+x)}-e^{\beta(\xi-4a+x)}+e^{\beta(\xi-6a+x)}-e^{-\beta(\xi+x)}\\
&-e^{-\beta(2a+\xi-x)}-e^{-\beta(2a-\xi+x)}+e^{-\beta(4a+\xi-x)}+e^{-\beta(4a-\xi+x)}\Big)\Big]\\
&+\frac{1}{(\kappa+1)\left(e^{-2a\beta}-1\right)^2}\Big[2a\kappa\left(e^{-2a\beta}+1\right)\Big(e^{-\beta(2a+\xi+x)}+e^{\beta(\xi-4a+x)}-e^{\beta(\xi-6a+x)}\\
&-\ e^{-\beta(\xi+x)}-e^{-\beta(2a+\xi-x)}+e^{-\beta(2a-\xi+x)}+e^{-\beta(4a+\xi-x)}-e^{-\beta(4a-\xi+x)}\Big)\Big]\Big]\\
&+2\beta\kappa\Bigg[\frac{1}{(\kappa+1)\left(e^{-2a\beta}-1\right)}\Big[\Big((2a-x)\Big(e^{-\beta(2a+\xi+x)}+e^{\beta(\xi-4a+x)}+e^{-\beta(2a+\xi-x)}+e^{-\beta(4a-\xi+x)}\Big)\\
&+\ x\Big(e^{\beta(\xi-6a+x)}+e^{-\beta(\xi+x)}+e^{-\beta(2a-\xi+x)}+e^{-\beta(4a+\xi-x)}\Big)\Big)(a-\xi)\Big]\\
&-\frac{1}{(\kappa+1)\left(e^{-2a\beta}-1\right)^2}\Big[a\Big(x\Big(e^{\beta(\xi-6a+x)}-e^{\beta(\xi+x)}+e^{-\beta(2a-\xi+x)}-e^{-\beta(4a+\xi-x)}\Big)\\
&-(2a-x)\Big(e^{-\beta(2a+\xi+x)}-e^{\beta(\xi-4a+x)}+e^{-\beta(2a+\xi-x)}-e^{-\beta(4a+\xi+x)}\Big)\Big)\left(e^{-2a\beta}+1\right)\Big]\Big]\Bigg\}\\
&+\frac{1}{\beta\left(e^{-2a\beta}-1\right)}\Big[0.5e^{\beta(\xi-2a+x)}-0.5e^{-\beta(\xi+x)}+0.5\operatorname{sign}(x-\xi)\Big(e^{-\beta|x-\xi|}-e^{-\beta(2a-|x-\xi|)}\Big)\\
&+\frac{\kappa}{\kappa e^{-4a\beta}-\kappa+4a\beta e^{-2a\beta}}\Big(e^{-\beta(2a+\xi+x)}+e^{\beta(\xi-4a+x)}-e^{\beta(\xi-6a+x)}-e^{-\beta(\xi+x)}-e^{-\beta(2a+\xi-x)}\\
&+\ e^{-\beta(2a-\xi+x)}+e^{-\beta(4a+\xi-x)}-e^{-\beta(4a-\xi+x)}\Big)\Big]\\
&+\frac{\left(e^{\beta(\xi-2a+x)}+e^{-\beta(\xi+x)}\right)(\xi-a+x)+\left(e^{-\beta|x-\xi|}+e^{-\beta(2a-|x-\xi|)}\right)(\xi-x+\operatorname{sign}(x-\xi)\,a)}{(\kappa+1)\left(e^{-2a\beta}-1\right)}\\
&+\frac{a}{2\kappa-2}\left[\frac{1}{\left(e^{-2a\beta}-1\right)^2}-\frac{\kappa-2}{(\kappa+1)\left(4e^{-2a\beta}-\left(e^{-2a\beta}+1\right)^2\right)}\right]\left(e^{-2a\beta}+1\right)\Big[e^{\beta(\xi-2a+x)}-e^{-\beta(\xi+x)}\\
&+\ \operatorname{sign}(x-\xi)\Big(e^{-\beta|x-\xi|}-e^{-\beta(2a-|x-\xi|)}\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&g_2(x,\xi,\beta)\\
&=\frac{\rho}{\pi}\Bigg\{\frac{\kappa-1}{(2\kappa+2)\left(e^{-2a\beta}-1\right)^2}\Big[e^{-\beta|x-\xi|}+e^{\beta(\xi-2a+x)}-e^{-\beta(2a+\xi+x)}-e^{\beta(\xi-4a+x)}+e^{-\beta(\xi+x)}\\
&+\ e^{-\beta(2a-|x-\xi|)}-e^{-\beta(2a+|x-\xi|)}-e^{-\beta(4a-|x-\xi|)}\Big]\\
&-\frac{\kappa-1}{(\kappa+1)\left(e^{-2a\beta}-1\right)\left(\kappa e^{-4a\beta}-\kappa+4a\beta e^{-2a\beta}\right)}\Big[\kappa\Big(e^{-\beta(\xi+x)}+e^{\beta(\xi-4a+x)}-e^{\beta(\xi-6a+x)}\\
&+e^{-\beta(2a-\xi+x)}-e^{-\beta(4a-\xi+x)}+e^{-\beta(2a+\xi-x)}-e^{-\beta(4a+\xi-x)}-e^{-\beta(2a+\xi+x)}\Big)-\beta\Big(x\Big(e^{\beta(\xi-6a+x)}\\
&+\ e^{-\beta(\xi+x)}+e^{-\beta(2a-\xi+x)}+e^{-\beta(4a+\xi-x)}\Big)+\Big(e^{-\beta(2a+\xi+x)}+e^{\beta(\xi-4a+x)}+e^{-\beta(2a+\xi-x)}\\
&+e^{-\beta(4a-\xi+x)}\Big)(2a-x)\Big)\Big]\Bigg\}
\end{aligned}
$$

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