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ORIGINAL PAPER

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Non-elliptical inclusions that achieve uniform internal strain fields in an elastic half-plane

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Abstract In existing literature, it remains an unexplored question whether any inclusion shape can achieve a uniform internal strain field in an elastic half-plane under either given uniform remote loadings or given uniform eigenstrains imposed on the inclusion. This paper examines the existence and construction of such single or multiple non-elliptical inclusions that achieve prescribed uniform internal strain fields in an elastic half-plane under given uniform anti-plane shear eigenstrains imposed on the inclusions. Such non-elliptical inclusion shapes in a half-plane can be determined by solving the original problem of an unknown holomorphic function in a multiply connected half-plane, which is transferred to an equivalent problem of an unknown holomorphic function in a multiply connected whole plane based on analytic continuation techniques. Extensive numerical examples are shown for single inclusion, multiple inclusions and two geometrically symmetrical inclusions, respectively. It is found that the inclusion shapes which achieve uniform internal strain fields depend on the given uniform eigenstrains, and the inclusion shapes that achieve uniform internal strain fields for arbitrarily given uniform eigenstrains do not exist. Moreover, specific conditions are derived on the given uniform eigenstrains and prescribed uniform internal strain fields for the existence of two geometrically symmetrical inclusions that achieve uniform internal strain fields.

1 Introduction

In the micromechanical analysis of composites, inclusions of special shapes that achieve uniform internal stress fields have received much attention due to practical significance that uniform internal stress fields do not induce stress peaks within the inclusions. On the other hand, the emerging 3D printing (or additive manufacturing) technique greatly simplifies the manufacture of the composites containing these special inclusions (especially of complicated shapes), and thus makes the design of such special inclusions much more meaningful. Historically, Eshelby [1] showed that an elliptical inclusion within an elastic whole plane can achieve a uniform internal stress field under uniform remote loadings or uniform eigenstrains imposed on the inclusion. Based on the complex variables method, Sendeckyj [2] and Ru and Schiavone [3] proved that elliptical shape is the only inclusion shape that achieves a uniform internal stress field inside the inclusion embedded in an elastic whole plane. On the other hand, for an elastic half-plane with a traction-free surface, it is known that even a circular inclusion [4,5] or an elliptical inclusion [6,7] can no longer achieve a uniform internal stress field under uniform remote loadings or uniform eigenstrains imposed on the inclusion. In addition, none of the common

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non-elliptical inclusions defined by simple polynomial conformal mappings (such as hypotrochoidal and polygonal inclusions [7]) can achieve uniform internal stress fields in a half-plane. In spite of this, however, it remains an unanswered question whether a non-elliptical inclusion does exist which achieves a uniform internal stress field in an elastic half-plane under either given uniform remote loadings or given uniform eigenstrains imposed on the inclusion.

For anti-plane shear, the answer for the above question is “yes.” Actually, the symmetrical inclusion pairs obtained in recent works (see Figure 1 of [8], Figure 3.1 of [9], Figures 2, 3 and 7 of [10], Figures 2a,c and 5d of [11]), which achieve uniform internal stress fields in an elastic whole plane under uniform remote anti-plane shear or uniform anti-plane shear eigenstrains, imply the existence of a single non-elliptical inclusion that achieves a uniform internal stress field in an elastic half-plane with a free surface. In particular, the shape of such inclusion is even not unique, depending on the specific uniform internal stress field inside the inclusion and the distance between the inclusion and the free surface.

The present work aims to investigate this issue more systematically, and particularly to examine the existence of multiple inclusions that achieve individually prescribed uniform internal strain fields in an elastic half-plane under given uniform anti-plane shear eigenstrains imposed on the inclusions. Basic formulation of the present problem is given in Sect. 2. In Sect. 3, based on the existence condition of the complex potential defined in a half-plane, the unknown shapes of the multiple inclusions are determined on using analytic continuation techniques, Cauchy’s integral formula, Faber series and Newton–Raphson method. In Sect. 4, extensive numerical examples are shown for a single inclusion, multiple inclusions and symmetrical inclusions which achieve various prescribed uniform internal strain fields, with an emphasis on the dependence of the inclusion shapes on the uniform eigenstrains imposed on the inclusions. Finally, the main results are summarized in Sect. 5.

2 Basic equations and problem description

2.1 Basic equations

In a Cartesian coordinate system (x_1, x_2, x_3), consider an isotropic elastic material under anti-plane shear deformation determined by the out-of-plane displacement along the x_3-axis, and then the anti-plane shear stresses $(\sigma_{13}, \sigma_{23})$ and the out-of-plane displacement w can be expressed by a complex potential $f(z)(z = x_1 + Ix_2)$ as [10]

$$\sigma_{23} + I\sigma_{13} = Gf'(z), \quad w = \mathrm{Im}\,[f(z)]\,, \tag{1}$$

where G represents the shear modulus, and the capital I is used to denote the imaginary unit in order to save the symbol i as a subscript. Additionally, the shear traction σ_{n3} on a directed curve from point A to B in the z-plane can be written in terms of $f(z)$ as [10]

$$\int_A^B \sigma_{n3} ds = -G\mathrm{Re}\,[f(z)]\big|_A^B \tag{2}$$

where *ds* is an element of arc length of the curve along its tangent.

2.2 Problem description

Shown in Fig. 1 is an infinite elastic half-plane (of shear modulus G) with n elastic inclusions (also of the same shear modulus G) bounded by the curves L_i $(i = 1 \ldots n)$, which undergo given uniform anti-plane stress-free shear eigenstrains $\varepsilon_{13}^{*(i)}$ and $\varepsilon_{23}^{*(i)}(i = 1 \ldots n)$, respectively. In particular, we assume that no anti-plane shear loadings are applied at infinity and on the free surface of the half-plane. Let S_i $(i = 1 \ldots n)$ and S_L denote the regions occupied by the inclusions and the remnant multiply connected half-plane, respectively. For a single inclusion in a whole plane, elliptical shape is the only possible inclusion shape that guarantees a uniform internal strain field, and the actual uniform internal strain field can be arbitrary within a certain admissible range determined by the aspect ratio and orientation of the elliptical inclusion. Here, since the shapes of multiple inclusions in the half-plane which achieve uniform internal strain fields are unknown, the internal uniform strain fields within the multiple inclusions could be prescribed only within a certain admissible range, and thus the problem is reduced to the determination of the unknown inclusion shapes. Particularly, we assume that the uniform internal strain field inside each of the inclusions can be different from those inside other inclusions.

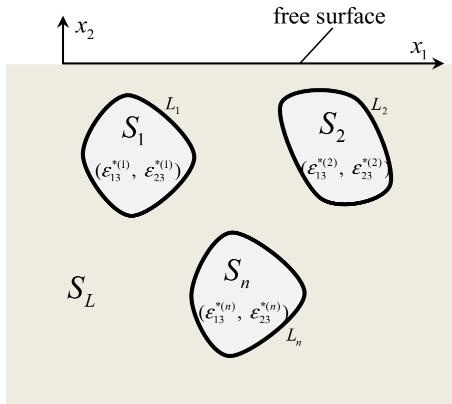

Fig. 1 Multiple inclusions in an elastic (lower) half-plane

Therefore, the complex potentials $f_i(z)$ $(i = 1...n)$ of the inclusions in S_i $(i = 1...n)$ have the form of

$$f_i(z) = \Gamma_i z + C_i, \quad i = 1...n \tag{3}$$

where Γ_i are some given complex constants and the prescribed uniform internal strain fields (given by $\Gamma_i/2$) could be restricted within a certain admissible range to ensure the existence of a solution, and C_i are some complex unknown constants to be determined. For the complex potential $f(z)$ of the multiply connected half-plane S_L, since no anti-plane shear loadings are applied at infinity, we can stipulate $\lim\limits_{|z|\to+\infty} f(z) = 0$.

The continuity conditions of the shear traction and out-of-plane displacement on the interfaces L_i $(i = 1...n)$ are described, according to Eqs. (1) and (2), as

$$\mathrm{Re}\,[f(t)] = \mathrm{Re}\,[f_i(t)]\,, \quad t \in L_i (i = 1...n), \tag{4}$$

$$\mathrm{Im}\,[f(t)] = \mathrm{Im}\left[f_i(t) + 2\Gamma_i^* t\right], \quad t \in L_i (i = 1...n), \tag{5}$$

which are equivalent to

$$f(t) = f_i(t) + \Gamma_i^* t - \overline{\Gamma}_i^* \bar{t}, \quad t \in L_i (i = 1...n), \tag{6}$$

with

$$\Gamma_i^* = \varepsilon_{23}^{*(i)} + I\varepsilon_{13}^{*(i)}, \quad i = 1...n. \tag{7}$$

Here the arbitrary real parts of the complex constants C_i $(i = 1...n)$ defined in Eq. (3) are chosen uniquely so that the continuity condition of shear traction (2) can be simplified into (4). Consequently, the real parts of complex constants C_i $(i = 1...n)$ will be determined uniquely. Substituting Eq. (3) into Eq. (6) leads to

$$\begin{aligned} &f(t) = A_i t + B_i \bar{t} + C_i, \quad t \in L_i (i = 1...n), \\ &A_i = \Gamma_i + \Gamma_i^*, \; B_i = -\overline{\Gamma}_i^* \end{aligned} \tag{8}$$

where, according to the statement after Eq. (3), A_i and B_i $(i = 1...n)$ are known constants determined by the given uniform anti-plane shear eigenstrains and the individually prescribed uniform internal strain fields, while C_i $(i = 1...n)$ will be determined as part of the solution.

In what follows, we will determine the unknown shapes of the inclusions based on the condition for the existence of such a holomorphic function $f(z)$ in the multiply connected lower half-plane S_L which meets the boundary conditions (8) on the interfaces L_i $(i = 1...n)$.

3 Solution procedure

3.1 Analytic continuation of the complex potential $f(z)$

In Fig. 2, S_U denotes the multiply connected upper half-plane with n holes bounded by the curves L_i $(i = n + 1...2n)$ which are symmetrical to the curves L_i $(i = 1...n)$ about the x_1-axis, respectively. Introduce a

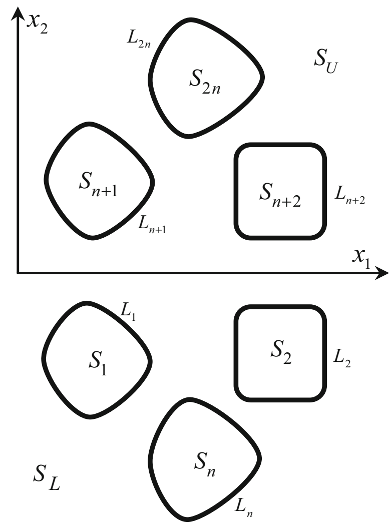

Fig. 2 A whole plane with $2n$ symmetrical holes

function defined in the two multiply connected half-planes S_L and S_U, separately,

$$g(z) = \begin{cases} f(z), & z \in S_L \\ -\overline{f}(z), & z \in S_U \end{cases}, \tag{9}$$

which is holomorphic in S_L and S_U, respectively. Since there is no anti-plane shear traction applied on the free surface of the half-plane S_L, one has

$$\sigma_{23} = G\left[f'(z) + \overline{f'(z)}\right]/2 = 0, \ x_2 = 0^-, \tag{10}$$

which leads to

$$g'^{(-)}(z) - g'^{(+)}(z) = f'^{(-)}(z) + \overline{f'}^{(+)}(z) = 0, \ x_2 = 0^-, \tag{11}$$

and thus the derivative of $g(z)$ defined by Eq. (9) is continuous across the x_1-axis, so $g(z)$ is holomorphic in the whole plane with $2n$ holes bounded by the curves L_i $(i = 1...2n)$. Then the existence of the holomorphic function $f(z)$ in the multiply connected lower half-plane S_L which meets the boundary conditions (8) is determined by the existence of the holomorphic function $g(z)$ in the multiply connected whole plane with $2n$ holes bounded by the curves L_i $(i = 1...2n)$ which meets the following boundary conditions:

$$\begin{aligned} &g(t) = A_i t + B_i \bar{t} + C_i, \quad t \in L_i (i = 1...n), \\ &g(t) = -\overline{A}_i t - \overline{B}_i \bar{t} - \overline{C}_i, \quad t \in L_{i+n} (i = 1...n). \end{aligned} \tag{12}$$

3.2 Existence of the complex potential $g(z)$

In order to ensure the existence of the function $g(z)$ holomorphic in the multiply connected whole plane with $2n$ holes bounded by the curves L_i $(i = 1...2n)$ (see Fig. 2), its boundary value $g(t)$ (see (12)) on the boundaries L_i $(i = 1...2n)$, according to Sokhotski–Plemelj theorem, should satisfy the following necessary and sufficient condition [12]:

$$\frac{1}{2\pi I}\sum_{j=1}^{2n}\oint_{L_j}\frac{g(t)}{t-z}dt = 0, \quad \forall z \in S_i (i = 1...2n). \tag{13}$$

Substituting Eq. (12) into Eq. (13) and then using Cauchy's integral formula, one has

$$A_i z + \frac{1}{2\pi I}\sum_{j=1}^{n}\left(B_j \oint_{L_j} \frac{\bar{t}}{t-z}dt - \overline{B}_j \oint_{L_{j+n}} \frac{\bar{t}}{t-z}dt\right) = -C_i, \quad \forall z \in S_i (i = 1\ldots n), \tag{14}$$

$$-\overline{A}_i z + \frac{1}{2\pi I}\sum_{j=1}^{n}\left(B_j \oint_{L_j} \frac{\bar{t}}{t-z}dt - \overline{B}_j \oint_{L_{j+n}} \frac{\bar{t}}{t-z}dt\right) = \overline{C}_i, \quad \forall z \in S_{i+n} (i = 1\ldots n). \tag{15}$$

Since the boundaries L_j and L_{j+n} $(j = 1\ldots n)$ are symmetrical about the x_1-axis, one can easily verify

$$\overline{\oint_{L_j} \frac{\bar{t}}{t-z}dt} = -\oint_{L_{j+n}} \frac{\bar{t}}{t-\bar{z}}dt, \quad j = 1\ldots n,$$
$$\overline{\oint_{L_{j+n}} \frac{\bar{t}}{t-z}dt} = -\oint_{L_j} \frac{\bar{t}}{t-\bar{z}}dt, \quad j = 1\ldots n. \tag{16}$$

According to Eq. (16), it is shown that the conjugate of the condition (14) is exactly equivalent to (15), so, in what follows, we will use only the condition (14) instead of simultaneous conditions (14) and (15).

Note that each of the integral expressions $(j = 1\ldots n)$ on the left side of condition (14) can be regarded as a holomorphic function of the argument z in the simply connected region S_i, and thus it can be expanded into a Faber series of the region S_i as [13,14]

$$\frac{1}{2\pi I}\oint_{L_j} \frac{\bar{t}}{t-z}dt = \sum_{k=0}^{+\infty} b_{ijk} P_{ik}(z - z_{0i}), \quad z \in S_i (i = 1\ldots n), \tag{17}$$

$$\frac{1}{2\pi I}\oint_{L_{j+n}} \frac{\bar{t}}{t-z}dt = \sum_{k=0}^{+\infty} d_{ijk} P_{ik}(z - z_{0i}), \quad z \in S_i (i = 1\ldots n), \tag{18}$$

where z_{0i} is a specific point in the region S_i (see mapping (19)) and $P_{ik}(z - z_{0i})$ is the k^{th}-order Faber polynomial defined in the region S_i particularly with $P_{i0}(z - z_{0i}) = 1$, while b_{ijk} and d_{ijk} are the coefficients of the related Faber series. Here, each of the undetermined simply connected regions S_i $(i = 1\ldots n)$ can be defined by a conformal mapping which maps the exterior of the boundary L_i of the region S_i in the z-plane to the exterior of the unit circle (denoted by $\sigma_i = e^{I\theta}$) in the ξ_i-plane [12],

$$z - z_{0i} = \omega_i(\xi_i) = R_i\left(\xi_i + \sum_{l=1}^{+\infty} a_{il}\xi_i^{-l}\right), \quad i = 1\ldots n \tag{19}$$

where the known complex constant z_{0i} and the known real constant R_i characterize the location and size of the i-th inclusion in the z-plane, while all the unknown complex coefficients a_{il} determine the actual shape of the inclusion. Here, it follows from the definition of mapping (19) that the derivative of $\omega_i(\xi_i)$ is required to have no zeros outside the unit circle in the ξ_i-plane. Particularly, the argument z on the curves L_{j+n} $(j = 1\ldots n)$ can be expressed, according to the symmetry of the curves L_j and L_{j+n} about the x_1-axis, by

$$t = \bar{z}_{0j} + \overline{\omega_j(\sigma_j)} = \bar{z}_{0j} + \overline{\omega}_j(\sigma_j^{-1}), \quad t \in L_{j+n}(j = 1\ldots n). \tag{20}$$

Then based on the definition of Faber series [14,15] and the mapping (19), the coefficients b_{ijk} in Eq. (17) and d_{ijk} in Eq. (18) are given by

$$j = i: b_{ijk} = \frac{1}{2\pi I}\oint_{|\sigma_i|=1} \overline{(z_{0i} + \omega_i(\sigma_i))}\sigma_i^{-k-1}d\sigma_i = \begin{cases} \bar{z}_{0i}, & k = 0 \\ R_i\bar{a}_{ik}, & k \geq 1 \end{cases}, \tag{21}$$

$$j \neq i: \ b_{ijk} = \frac{-1}{4\pi^2} \oint_{|\sigma_i|=1} \oint_{|\sigma_j|=1} \frac{\overline{\left(z_{0j} + \omega_j(\sigma_j)\right)} \omega_j'(\sigma_j)\sigma_i^{-k-1}}{z_{0j} + \omega_j(\sigma_j) - z_{0i} - \omega_i(\sigma_i)} d\sigma_j \cdot d\sigma_i$$

$$= \frac{-1}{4\pi^2} \oint_{|\sigma_i|=1} \oint_{|\sigma_j|=1} \frac{\overline{\omega_j(\sigma_j)} \omega_j'(\sigma_j)\sigma_i^{-k-1}}{z_{0j} + \omega_j(\sigma_j) - z_{0i} - \omega_i(\sigma_i)} d\sigma_j \cdot d\sigma_i, \ k \geq 0, \tag{22}$$

$$d_{ijk} = \frac{-1}{4\pi^2} \oint_{|\sigma_i|=1} \oint_{|\sigma_j|=1} \frac{\left(z_{0j} + \omega_j(\sigma_j)\right) \overline{\omega}_j'(\sigma_j^{-1})\sigma_j^{-2}\sigma_i^{-k-1}}{\overline{z}_{0j} + \overline{\omega}_j(\sigma_j^{-1}) - z_{0i} - \omega_i(\sigma_i)} d\sigma_j \cdot d\sigma_i$$

$$= \frac{-1}{4\pi^2} \oint_{|\sigma_i|=1} \oint_{|\sigma_j|=1} \frac{\omega_j(\sigma_j) \overline{\omega}_j'(\sigma_j^{-1})\sigma_j^{-2}\sigma_i^{-k-1}}{\overline{z}_{0j} + \overline{\omega}_j(\sigma_j^{-1}) - z_{0i} - \omega_i(\sigma_i)} d\sigma_j \cdot d\sigma_i, \ k \geq 0. \tag{23}$$

Then substituting Eqs. (17) and (18) into Eq. (14) and using the formula $P_{i0}(z - z_{0i}) = 1$ and $P_{i1}(z - z_{0i}) = (z - z_{0i})/R_i$ [13,14], we obtain

$$A_i R_i P_{i1}(z - z_{0i}) + \sum_{k=1}^{+\infty} \left(\sum_{j=1}^{n} \left(B_j b_{ijk} - \overline{B}_j d_{ijk}\right) \right) P_{ik}(z - z_{0i})$$

$$= \sum_{j=1}^{n} \left(\overline{B}_j d_{ij0} - B_j b_{ij0}\right) - A_i z_{0i} - C_i, \quad \forall z \in S_i (i = 1 \ldots n). \tag{24}$$

In order to satisfy Eq. (24) for any given z in the region S_i $(i = 1 \ldots n)$, clearly the sufficient and necessary conditions are

$$A_i R_i + \sum_{j=1}^{n} \left(B_j b_{ij1} - \overline{B}_j d_{ij1}\right) = 0,$$

$$\sum_{j=1}^{n} \left(B_j b_{ijk} - \overline{B}_j d_{ijk}\right) = 0 \, (k \geq 2), \quad i = 1 \ldots n, \tag{25}$$

$$\sum_{j=1}^{n} \left(\overline{B}_j d_{ij0} - B_j b_{ij0}\right) - A_i z_{0i} - C_i = 0, \quad i = 1 \ldots n. \tag{26}$$

Here in Eqs. (25) and (26), the loading parameters A_i and B_i $(i = 1 \ldots n)$ defined in (8), determined by the given uniform eigenstrains and the prescribed uniform internal strain fields, and the geometry parameters z_{0i} and R_i $(i = 1 \ldots n)$ are all known, and the unknowns are the coefficients a_{il} $(i = 1 \ldots n, l = 1 \ldots +\infty)$ introduced in the mapping (19) which determine the actual shapes of the inclusions.

Generally speaking, the conformal mappings for the unknown inclusion shapes that achieve uniform internal strain fields could be given by infinite series and cannot be exactly expressed by a finite polynomial, but in a practical sense, the truncation of infinite series to a finite polynomial usually offers good approximations to the conformal mappings. Therefore, in what follows, the infinite series form of the conformal mapping (19) of each region S_i will be truncated into an N_i-order polynomial of N_i unknown coefficients a_{il} $(i = 1 \ldots n, l = 1 \ldots N_i)$, and thus the infinite number of nonlinear equations (25) (for $i = 1 \ldots n$, and $k \geq 2$) is truncated into a finite number of nonlinear equations (25) with $k = 2 \ldots N_i$ $(i = 1 \ldots n)$, respectively. Numerical methods will be employed to obtain these $\sum_{i=1}^{n} N_i$ coefficients by solving the $\sum_{i=1}^{n} N_i$ Eq. (25). Once the shapes of the multiple inclusions are obtained by solving the $\sum_{i=1}^{n} N_i$, Eq. (25), the unknown constants C_i $(i = 1 \ldots n)$ introduced in Eq. (3) can be determined uniquely from Eq. (26).

3.3 Newton–Raphson iteration

By defining two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{F}(\boldsymbol{\alpha})$ on the real and imaginary parts of the truncated coefficients a_{il} $(i = 1 \ldots n, l = 1 \ldots N_i)$,

$$\boldsymbol{\alpha}=\begin{bmatrix}\mathrm{Re}(a_{11})\\ \mathrm{Im}(a_{11})\\ \vdots\\ \mathrm{Re}(a_{nN_n})\\ \mathrm{Im}(a_{nN_n})\end{bmatrix},\ \boldsymbol{F}(\boldsymbol{\alpha})=\begin{bmatrix}\mathrm{Re}\,(A_1R_1)\\ \mathrm{Im}\,(A_1R_1)\\ 0\\ 0\\ \vdots\\ \mathrm{Re}\,(A_nR_n)\\ \mathrm{Im}\,(A_nR_n)\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}\sum_{j=1}^{n}\mathrm{Re}\left(B_jb_{1j1}-\overline{B}_jd_{1j1}\right)\\ \sum_{j=1}^{n}\mathrm{Im}\left(B_jb_{1j1}-\overline{B}_jd_{1j1}\right)\\ \vdots\\ \sum_{j=1}^{n}\mathrm{Re}\left(B_jb_{njN_n}-\overline{B}_jd_{njN_n}\right)\\ \sum_{j=1}^{n}\mathrm{Im}\left(B_jb_{njN_n}-\overline{B}_jd_{njN_n}\right)\end{bmatrix}, \tag{27}$$

the truncated real form of Eq. (25) can be rewritten as

$$\boldsymbol{F}(\boldsymbol{\alpha})=\boldsymbol{0}, \tag{28}$$

and the related Jacobian matrix $[\partial\boldsymbol{F}(\boldsymbol{\alpha})/\partial\boldsymbol{\alpha}]$ can be easily obtained based on the corresponding mapping (19) and the expressions (21)–(23). The iterative process is then given by

$$\boldsymbol{\alpha}^{(p+1)}=\boldsymbol{\alpha}^{(p)}-\left[\left.\frac{\partial\boldsymbol{F}(\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(p)}}\right]^{-1}\boldsymbol{F}(\boldsymbol{\alpha}^{(p)}),\quad p=0,1,\ldots \tag{29}$$

where the superscript "−1" indicates the inverse of the Jacobian matrix, and $\boldsymbol{\alpha}^{(p)}$ represents the value of the vector $\boldsymbol{\alpha}$ after the p-th iteration.

To guarantee convergence of the iterative process (29), here, the initial value $\boldsymbol{\alpha}^{(0)}$ with the known geometry parameters z_{0i} and R_i $(i=1\ldots n)$ will be given by, say, n disjoint ellipses (or circles) in the z-plane. It is expected that all of the inclusions would be elliptical (or circular) if the effects of the free surface on the inclusions and interaction between the inclusions are ignored, so here n disjoint ellipses (or circles) are given as the initial value to outline the rough shapes of the inclusions, and the iterative process describes how these rough inclusion shapes change to the final required inclusion shapes under the effects of the free surface on the inclusions and interaction between the inclusions. If the iterative process (29) does not converge for any reasonable initial value $\boldsymbol{\alpha}^{(0)}$, it implies that the prescribed uniform internal strain fields cannot actually be achieved under the given uniform eigenstrains and geometry conditions. In addition, even a convergent solution will be considered inadmissible if either the corresponding boundaries $L_i\,(i=1\ldots n)$ intersect in the z-plane or the derivative of any of the corresponding mappings (19) has zero(s) outside the unit circle in the ξ_i-plane.

For the present general problem of multiple inclusions, we have not achieved a simple sufficient and necessary condition imposed on the given uniform eigenstrains, geometry conditions and prescribed uniform internal strain fields, which guarantees the existence and uniqueness of the required inclusion shapes. However, for given uniform eigenstrains, geometry conditions and prescribed uniform internal strain fields, our numerical results indicated that the solution is unique because the iteration process always converges to the same inclusion shapes for different reasonable initial values.

4 Numerical examples

Our extensive numerical examples (including all examples described below) confirmed that moderately large numbers $N_i\,(7\le N_i\le 12)(i=1\ldots n)$ are sufficient to achieve a reasonably accurate convergent solution with relative errors less than 1 %.

4.1 A single inclusion in a half-plane

In the present method, when all of the inclusions are far away from the free surface of the half-plane, all of the coefficients d_{ijk} defined by (23) tend to be zero and the present solution for the half-plane converges to that for a whole plane studied in [11]. Particularly, letting all of the coefficients d_{ijk} in (25) be zero, and then substituting (21) into (25), we obtain the solution for a single inclusion ($n = 1$) that achieves a uniform internal strain field in a whole plane, as given by

$$a_{11} = -\overline{A}_1/\overline{B}_1, \quad a_{1l} = 0, \quad l = 2, 3, \ldots \tag{30}$$

where the condition $|a_{11}| < 1$ must be met for a correct conformal mapping (19). Equation (30) indicates that a single inclusion with a uniform internal strain field in a whole plane must be elliptical, and, on the other hand, it gives the admissible range of the uniform internal strain field inside a single elliptical inclusion in a whole plane by

$$A_1/B_1 = g_1, \ \ |g_1| < 1 \tag{31}$$

which, according to Eq. (8), can be expressed in a detailed form as

$$\Gamma_1 = -\Gamma_1^* - g_1\overline{\Gamma}_1^*, \ \ |g_1| < 1. \tag{32}$$

Therefore, for given uniform eigenstrains (determined by Γ_1^*) imposed on a single elliptical inclusion in a whole plane, the uniform internal strain field (determined by Γ_1) inside the elliptical inclusion may vary within a certain admissible range defined by the above arbitrary complex constant g_1 of absolute value less than unity which depends on the aspect ratio and orientation of the elliptical inclusion.

For a single inclusion in a half-plane, we shall still use the complex parameter $g_1 = A_1/B_1$ to define the prescribed uniform internal strain field in a similar way as formula (32). More precisely, for given uniform eigenstrains imposed on the inclusion, the complex parameter Γ_1, which determines the uniform internal strain field inside the inclusion, is now determined by the complex parameter g_1. Unlike the single-inclusion problem for a whole plane in which only an elliptical inclusion can achieve a uniform internal strain field, our results show that a single inclusion that achieves a uniform internal strain field in a half-plane is certainly non-elliptical. Shown in Fig. 3 are a few examples of a single inclusion with different given uniform eigenstrains in a half-plane which achieves a prescribed uniform internal strain field.

It can be readily seen that the non-elliptical shapes of a single inclusion in a half-plane shown in Fig. 3a–c are essentially equivalent to those of two symmetrical inclusions in a whole plane shown in Figure 1 of [8], Figure 3.1 of [9] and Figures 2, 3 and 7a of [10], respectively. Actually, for example, for a symmetrical inclusion pair that is symmetrical about the x_1-axis in a whole plane with symmetrical uniform eigenstrains $\varepsilon_{13}^{*(1)} = \varepsilon_{13}^{*(2)}$ and $\varepsilon_{23}^{*(1)} = -\varepsilon_{23}^{*(2)}$ (equivalently $\Gamma_1^* = -\overline{\Gamma}_2^*$), the shear stress σ_{23} will vanish along the entire midline which is thus equivalent to the free surface of a half-plane. Therefore, a symmetrical inclusion pair in a whole plane is equivalent to a single inclusion in a half-plane. Note that the problem studied by Liu [8] is a standard eigenstrain problem, while the problems studied by Kang et al. [9] and Wang [10], which considered the remote anti-plane shear loadings without eigenstrains, can be transformed into standard eigenstrain problems. However, in all the eigenstrain (or transformed eigenstrain) problems studied by Liu [8], Kang et al. [9] and Wang [10], the nonzero uniform eigenstrains imposed on any two inclusions in a whole plane are limited to either $\varepsilon_{13}^{*(1)} = \varepsilon_{13}^{*(2)}$ and $\varepsilon_{23}^{*(1)} = \varepsilon_{23}^{*(2)}$ (equivalently $\Gamma_1^* = \Gamma_2^*$) or $\varepsilon_{13}^{*(1)} \neq \varepsilon_{13}^{*(2)}$ and $\varepsilon_{23}^{*(1)} \neq \varepsilon_{23}^{*(2)}$ (equivalently $\mathrm{Im}(\Gamma_1^*) \neq \mathrm{Im}(\Gamma_2^*)$ and $\mathrm{Re}(\Gamma_1^*) \neq \mathrm{Re}(\Gamma_2^*)$). Therefore, when the uniform eigenstrains imposed on a single inclusion in a half-plane are given as $\varepsilon_{13}^{*(1)} \neq 0$ and $\varepsilon_{23}^{*(1)} \neq 0$ (equivalently $\mathrm{Im}(\Gamma_1^*) \neq 0$ and $\mathrm{Re}(\Gamma_1^*) \neq 0$), the required inclusion shape of the present problem (such as the shape shown in Fig. 3d) cannot be derived from the solutions of two symmetrical inclusions in a whole plane studied by Liu [8], Kang et al. [9], and Wang [10].

It is expected that the shape of a single inclusion in a half-plane which achieves a uniform internal strain field will be impacted by the distance between the inclusion and the free surface of the half-plane, so it is of particular interest to study the dependence of the inclusion shape on the distance between the inclusion and the free surface. Figure 4 shows a few examples of a single inclusion which achieves a uniform internal strain field in a half-plane when the distance between the inclusion and the free surface is extremely small, while Fig. 5 shows the convergence of a single inclusion shape which achieves a uniform internal strain field in a half-plane with increasing distance between the inclusion and the free surface.

It should be noted that the specific elliptical inclusion shown in Figs. 4 and 5 is constructed to achieve the uniform internal strain field (in a whole plane, not in a half-plane) defined by the same parameter g_1 and the same uniform eigenstrains (see formula (32)). It can be seen from Fig. 4 that for real parameters g_1 and uniform eigenstrains satisfying $\varepsilon_{13}^{*(1)} \neq 0$ with $\varepsilon_{23}^{*(1)} = 0$ (equivalently $\mathrm{Im}(\Gamma_1^*) \neq 0$ with $\mathrm{Re}(\Gamma_1^*) = 0$), the single inclusion that achieves a uniform internal strain field in a half-plane does not converge to a half-elliptical shape when the distance between the inclusion and the free surface of the half-plane reduces to zero.

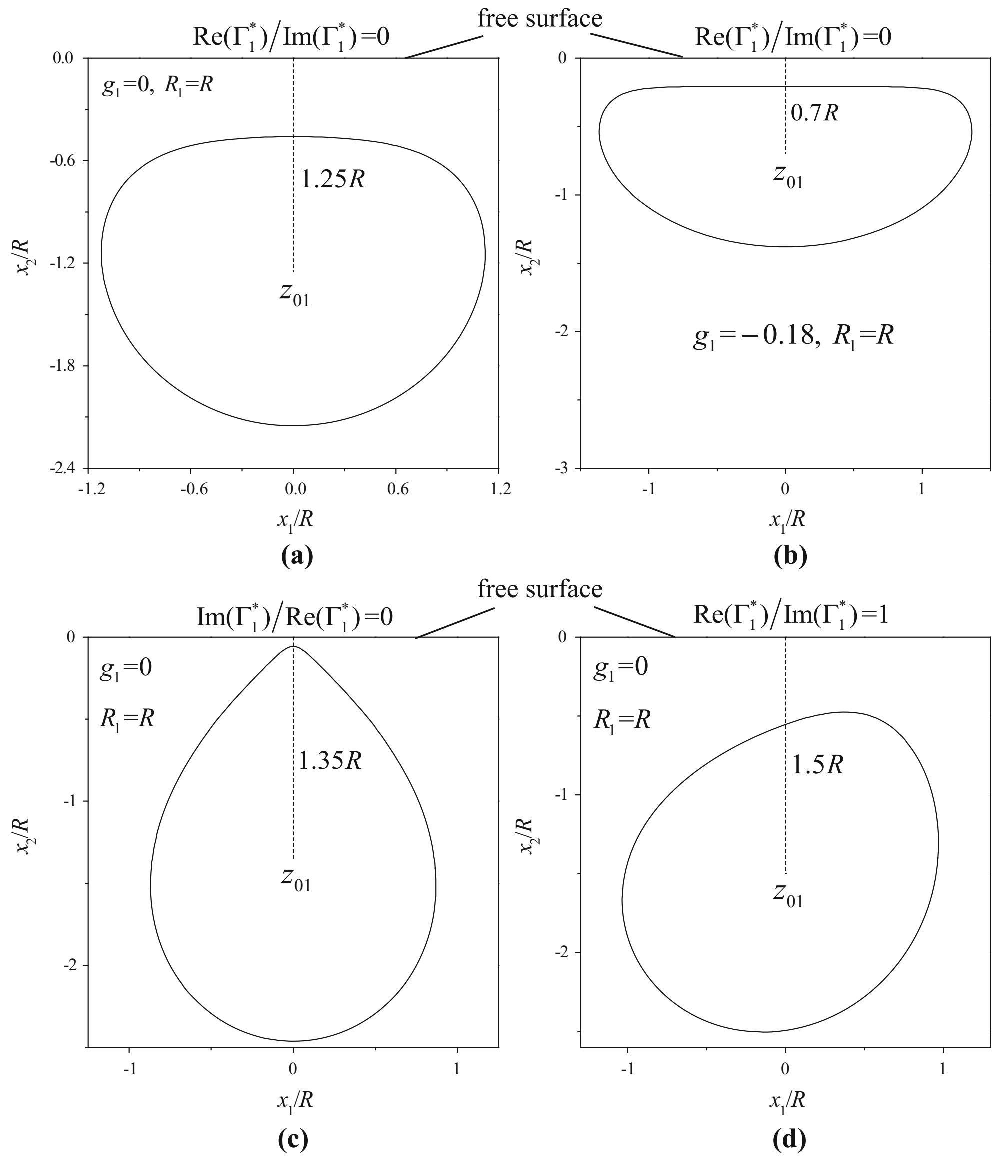

Fig. 3 A single inclusion that achieves a uniform internal strain field in a half-plane

Therefore, the results shown in Fig. 4 indicate that a single non-elliptical inclusion that achieves a uniform internal strain field in a half-plane will not converge to a half of a single elliptical inclusion in a whole plane which achieves the same uniform internal strain field under the same uniform eigenstrains when the distance between the inclusion and the free surface of the half-plane reduces to zero. It is shown from Fig. 5 that a single non-elliptical inclusion that achieves a uniform internal strain field in a half-plane will converge to a single elliptical inclusion in a whole plane (with relative error less than 5 %) which achieves the same uniform internal strain field under the same uniform eigenstrains when the distance between the non-elliptical inclusion and the free surface of the half-plane is at least a few times (say, three times) the size of the non-elliptical inclusion.

It is of great interest to see whether the uniform internal strain field for a single inclusion in a half-plane could be beyond or entirely limited to the admissible range of the uniform internal strain field defined by $|g_1| < 1$ for a single elliptical inclusion in a whole plane. Actually, Figs. 3, 4 and 5 already give a few examples of a single inclusion in a half-plane which achieves prescribed uniform internal strain fields defined by some $|g_1| < 1$ (see formula (32)). Here, shown in Fig. 6 are some additional examples of a single inclusion in a half-plane which achieves prescribed uniform internal strain fields defined by several other $|g_1| < 1$ (see formula (32)).

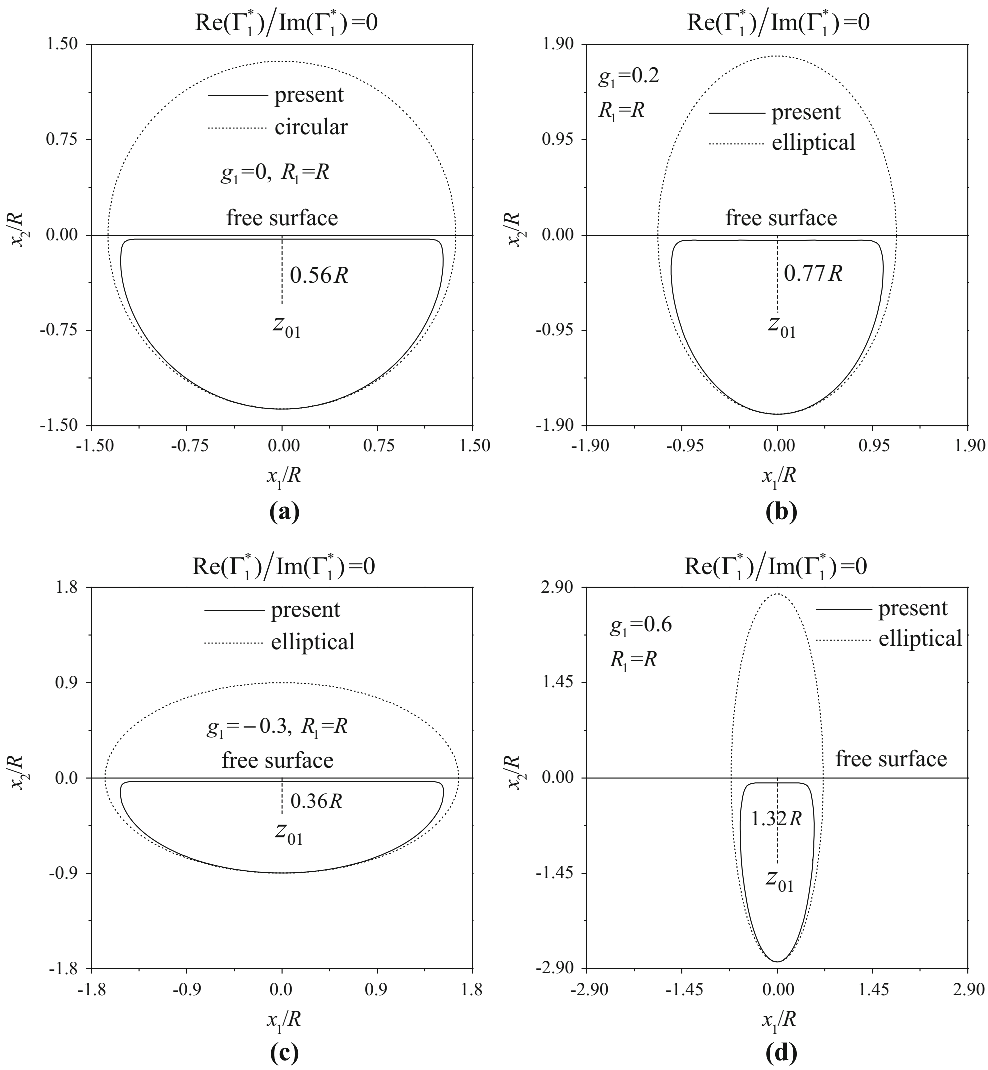

Fig. 4 A single inclusion that achieves a uniform internal strain field in a half-plane when the distance between the inclusion and the free surface of the half-plane is extremely small

Our results shown in Figs. 3, 4, 5 and 6 indicate that any admissible uniform internal strain field with $|g_1| < 1$ for a single elliptical inclusion in a whole plane is achievable by a single non-elliptical inclusion in a half-plane. However, our extensive numerical examples also indicated that it is almost impossible to construct a single non-elliptical inclusion in a half-plane which achieves a uniform internal strain field (with $|g_1| > 1$) beyond the admissible range $|g_1| < 1$ for a single elliptical inclusion in a whole plane. Thus, the present work suggested that the admissible range of the uniform internal strain field for a single inclusion in a half-plane is almost identical to the admissible range of the uniform internal strain field for a single elliptical inclusion in a whole plane.

4.2 Multiple inclusions in a half-plane

For multiple inclusions in a half-plane, we shall also use the complex parameters $g_i = A_i/B_i$ $(i = 1\ldots n)$ to define their prescribed uniform internal strain fields in a similar way as formula (32). Clearly, for given uniform eigenstrains (determined by $\Gamma_i^*, i = 1\ldots n$) imposed on each of the multiple inclusions, the prescribed uniform internal strain field inside each of the multiple inclusions is given by

$$\Gamma_i = -\Gamma_i^* - g_i\overline{\Gamma}_i^*,\ i = 1\ldots n. \tag{33}$$

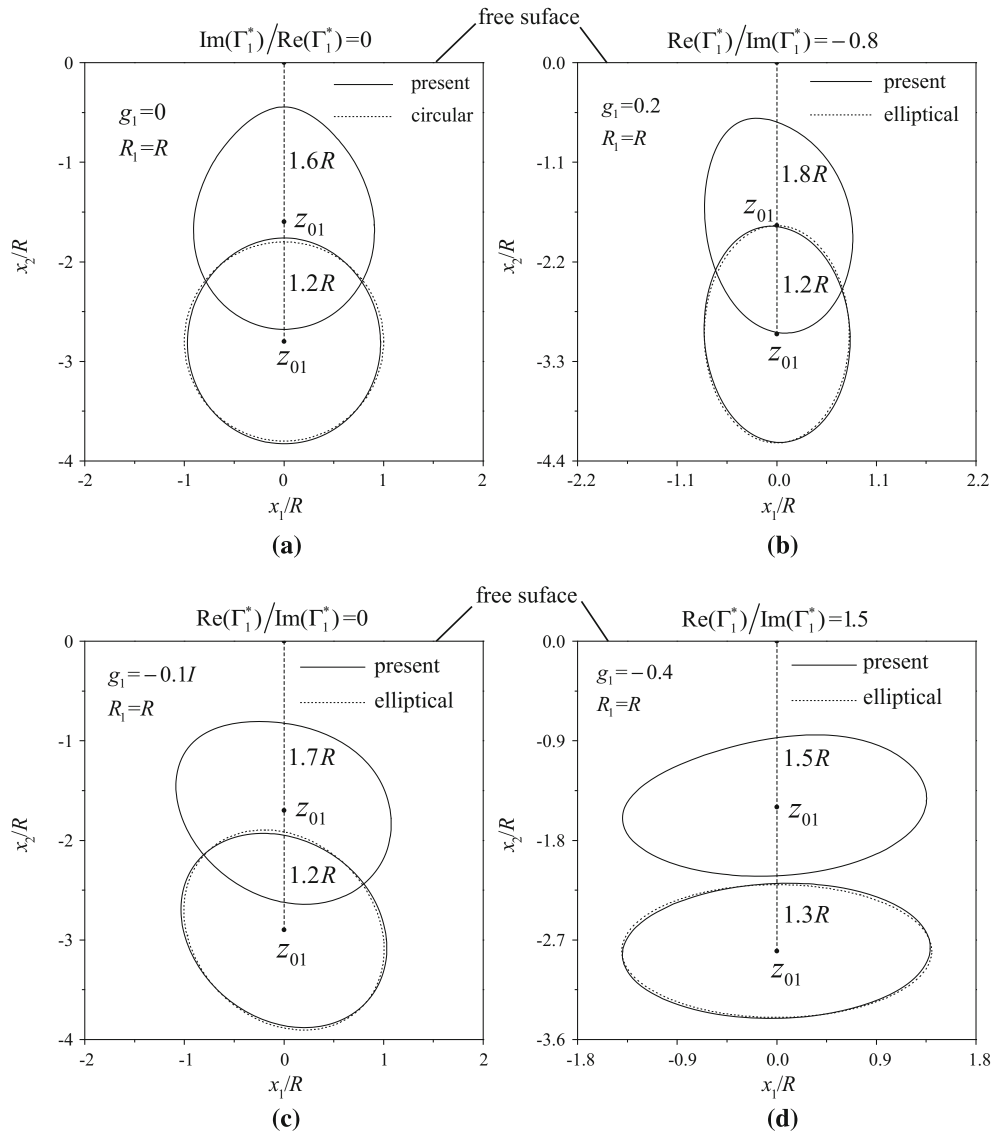

Fig. 5 Convergence of the shape of a single inclusion that achieves a uniform internal strain field in a half-plane with increasing distance between the inclusion and the free surface of the half-plane

As the first example, Fig. 7 gives a comparison between the previous results of multiple inclusions that achieve uniform internal strain fields in a whole plane (see [11]) and our present results of multiple inclusions that achieve uniform internal strain fields in a half-plane when the distance between the multiple inclusions and the free surface of the half-plane increases.

It should be pointed out that although the problem studied in [11] considers only the remote shear loadings without eigenstrains, it can be transformed equivalently into a standard eigenstrain problem in which the uniform eigenstrains imposed on each inclusion are determined by the remote loading and the elastic constants of the inclusion and matrix. It is shown in Fig. 7 that a inclusion pair which achieves uniform internal strain fields in a half-plane converges to that which achieves the same uniform internal strain fields in a whole plane (with relative errors less than 5 %) when the distance between the inclusion pair and the free surface of the half-plane is a few times (say, three times) the size of the inclusion pair.

It is concluded in Sect. 4.1 that any admissible uniform internal strain field defined by $|g_1| < 1$ for a single elliptical inclusion in a whole plane is achievable by a single inclusion in a half-plane. Therefore, it is of particular interest to see whether this conclusion for a single inclusion in a half-plane could still hold for multiple inclusions ($n \geq 2$) in a half-plane. Fig. 8 shows multiple inclusions with various uniform eigenstrains

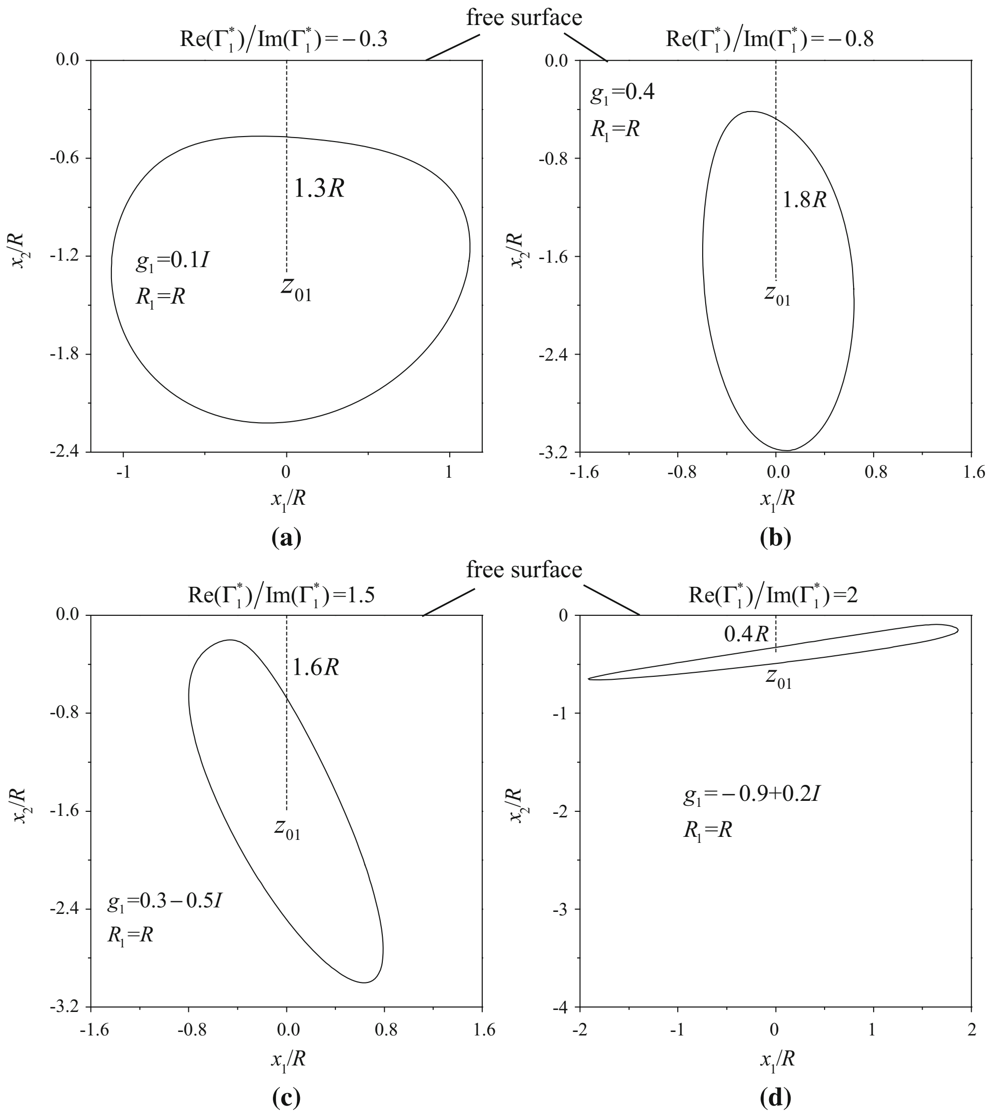

Fig. 6 A single inclusion in a half-plane which achieves various uniform internal strain fields within the admissible range of the uniform internal strain field for a single elliptical inclusion in a whole plane

in a half-plane which achieve various prescribed uniform internal strain fields (with $|g_i| < 1$) that are admissible for a single elliptical inclusion in a whole plane.

Our results shown in Fig. 8 indicate that any admissible uniform internal field (defined by $|g_1| < 1$) for a single elliptical inclusion in a whole plane is achievable for multiple inclusions in a half-plane.

It is suggested in Sect. 4.1 that the uniform internal strain field for a single inclusion in a half-plane can be hardly beyond the admissible range of the uniform internal strain field (defined by $|g_1| < 1$) for a single elliptical inclusion in a whole plane. However, our results will show that this conclusion for a single inclusion in a half-plane will be no longer valid for multiple inclusions ($n \geq 2$) in a half-plane. Shown in Fig. 9 are some examples of multiple inclusions subjected to various uniform eigenstrains in a half-plane, some of which achieve(s) prescribed uniform internal strain fields with $|g_i| > 1$ that are slightly outside the admissible range defined by $|g_1| < 1$ for a single elliptical inclusion in a whole plane.

Our extensive numerical examples showed that it seems almost impossible to construct such multiple inclusions in a half-plane that all of the inclusions achieve uniform internal strain fields with $|g_i| > 1$ simultaneously. Our numerical examples also indicated that when the norm of one prescribed parameter g_i is relatively large (such as $|g_i| > 2$), it is very difficult to construct an inclusion in a half-plane which achieves such a uniform internal strain field much beyond the admissible range $|g_i| < 1$ for a single elliptical inclusion in a

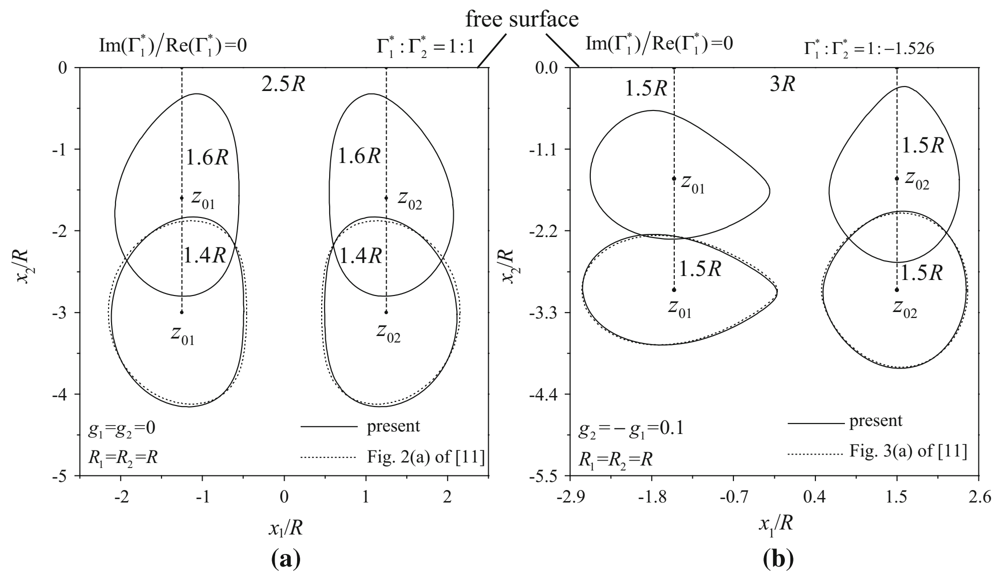

Fig. 7 Convergence of the shapes of multiple inclusions that achieve uniform internal strain fields in a half-plane with increasing distance between the inclusions and the free surface of the half-plane

whole plane. Thus, the present work suggested that only some of multiple inclusions in a half-plane could simultaneously achieve uniform internal strain fields which are moderately beyond the admissible range of the uniform internal strain field for a single elliptical inclusion in a whole plane.

4.3 Two symmetrical inclusions in a half-plane

In the z-plane, consider two closed curves L_1 and L_2 which are symmetrical to each other about a certain line parallel to the x_2-axis. For any two symmetrical points z_1 and z_2 about the line of symmetry, one can verify that

$$\overline{\frac{1}{2\pi I}\oint_{L_1}\frac{\bar{t}}{t-z_1}dt} = \text{constant} - \frac{1}{2\pi I}\oint_{L_2}\frac{\bar{t}}{t-z_2}dt, \quad \forall z_1 \in S_1, \tag{34}$$

$$\overline{\frac{1}{2\pi I}\oint_{L_1}\frac{\bar{t}}{t-z_1}dt} = -\frac{1}{2\pi I}\oint_{L_2}\frac{\bar{t}}{t-z_2}dt, \quad \forall z_1 \notin S_1 \tag{35}$$

where S_1 denotes the finite region bounded by the curve L_1. Using Eqs. (34) and (35), we can derive the condition on the uniform eigenstrains and prescribed uniform internal strain fields which guarantees the existence of even-number symmetrical inclusions in a half-plane that are symmetrical about a line parallel to the x_2-axis.

Here, we give an example of two such symmetrical inclusions in a half-plane which achieve uniform internal strain fields. For two symmetrical inclusions bounded by the curves L_1 and L_2 in a half-plane which are symmetrical about a certain line parallel to the x_2-axis, the related curves L_3 and L_4 in Eq. (14) ($n = 2$) are also symmetrical about the line, and then conjugating the two sides of Eq. (14) ($n = 2$) and using Eqs. (34) and (35), one gets

$$\begin{aligned}
&-\overline{A}_1 z - \frac{\overline{B}_2}{2\pi I}\oint_{L_1}\frac{\bar{t}}{t-z}dt - \frac{\overline{B}_1}{2\pi I}\oint_{L_2}\frac{\bar{t}}{t-z}dt \\
&\quad + \frac{B_2}{2\pi I}\oint_{L_3}\frac{\bar{t}}{t-z}dt + \frac{B_1}{2\pi I}\oint_{L_4}\frac{\bar{t}}{t-z}dt = -D_1, \ \forall z \in S_2, \\
&-\overline{A}_2 z - \frac{\overline{B}_2}{2\pi I}\oint_{L_1}\frac{\bar{t}}{t-z}dt - \frac{\overline{B}_1}{2\pi I}\oint_{L_2}\frac{\bar{t}}{t-z}dt \\
&\quad + \frac{B_2}{2\pi I}\oint_{L_3}\frac{\bar{t}}{t-z}dt + \frac{B_1}{2\pi I}\oint_{L_4}\frac{\bar{t}}{t-z}dt = -D_2, \ \forall z \in S_1
\end{aligned} \tag{36}$$

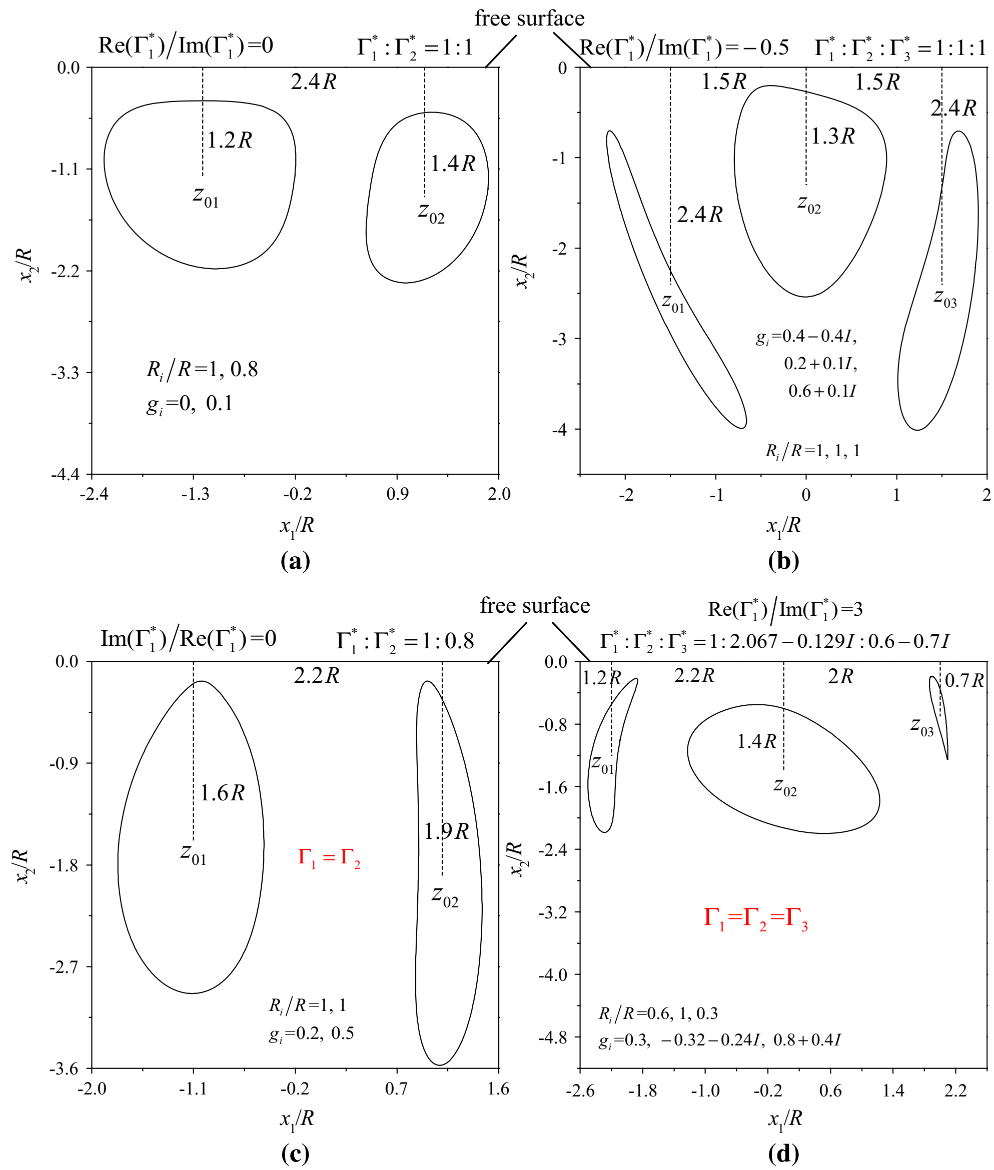

Fig. 8 Multiple inclusions in a half-plane which achieve various uniform internal strain fields within the admissible range of the uniform internal strain field for a single elliptical inclusion in a whole plane

where D_1 and D_2 are two new unknown constants. Considering that Eq. (36) has to be equivalent to Eq. (14) ($n = 2$), we require

$$\frac{\overline{B}_1}{B_2} = \frac{B_2}{\overline{B}_1},\ \frac{A_1}{B_1} = \frac{\overline{A}_2}{\overline{B}_2}. \tag{37}$$

Furthermore, based on the condition (37) with detailed relations in Eq. (8), the parameters Γ_i^* $(i = 1, 2)$, which determine the uniform eigenstrains imposed on the two symmetrical inclusions, and the complex parameters $g_i = A_i/B_i$ $(i = 1, 2)$ should satisfy

$$\Gamma_2^* = \pm\overline{\Gamma}_1^*,\ g_2 = \overline{g}_1. \tag{38}$$

Shown in Figs. 10 and 11 are a series of two symmetrical inclusions with prescribed uniform internal strain fields in a half-plane based on the condition (38).

Since the condition $\Gamma_2^* = \overline{\Gamma}_1^*$ (or equivalently $\varepsilon_{13}^{*(1)} = -\varepsilon_{13}^{*(2)}$ and $\varepsilon_{23}^{*(1)} = \varepsilon_{23}^{*(2)}$) indicates that the uniform eigenstrains imposed on two certain inclusions are symmetrical about a midline parallel to the x_2-axis, it is expected that the two inclusions which achieve uniform internal strain fields will be symmetrical

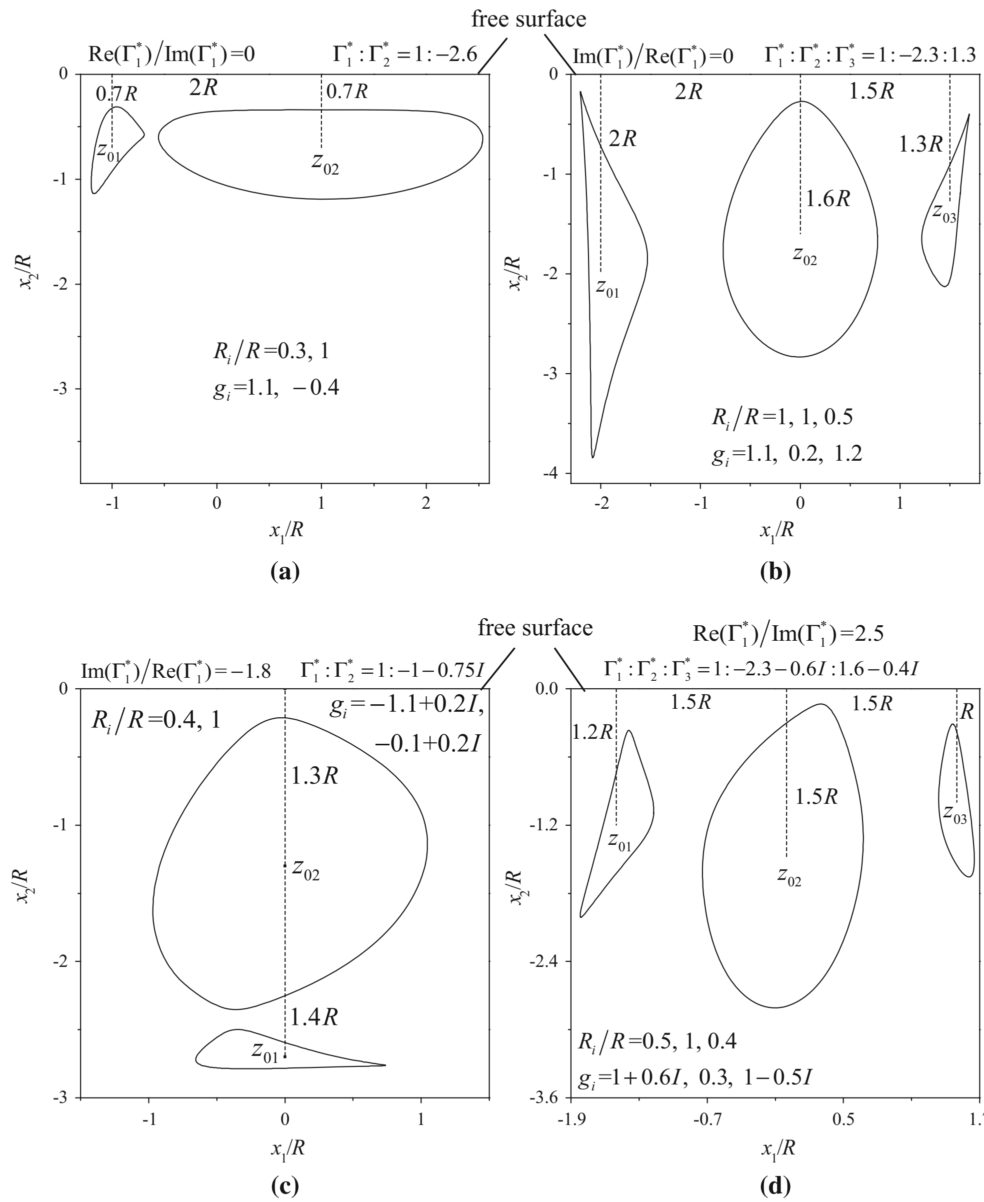

Fig. 9 Multiple inclusions in a half-plane which achieve uniform internal strain fields moderately beyond the admissible range of the uniform internal strain field for a single elliptical inclusion in a whole plane

about the midline (see Fig. 10). However, as shown in Fig. 11, when the uniform eigenstrains (satisfying $\Gamma_2^* = -\overline{\Gamma}_1^*$ or equivalently $\varepsilon_{13}^{*(1)} = \varepsilon_{13}^{*(2)}$ and $\varepsilon_{23}^{*(1)} = -\varepsilon_{23}^{*(2)}$) imposed on two certain inclusions are anti-symmetrical about a midline parallel to the x_2-axis, the two inclusions that achieve uniform internal strain fields could also be geometrically symmetrical about the midline. In particular, for symmetrical uniform eigenstrains, such symmetrical inclusion pairs (see Fig. 10) offer interesting examples of non-elliptical inclusion shapes that achieve uniform internal strain fields in a quarter plane with two mutually perpendicular free surfaces.

4.4 Dependence of the inclusion shapes on the given eigenstrains

Unlike the single-inclusion problem in a whole plane in which an arbitrary elliptical inclusion always enjoys a uniform internal strain field for any arbitrary uniform eigenstrains imposed on the inclusion, it is most

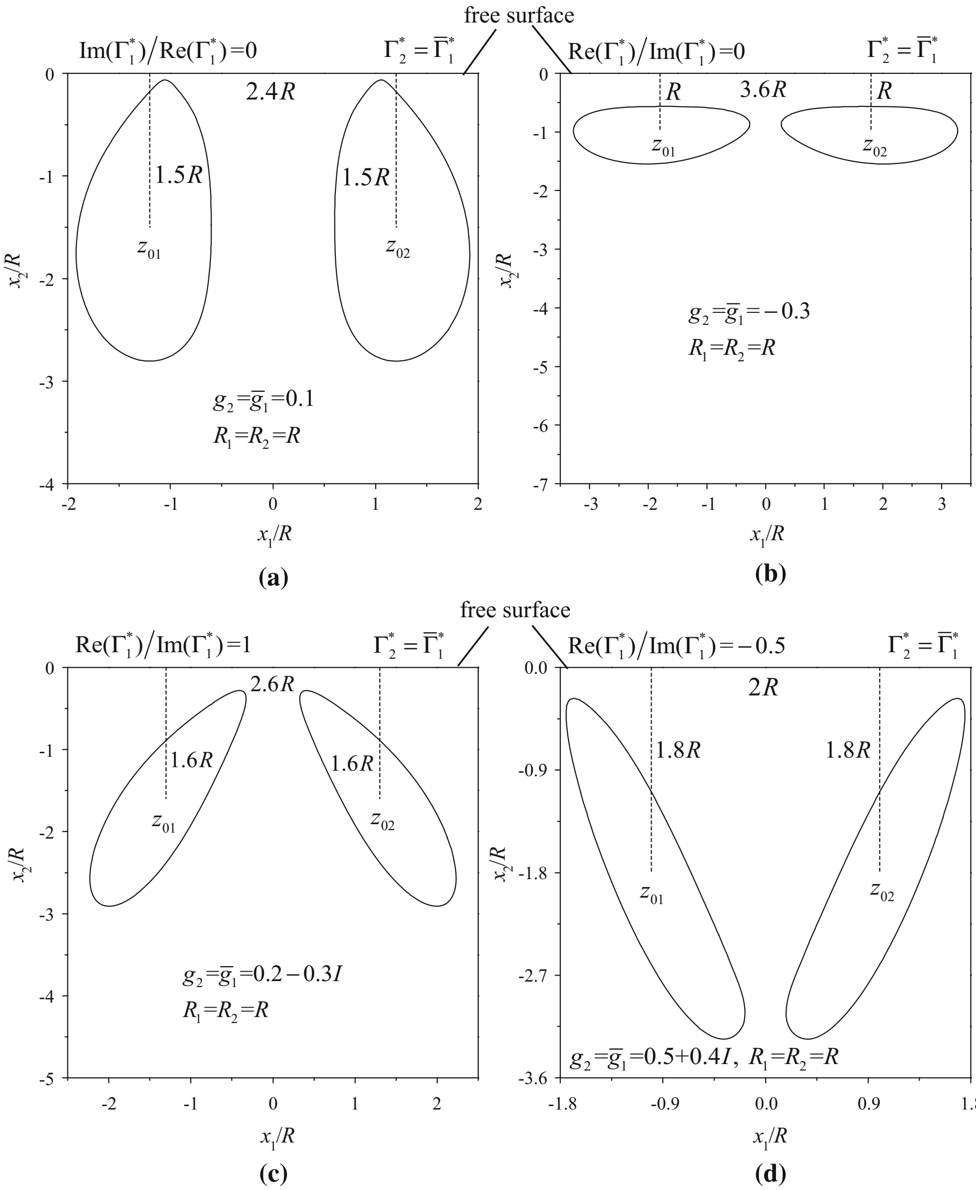

Fig. 10 Two symmetrical inclusions with various uniform internal strain fields in a half-plane under the condition $\Gamma_2^* = \overline{\Gamma}_1^*$

likely that the shapes of the inclusions in a half-plane shown in Sects. 4.1–4.3 (specially see Fig. 3) depend on specific given uniform eigenstrains imposed on the inclusions, and their internal strain fields may be no longer uniform for other different uniform eigenstrains. Therefore, it is of particular interest to see if any possible inclusion shapes exist which can always achieve uniform internal strain fields for arbitrary uniform eigenstrains imposed on the inclusions. Note that the inclusion shapes determined by Eq. (25) can be rewritten as

$$\frac{A_i}{B_i}R_i + \sum_{j=1}^{n}\left(\frac{B_j}{B_i}b_{ij1} - \frac{\overline{B}_j}{B_i}d_{ij1}\right) = 0,$$
$$\sum_{j=1}^{n}\left(\frac{B_j}{B_i}b_{ijk} - \frac{\overline{B}_j}{B_i}d_{ijk}\right) = 0\ (k \geq 2),\ \ i = 1\ldots n. \tag{39}$$

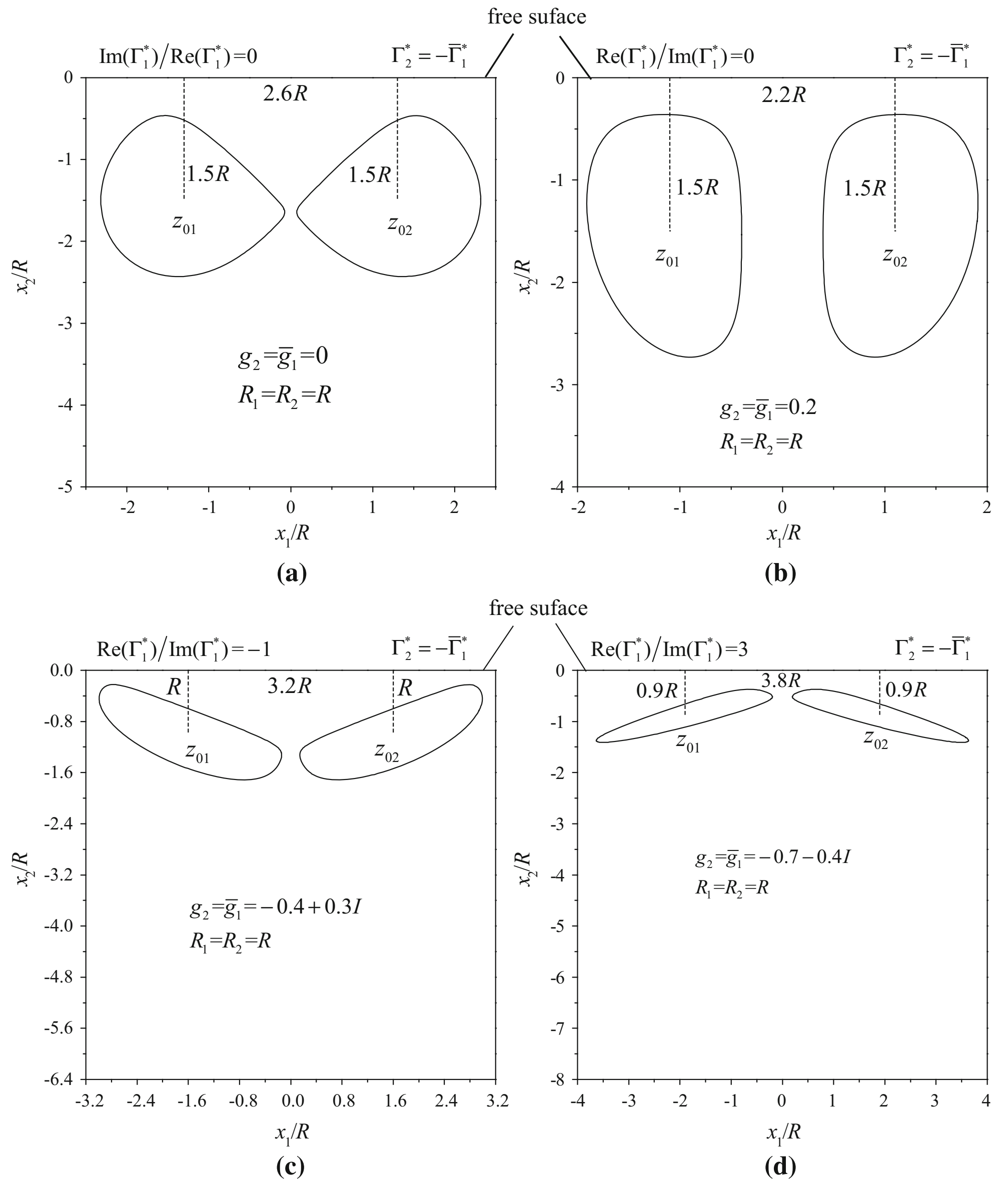

Fig. 11 Two symmetrical inclusions with various uniform internal strain fields in a half-plane under the condition $\Gamma_2^* = -\overline{\Gamma}_1^*$

If the prescribed uniform internal strain fields are given in such way that all ratios A_i/B_i, B_j/B_i and $\overline{B}_j/B_i$ $(i, j = 1 \ldots n)$ in Eq. (39) (or equivalently all ratios A_i/B_i, B_j/B_i and $\overline{B}_j/B_j$) are independent of the uniform eigenstrains imposed on the inclusions, then the corresponding inclusion shapes will be independent of the uniform eigenstrains.

According to Eq. (8), one can get

$$B_j/B_i = \overline{\Gamma_j^*/\Gamma_i^*}; \quad i, j = 1 \ldots n, \tag{40}$$

$$\overline{B}_j/B_j = \Gamma_j^*/\overline{\Gamma}_j^*, \quad j = 1 \ldots n \tag{41}$$

where, clearly, the ratios B_j/B_i and $\overline{B}_j/B_j$ $(i, j = 1 \ldots n)$ are certainly dependent on the uniform eigenstrains determined by Γ_j^* $(j = 1 \ldots n)$, and also the uniform eigenstrains will be determined uniquely by given ratios B_j/B_i and $\overline{B}_j/B_j$ $(i, j = 1 \ldots n)$. Therefore, it is concluded that there exist no single or multiple inclusion shapes that can achieve a uniform internal strain field in a half-plane for arbitrarily given uniform eigenstrains imposed on the inclusions. More precisely, for example, the inclusions shown in all of previous figures (see Sects. 4.1–4.3) are only based on specific given uniform eigenstrains, and they will no longer achieve uniform internal strain fields when the eigenstrains imposed on at least one of the inclusions are changed. In addition, note that when all of the parameters d_{ijk} in (39) tend to be zero, the present solution for a half-plane converges to that for a whole plane, and it follows from (40) that there indeed exist multiple inclusions in a whole plane which achieve uniform internal strain fields for varying uniform eigenstrains provided that the ratios between the uniform eigenstrains imposed on all inclusions keep unchanged.

5 Conclusions

The present work aims to answer an unexplored question whether any inclusion shape can achieve a uniform internal strain field in an elastic half-plane under either given uniform remote loadings or given uniform eigenstrains imposed on the inclusion. Such non-elliptical inclusions with given uniform anti-plane shear eigenstrains which achieve prescribed uniform internal strain fields in a half-plane are constructed by solving the original problem of an unknown holomorphic function in a multiply connected half-plane which, based on an analytic continuation, is transferred to an equivalent problem of an unknown holomorphic function in a multiply connected whole plane. Numerical examples are given to verify the validity and accuracy of the present method, and the dependence of the inclusion shapes on the given uniform eigenstrains is examined. Among others, some conclusions can be drawn as follows:

(i) Single or multiple non-elliptical inclusion shapes exist which achieve uniform internal strain fields in an elastic half-plane under given uniform anti-plane eigenstrains imposed on the inclusions. However, such inclusion shapes depend on the given uniform eigenstrains, and the inclusion shapes in a half-plane which achieve uniform internal strain fields for arbitrarily given uniform eigenstrains do not exist.

(ii) The effect of the free surface on the inclusion shapes that achieve uniform internal strain fields is almost ignorable, and thus the half-plane can be treated approximately as a whole plane when the distance between the inclusions and the free surface increases up to more than three times the size of the inclusions.

(iii) Two symmetrical inclusions in a half-plane, which are geometrically symmetrical about a midline perpendicular to the free surface of the half-plane, can be constructed to achieve uniform internal strain fields if the given uniform eigenstrains imposed on the two inclusions are symmetrical or anti-symmetrical about the midline. In particular, for symmetrical uniform eigenstrains, such symmetrical inclusion pairs offer interesting examples of non-elliptical inclusion shapes that achieve uniform internal strain fields in a quarter plane with two mutually perpendicular free surfaces.

Finally, it should be mentioned that the present method and results are limited to anti-plane shear. A similar problem in plane stress or plane strain about the existence of non-elliptical inclusions that achieve uniform internal stress fields in an elastic half-plane is a real challenge for further study.

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