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ORIGINAL PAPER

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Static response bounds of Timoshenko beams with spatially varying interval uncertainties

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Abstract Response variability of Timoshenko beams with uncertain Young's modulus subjected to deterministic static loads is analyzed. The uncertain material property is idealized within a non-probabilistic context by using an interval field model recently proposed by the first two authors. Such a model is able to quantify the dependency between adjacent values of an interval uncertainty by means of a real, deterministic, symmetric, nonnegative, bounded function conceived as the non-probabilistic counterpart of the autocorrelation function characterizing random fields. In order to analyze the effects of Young's modulus uncertainty on the static response of the Timoshenko beam, a finite difference discretization of the coupled interval ordinary differential equations of equilibrium is performed. Then, approximate explicit expressions of the lower bound and upper bound of the interval response are derived. Numerical results showing the effects of interval material uncertainty on the static response of a simply supported beam under uniformly distributed load are presented.

1 Introduction

Engineering experience has demonstrated that uncertainties inherent in structural parameters, such as material or geometric properties, may affect to a large extent the structural response. In the last decades, several procedures have been developed to incorporate uncertainties in structural analysis and then predict the corresponding variability of the response. In this context, a crucial issue is taking into account the spatial character of the uncertain physical properties. Probabilistic methods handle the spatial dependency of nondeterministic properties involved in structural engineering problems by using the well-established concept of random field (see, e.g., [1]).

Recently, Moens et al. [2] introduced the so-called *interval field* as a natural extension of the random field concept (see also [3]). The interval field is conceived as a means of representing spatially variable uncertain properties in the context of a widely used non-probabilistic uncertainty model known as interval model. Such

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an uncertainty model, originally developed from the *classical interval analysis* (*CIA*) [4], turns out to be a very useful tool when only range information is known and available experimental data are insufficient to define a reliable probabilistic distribution of the uncertain parameters. Starting from the pioneering work by Moens et al. [2], more recently a novel interval field model based on the so-called *improved interval analysis via extra unitary interval* (*IIA via EUI*) [5] has been proposed [6,7]. The *IIA via EUI* is a very effective tool to limit the overestimation of the interval solution width due to the *dependency phenomenon* [8] affecting the *CIA*. The key idea behind the interval field model based on the *IIA via EUI* is to describe the dependency between interval values of an uncertain property at various locations by means of a real, deterministic, symmetric, nonnegative, bounded function playing the same role of the autocorrelation function in random field theory.

The aim of the present study is to analyze the response variability of a Timoshenko beam with uncertain Young's modulus subjected to transversally distributed deterministic static loads. The uncertain elastic modulus is idealized as an interval field adopting the model based on the *IIA* via *EUI* [6,7]. In order to analyze the effects of Young's modulus uncertainty on the bending behavior of the beam, the regions of the interval transverse displacement and rotation fields have to be determined. This task is pursued in the paper by applying an efficient procedure which basically requires the following steps: (i) to perform a finite difference discretization of the coupled interval differential equations governing the interval transverse displacement and rotation fields; (ii) to evaluate in approximate explicit form the bounds of the interval response fields at the grid points by applying the *IIA* in conjunction with the *Interval Rational Series Expansion* (*IRSE*) [6,7], which is a modified explicit expression of the Neumann series expansion for the inverse of an interval matrix with small rank-r modifications.

A numerical application concerning a simply supported Timoshenko beam with interval Young's modulus under uniformly distributed load is presented. Numerical results demonstrate the key role played by the spatial dependency of the uncertain material property on beam response variability. Appropriate comparisons with the results provided by the Euler–Bernoulli theory for beams of different slenderness are also included.

2 Interval Young's modulus field

Let us consider a Timoshenko beam of length L with uncertain Young's modulus of the material. Under the assumption that only range information on the uncertain material property is available, the interval model of uncertainty is adopted. Such a model, originally developed from the interval analysis [4], treats the uncertain parameters as interval quantities with given lower bound (LB) and upper bound (UB). Unfortunately, interval variables are unable to describe the inherent spatial character of uncertain physical parameters like material or geometrical properties. In view of this observation, the variability of the uncertain Young's modulus within the 1D domain $0 \leq x \leq L$ is herein described by the following interval function:

$$E^I(x) = \left[\underline{E}(x), \bar{E}(x)\right] = E_0\left[1 + B^I(x)\right], \quad x \in [0, L], \tag{1}$$

where the apex I denotes interval quantities; $\underline{E}(x)$ and $\bar{E}(x)$ are the LB and UB. Like interval variables [4], the interval function $E^I(x)$ is characterized by a midpoint value, $E_0 \in \mathbb{R}$, herein taken constant over the whole domain $[0, L]$, and a deviation amplitude $\Delta E(x)$ given, respectively, by

$$\text{mid}\left\{E^I(x)\right\} = \frac{\bar{E}(x) + \underline{E}(x)}{2} \equiv E_0; \quad \Delta E(x) = \frac{\bar{E}(x) - \underline{E}(x)}{2} \tag{2a,b}$$

where mid$\{\bullet\}$ denotes the midpoint of the interval quantity between curly brackets. Furthermore, in Eq. (1) $B^I(x) = \left[\underline{B}(x), \bar{B}(x)\right]$ is a dimensionless interval function having zero midpoint, mid$\left\{B^I(x)\right\} = 0$, and deviation amplitude $\Delta B(x) < 1$.

The interval function $E^I(x)$ in Eq. (1) is herein referred to as *interval field* since, in analogy to the random field, it may be considered as a spatially dependent interval variable. From a physical point of view, the uncertain Young's modulus may be represented by any function ranging between the LB and UB, $\underline{E}(x)$ and $\bar{E}(x)$. The interval field concept has been first introduced by Moens et al. [2] to provide a more realistic description of spatially varying uncertain-but-bounded parameters. Recently, the first two authors proposed a novel interval field model [6,7] based on the so-called *improved interval analysis* (*IIA*) *via extra unitary interval* (*EUI*) (see [5] and Appendix A). According to this model, it is assumed that the spatial dependency of the interval field $E^I(x)$ is governed by a real, deterministic, symmetric, nonnegative, bounded function, $\Gamma_B(x,\xi)$, defined as:

$$\Gamma_B(x,\xi) = \text{mid}\left\{B^I(x)B^I(\xi)\right\} \equiv \frac{\text{mid}\left\{E^I(x)E^I(\xi)\right\}}{(E_0)^2} - 1, \quad x,\xi \in [0,L]. \tag{3}$$

The key idea is to regard the midpoint operator as the equivalent of the stochastic average operator, so that the function $\Gamma_B(x,\xi)$, named *spatial dependency function*, may be viewed as the non-probabilistic counterpart of the autocorrelation function characterizing probabilistically a random field. Then, starting from the spectral decomposition of the *spatial dependency function*, a Karhunen-Loève-like expansion (see, e.g., [9]) of the interval field $E^I(x)$ in Eq. (1) can be performed:

$$\Gamma_B(x,\xi) = \sum_{i=1}^{\infty}\lambda_i\psi_i(x)\psi_i(\xi) \Rightarrow E^I(x) = E_0\left[1+B^I(x)\right] = E_0\left[1+\sum_{i=1}^{N}\sqrt{\lambda_i}\;\psi_i(x)\hat{e}_i^I\right], \quad x \in [0,L], \tag{4}$$

where the first N terms are retained and $\hat{e}_i^I \triangleq [-1,+1]$ denotes the so-called *EUI* (see Appendix A) introduced in the context of the *IIA* [5] to limit the overestimation affecting the *classical interval analysis* (*CIA*) due to the *dependency phenomenon* [8]. Furthermore, $\lambda_i (i = 1, 2, \ldots)$ is the i-th *eigenvalue* of the bounded, symmetric, nonnegative function, $\Gamma_B(x,\xi)$, and $\psi_i(x)$ is the corresponding *eigenfunction*, solutions of the following homogeneous Fredholm integral equation of the second kind:

$$\int_0^L \Gamma_B(x,\xi)\psi_i(x)\mathrm{d}x = \lambda_i\psi_i(\xi); \quad \int_0^L \psi_i(x)\psi_j(x)\mathrm{d}x = \delta_{ij}, \tag{5a,b}$$

where δ_{ij} is the Kronecker delta. It is worth emphasizing that the *EUI* does not follow the rules of the *CIA*, and therefore, it is different from the well-known *classical unitary interval* $e^I \triangleq [-1,+1]$ (see Appendix A). In particular, multiplication between *EUIs* is defined in such a way that the *EUIs* may be viewed as "orthonormal" within the interval context:

$$\begin{aligned}&\text{mid}\{\hat{e}_i^I \times \hat{e}_i^I\} = \text{mid}\{[1,1]\} = 1;\\ &\text{mid}\{\hat{e}_i^I \times \hat{e}_j^I\} = \text{mid}\{[-1,+1]\} = 0.\end{aligned} \tag{6a,b}$$

In view of Eqs. (6a,b), the following relationship is satisfied:

$$B^I(x) = \sum_{i=1}^{N}\sqrt{\lambda_i}\;\psi_i(x)\hat{e}_i^I \Rightarrow \text{mid}\left\{\left(B^I(x)\right)^2\right\} = \sum_{i=1}^{\infty}\lambda_i\psi_i^2(x) \equiv \Gamma_B(x,x), \tag{7}$$

which does not hold in the context of the *CIA*.

The LB and UB, $\underline{E}(x)$ and $\bar{E}(x)$, of the interval Young's modulus $E^I(x)$ in Eq. (4) can be defined as:

$$\underline{E}(x) = E_0\left[1-\Delta B(x)\right]; \quad \bar{E}(x) = E_0\left[1+\Delta B(x)\right], \quad x \in [0,L] \tag{8a,b}$$

with

$$\Delta B(x) = \frac{\Delta E(x)}{E_0} = \sum_{i=1}^{N}\left|\sqrt{\lambda_i}\;\psi_i(x)\right|, \quad x \in [0,L], \tag{9}$$

where $|\bullet|$ denotes the absolute value of $\bullet$.

3 Timoshenko beam with interval Young's modulus

The aim of this section is to analyze the variability of the response of the Timoshenko beam with uncertain Young's modulus, idealized as the interval field $E^I(x)$ described in the previous section. The beam is subjected to a deterministic transversally distributed load $p_z(x)$ (see Fig. 1). The material is assumed to be linear elastic isotropic. The beam is referred to a Cartesian coordinate system $Oxyz$, where the x-axis represents the centroidal axis, the y- and z-axes are principal axes of the cross section, and xz is the bending plane, as shown in Fig. 1.

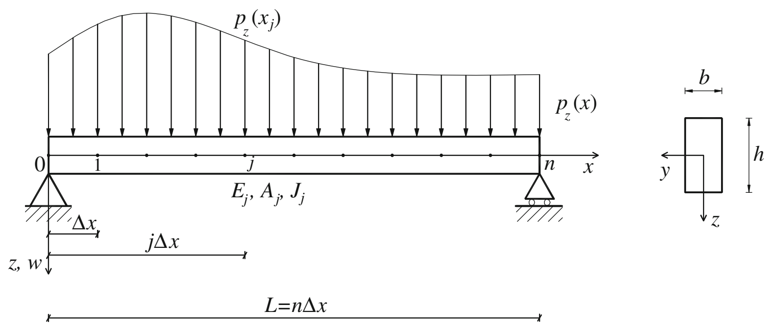

Fig. 1 Timoshenko beam with uncertain Young's modulus

The bending behavior of the Timoshenko beam with interval Young's modulus $E^I(x)$ is governed by the following two coupled interval ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{G^I(x)A_*(x)\left[\frac{\mathrm{d}w^I(x)}{\mathrm{d}x}-\varphi^I(x)\right]\right\}+p_z(x)=0;$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[E^I(x)J(x)\frac{\mathrm{d}\varphi^I(x)}{\mathrm{d}x}\right]+G^I(x)A_*(x)\left[\frac{\mathrm{d}w^I(x)}{\mathrm{d}x}-\varphi^I(x)\right]=0, \quad \text{(10a,b)}$$

where $w^I(x)$ denotes the z-component of the displacement of a point at x on the centroidal axis; $\varphi^I(x)$ is the interval rotation about the y-axis taken as positive if clockwise; $A_*(x) = K_s A(x)$ with K_s being the shear correction factor and $A(x)$ the cross-sectional area; $J(x)$ is the moment of inertia of the cross section; $G^I(x) = E^I(x)/2(1+\nu)$ is the interval transversal modulus of elasticity with ν denoting the Poisson ratio herein taken deterministic. Equations (10a,b) need to be supplemented by appropriate kinematic and static boundary conditions herein assumed deterministic.

3.1 Finite difference discretization

Within the interval framework, the solution of Eqs. (10a,b) involves finding the narrowest intervals containing all possible transverse displacement and rotation fields, $w(x)$ and $\varphi(x)$, as the interval Young's modulus $E^I(x)$ ranges between its LB and UB (see Eqs. (8a,b)). Specifically, the aim of the analysis is the evaluation of the spatially dependent LB and UB of the interval fields $w^I(x)$ and $\varphi^I(x)$. In the present study, approximate explicit expressions of such bounds are obtained by performing a finite difference (FD) discretization of the governing equations and applying the *IIA via EUI* [5].

Let the beam domain $[0, L]$ be subdivided into n intervals Δx, so that the abscissa of the j-th grid point is identified by $x_j = j\Delta x$ with $j = 0, 1, 2, \ldots, n$. Upon replacing the definition of the interval Young's modulus field (4) and the central finite difference approximations of first- and second-order derivatives, the equilibrium Eq. (10a,b) takes the following discretized form:

$$\begin{aligned}
& c_s\left[\left(d_{j+1}-d_{j-1}+4d_j\right)w^I_{j+1}+\left(d_{j-1}-d_{j+1}+4d_j\right)w^I_{j-1}-8d_j w^I_j\right] \\
& -2c_s\Delta x\left[\left(d_{j+1}-d_{j-1}\right)\varphi^I_j-d_j\varphi^I_{j-1}+d_j\varphi^I_{j+1}\right] \\
& +\sum_{i=1}^{N}\left\{c_s\left[\left(s^{(i)}_{j+1}-s^{(i)}_{j-1}+4s^{(i)}_j\right)w^I_{j+1}+\left(s^{(i)}_{j-1}-s^{(i)}_{j+1}+4s^{(i)}_j\right)w^I_{j-1}-8s^{(i)}_j w^I_j\right]\right. \\
& \left.-2c_s\Delta x\left[\left(s^{(i)}_{j+1}-s^{(i)}_{j-1}\right)\varphi^I_j-s^{(i)}_j\varphi^I_{j-1}+s^{(i)}_j\varphi^I_{j+1}\right]\right\}\hat{e}^I_i+F_j=0; \\
& \left[\left(k_{j+1}-k_{j-1}+4k_j\right)\varphi^I_{j+1}+\left(k_{j-1}-k_{j+1}+4k_j\right)\varphi^I_{j-1}-8k_j\varphi^I_j\right] \\
& +2c_s\Delta x d_j\left[w^I_{j+1}-w^I_{j-1}\right]-4\Delta x^2 c_s d_j\varphi^I_j
\end{aligned}$$

$$+\sum_{i=1}^{N}\left\{\left[\left(g_{j+1}^{(i)}-g_{j-1}^{(i)}+4g_{j}^{(i)}\right)\varphi_{j+1}^{I}+\left(g_{j-1}^{(i)}-g_{j+1}^{(i)}+4g_{j}^{(i)}\right)\varphi_{j-1}^{I}-8g_{j}^{(i)}\varphi_{j}^{I}\right]\right.$$
$$\left.+2c_{s}\Delta x s_{j}^{(i)}\left[w_{j+1}^{I}-w_{j-1}^{I}\right]-4\Delta x^{2}c_{s}s_{j}^{(i)}\varphi_{j}^{I}\right\}\hat{e}_{i}^{I}=0, \tag{11a,b}$$

where

$$\begin{aligned} &c_s=\frac{K_s}{2(1+\nu)}; \quad F_j=4\Delta x^2 p_z(x_j)=4\Delta x^2 p_{z,j};\\ &d_j=E_0A(x_j)=E_0A_j; \quad k_j=E_0J(x_j)=E_0J_j;\\ &s_j^{(i)}=E_0A(x_j)\sqrt{\lambda_i}\psi_i(x_j)=d_j\sqrt{\lambda_i}\psi_{ij}; \quad g_j^{(i)}=E_0J(x_j)\sqrt{\lambda_i}\psi_i(x_j)=k_j\sqrt{\lambda_i}\psi_{ij}. \end{aligned} \tag{12a-f}$$

The set of linear interval coupled equations (11a,b) can be rewritten in matrix form as:

$$\mathbf{K}^I\mathbf{u}^I=\left(\mathbf{K}_0+\Delta\mathbf{K}_B^I\right)\mathbf{u}^I=\mathbf{F}, \tag{13}$$

where $\mathbf{u}^I=\left[\,\mathbf{w}^I\;\boldsymbol{\varphi}^I\,\right]^T$ is the q-vector collecting the interval displacements and rotations, w_j^I and φ_j^I, at the grid points x_j, whose order q can be defined once the boundary conditions are imposed; $\mathbf{F}$ is the vector of order q listing the forces at the grid points F_j. Notice that the interval coefficient matrix $\mathbf{K}^I$ in Eq. (13) is expressed as sum of two terms: the first one, $\mathbf{K}_0$, is the $q\times q$ coefficient matrix pertaining to the nominal beam with nominal value (or midpoint) E_0 of the Young's modulus; the second term, $\Delta\mathbf{K}_B^I$, is an interval matrix accounting for the uncertain nature of the elastic modulus. Such a matrix can be expressed as superposition of N deterministic matrices multiplied by the corresponding *EUI*, $\hat{e}_i^I$, i.e.:

$$\Delta\mathbf{K}_B^I=\sum_{i=1}^{N}\Delta\mathbf{K}_{B,i}\hat{e}_i^I. \tag{14}$$

3.2 Bounds of the interval response

Since the square interval coefficient matrix $\mathbf{K}^I$ is regular, that is, each matrix $\mathbf{K}\in\mathbf{K}^I$ is non-singular [10], the solution $\mathbf{u}^I$ of Eq. (13) exists for all $\mathbf{K}\in\mathbf{K}^I$. An approximate explicit expression of the interval vector $\mathbf{u}^I$ is herein derived by applying the so-called *Interval Rational Series Expansion* (*IRSE*), recently proposed for evaluating the inverse of an interval matrix with small rank-r modifications [6,7]. To this aim, first the interval coefficient matrix $\mathbf{K}^I$ is decomposed as follows:

$$\mathbf{K}^I=\mathbf{K}_0+\Delta\mathbf{K}_B^I=\mathbf{K}_0+\sum_{i=1}^{N}\sum_{\ell=1}^{q}\mathbf{k}_{B,i\ell}\mathbf{v}_\ell^T\hat{e}_i^I, \tag{15}$$

where $\mathbf{k}_{B,i\ell}$ is the ℓ-th column of the matrix $\Delta\mathbf{K}_{B,i}$ in Eq. (14) and $\mathbf{v}_\ell$ is a column vector of order q containing all zeros except the ℓ-th element which is equal to 1. According to the previous equations, the interval matrix $\mathbf{K}^I$ is expressed as sum of the nominal value $\mathbf{K}_0$ plus an interval deviation $\Delta\mathbf{K}_B^I$ given as superposition of rank-one matrices $\mathbf{k}_{B,i\ell}\mathbf{v}_\ell^T\hat{e}_i^I$. Such decomposition enables us to apply the *IRSE* which, under the assumption of small dimensionless deviation amplitude of the interval elastic modulus, i.e., $\Delta B(x)\ll 1$ for all $x\in[0,L]$, can be truncated to first-order terms, i.e.:

$$\left(\mathbf{K}^I\right)^{-1}\approx\mathbf{K}_0^{-1}-\sum_{i=1}^{N}\sum_{\ell=1}^{q}\frac{\hat{e}_i^I}{1+\hat{e}_i^I d_{B,i\ell}}\mathbf{D}_{B,i\ell}=\mathbf{K}_0^{-1}+\sum_{i=1}^{N}\sum_{\ell=1}^{q}\left(a_{0,i\ell}+\Delta a_{i\ell}\hat{e}_i^I\right)\mathbf{D}_{B,i\ell}, \tag{16}$$

where

$$a_{0,i\ell}=\frac{d_{B,i\ell}}{1-d_{B,i\ell}^2}; \quad \Delta a_{i\ell}=\frac{1}{1-d_{B,i\ell}^2} \tag{17a,b}$$

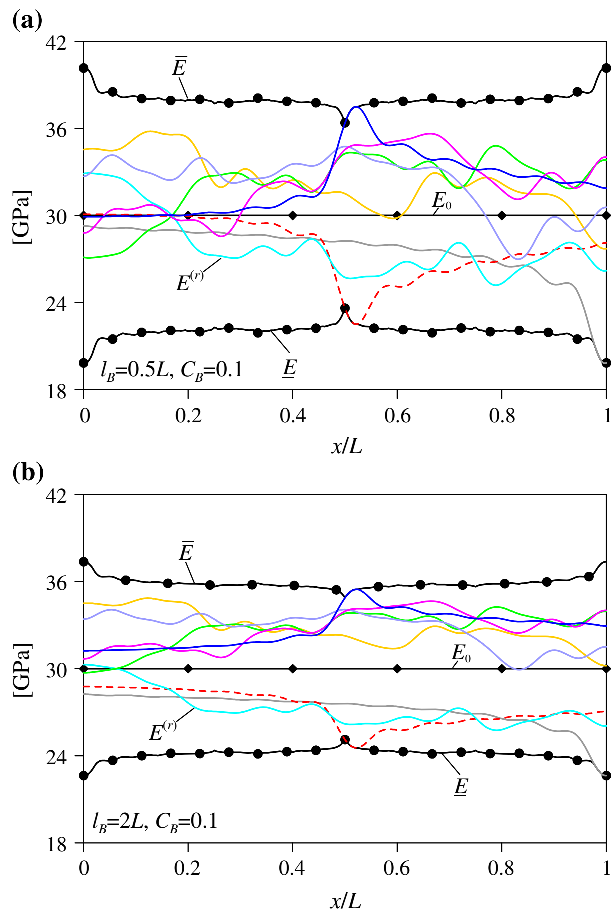

Fig. 2 Bounds, $\underline{E}$ and $\bar{E}$, and some samples ($E^{(r)}$) ($r = 1, 2, \ldots$) of the interval Young's modulus field for $C_B = 0.1$ and two different values of the parameter l_B : $l_B = 0.5L$ (**a**) and $l_B = 2L$ (**b**)

denote the midpoint and deviation amplitude of the generic series term rewritten in *affine form* and

$$d_{B,i\ell} = \left| \mathbf{v}_\ell^T \mathbf{K}_0^{-1} \mathbf{k}_{B,i\ell} \right|; \quad \mathbf{D}_{B,i\ell} = \mathbf{K}_0^{-1} \mathbf{k}_{B,i\ell} \mathbf{v}_\ell^T \mathbf{K}_0^{-1}. \tag{18a,b}$$

It has to be mentioned that Eq. (16) holds if and only if $d_{B,i\ell} < 1$. Then, based on Eq. (16), the approximate solution $\mathbf{u}^I \in \mathbb{IR}^q$ of the set of linear interval Eq. (13) can be expressed as sum of the midpoint value, mid$\{\mathbf{u}^I\}$, plus the interval deviation, dev$\{\mathbf{u}^I\}$, i.e.:

$$\mathbf{u}^I = (\mathbf{K}^I)^{-1}\mathbf{F} \approx \text{mid}\{\mathbf{u}^I\} + \text{dev}\{\mathbf{u}^I\}, \tag{19}$$

where

$$\text{mid}\{\mathbf{u}^I\} = \left(\mathbf{K}_0^{-1} + \sum_{i=1}^{N} \sum_{\ell=1}^{q} a_{0,i\ell} \mathbf{D}_{B,i\ell} \right) \mathbf{F}; \quad \text{dev}\{\mathbf{u}^I\} = \sum_{i=1}^{N} \sum_{\ell=1}^{q} \Delta a_{i\ell} \mathbf{D}_{B,i\ell} \mathbf{F} \hat{e}_i^I. \tag{20a,b}$$

In Eq. (20b), dev $\{\bullet\}$ denotes the interval deviation of the interval quantity between curly brackets.

Based on the approximate explicit expression in Eq. (19) and applying the *IIA via EUI*, the LB and UB, $\underline{\mathbf{u}}$ and $\bar{\mathbf{u}}$, of the interval response vector $\mathbf{u}^I$ can be evaluated as follows:

$$\underline{\mathbf{u}} = \mathbf{u}_\text{m} - \Delta\mathbf{u}; \quad \bar{\mathbf{u}} = \mathbf{u}_\text{m} + \Delta\mathbf{u}, \tag{21a,b}$$

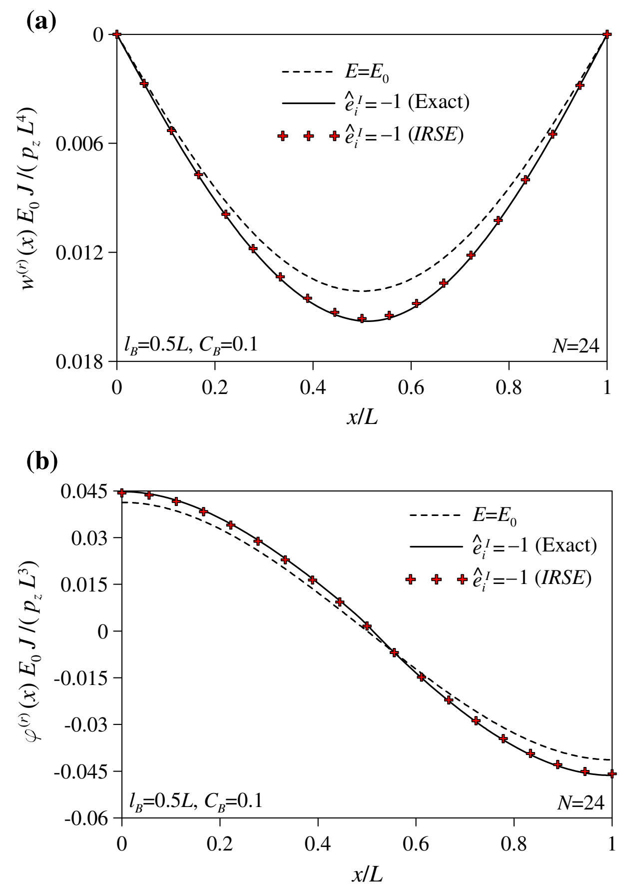

Fig. 3 Samples of the normalized interval (**a**) displacement and (**b**) rotation fields along the simply supported Timoshenko beam: comparison between the "exact" FD solution and the *IRSE* approximation ($l_B = 0.5L$, $C_B = 0.1$)

where $\mathbf{u}_\mathrm{m}$ is the midpoint value vector given by Eq. (20a), while $\Delta\mathbf{u}$ is the deviation amplitude vector defined as:

$$\Delta\mathbf{u} = \sum_{i=1}^{N}\left|\sum_{\ell=1}^{q}\Delta a_{i\ell}\mathbf{D}_{B,i\ell}\mathbf{F}\right| \tag{22}$$

with the symbol $|\bullet|$ denoting the component-wise absolute value.

Within the interval framework, a very useful tool for analyzing the propagation of the uncertain properties to the structural response is the *coefficient of interval uncertainty* which for the j-th component of the interval solution vector $\mathbf{u}^I \in \mathbb{IR}^q$ is defined as follows:

$$\text{c.i.u.}[u_j^I] = \frac{\Delta u_j}{\left|u_{\mathrm{m},j}\right|}. \tag{23}$$

Such coefficient provides a measure of the dispersion of the interval response quantity u_j^I around the midpoint value $u_{\mathrm{m},j}$, and therefore, it may be viewed as the non-probabilistic counterpart of the *coefficient of variation* used in stochastic analysis.

4 Numerical application

In this section, the presented procedure is applied to investigate the effects of Young's modulus uncertainty on the response of a simply supported beam under uniformly distributed transversal load p_z. The beam has span

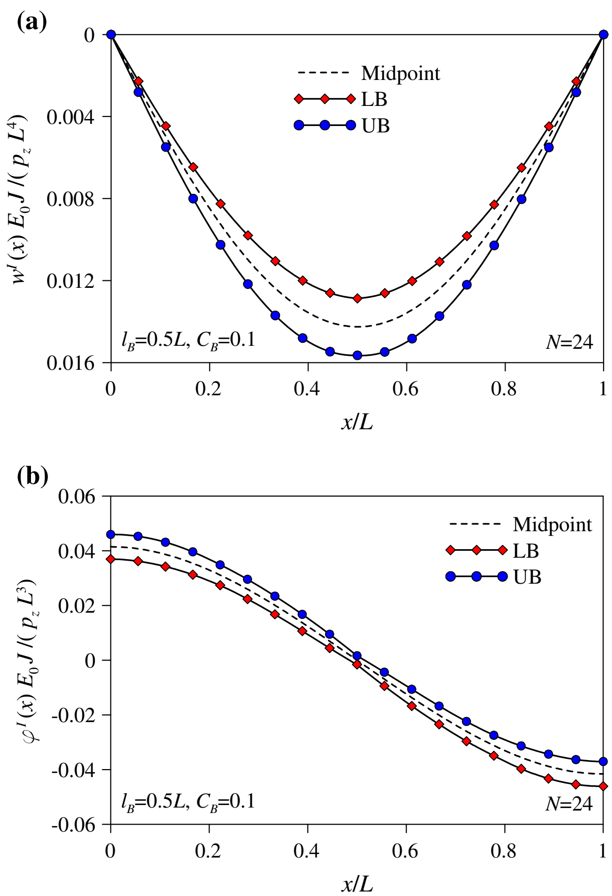

Fig. 4 Proposed LB, UB and midpoint of the normalized interval (**a**) displacement and (**b**) rotation fields along the simply supported Timoshenko beam ($l_B = 0.5L$, $C_B = 0.1$)

length $L = 5$ m and rectangular cross section with $b = 0.5$ m and $h = 1$ m (see Fig. 1). The Poisson ratio of the material is set equal to $\nu = 0.2$. The Young's modulus is modeled as an interval field $E^I(x)$ with constant midpoint $E_0 = 30$ GPa and spatial variability described by the exponential function:

$$\Gamma_B(x, \xi) = C_B^2 \exp\left(-\frac{|x-\xi|}{l_B}\right), \tag{24}$$

where C_B is a coefficient affecting the deviation amplitude of the interval field, while the parameter l_B governs the spatial dependency. When not otherwise specified, these parameters are selected as $C_B = 0.1$ and $l_B = 0.5L$. The function $\Gamma_B(x, \xi)$ is decomposed by applying Eq. (7) retaining the first $N = 24$. The interval ordinary differential equations governing the response are discretized by the FDM using a uniform grid with $n = 360$ subdivisions (see Eqs. (11a,b)).

Figure 2 displays the UB and LB, $\bar{E}(x)$ and $\underline{E}(x)$ (see Eqs. (8a,b)), along with some samples $E^{(r)}(x)$ of the interval Young's modulus field $E^I(x)$ (see Eq.(4)) for two different values of the parameter l_B, say $l_B = 0.5L$ and $l_B = 2L$. Each sample corresponds to a different combination of the bounds of the *EUIs*, $\hat{e}_i^I$, appearing in the definition of the interval field (see Eq.(4)). For instance, the sample denoted by the red dashed line is obtained setting all the *EUIs* at their LB, that is, $\hat{e}_i^I = -1$ $(i = 1, 2, \ldots, N)$. By inspection of Fig. 2, it is observed that the samples of the interval Young's modulus are strongly affected by the parameter l_B. In particular, samples pertaining to the value $l_B = 0.5L$ (see Fig. 2a) are more irregular than those corresponding to $l_B = 2L$ (see Fig. 2b). Indeed, as the parameter l_B increases, the uncertain material property tends to be totally spatially dependent. In the limit as $l_B \to \infty$, the interval field reduces to a single interval variable over the whole domain $[0, L]$ defined as $E^I = E_0(1 + C_B\hat{e}^I)$ [7]. Conversely, as $l_B \to 0$ the uncertain Young's modulus becomes spatially independent and the interval field ideally reduces to a series of independent interval

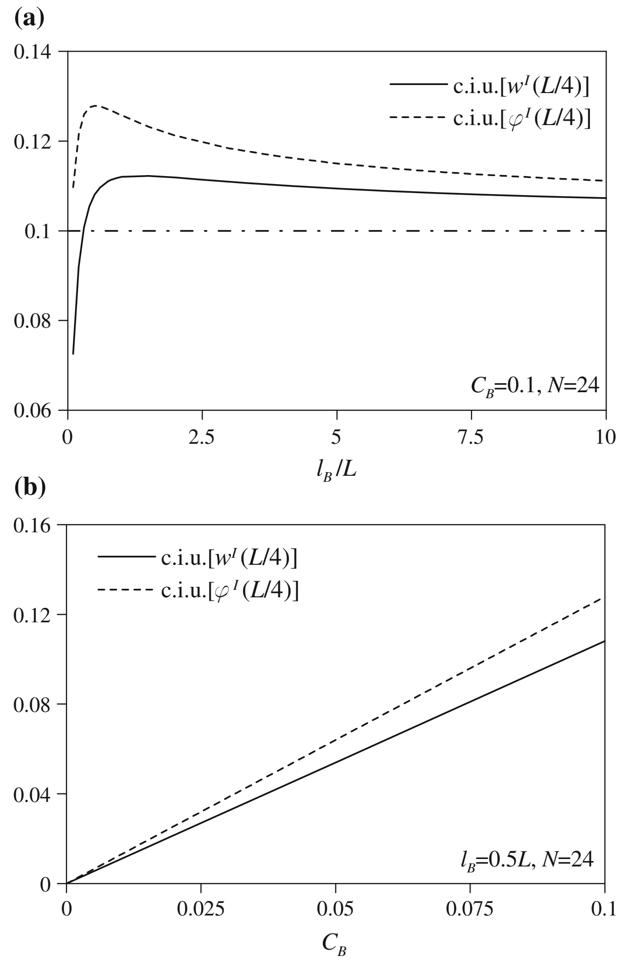

Fig. 5 *Coefficient of interval uncertainty* of the interval transverse displacement and rotation at the abscissa $x = L/4$ versus: (**a**) the ratio l_B/L; (**b**) the coefficient C_B

variables, one for each abscissa within the domain [0, L]. Finally, as expected, Fig. 2 shows that the samples of the interval Young's modulus are always enclosed by the bounds $\bar{E}(x)$ and $\underline{E}(x)$.

In order to assess the accuracy of the approximate explicit expression of the interval response vector $\mathbf{u}^I \in \mathbb{IR}^q$ obtained by applying the *IRSE*, a realization of the interval Young's modulus $E^I(x)$ with $\hat{e}_i^I = -1$ ($i = 1, 2, \ldots, N$) (see the red dashed line in Fig. 2a) is considered. Figure 3a, b shows the corresponding samples of the normalized interval displacement and rotation fields, $w^I(x)E_0J/(p_zL^4)$ and $\varphi^I(x)E_0J/(p_zL^3)$. The approximate solution provided by the *IRSE* truncated to first-order terms (Eq.(16)) is compared with the FD solution obtained by numerical inversion of the coefficient matrix in Eq.(13). It is observed that the *IRSE* is capable of predicting with great accuracy the deviation of the response from the nominal value due to the variation of the elastic modulus.

Figure 4a, b displays the LB, UB and midpoint value of the normalized interval displacement and rotation fields obtained by the proposed procedure. Notice that, as a consequence of Young's modulus uncertainty, both the transverse displacement and rotation along the beam may range within a wide region enclosed by the LB and UB. A more meaningful measure of the dispersion of the response around the midpoint value is given by the c.i.u. defined in Eq. (23). In Fig. 5a, b, the c.i.u. of the transverse displacement and rotation at $x = L/4$ versus the ratio l_B/L and the coefficient C_B are plotted. Inspection of Fig. 5a reveals that both the coefficients almost everywhere are greater than $C_B = 0.1$ so that propagation of the uncertainty to the response implies amplification. Such amplification is more significant for the rotation since the associated c.i.u. is greater than the one of the displacement (see also Fig. 5b). Furthermore, it is observed that the c.i.u. of both displacement and rotation tends to the value $C_B = 0.1$ pertaining to the beam with totally spatially

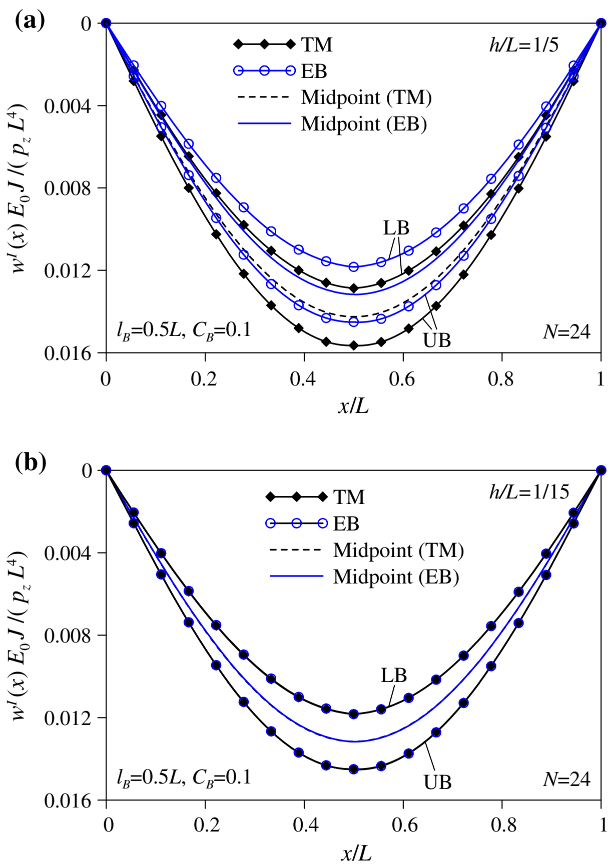

Fig. 6 Proposed LB, UB and midpoint of the normalized interval displacement field along the simply supported beam: comparison between the TM and EB beam models for (**a**) $h/L = 1/5$, (**b**) $h/L = 1/15$ ($l_B = 0.5L$, $C_B = 0.1$)

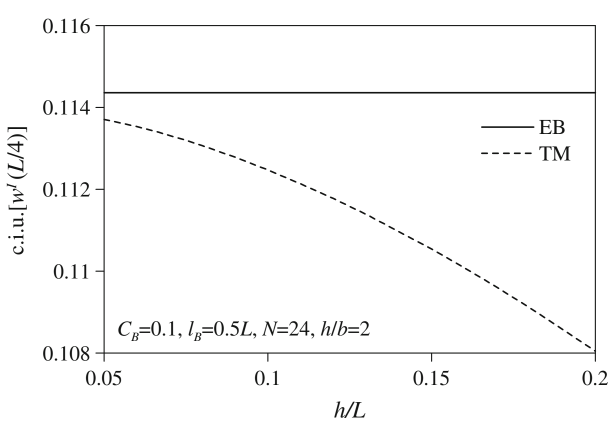

Fig. 7 *Coefficient of interval uncertainty* of the interval transverse displacement at the abscissa $x = L/4$ versus h/L : comparison between the TM and EB beam models ($l_B = 0.5L$, $C_B = 0.1$, $h/b = 2$)

dependent Young's modulus. Therefore, assuming total spatial dependency of the uncertain elastic modulus may lead to underestimate the dispersion of the response around the midpoint value.

As known, within a deterministic context, as the ratio h/L decreases, the solution provided by the Timoshenko (TM) beam theory approaches the one of the classical Euler–Bernoulli (EB) theory. The proposed approach is here applied to compare the solutions given by the two beam theories in the presence of uncertain Young's modulus described by an interval field. Figure 6 shows the proposed estimates of the bounds and midpoint value of the normalized interval transverse displacement obtained by applying the TM and EB theories for two different values of the ratio between the thickness and length of the beam, say $h/L = 1/5$ and $h/L = 1/15$. As expected, the EB theory underestimates the interval solution for $h/L = 1/5$, while the two beam models yield nearly the same bounds and midpoint values of the interval deflection as the slenderness increases.

To gain a deeper insight into the propagation of Young's modulus uncertainty in the context of the TM and EB beam theories, in Fig. 7 the c.i.u. of the interval displacement at $x = L/4$ versus the ratio h/L is plotted. The ratio between the thickness and width of the cross section of the beam is kept constant, i.e., $h/b = 2$. It can be observed that the c.i.u. pertaining to the EB theory does not depend on the ratio h/L. Conversely, the dispersion of the interval deflection around the midpoint value predicted by the TM theory decreases with the ratio h/L. As expected, for small values of h/L the estimates of the c.i.u. provided by the TM and EB theories tend to the same value.

5 Conclusions

Within a non-probabilistic framework, the response variability of Timoshenko beams due to Young's modulus uncertainty has been analyzed. Deterministic static loads have been considered. The uncertain material property has been idealized as an *interval field* adopting a model recently proposed by the first two authors relying on the *improved interval analysis* via *extra unitary interval*. In the context of the well-known interval model of uncertainty, an interval field is conceived as being able of describing spatial dependency of physical properties which are unknown but assumed to lie within prescribed intervals. The regions of the interval transverse displacement and rotation fields of the Timoshenko beam with interval modulus of elasticity have been determined by applying a procedure based on the finite difference discretization of the governing coupled interval differential equations. Approximate explicit expressions of the bounds have been obtained by applying the *Interval Rational Series Expansion* for evaluating the inverse of an interval matrix with modifications.

Numerical results have demonstrated that the presented approach is capable of predicting the response variability of the Timoshenko beam under spatially varying interval uncertainty. In particular, it has been observed that the spatial dependency of the interval Young's modulus has a remarkable effect on the interval transverse displacement and rotation fields of the beam. Furthermore, it has been shown that the slenderness affects the propagation of uncertainty to the response of the Timoshenko beam. Conversely, by applying the Euler–Bernoulli theory the dispersion of the interval deflection around the midpoint value has been found to be independent of the slenderness of the beam.

APPENDIX A: Improved interval analysis via extra unitary interval

Aim of this Appendix is to introduce the fundamentals of the *Improved Interval Analysis* (*IIA*) developed by Muscolino and Sofi [5] to limit the overestimation affecting the *classical interval analysis* (*CIA*) due to the *dependency phenomenon* [8].

It is recalled that an interval variable is denoted by $\alpha^I \triangleq [\underline{\alpha}, \bar{\alpha}] \in \mathbb{IR}$ such that $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$, $\mathbb{IR}$ being the set of all real interval numbers [4,8]. The symbols $\underline{\alpha}$ and $\bar{\alpha}$ denote the lower bound (LB) and upper bound (UB) of the interval, respectively, while the apex I characterizes the interval variables.

The *IIA* is based on the definition of the so-called *extra unitary interval* (*EUI*) $\hat{e}_i^I \triangleq [-1, +1]$ where the subscript i indicates that the *EUI* is associated with the i-th uncertain-but-bounded parameter α_i^I. The *EUI* is defined in such a way that the following properties hold:

$$\hat{e}_i^I - \hat{e}_i^I = [0, 0]; \quad \hat{e}_i^I \times \hat{e}_i^I = \left(\hat{e}_i^I\right)^2 = [1, 1]; \quad \hat{e}_i^I \times \hat{e}_j^I = \hat{e}_{ij}^I = [-1, +1], i \neq j;$$
$$\hat{e}_i^I / \hat{e}_i^I = [1, 1]; \quad x_i \hat{e}_i^I \pm y_i \hat{e}_i^I = (x_i \pm y_i)\, \hat{e}_i^I; \quad x_i \hat{e}_i^I \times y_i \hat{e}_i^I = x_i y_i \left(\hat{e}_i^I\right)^2 = x_i y_i\, [1, 1]. \quad \text{(A.1a-f)}$$

In these equations, $[1, 1] = 1$ is the so-called unitary *thin interval*. It is recalled that a thin interval occurs when $\underline{\alpha} = \bar{\alpha}$ and it is defined as $\alpha^I \triangleq [\underline{\alpha}, \underline{\alpha}]$, so that $\alpha \in \mathbb{R}$. Notice that, the *EUI* differs from the *classical unitary interval* (*CUI*), $e^I \triangleq [-1, +1]$, which follows the rules of the *CIA*, i.e.:

$$e^I - e^I = [-2,\ +2]\,;\ e^I \times e^I = [-1,\ +1]\,;$$
$$e^I / e^I \text{ does not exist because } 0 \in [-1, +1]\,;$$
$$x_i e^I \pm y_i e^I = [-x_i - y_i,\ x_i + y_i]\,;\quad x_i e^I \times\ y_i e^I = [-x_i y_i,\ x_i y_i]\,. \tag{A. 2a-e}$$

In the context of the *IIA,* following the philosophy of the *affine arithmetic* [11], the i-th interval variable α_i^I can be expressed in the so-called *affine form*, i.e.:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i \hat{e}_i^I\,, \tag{A.3}$$

where $\hat{e}_i^I$ is the *EUI* associated with α_i^I and

$$\alpha_{0,i} = \frac{1}{2}\left(\underline{\alpha}_i + \bar{\alpha}_i\right)\,;\quad \Delta\alpha_i = \frac{1}{2}\left(\bar{\alpha}_i - \underline{\alpha}_i\right) \tag{A.4}$$

denote the midpoint and the deviation amplitude of α_i^I, respectively.

Unlike the *CUI*, e^I, the *EUI*, $\hat{e}_i^I$, is associated with the i-th uncertain-but-bounded parameter. This allows us to keep track of the dependencies between interval variables throughout calculations, thus limiting the overestimation due to the *dependency phenomenon*. Furthermore, the use of the *EUI* enables to eliminate some physically inconsistent results of the *CIA* such as the subcancellation property (see Eq. (A.2a)).

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