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ORIGINAL PAPER

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New integrable problems in the dynamics of particle and rigid body

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Abstract In the present article, a new two-dimensional integrable system containing 17 free parameters is introduced. For giving certain values for these parameters, new integrable problems can be constructed, which generalize some known previous problems, and in some cases, we can restore some previous integrable problems. Two new integrable problems are announced, describing the motion in an Euclidean plane and on a pseudo-sphere. In the irreversible case, a new integrable problem in rigid body dynamics, which generalizes Goriachev–Chaplygin's case (Varshav Univ Izvest 3:1–13, 1916), Yehia's case (Mech Res Commun 23:423–427, 1996) and Elmandouh's case (Acta Mech 226:2461–2472, 2015), is announced.

1 Preliminaries

It is well known that, in general, Hamiltonian systems are non-integrable and integrable ones are a rare exception among them. For them, their behavior can be investigated globally in an infinite time interval. They can also be used, through perturbation theories, to give certain conclusions about the motion of non-integrable systems near to them.

In the majority of all known integrable systems in mechanics the second integral, which is used for proving the integrability and solving equations of motion, is a polynomial in velocity variables. The classification of such systems attracted the attention of many scientists interested in this branch. Bertrand [1] was the first who tried to construct all plane systems with linear, quadratic and cubic integrals, a long time ago. He was shortly followed by Darboux [2] who studied the construction of integrable problems with quadratic integral (see also [3]).

Kowalevski's [4] integrable case in the dynamics of a rigid body which moves under the action of a uniform gravitational field seems probably the first integrable case of a mechanical system having a complementary quartic integral in velocities. After a decade or so, another case describing the motion of a rigid body in a liquid was introduced by Chaplygin [5]. It should be noted that, until now, these two cases are the only examples of integrable mechanical systems having complementary quartic integral on a two-dimensional curved manifolds.

In the last four decades or so, a limited number of integrable problems characterizing the motion of a particle in the Euclidean plane were introduced (see, e.g., [6–19]). Most of these cases were collected in Hietarienta's review [20].

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1.1 Equations of motion of a rigid body

The problem of a rigid body is a rich problem with applications in various branches of science such as astronomy and physics (see, e.g., [21–23]). Consequently, it is a good model for investigation.

Let a rigid body with a fixed point be in motion under the influence of the resultant of potential and gyroscopic forces. The potential force is characterized by $V(\gamma)$, and the gyroscopic force is determined by $\mathbf{l} = (0, 0, l_3)$. Such motion is described by the Lagrangian (see, e.g., [25])

$$L = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\omega}\mathbf{I} + \mathbf{l} \cdot \boldsymbol{\omega} - V, \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity and where $\mathbf{I} = \mathbf{diag}(A, B, C)$ is the inertia matrix of the body. The equations of motion for Lagrangian (1) can be expressed as (see, e.g., [24–26])

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \gamma}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0}, \tag{2}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is a unit vector which is fixed upward in space and $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = \frac{\partial}{\partial \boldsymbol{\gamma}}(\mathbf{l} \cdot \boldsymbol{\gamma}) - \left(\frac{\partial}{\partial \boldsymbol{\gamma}} \cdot \mathbf{l}\right)\boldsymbol{\gamma}. \tag{3}$$

It is evident that Eqs. (2) have three integrals of motion, which are called classical integrals. They are

1. Jacobi integral:

$$I_1 = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\omega}\mathbf{I} + V = h, \tag{4}$$

   where h is the numerical value of the Jacobi integral.
2. Geometric integral:

$$I_2 = \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = 1. \tag{5}$$

3. Cyclic integral:

$$I_3 = (\boldsymbol{\omega}\mathbf{I} + \mathbf{l}) \cdot \boldsymbol{\gamma} = f, \tag{6}$$

where f is an arbitrary constant that represents the value of the cyclic integral.

According to Jacobi's theorem on the last integrating multiplier [3], four integrals of motion are required to prove the integrability of Eqs. (2). Consequently, one integral that is independent of the three integrals (4), (5) and (6) is sufficient to establish the integrability.

It is clear that the present problem has three degrees of freedom, which can decrease to two due to the presence of the cyclic variable ψ. Therefore, one can use the Routh procedure to ignore the cyclic variable and describe the problem by using the Routhian

$$R = \frac{1}{2}\left[\frac{\dot{\gamma}_3^2}{1-\gamma_3^2} + \frac{C(1-\gamma_3^2)\dot{\varphi}^2}{A-(A-C)\gamma_3^2}\right] + \frac{fC\gamma_3 + Al_3(1-\gamma_3^2)}{A\left[A-(A-C)\gamma_3^2\right]}\dot{\varphi} - \frac{1}{A}\left(V + \frac{(f-l_3\gamma_3)^2}{2[A-(A-C)\gamma_3^2]}\right). \tag{7}$$

The new integrable case in the rigid body dynamic is always characterized by the two scalar and vector functions V and $\boldsymbol{\mu}$. The reason for this is that they are invariant under all possible gauge transformations. To clarify that, one can add $\frac{\mathrm{d}}{\mathrm{d}t}N(\gamma_1, \gamma_2, \gamma_3)$ to the Lagrangian (1). The linear terms in velocity in the Lagrangian will be changed, and these changes do not affect the two functions V and $\boldsymbol{\mu}$. Consequently, the equations of motion (2) still remain unchanged.

2 Formulation of the problem

In [27], two-dimensional integrable problems (not necessarily plane) that admit a complementary integral polynomial in velocities were presented, and have been reformulated in [28]. This method has been used in constructing several integrable problems with complementary integral quadratic (see, e.g., [29,30]), cubic (see, e.g., [31]) and quartic (see, for example, [32–34]). The utilization of this method is restricted to two-dimensional mechanical systems. Many examples belong to this type such as the problem of motion with n degrees of freedom having $n-2$ cyclic coordinates and the problem of motion of a particle on a smooth surface under a variety of forces. Further example is the motion of a rigid body about its fixed point under the action of a combination of potential and gyroscopic forces which permit to cyclic coordinate to exist (see, e.g., [35,36]). Such systems are described by the Lagrangian

$$L = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + a_1\dot{q}_1 + a_2\dot{q} - V, \tag{8}$$

where a_{ij}, a_i, V are functions in both variables q_1, q_2 and dots denote differentiation with respect to time t. As outlined in [27], using point transformation to isometric coordinates and performing a time transformation

$$\mathrm{d}t = \Lambda \mathrm{d}\tau, \tag{9}$$

the Lagrangian (8) is reduced to

$$L = \frac{1}{2}(\acute{x}^2 + \acute{y}^2) + l_1\acute{x} + l_2\acute{y} + U, \tag{10}$$

where $U = \Lambda(h - V)$ and dashes denote derivatives with respect to τ. Lagrange equations for the Lagrangian (10) can be written in the form

$$x'' + \Omega y' = \frac{\partial U}{\partial x}, \qquad y'' - \Omega x' = \frac{\partial U}{\partial y}, \tag{11}$$

where $\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}$. The Jacobi integral for this system is written in the form

$$I_1 = \frac{1}{2}(x'^2 + y'^2) - U = 0. \tag{12}$$

The complementary integral for this problem is assumed to be quartic in the velocity variables. Following [43], the complementary integral is written in the form

$$\begin{aligned} I_2 = {} & \acute{x}^4 + \kappa f(y)\,\acute{x}^3 + [F_{xx} + \frac{3}{8}\kappa^2 f^2]\acute{x}^2 - F_{xy}\acute{x}\acute{y} + \kappa\left[-G_x + \frac{\kappa^2}{16}f^3\right]\acute{x} + \kappa\left[G_y + \frac{1}{2}\frac{df}{\mathrm{d}y}F_x\right]\acute{y} \\ & - \int\left(Q_2\frac{\partial U}{\partial x} - \Omega P_1\right)\mathrm{d}y - \int\left[2P_2\frac{\partial U}{\partial x} + Q_2\frac{\partial U}{\partial y} + 2U\frac{\partial Q_2}{\partial y} + \Omega Q_1 - 4\Omega U Q_3\right]_0 \mathrm{d}x, \end{aligned} \tag{13}$$

where $[\cdot]_0$ means that the expression in the bracket is computed for y taking an arbitrary constant value y_0 (say). The functions F and G are two arbitrary functions in both variables x, y, and f is a function in y while κ is an arbitrary parameter which plays an important rules as will be seen later. Differentiating (13) with respect to τ and using the Jacobi integral again, we obtain a nonlinear system of partial differential equations (for more details, see [43]):

$$U = \frac{1}{4}\nabla^2 F, \tag{14}$$

$$\kappa\left[4\nabla^2 G + 3f\frac{\partial}{\partial x}\nabla^2 F + 2\frac{\mathrm{d}^2 f}{\mathrm{d}y^2}F_x + 4\frac{\mathrm{d}f}{\mathrm{d}y}F_{xy}\right] = 0, \tag{15}$$

$$\kappa\left\{\left[\kappa^2 f + 8\left(2G_y + \frac{\mathrm{d}f}{\mathrm{d}y}F_x\right)\right]\nabla^2 F_x + 8\left[\frac{\mathrm{d}f}{\mathrm{d}y} + 2G_x + 4G_{yy} - \frac{\mathrm{d}f}{\mathrm{d}y}F_{xy}\right]\nabla^2 F\right\} = 0, \tag{16}$$

$$\begin{aligned} & \kappa^2\left[\frac{\mathrm{d}^2 f}{\mathrm{d}y^2}\left(G_y + \frac{\mathrm{d}f}{\mathrm{d}y}F_x\right) + \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2 F_{xy} - \frac{\mathrm{d}f}{\mathrm{d}y}(G_{xx} - G_{yy}) - \frac{3}{2}f\frac{\mathrm{d}f}{\mathrm{d}y}\nabla^2 F_x - \frac{3}{4}f^2\nabla^2 F_{xy}\right] \\ & \quad + F_{xy}F_{yyyy} - F_{xy}F_{xxxx} + 2[F_{yyyx} - F_{xx}F_{yxxx}] + 3[F_{xyy}F_{yyy} - F_{xxy}F_{xxx}] = 0. \end{aligned} \tag{17}$$

It is clear that any solution of Eqs. (14)–(17) can be used to construct a time-irreversible integrable systems with complementary integral quartic in velocities. Also, these systems are conditional since they are valid only on the zero-level Jacobi integral. Therefore, the inverse of a time transformation (9) is required.

3 Construction of reversible systems

The problem under consideration becomes time reversible when $\kappa = 0$. In this case, the Lagrangian (10) and its conditional quartic integral (13) are reduced to

$$L = \frac{1}{2}(\acute{x}^2 + \acute{y}^2) + \frac{1}{4}\nabla^2 F \tag{18}$$

and

$$I_2 = \acute{x}^4 + F_{xx}\,\acute{x}^2 - F_{xy}\,\acute{x}\acute{y} + \frac{1}{4}\int F_{xy}\frac{\partial}{\partial x}\nabla^2 F dy - \frac{1}{4}\int \left[2F_{xx}\frac{\partial}{\partial x}\nabla^2 F - F_{xy}\frac{\partial}{\partial y}\nabla^2 F + 2F_{xyy}\nabla^2 F\right]_0 \mathrm{d}x, \tag{19}$$

where the function F satisfies

$$F_{xy}F_{yyyy} - F_{xy}F_{xxxx} + 2[F_{yy}F_{yyyx} - F_{xx}F_{yxxx}] + 3[F_{xyy}F_{yyy} - F_{xxy}F_{xxx}] = 0. \tag{20}$$

This equation was introduced in [28] for the first time, and it is called *resolving equation*. It is also used to construct new integrable problems describing the motion of a particle in the Euclidean plane [37]. Until now, it is not known whether this equation is solvable, in the sense that its general solution can be constructed. As outlined in [28,38], it is evident that the two variables x, y are not suitable variables to find the solution of Eq. (20), and so, let us insert the two variables p, q instead of the present variables through the following point transformation:

$$x = \int \frac{\mathrm{d}q}{\sqrt[4]{a(q-q_1)(q-q_2)(q-q_3)(q-q_4)}}, \quad y = \int \frac{\mathrm{d}p}{\sqrt[4]{b(p-p_1)(p-p_2)(p-p_3)(p-p_4)}}. \tag{21}$$

In [28], $F(p,q)$ is assumed to have the structure

$$F(p,q) = u_0(p) + u_1(q) + k_1 pq, \tag{22}$$

where k_1 is an arbitrary constant. This choice of F leads to construct a new integrable problem describing the motion of a particle in the Euclidean plane and two new integrable problems in rigid body dynamics generalizing special versions of the Kowalevski and Chaplygin cases (for more details, see [28]). In another article [39], the function F is assumed to have the formula

$$F(p,q) = u_0(p) + u_1(q) + k_1 pq + k_2 p^2 q^2, \tag{23}$$

where k_1 and k_2 are free parameters. Notice that this structure (23) of F enables the author to construct new integrable problem in the dynamics of particle and rigid body.

In the present work, we aim to find another structure for the function F containing the two formulas (22) and (23) as a special case, and this structure is considered acceptable if it provides us with new integrable problems. Let us assume that the function $F(p,q)$ has the structure

$$\begin{aligned} F(p,q) = {} & u_0(p) + u_1(q) + p(\beta_3 + \beta_4 q + \beta_5 q^2) + \sqrt[4]{a(q-q_1)(q-q_2)(q-q_3)(q-q_4)} \\ & \times \left[\beta_0 + \beta_1 \sqrt[4]{b(p-p_1)(p-p_2)(p-p_3)(p-p_4)}\right] + \sqrt{p+\beta_6}(\beta_7 + \beta_8 q), \end{aligned} \tag{24}$$

where β_i are free parameters. It is clear that the function F in (24) is reduced to (22) when $\beta_3 = \beta_5 = a = \beta_7 = \beta_8 = 0$, $\beta_4 = k_1$, and it has the same structure as (23) if $\beta_0 = \beta_3 = \beta_5 = \beta_7 = \beta_8 = p_i = q_i = 0$, $\beta_4 = k_1$, $\beta_1 = \frac{k_2}{\sqrt[4]{ab}}$. Inserting the transformation (21) into Eq. (20) and using the expression (24), we obtain a system of ordinary differential equations. After some manipulations, this system of ordinary differential equations is solved, and thus, the function $F(p, q)$ takes the form

$$F = \int \frac{C_0 - \int \frac{C_1 p + C_2}{\sqrt{p-p_1}\sqrt[4]{(p-p_3)^3(p+2p_1+p_3)^3}}\mathrm{d}p}{\sqrt{p-p_1}\sqrt[4]{(p-p_3)(p+2p_1+p_3)}}\mathrm{d}p + C_7(\beta - \mu q)\sqrt{-\mu q^2 + 2\beta q + \alpha}] + \\ + \sqrt{p-p_1}\left[C_4\sqrt{-\mu q^2 + 2\beta q + \alpha} + C_5(\beta - \mu q)\right] + p[C_6(-2\mu^2 q^2 + (4\beta q + \alpha)\mu - \beta^2) \\ + 2\int \frac{\mathrm{d}q}{\sqrt{-\mu q^2 + 2\beta q + \alpha}}\left[C_3 + C_8\int \frac{\mathrm{d}q}{\sqrt{-\mu q^2 + 2\beta q + \alpha}}\right], \tag{25}$$

where C_i , β and α are free parameters, which are introduced instead of the original ones for suitability. Using (21) and inserting (25) in the expression (18), we get

$$L = \frac{1}{2}\left[\frac{\acute{q}^2}{-\mu q^2 + 2\beta q + \alpha} + \frac{\acute{p}^2}{4\mu(p-p_1)\sqrt{(p-p_3)((p+2p_1+p_3)}}\right] + \frac{K_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}} \\ + \left(\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}} - 1\right)[K_2 + \sqrt{p-p_1}[a_2(\beta - \mu q) - a_1\sqrt{-\mu q^2 + 2\beta q + \alpha}]] \\ + \left(\frac{2p^2 + 2p_1 p - p_1^2 - 2p_1 p_3 - p_3^2}{2\sqrt{(p-p_3)(p+2p_1+p_3)}} - p\right)[a_3(-2\mu^2 q^2 + [4\beta q + \alpha]q - \beta^2) - a_4(\beta - \mu q) \\ \times \sqrt{-\mu q^2 + 2\beta q + 2\alpha}], \tag{26}$$

where K_1, K_2 and a_i are free parameters. Its Jacobi integral becomes

$$I_1 = \frac{1}{2}\left[\frac{\acute{q}^2}{-\mu q^2 + 2\beta q + \alpha} + \frac{\acute{p}^2}{4\mu(p-p_1)\sqrt{(p-p_3)((p+2p_1+p_3)}}\right] - \frac{K_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}} \\ - \left(\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}} - 1\right)[K_2 + \sqrt{p-p_1}[a_2(\beta - \mu q) - a_1\sqrt{-\mu q^2 + 2\beta q + \alpha}]] \\ - \left(\frac{2p^2 + 2p_1 p - p_1^2 - 2p_1 p_3 - p_3^2}{2\sqrt{(p-p_3)(p+2p_1+p_3)}} - p\right)[a_3(-2\mu^2 q^2 + [4\beta q + \alpha]q - \beta^2) - a_4(\beta - \mu q) \\ \times \sqrt{-\mu q^2 + 2\beta q + 2\alpha}] \\ = 0. \tag{27}$$

Its conditional quartic integral is expressed in the form

$$I_2 = I_2(p, q, \acute{p}, \acute{q}; \mu, \beta, \alpha, p_1, p_3, K_1, K_2, a_1, a_2, a_3, a_4) \\ = \frac{\mu^2\acute{q}^2}{-\mu q^2 + 2\beta q + \alpha} + 2\mu^2\left\{a_3(-2\mu q^2 + 4\beta\mu q + \alpha\mu - \beta^2) + 2a_2\sqrt{p-p_1}(\beta - \mu q) - \beta^2) + 2K_2\right. \\ \left. + 2a_2\sqrt{p-p_1}(\beta - \mu q) + 2\sqrt{-\mu q^2 + 2\beta q + \alpha}[a_4 p(\beta - \mu q) + a_1\sqrt{p-p_1}]\right\}\frac{\acute{q}}{-\mu q^2 + 2\beta q + \alpha} \\ + 2\mu^2\left\{a_3(\beta - \mu q) - \frac{a_2}{\sqrt{p-p_1}} + \frac{a_4\sqrt{p-p_1}(-2\mu q^2 + 4\beta\mu q + \alpha\mu - \beta^2) + a_1(\beta - \mu q)}{\mu\sqrt{p-p_1}\sqrt{-\mu q^2 + 2\beta q + \alpha}}\right\} \\ \times \acute{p}\acute{q} + R(p, q), \tag{28}$$

where

$$
\begin{aligned}
R(p,q)=&2p_1p_3\mu^3(a_3^2\mu-a_4^2)(-\mu q^2+2\beta q+\alpha)^2-\frac{a_4^2}{2}\mu(\alpha\mu+\beta^2)^2(p-p_1)\sqrt{(p-p_1)(p+2p_1+p_3)}\\
&-2\mu^2(\alpha\mu+\beta)[a_3a_4(p-p_1)(\beta-\mu q)\sqrt{(p-p_3)(p+2p_1+p_3)(-\mu q^2+2\beta q+\alpha)}-p_1p_3\\
&\times(\mu a_3^2-a_4^2)(\mu q^2-2\beta q-\alpha)]+\frac{\mu}{4}(\alpha\mu+\beta^2)[4\mu^2(2p^2-3p_1^2-p_3^2)(a_3^2(-\mu q^2+2\beta q+\alpha)\\
&+a_4^2q^2)+a_4^2(-\beta^2p_1p_3-\mu a_4^2(8p^2(2\beta q+\alpha)-8\beta q(3p_1^2+p_3^2)-\alpha(12p_1^2-p_1p_3+4p_3^2)]\\
&-2a_3a_4\mu^3(-\mu q^2+2\beta q+\alpha)^{\frac{3}{2}}(\mu q-\beta)(2p^2-3p_1^2-2p_1p_3-p_3^2)+\frac{\mu}{8}(\alpha\mu+\beta)^2[16\mu K_1a_3\\
&\times a_4^2(\alpha\mu+\beta)(4p^2-3p_1^2-p_3^2)]-2\mu^2(-\mu q^2+2\beta q+\alpha)[(p+p_1)(a_1^2-\mu a_2^2)+2\mu K_1a_3]\\
&+\frac{4\mu^2(\beta-\mu q)}{\sqrt{p-p_1}}\sqrt{(p-p_3)(p+2p_1+p_3)}[a_3a_2\mu(p-p_1)-a_1a_4p]-4\mu^3(-\mu q^2+2\beta q+\alpha)^{\frac{3}{2}}\\
&\times\sqrt{(p-p_1)(p+2p_1+p_3)}[a_3a_4(\beta-\mu q)\sqrt{p-p_1}-a_3a_1-a_2a_4]-\frac{\mu^2(\alpha\mu+\beta^2)}{\sqrt{p-p_1}}\\
&\times\sqrt{-\mu q^2+2\beta q+\alpha}[a_3a_4(\beta-\mu q)(2p^2-3p_1^2-p_3^2)\sqrt{p-p_1}-2(p^2-p_1^2)(a3a_1+a_4a_2)]\\
&+2\mu^4(p-p_1)\sqrt{(p-p_3)(p+2p_1+p_3)}[a_3^2(\mu^2q^4-4\beta\mu q^3-2(\alpha\mu-2\beta^2)q^2+4\alpha\beta q+\alpha^2)\\
&+a_4^2(-\mu q^4+4\beta q^3+\alpha q^2)]-2\mu^3\left[2q\sqrt{-\mu q^2+2\beta q+\alpha}(a_4K_1-a_2a_1(p+p_1))+a_4^2(p-p_1)\right.\\
&\left.\times(2\beta q+\alpha)^2\sqrt{(p-p_3)(p+2p_1+p_3)}\right]+4\beta\mu^2\sqrt{-\mu q^2+2\beta q+\alpha}[a_4K_1-a_1a_2(p+p_1)]\\
&-2\mu\sqrt{(p-p_3)(p+2p_1+p_3)}\left[2a_1a_2(\mu q-\beta)+\frac{\mu}{2}(a_1^2-\mu a_2^2)q(2\beta-\mu q)+\mu^2\alpha a_2^2+a_1^2\beta^2\right]\\
&+\frac{2p_1a_4\mu^4}{\sqrt{p-p_1}}\left[a_3p_3(\alpha\mu+\beta^2)\sqrt{p-p_1}\sqrt{-\mu q^2+2\beta q+\alpha}+2a_1(-\mu q^2+2\beta q+\alpha)\right.\\
&\left.\times\sqrt{(p-p_3)(p+2p_1+p_3)}\right]-2\mu(p-p_1)^{\frac{3}{2}}(\beta-\mu q)[2\mu^2a_2a_3(-\mu q^2+2\beta q+\alpha)-a_1a_3\\
&\times(-2\mu^2q^2+4\beta\mu q+\alpha\mu-\beta^2)]-4\mu^3(p+p_1)\sqrt{p-p_1}(a_3a_1+a_4a_2)(-\mu q^2+2\beta q+\alpha)^{\frac{3}{2}}\\
&-2\mu(\alpha\mu+\beta^2)\left[\mu(p-p_1)(-\mu q^2+2\beta q+\alpha)(a_3^2\mu-a_4^2)\sqrt{(p-p_3)}\sqrt{p+2p_1+p_3}-a_1^2p\right]\\
&-\mu^3(-\mu q^2+2\beta q+\alpha)^2(2p^2-3p_1^2-p_3^2)(\mu a_3^2-a_4^2)-2\mu(\alpha\mu+\beta^2)[\mu(2a_1a_3+a_4a_2)\\
&\times\sqrt{-\mu q^2+2\beta q+\alpha}+a_4a_1(\beta-\mu q)\sqrt{(p-p_3)(p+2p_1+p_3)}.
\end{aligned}
\tag{29}
$$

3.1 The generic unconditional system

The Lagrangian (26) is integrable on its zero level of energy integral (27), and so it is named a conditional system. The parameters K_i and a_j are energy-like parameters. Let us introduce new parameters instead of them:

$$a_1=b_1+n_1h,\quad a_2=b_2+n_2h,\quad a_3=b_3+n_3h,\quad a_4=b_4+n_4h,\quad K_1=b_5+n_5h,\quad K_2=b_6+n_6h, \tag{30}$$

where b_i and n_i are arbitrary parameters. We perform the inverse of a time transformation (9) with a conformal factor Λ, which is given by

$$\Lambda = \sqrt{p-p_1}\left[\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}-1\right]\left[n_1\sqrt{-\mu q^2+2\beta q+\alpha}+4n_2(\beta-\mu q)\right]$$
$$+\frac{n_5}{\sqrt{(p-p_3)(p+2p_1+p_3)}}+n_6\left[\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}-1\right]$$
$$+\left[\frac{2p^2-2p_1p-p_1^2-2p_1p_3-p_3^2}{2\sqrt{(p-p_3)(p+2p_1+p_3)}}-p\right]\left(n_3\left[-\mu^2q^2+\left(2\beta q+\frac{\alpha}{2}\right)\mu-\frac{\beta^2}{2}\right]\right.$$
$$\left.-n_4(\beta-\mu q)\sqrt{-\mu q^2+2\beta q+\alpha}\right). \tag{31}$$

The Lagrangian (26) takes the form

$$L = \frac{\Lambda}{2}\left[\frac{\dot{q}^2}{-\mu q^2+2\beta q+\alpha}+\frac{\dot{p}^2}{4\mu(p-p_1)\sqrt{(p-p_3)((p+2p_1+p_3)}}\right]$$
$$+\frac{1}{\Lambda}\left\{\frac{K_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}+\left(\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}-1\right)\left[K_2+\sqrt{p-p_1}\right.\right.$$
$$\left.\times\left[a_2(\beta-\mu q)-a_1\sqrt{-\mu q^2+2\beta q+\alpha}\right]\right]+\left(\frac{2p^2+2p_1p-p_1^2-2p_1p_3-p_3^2}{2\sqrt{(p-p_3)(p+2p_1+p_3)}}-p\right)$$
$$\left.\times\left[a_3(-2\mu^2q^2+[4\beta q+\alpha]q-\beta^2)-a_4(\beta-\mu q)\times\sqrt{-\mu q^2+2\beta q+2\alpha}\right]\right\}+h. \tag{32}$$

The uconditional energy integral can be written in the form

$$I_1 = \frac{\Lambda}{2}\left[\frac{\dot{q}^2}{-\mu q^2+2\beta q+\alpha}+\frac{\dot{p}^2}{4\mu(p-p_1)\sqrt{(p-p_3)((p+2p_1+p_3)}}\right]$$
$$-\frac{1}{\Lambda}\left\{\frac{K_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}+\left(\frac{p+p_1}{\sqrt{(p-p_3)(p+2p_1+p_3)}}-1\right)\left[K_2+\sqrt{p-p_1}\right.\right.$$
$$\left.\times\left[a_2(\beta-\mu q)-a_1\sqrt{-\mu q^2+2\beta q+\alpha}\right]\right]+\left(\frac{2p^2+2p_1p-p_1^2-2p_1p_3-p_3^2}{2\sqrt{(p-p_3)(p+2p_1+p_3)}}-p\right)$$
$$\left.\times\ [a_3(-2\mu^2q^2+[4\beta q+\alpha]q-\beta^2)-a_4(\beta-\mu q)\times\sqrt{-\mu q^2+2\beta q+2\alpha}]\right\}$$
$$= h. \tag{33}$$

Its unconditional complementary integral becomes

$$I_2 = I_2(p, q, \Lambda\dot{p}, \Lambda\dot{q}; \mu, \beta, \alpha, p_1, p_3, b_5+n_5h, b_6+n_6h, b_1+n_1h, b_2+n_2h, b_3+n_3h, b_4+n_4h). \tag{34}$$

Note that the presence of the arbitrary parameters h in the Lagrangian (32) is insignificant and can be ignored. The same arbitrary constant h is now interpreted as the value of the energy integral (33). It is more suitable that the energy's constant h in (34) should be replaced by its expression in (33). The system (32)–(34) describes a new two-dimensional integrable mechanical system in which the complementary integral is quartic in the velocities. It contains **17** free parameters

$$\mu, \beta, \alpha, p_1, p_3, n_1, n_2, n_3, n_4, n_5, n_6, b_1, b_2, b_3, b_4, b_5, b_6.$$

The first eleven parameters constitute the structure of the line element on the configuration manifold as seen from Eq. (32), and the other six parameters enter in the potential part of the Lagrangian.

Until now, the full physical interpretation of this system is unknown. One of the most important advantages for this system is the structure of a configuration manifold containing a large set of free parameters. This structure widens the range of its applications to various problems such as the problem of motion in the Euclidean plane, the hyperbolic plane and different types of curved two-dimensional manifolds (for example, the problem of rigid body dynamics). The Gaussian curvature of the configuration manifolds plays a significant rule in this study. Therefore, let us give the expression of Gaussian curvature of the configuration manifold of the Lagrangian (32),

$$\chi = \frac{1}{4\Lambda}\left[2\Lambda\frac{\partial\Lambda}{\partial p}\frac{\partial F}{\partial p} + 4F\Lambda\frac{\partial^2\Lambda}{\partial^2 p} - \frac{\partial\Lambda}{\partial q}\frac{\partial G}{\partial q} - 2G\frac{\partial^2\Lambda}{\partial^2 q}\right], \tag{35}$$

where

$$F(p) = \mu(p-p_1)\sqrt{(p-p_3)(p+2p_1+p_3)}, \quad G(q) = -\mu q^2 + 2\beta q + \alpha. \tag{36}$$

It is well known that the Gaussian curvature for a sphere is $\chi = \frac{1}{a^2}$. For the Euclidean space, the Gaussian curvature vanishes ($\chi = 0$). Also, for the Gauss–Bolyai–Lobachevsky space, the Gaussian curvature is $\chi = -\frac{1}{a^2}$.

3.2 New integrable problems

1. The first case is constructed by setting $\alpha = \mu = 1$, $\beta = 0$, $p_1 = -p_3 = -1$ and using the point transformation $p = -\cos 2y$, $q = \cos x$ in the expression (32); we obtain after some manipulations

$$L = \frac{\Lambda_1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{\Lambda_1}\left[\frac{-\rho_1}{\cos^2 y} + \sin y[\rho_2\cos x - \rho_3\sin x] + \rho_4[\cos^2 x + \cos^2 y \right.$$
$$\left. - 2\cos^2 y\cos^2 x] - \rho_5\sin x\cos x\cos 2y\right] + h, \tag{37}$$

where

$$\Lambda_1 = \delta_0 + \frac{\delta_1}{\cos^2 y} + \sin y(\delta_2\cos x + \delta_3\sin x) - \delta_4(\cos^2 y - \cos^2 x - 2\cos^2 x\cos^2 y)$$
$$+ \delta_5\sin x\cos x\cos 2y, \tag{38}$$

where δ_i and ρ_i are free parameters, which are introduced instead of the original ones for simplicity. This system is new. It generalizes the case that is introduced by Yehia in [39] by adding four parameters ρ_3, ρ_5, δ_3 and δ_5. Using (35), one can evaluate the Gaussian curvature that vanishes when $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$ and $\delta_0 = 1$. The Lagrangian (37) becomes

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\rho_1}{\cos^2 y} + \sin y[\rho_2\cos x - \rho_3\sin x] + \rho_4[\cos^2 x + \cos^2 y$$
$$- 2\cos^2 y\cos^2 x] - \rho_5\sin x\cos x\cos 2y + h, \tag{39}$$

This special case is also new. It adds two parameters, ρ_3 and ρ_5, to the case that is presented by Yehia in [39]. It also generalizes a special version of Bozis's case by inserting three parameters, ρ_3, ρ_4 and ρ_5. The complementary quartic integral for the present case can be expressed in the form

$$I_2 = \dot{x}^2\left(\dot{y}^2 + \frac{2\rho_1}{\cos^2 y}\right) - 2\left[(2\rho_4\sin x\cos x + \rho_5(1-2\cos^2 x))\sin y + (\rho_2\sin x + \rho_3\cos x)\right]$$
$$\times \cos y\dot{x}\dot{y} + \cos^4 y\left[4\cos^3 x((\rho_4^2 - \rho_5^2)\cos x + 2\rho_4\rho_5\sin x) - 4\cos x((\rho_4^2 - \rho_5^2)\cos x\right.$$
$$\left. + \rho_4\rho_5\sin x) - \rho_5^2\right] - \cos^2 y[4(\rho_4^2 - \rho_5^2)\cos^4 x + 4(2\rho_4\rho_5\sin x + (\rho_2\rho_4 + \rho_3\rho_5)\sin y)$$
$$\times \cos^3 x + \left(4(\rho_2\rho_5 - \rho_3\rho_4)\sin x\sin y + \rho_2^2 - \rho_3^2 - 4\rho_4^2 + 4\rho_5^2\right)\cos^2 x - 2(\rho_2(\rho_3\sin x$$
$$+ 2\rho_4\sin y) + \rho_5(\rho_3\sin y + 2\rho_4\sin x))\cos x - 2\rho_2\rho_5\sin x\sin y - \rho_2^2 - \rho_5^2]$$
$$+ 4\rho_1\cos x(\rho_5\sin x + \rho_4\cos x). \tag{40}$$

Another new integrable problem like the types of [40] can be constructed by performing the transformation $(x, y) \to (ix, iy)$ in the Lagrangian (39), and we get

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\rho_1}{\cosh^2 y} - \sinh y[a\cosh x + \rho_3\sinh x] - \rho_4[\cosh^2 x + \cosh^2 y$$
$$- 2\cosh^2 y\cosh^2 x] + b\sinh x\cosh x\cosh 2y + h, \tag{41}$$

where the two constants a, b are introduced instead of ρ_2, ρ_5 to make the potential of the force real-valued. It is easy to construct the complementary integral for the present case by applying the same transformation to the integral (40).

2. Let $p_1 = -p_3 = 1$, $\alpha = 1$, $\beta = 0$, $\mu = 1$. Under the coordinate transformation $q = \cos x$, $p = 1 - 2\tanh^2 y$, the Lagrangian (32) takes the following form after some manipulations:

$$L = \frac{\Lambda_2}{2}\left(\cosh^2 y\dot{x}^2 + \dot{y}^2\right) + \frac{1}{\Lambda_2}\left[\frac{\rho_1}{\cosh^2 y} + \frac{\sinh y}{\cosh^3 y}(\rho_2\cos x + \rho_3\sin x) + \frac{\cosh^2 y - 2}{2\cosh^4(y)}\right.$$
$$\left.\times\ (\rho_4\cos 2x + \rho_5\sin 2x)\right] + h, \tag{42}$$

where

$$\Lambda_2 = \delta_0 + \frac{\delta_1}{\cosh^2 y} + \frac{\cosh^2 y - 2}{2\cosh^4 y}(\delta_4\cos 2x + \delta_5\sin 2x) + \frac{\sinh y}{\cosh^3 y}(\delta_2\cos x + \delta_3\sin x), \tag{43}$$

where ρ_i and δ_i are arbitrary parameters, which are introduced instead of the original ones for simplicity. This problem characterizes a new integrable problem. The Gaussian curvature (35) for the present problem takes the form

$$\chi_2 = \frac{-2}{\Lambda_2}. \tag{44}$$

It is evident that the Gaussian curvature (44) takes a negative constant value when $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, $\delta_0 > 0$, and consequently, the configuration manifold for the Lagrangian (42) represents a metric of a pseudo-sphere. Under these conditions, the Lagrangian (42) can be written in the form

$$L = \frac{1}{2}(\cosh^2 y\dot{x}^2 + \dot{y}^2) + \frac{\rho_1}{\cosh^2 y} + \frac{\sinh y}{\cosh^3 y}(\rho_2\cos x + \rho_3\sin x) + \frac{\cosh^2 y - 2}{2\cosh^4(y)}$$
$$\times\,(\rho_4\cos 2x + \rho_5\sin 2x)] + h. \tag{45}$$

Its quartic integral can be written in the form

$$\begin{aligned}
I_2 = {} & \cosh^8 y\dot{x}^4 + [\rho_4\cos 2x + \rho_5\sin 2x]\dot{y}^2 + 2\cosh y\dot{x}\dot{y}[\cosh y(\rho_2\sin x - \rho_3\cos x)\\
& + \sinh y\cosh y(\rho_4\sin 2x - \rho_5\cos 2x)] + \cosh^2 y\dot{x}^2[(3 - \cosh^2 y)(\rho_4\cos 2x\\
& + \rho_5\sin 2x) - 2(\rho_1\cosh^2 y + \sinh y\cosh y(\rho_2\cos x + \rho_3\sin x))] + \frac{1}{2}\frac{\sinh^2 y}{\cosh^4 y}\\
& \times[(\rho_4^2 - \rho_5^2)\cos 4x + 2\rho_4\rho_5\sin 4x] + \frac{\sinh y}{\cosh^3 y}[(\rho_2\rho_4 - \rho_3\rho_5)\cos 3x + (\rho_2\rho_5\\
& + \rho_3\rho_4)\sin 3x] - \frac{(\rho_4\cos 2x + \rho_5\sin 2x)}{\cosh^4 y}[2\rho_1\cosh^2 y + 2\sinh y\cosh y(\rho_2\cos x\\
& + \rho_3\sin x) + (\cosh^2 y - 2)(\rho_4\cos 2x + \rho_5\sin 2x)] + \frac{1}{2\cosh^2 y}[(\rho_2^2 - \rho_3^2)\\
& \times\cos 2x + 2\rho_2\rho_3\sin 2x] - \frac{\sinh y}{\cosh^3 y}[(\rho_2\rho_5 - \rho_3\rho_4)\sin x + (\rho_2\rho_4 + \rho_3\rho_5)\cos x]\\
& + \frac{(\rho_4^2 + \rho_5^2)}{16\cosh^4 y}[\sinh^4 y - 6\sinh^2 y + 1] + \frac{(\rho_2^2 + \rho_3^2)}{4\cosh^2 y}[\sinh^2 y - 1].
\end{aligned} \tag{46}$$

This case is new. It describes the motion of a particle on a pseudo-sphere. From another point of view, the Gaussian curvature (44) takes a positive constant value when $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, $\delta_0 < 0$, and consequently, the configuration manifold for the Lagrangian (42) represents a metric of a standard sphere. Alternatively, this case can be obtained by performing the point transformation $(x, y) = (i\varphi, i(\frac{\pi}{2} - \theta))$ to express the Lagrangian (45) in the usual spherical coordinates, and we get

$$L = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - \frac{\rho_1}{\sin^2\theta} - \frac{\cos\theta}{\sin^3\theta}[a\cosh\varphi - \rho_3\sinh\varphi] + \frac{1+\cos^2\theta}{\sin^4\theta} \times [\rho_4\cosh 2\varphi + b\sinh 2\varphi]. \tag{47}$$

It is clear that the potential function is not periodic in the longitudinal variable φ, and so, the given integrable problem has only limited use in real problems. The complementary integral for the present case can be formulated by applying the same transformation to (46).

Also, some previously known integrable problems can be reconstructed as special cases of (34) for certain values of the parameters. Let us illustrate that by introducing a case as an example. This case is constructed by setting $\alpha = 0$, $\beta = \frac{\nu}{2}$, $\mu = -1$, $p_1 = p_3 = 0$ in the Lagrangian (34) and performing the coordinate transformation $p = \mathrm{e}^{-x-y}$ and $q = \frac{1}{2}\mathrm{e}^{\frac{x-y}{2}} - \frac{\nu}{2} + \frac{\nu^2}{8\mathrm{e}^{\frac{x-y}{2}}}$. The Lagrangian (32) takes the following form after some manipulations:

$$L = \frac{1}{2}\Lambda_3(\dot{x}^2 + \dot{y}^2) - \frac{1}{\Lambda_3}[a_0 + a_1\mathrm{e}^{-2x} + a_2\mathrm{e}^{-2y} + a_3\mathrm{e}^{x+y} + a_4\mathrm{e}^{2(x+y)}] + h, \tag{48}$$

where

$$\Lambda_3 = b_0 + b_1\mathrm{e}^{-2x} + b_2\mathrm{e}^{-2y} + b_3\mathrm{e}^{x+y} + b_4\mathrm{e}^{2(x+y)}. \tag{49}$$

Its quartic integral can be written in the form

$$I_2 = \Lambda_3^4\dot{x}^2\dot{y}^2 + 2\Lambda_3^2[a_2\mathrm{e}^{-2y}\dot{x}^2 + a_1\mathrm{e}^{-2x}\dot{y}^2 + (a_3\mathrm{e}^{x+y} + d\mathrm{e}^{2x+2y})\dot{x}\dot{y}] + \mathrm{e}^{2x+2y}(a_3 + a_4\mathrm{e}^{x+y})^2 + 2d(a_2\mathrm{e}^{2x} + a_1\mathrm{e}^{2y}) + 4a_2a_1\mathrm{e}^{-2x-2y}. \tag{50}$$

This case is completely found in [39].

4 Irreversible case

Now, we return again to the irreversible case. As outlined in section two, the basic equations that are used to construct a time-irreversible system with complementary quartic integral have been formulated in the general setting. Until now, these equations remain unsolved in general, but they will be solved for certain values of parameters which lead to a rigid body dynamics, especially to Kowalevski's configuration space. In other words, the solution of Eqs. (15)–(17) is composed of the solution in the reversible case plus some additional terms under certain conditions leading to the metric of Kowalevski's type of rigid body dynamics. To keep these equations (15)–(17) tractable, we will assume that the function F has the same structure as in the reversible case (25) and the other function G we will postulate to be written in the form

$$G(p,q) = w_0(p) + w_1(q) + p(\delta_3 + \delta_4 q + \delta_5 q^2) + \sqrt[4]{a(q-q_1)(q-q_2)(q-q_3)(q-q_4)} \times \left[\delta_0 + \delta_1\sqrt[4]{b(p-p_1)(p-p_2)(p-p_3)(p-p_4)}\right] + \sqrt{p+\delta_6}(\delta_7 + \delta_8 q), \tag{51}$$

where δ_i are free parameters. Taking into consideration the transformation (21) and the two expressions for F and G, Eqs. (15)–(17) are reduced to a system of ordinary differential equations involving u_0, u_1, w_0, w_1, f, and they are solved for the following combination of parameters:

$$p_1 = p_3 = \alpha = \mu = 1, \beta = 0, \tag{52}$$

and the point transformation

$$q = \cos\varphi, \quad p = \frac{\cos^4\theta}{1-\cos^2\theta} + 1\,, \tag{53}$$

where θ is the angle of nutation and φ is the angle of proper rotation. Inserting (52), (53) and the two expressions (25), (51) into Eqs. (15)–(17), we get, after some manipulations which are not writable in a suitable size, a new integrable problem in a rigid body dynamic. Its Lagrangian can be written in the form

$$L = \frac{1}{2}\left[\dot{\theta}^2 + \frac{\sin^2\theta}{2-\cos^2\theta}\dot{\varphi}^2\right] + \frac{\sin^2\theta}{2-\cos^2\theta}\left(K + \frac{\nu[1+\sin^2\theta\cos^2\varphi]}{\sin^2\theta\sin^2\varphi}\right)\dot{\varphi} - \frac{1}{2}\left\{\frac{\lambda}{2\cos^2\theta}\right.$$
$$+ \sin\theta[a\sin\varphi + b\cos\varphi] + \frac{1}{2}\sin^2\theta[d\sin 2\varphi - 2c\cos 2\varphi] - \frac{\nu^2\cos^2\theta}{\sin^4\theta\sin^4\varphi}[1+\sin^2\theta\cos 2\varphi]$$
$$\left. - \frac{\nu K\cos^2\theta}{\sin^2\theta\sin^2\varphi} + \frac{\cos^2\theta}{2(2-\cos^2\theta)}\left(K + \frac{\nu[1+\sin^2\theta\cos^2\varphi]}{\sin^2\theta\sin^2\varphi}\right)^2\right\}, \tag{54}$$

where K, σ, a, b, c, d and λ are free parameters. Its Jacobi integral becomes

$$I_1 = \frac{1}{2}\left[\dot{\theta}^2 + \frac{\sin^2\theta}{2-\cos^2\theta}\dot{\varphi}^2\right] + \frac{1}{2}\left\{\frac{\lambda}{2\cos^2\theta} + \sin\theta[a\sin\varphi + b\cos\varphi] + \frac{1}{2}\sin^2\theta[d\sin 2\varphi - 2c\cos 2\varphi]\right.$$
$$\left. - \frac{\nu K\cos^2\theta}{\sin^2\theta\sin^2\varphi} - \frac{\nu^2\cos^2\theta}{\sin^4\theta\sin^4\varphi}[1+\sin^2\theta\cos 2\varphi] + \frac{\cos^2\theta}{2(2-\cos^2\theta)}\left(K + \frac{\nu[1+\sin^2\theta\cos^2\varphi]}{\sin^2\theta\sin^2\varphi}\right)^2\right\}$$
$$= h, \tag{55}$$

where h is the value of the numerical value of the Jacobi integral. The complementary integral can be expressed after utilizing the expression (55) to remove h as

$$I_2 = \dot{\theta}^4 + \frac{\sin^4\theta}{(2-\cos^2\theta)^4}\dot{\varphi}^4 + \left\{\frac{2\sin^2\theta}{(2-\cos^2\theta)^2}\dot{\varphi}(\cos^2\theta\dot{\varphi} + 4(K-\nu)) + 2\sin\theta(a\sin\varphi + b\cos\varphi)\right.$$
$$\left. - \cos^2\theta[d\sin 2\varphi - 2c\cos 2\varphi] + \frac{\lambda\sin^2\theta}{\cos^2\theta} - \frac{2(K-\nu)^2}{(2-\cos^2\theta)^2}\left[\cos^4\theta - 3\cos^2\theta + 4\right]\right\}\dot{\theta}^2$$
$$+ \frac{2\cos\theta}{\cos^2\theta - 2}\dot{\theta}[\sin\theta(2c\sin 2\varphi + d\cos 2\varphi](\cos^2\theta\dot{\varphi} + 2K - 2\nu) - 2(K - \nu - \sin^2\theta\dot{\varphi})(b\sin\varphi$$
$$- a\cos\varphi] + \frac{2\cos^2\theta}{(2-\cos^2\theta)}\dot{\varphi}^2\left\{\cos^2\theta\sin^2\theta(d\sin 2\varphi - 2c\cos 2\varphi) - \frac{\lambda\sin^4\theta}{\cos^2\theta} - 2\sin^3\theta(a\sin\varphi\right.$$
$$\left. + b\cos\varphi) + \frac{2\sin^2\theta}{(2-\cos^2\theta)^2}(\cos^4\theta - 7\cos^2\theta + 4)(K-\nu)^2\right\} + \frac{4(K-\nu)\sin^2\theta\cos^2\theta}{(2-\cos^2\theta)^2}\dot{\varphi}$$
$$\times\left\{d\sin 2\varphi - 2c\cos 2\varphi + \frac{a\sin\varphi + b\cos\varphi}{\sin\theta} + \frac{\lambda}{\cos^4\theta} - \frac{(K-\nu)^2}{(2-\cos^2\theta)^2}\right\} - \frac{\lambda(K-\nu)^2}{\cos^2\theta(2-\cos^2\theta)^2}$$
$$\times(\cos^6\theta - 4\cos^4\theta + 3\cos^2\theta + 4) - \frac{(K-\nu)^4\cos^2\theta\sin^2\theta}{(2-\cos^2\theta)^4}[\cos^4\theta - 5\cos^2\theta + 8] + \frac{\lambda^2\sin^4\theta}{4\cos^4\theta}$$
$$- \frac{\cos^4\theta}{4}[4(a^2+b^2) - (4c^2+d^2)\cos^2\theta] - \frac{\sin\theta}{\cos^2\theta(2-\cos^2\theta)}\left\{\left[(ad-2bc)\cos^8\theta\right.\right.$$
$$+ \left[-4da + 2b\left[(K-\nu)^2 + 4c + \frac{\lambda}{2}\right]\right]\cos^6\theta + \left[4da - 6b\left[(K-\nu)^2 + \frac{4c}{3} + \frac{5\lambda}{6}\right]\right]\cos^4\theta + 8b\lambda$$
$$\times\left.\cos^2\theta - 4b\lambda\right]\cos\varphi + 2\left[\left(\frac{bd}{2} + ac\right)\cos^8\theta + \left[\left(\frac{\lambda}{2} + (K-\nu)^2 - 4c\right)a - 2bd\right]\cos^6\theta\right.$$
$$\left.\left. + \left[2bd + \left[4K\nu - \frac{5\lambda}{2} - 3\nu^2 - 3K^2 + 4c\right]a\right]\cos^4\theta + 4a\lambda\cos^2\theta - 2a\lambda\right]\sin\varphi\right\}.$$

Comparing the Routhian (7) and the Lagrangian (54), we obtain

$$f = 0,$$
$$l_3 = K + \frac{\nu[1+\sin^2\theta\cos^2\varphi]}{\sin^2\theta\sin^2\varphi},$$
$$V = \sin\theta[a\sin\varphi + b\cos\varphi] + \frac{1}{2}\sin^2\theta[d\sin 2\varphi - 2c\cos 2\varphi] - \frac{\nu^2\cos^2\theta}{\sin^4\theta\sin^4\varphi}[1+\sin^2\theta\cos 2\varphi]$$
$$+ \frac{\lambda}{2\cos^2\theta} - \frac{\nu K\cos^2\theta}{\sin^2\theta\sin^2\varphi}. \tag{56}$$

This case describes a new integrable problem in a rigid body dynamics. It also generalizes some previous results in this field. To clarify the comparison with other results, we now write this case in terms of traditional Euler–Poisson variables as the following

Theorem 1 *Assume the inertia matrix for a rigid body is* $\mathbf{I} = \mathrm{diag}(2C, 2C, C)$ *and let the scalar and vector potentials* V *and* $\boldsymbol{\mu}$ *be given by*

$$V = C\left[a\gamma_1 + b\gamma_2 + d\gamma_1\gamma_2 + c(\gamma_1^2 - \gamma_2^2) - \frac{\nu K}{\gamma_1^2}\gamma_3^2 - \frac{\nu^2(2\gamma_2^2+\gamma_3^2)}{2\gamma_1^4}\gamma_3^2 + \frac{\lambda}{2\gamma_3^2}\right], \tag{57}$$

and

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = C\left(\frac{-2\nu\gamma_3}{\gamma_1^3}(1+\gamma_2^2), \frac{2\nu\gamma_2\gamma_3}{\gamma_1^2}, K + \frac{\nu(1+\gamma_2^2)}{\gamma_1^2}\right), \tag{58}$$

or, equivalently,

$$\mathbf{l} = C\left(0, 0, K + \frac{\nu(1+\gamma_2^2)}{\gamma_1^2}\right),$$

where a, b, c, d, λ, K *and* ν *are free parameters. Then, the Euler–Poisson equations* (2) *with* (57) *and* (58) *are integrable on the zero level of the cyclic integral*

$$I_1 = 2p\gamma_1 + 2q\gamma_2 + \left(r + K + \frac{\nu(1+\gamma_2^2)}{\gamma_1^2}\right)\gamma_3.$$

The complementary integral takes the form

$$I_2 = \left[p^2 - q^2 - a\gamma_1 + b\gamma_2 + c\gamma_3^2 - \frac{\lambda(\gamma_1^2-\gamma_2^2)}{2\gamma_3^2}\right]^2 + \left[2pq - a\gamma_2 - b\gamma_1 + \frac{d}{2}\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}\right]^2$$
$$+ (K-\nu)\left[(r-K+\nu)\left[2(p^2+q^2)+\lambda(1+\frac{1}{\gamma_3^2})\right] - 2\gamma_3\left[(2c\gamma_1+d\gamma_2)p+q(d\gamma_1-2c\gamma_2)\right]+\frac{2\nu\lambda}{\gamma_1^2}\right]$$
$$- \frac{4\gamma_3}{\gamma_1^2}(K\gamma_1^2+\nu\gamma_2^2)(ap+bq) + \frac{2\nu\gamma_3^2}{\gamma_1^2}(a\gamma_1+b\gamma_2)\left[\frac{\nu(\gamma_1^2-\gamma_2^2)}{\gamma_1^2} - 2K\right] + \frac{\nu\gamma_3^2}{\gamma_1^4}[\nu(2\gamma_1^2+\gamma_3^2)$$
$$- 2K\gamma_1^2](r^2 + 2(c(\gamma_1^2-\gamma_2^2)+d\gamma_1\gamma_2)) - \frac{\nu(\nu+r\gamma_1^2)}{\gamma_1^4}\left[2\gamma_3^2(p^2+q^2) - \lambda(\gamma_1^2+\gamma_2^2)\right] + \frac{2\nu\gamma_3^3}{\gamma_1^2}$$
$$\times[p(2c\gamma_1+d\gamma_2)+q(d\gamma_1-2c\gamma_2)] - \frac{2\nu^2\gamma_3^2}{\gamma_1^6}r[K\gamma_1^2 - \nu(\gamma_1^2+\gamma_3^2)] + \frac{\nu(K-\nu)^2\gamma_3^2}{\gamma_1^4}[2K\gamma_1^2$$
$$+ \nu(\gamma_3^2+2\gamma_2^2)] + \frac{\nu^4\gamma_3^4}{\gamma_1^8}. \tag{59}$$

Table 1 Comparison with previous results

Authors	Conditions on parameters	References
Elmandouh	$a = b = 0$	[43]
Yehia	$\nu = 0$	[42]
Goriachev	$K = \nu = 0$	[41]
Chaplygin	$c = d = \nu = K = \lambda = 0$	[5]

It contains seven free parameters. The comparison between the present integrable case and the related previous cases is summarized in Table 1.

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References

1. Bertrand, J.: Quelques-unes des forms les plus simple que puissant presenter les integrables des equations differentialles du movement d'un point materiel. J. Math. Pure Appl. **2**, 113–153 (1857)
2. Darboux, G.: Sur un problem de mecanique. Arch. Neerl. **6**, 371–377 (1901)
3. Whittaker, E.T.: A Treatise on Analytical Dynamics of Particles and Rigid Bodies. Dover, New York (1944)
4. Kowalevski, S.V.: Sur le problem de la rotation d'un crops solid autor d'un point fixe. Acta Math. **12**, 177–232 (1889)
5. Chaplygin, S.A.: A new partial solution of the problem of motion of a rigid body in a liquid. Trudy Otdel. Fiz. Nauk Obsh. Luib. Est. **11**, 7–10 (1903)
6. Winternitz, P., Smorodisky, J.A., Uhlir, M., Fris, I.: On symmetry groups in classical and quantum mechanics. Jadernaja Fiz. **44**, 625–635 (1967). (English trans: Soviet J. Nucl. Phys. **4**, 444–450, 1967)
7. Hall, L.S.: A theory of exact and approximate configurational invariants. Phys. D **8**, 90–116 (1983)
8. Yehia, H.M.: Two dimensional conservative mechanical systems with quartic second integral. Regul. Chaotic. Dyn. **11**, 103–122 (2006)
9. Grammaticos, B., Dorizzi, B., Ramani, A.: Hamiltonian with high-order integrals and the "weak-Painleve" concept. J. Math. Phys. **25**, 3470–3473 (1984)
10. Evans, N.W.: On Hamiltonian systems in two degrees of freedom with invariants quartic in the momenta of form $p_1^2 p_2^2$. J. Math. Phys. **31**, 600–604 (1990)
11. Karlovini, M., Pucacoo, G., Rosquist, K., Samuelsson, L.: A unified treatment of quartic invariants at fixed and arbitrary energy. J. Math. Phys. **43**, 4041–4059 (2002)
12. Ranada, M., Santander, M.: Super integrable systems on the two dimensional sphere S^2 the hyperbolic plane H_2. J. Math. Phys. **43**, 431–451 (1999)
13. Bozis, G.: Compatibility conditions for a non-quadratic integral of motion. Clest. Mech. **28**, 367–380 (1982)
14. Bozis, G.: Two dimensional integrable potentials with quartic invariants. J. Phys. A Math. Gen. **25**, 3329–3351 (1992)
15. Wojciechowska, M., Wojciechowski, S.: New integrable potentials of two degrees of freedom. Phys. Lett. A. **105**, 11–14 (1984)
16. Yehia, H.M., Bedweihy, N.: Certain generalizations of Kovalevskaya's case. Mansoura Sci. Bull. **14**, 373–386 (1987)
17. Sen, T.: A class of integrable potentials. J. Math. Phys. **28**, 2841–2850 (1987)
18. Sen, T.: Integrable potentials with cubic and quartic invariants. Phys. Lett. A. **122**, 100–106 (1987)
19. Selivanova, E.N.: New families of conservative systems on S^2 possessing an integral of fourth degree in momenta. Ann. Glob. Anal. Geom. **17**, 201–219 (1999)
20. Hietarinta, J.: Direct methods for the search of the second invariant. Phys. Rep. **147**, 87–154 (1987)
21. Volterra, V.: Sur la theories des variations des latitudes. Acta Math. **22**, 201–357 (1899)
22. Gluhovsky, A., Christopher, T.: The structure of energy conserving low-order models. Phys. Fluid **11**, 334–337 (1999)
23. Hughes, P.C.: Spacecraft Attitude Dynamics. Wiely, New York (1986)
24. Kharlamov, M.P.: Symmetry in systems with gyroscopic forces. Mekh. Tverd. Tela **15**, 87–93 (1983)
25. Yehia, H.M.: On the motion of a rigid body acted upon by potential and gyroscopic forces: I. The equations of motion and their transformations. J. Mecan. Theor. Appl. **5**, 747–754 (1986)
26. Leimanis, E.: The General Problem of Motion of Coupled Rigid Bodies About a Fixed Point. Springer, Berlin (1965)
27. Yehia, H.M.: On the integrability of certain problems in particle and rigid body in an axisymmetric field. J. Mecan. Theor. Appl. **5**, 55–71 (1986)
28. Yehia, H.M.: The master integrable two dimensional systems with a quartic second integral. J. Phys. A Math. Gen. **39**, 5807–5824 (2006)
29. Yehia, H.M.: Generalized natural mechanical systems of two degrees of freedom with quadratic integrals. J. Phys. A: Math. Gen. **25**, 197–221 (1992)
30. Yehia, H.M.: Atlas of two dimensional conservative Lagrangian mechanical systems with a second quadratic integral. J. Math. Phys. **48**, 082902 (2007)
31. Yehis, H.M.: On certain two dimensional conservative mechanical system with cubic second integral. J. Phys. A Math. Gen. **35**, 9469–9487 (2002)

32. Yehia, H.M., Elmandouh, A.A.: New integrable systems with quartic integral and new generalizations of Kovalevskaya and Goriachev cases. Regul. Chaot. Dyn. **13**, 57–69 (2008)
33. Yehia, H.M., Elmandouh, A.A.: New conditional integrable cases of motion of a rigid body with Kovalevskaya's configuration. J. Phys. A Math. Gen. **44**, 1–8 (2011)
34. Yehia, H.M., Elmandouh, A.A.: A new integrable problem with a quartic integral in the dynamics of a rigid body. J. Phys. A Math. Gen. **46**, 1–8 (2013)
35. Yehia, H.M.: Equivalent problem in rigid body dynamics I. Clest. Mech. **41**, 275–288 (1988)
36. Yehia, H.M.: Equivelant problems in rigid body dynamics II. Clest. Mech. **41**, 289–295 (1988)
37. Galajinsky, A., Lechtenfeld, O.: On two-dimensional integrable models with a cubic or quartic integral of motion. JHEP **9**, 1–12 (2013)
38. Yehia, H.M., Hussein, A.M.: New Families of Integrable Two-dimensional Systems with Quartic Second Integrals. arXiv:1308.1442
39. Yehia, H.M.: A new 2D integrable system with a quartic invariant. J. Phys. A Math. Theor. **45**, 395209 (2012)
40. Wojciechowski, S.: On the integrability of the Calogero–Moser system in an external quartic potential and other many-body systems. Phys. Lett. A **102**, 85–93 (1984)
41. Goriachev, G.D.: New integrable cases of Euler's dynamical equations. Varshav. Univ. Izvest. **3**, 1–13 (1916)
42. Yehia, H.M.: New integrable problems in the dynamics of rigid bodies with the Kovalevskaya configuration. I—The case of axisymmetric forces. Mech. Res. Commun. **23**, 423–427 (1996)
43. Elmandouh, A.A.: New integrable problems in rigid body dynamics with quartic integrals. Acta Mech. **226**, 2461–2472 (2015)