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Lian-Zhi Yang · Yang Gao · Ernian Pan · Natalie Waksmanski

An exact closed-form solution for a multilayered one-dimensional orthorhombic quasicrystal plate

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Abstract By extending the pseudo-Stroh formalism to multilayered one-dimensional orthorhombic quasicrystal plates, we derive an exact closed-form solution for simply supported plates under surface loadings. The propagator matrix method is introduced to efficiently and accurately treat the multilayered cases. As a numerical example, a sandwich plate made of quasicrystals and crystals with two different stacking sequences is investigated. The displacement and stress fields for these two stacking sequences are presented, which clearly demonstrate the importance of the stacking sequences on the induced physical quantities. Our exact closed-form solution should be of particular interest to the design of one-dimensional quasicrystal laminated plates. The numerical results can be further used as benchmarks to various numerical methods, such as the finite element and finite difference methods, on the analysis of laminated composites made of one-dimensional quasicrystals.

1 Introduction

Since the discovery of icosahedral quasicrystals (QCs) in Al–Mn alloys in the early 1980s [1], great progress has been made in experimental and theoretical analyses of QCs [2,3]. As a new structure of solid matter, QCs have a long-range quasiperiodic translational order and a long-range orientational order [4]. Based on the quasiperiodic directions of QCs, there are three kinds of QCs which are classified as one-, two- and three-dimensional QCs [5]. A one-dimensional (1D) QC refers to a three-dimensional (3D) solid where its atomic arrangement is periodic in a plane and quasiperiodic in the direction normal to the plane. A two-dimensional (2D) QC is defined as a 3D body where its atomic arrangement is quasiperiodic in a plane and periodic in the direction normal to the plane. A 3D QC behaves in such a way that the atomic arrangement presents quasiperiodicity in all three directions. In the past few decades, 3D [1], 2D [6] and 1D [7] QCs with thermodynamical stability were successfully discovered. Due to their low friction coefficient, low adhesion, high wear resistance and low level of porosity, QCs have been increasingly investigated and utilized in industry. For instance, they can be used as coatings or films of metals [8], as strengthening phases to reinforce alloys [9], among others.

Soon after the discovery of QCs, Bak [10,11] and Levine et al. [12] developed the elastic energy theory of QCs based on the Landau–Lifshitz phenomenological theory of elementary excitation of condensed matter.

L.-Z. Yang
School of Civil and Environmental Engineering, University of Science and Technology Beijing, Beijing 100083, China

L.-Z. Yang · Y. Gao (✉)
College of Science, China Agricultural University, Beijing 100083, China
E-mail: gaoyangg@gmail.com
Tel.: +86-10-6273-6411

E. Pan · N. Waksmanski
Department of Civil Engineering, University of Akron, Akron, OH 44325-3905, USA

In the elastic theory of QCs, there exist two lower frequency excitations: phonon and phason. While phonons are related to translations of atoms (standard elasticity), phasons are related to rearrangements of atomic configurations. The introduction of the phason gives a macroscale description of the quasiperiodicity of QCs. Since QCs at room temperature are brittle solids [5], Ding et al. [13] summarized the generalized linear elasticity of QCs, which provides us with a fundamental theory to describe the elastic behavior of QCs based on the notion of a continuum model. The material constants involved in the constitutive relation for some QCs have been experimentally measured by various methods including X-ray diffraction and neutron scattering [14–16]. Recent reviews on the linear elasticity theory of QCs can be found in Refs. [5,17,18].

Due to the introduction of the phason field, the equations in QC elasticity are much more complicated than those in classical elasticity, and exact closed-form solutions are difficult to obtain in most cases. For 1D QCs, many efforts have been made involving defect problems, such as dislocation and crack problems in infinite space [19–23]. Based on the general solutions of 1D QCs [24,25], Gao et al. [26,27] solved exactly plane problems for both a QC beam and QC plate by introducing a refined theory. Recently, static and transient bending of 1D QC plates was studied by a mesh-free method [28], and an exact closed-form solution for a half-infinite plane crack in an infinite space of 1D hexagonal QC under thermal loading was derived by Li [29]. However, an exact closed-form solution for 3D static problems of 1D orthorhombic QCs in a finite domain has not been studied yet to the best of the authors' knowledge.

It is well known that the Stroh formalism [19,30–32] is powerful and convenient in dealing with plane crack and dislocation problems of QCs. The pseudo-Stroh formalism given by Pan [33] has been successfully used to obtain the solutions for the graded and multilayered plates [34,35]. In this paper, the powerful pseudo-Stroh formalism is extended to find the general solution for simply supported 1D orthorhombic QC plates of finite size. The propagator matrix method is then introduced to treat the corresponding multilayered case. In so doing, the final exact closed-form solution, derived for multilayered 1D QC plates under surface loadings, is concise and elegant. As numerical illustrations, a multilayered plate made of QCs and crystals with different stacking sequences under a surface loading on the top of the plate are investigated.

2 Problem description and basic equations

A 1D QC is defined as a 3D body where its atomic arrangement is quasiperiodic along x_3-direction and periodic in the $x_1 - x_2$ plane referred to a coordinate system (x_1, x_2, x_3). According to Wang et al. [36], there are thirty-one possible point groups in 1D QCs, which are divided into ten Laue classes and six systems, namely triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal systems. In this work, a 1D orthorhombic QC with the point group 2_h2_h2, $mm2$, 2_hmm_h, mmm_h is considered, and the solutions for this system can be used for other systems (such as tetragonal, trigonal and hexagonal systems) of 1D QCs by simply changing the elastic constant matrices.

We consider a simply supported multilayered 1D orthorhombic QC plate with horizontal dimensions $x_1 \times x_2 = L_1 \times L_2$ and a total thickness $x_3 = H$, as shown in Fig. 1. The origin of the coordinate system is at one of the four corners on the bottom surface such that the plate is in the positive x_3 region. We let j denote the jth layer of the layered plate. For layer j, its lower and upper interfaces are defined, respectively, as $x_3^{(j)}$ and $x_3^{(j+1)}$. Thus, for an N-layered plate, it is obvious that $x_3^{(1)} = 0$ and $x_3^{(N+1)} = H$.

The deformed state of the QCs requires a combined consideration of interrelated phonon and phason fields. In 1D QCs, a phason displacement field w_3 exists in addition to the phonon displacement u_i $(i = 1, 2, 3)$. According to the linear elastic theory of QCs [13], the strain-displacement relations for 1D QCs are given by

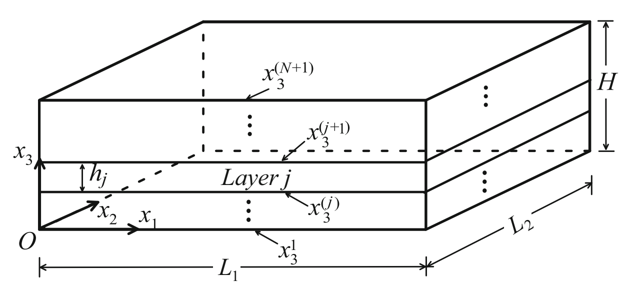

Fig. 1 Multilayered 1D orthorhombic QC plate

$$\varepsilon_{ij} = \left(\partial_j u_i + \partial_i u_j\right)/2, \quad w_{3j} = \partial_j w_3 \tag{1}$$

where $j = 1, 2, 3$, with repeated indices implying summation. Also in Eq. (1), $\partial_j = \partial/\partial x_j$, and ε_{ij} and w_{3j} denote the phonon and phason strains, respectively.

In the absence of body forces, the force balance laws require

$$\sigma_{ij,j} = 0, \;\; H_{3j,j} = 0 \tag{2}$$

where σ_{ij} and H_{3j} denote, respectively, the phonon and phason stresses. It should be noted that in equilibrium Eq. (2), the conservative component of the inner self-action shown in previous studies [37,38] in the phason field is not taken into account. This may cause non-physical results as evidenced by Colli and Mariano [39]. In the present work, however, the common assumptions without self-action in the phason field are accepted as reported in recent progresses for static problems [5,27,31] and dynamic problems [18,32].

For 1D orthorhombic QCs, the linear constitutive equations take the following form [5]:

$$\begin{aligned}
\sigma_{11} &= C_{11}\partial_1 u_1 + C_{12}\partial_2 u_2 + C_{13}\partial_3 u_3 + R_1\partial_3 w_3, \\
\sigma_{22} &= C_{12}\partial_1 u_1 + C_{22}\partial_2 u_2 + C_{23}\partial_3 u_3 + R_2\partial_3 w_3, \\
\sigma_{33} &= C_{13}\partial_1 u_1 + C_{23}\partial_2 u_2 + C_{33}\partial_3 u_3 + R_3\partial_3 w_3, \\
\sigma_{23} &= \sigma_{32} = C_{44}\left(\partial_3 u_2 + \partial_2 u_3\right) + R_5\partial_2 w_3, \\
\sigma_{31} &= \sigma_{13} = C_{55}\left(\partial_3 u_1 + \partial_1 u_3\right) + R_6\partial_1 w_3, \\
\sigma_{12} &= \sigma_{21} = C_{66}\left(\partial_2 u_1 + \partial_1 u_2\right), \\
H_{33} &= R_1\partial_1 u_1 + R_2\partial_2 u_2 + R_3\partial_3 u_3 + K_3\partial_3 w_3, \\
H_{32} &= R_5\left(\partial_3 u_2 + \partial_2 u_3\right) + K_2\partial_2 w_3, \\
H_{31} &= R_6\left(\partial_3 u_1 + \partial_1 u_3\right) + K_1\partial_1 w_3
\end{aligned} \tag{3}$$

where C_{ij}, C_{44}, C_{55}, C_{66} represent the elastic constants in the phonon field, K_1, K_2, K_3 are the elastic constants in the phason field, and R_1, R_2, R_3, R_5, R_6 are the phonon–phason coupling elastic constants.

For the multilayered QC plate, the simply supported displacement boundary conditions can be written as

$$\begin{aligned}
x_1 &= 0 \;\; \text{and} \;\; L_1 : u_2 = u_3 = w_3 = 0; \\
x_2 &= 0 \;\; \text{and} \;\; L_2 : u_1 = u_3 = w_3 = 0.
\end{aligned} \tag{4}$$

Along the interface of the layers, the displacements and traction forces are assumed to be continuous, i.e.,

$$\begin{cases} (u_i)_j = (u_i)_{j+1}, (w_3)_j = (w_3)_{j+1}, \\ (\sigma_{i3})_j = (\sigma_{i3})_{j+1}, (H_{33})_j = (H_{33})_{j+1}, \end{cases} \text{at the interface between layers} \;\; j \;\; \text{and} \;\; j+1. \tag{5}$$

3 General solutions for 1D orthorhombic QC plates

For a homogeneous 1D QC plate under simply supported lateral boundary conditions, the solution of the displacement vector can be assumed in the following form:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_3 \end{Bmatrix} = \sum_{p,q} \mathrm{e}^{sx_3} \begin{Bmatrix} a_1 \cos px_1 \sin qx_2 \\ a_2 \sin px_1 \cos qx_2 \\ a_3 \sin px_1 \sin qx_2 \\ a_4 \sin px_1 \sin qx_2 \end{Bmatrix} \tag{6}$$

where

$$p = n\pi/L_1, \quad q = m\pi/L_2, \tag{7}$$

with n and m being two positive integers. Also in Eq. (6), s and (a_1, a_2, a_3, a_4) are eigenvalue and the corresponding eigenvector to be determined. It can be seen that the displacement vector satisfies the simply supported displacement boundary conditions in Eq. (4). In general, summations for n and m over suitable ranges are implied whenever the sinusoidal term appears. To simplify the notation, we henceforth drop the summation over p and q (or over n and m).

Substitution of **u** from Eq. (6) in the constitutive Eq. (3) yields the following traction vector:

$$\mathbf{t} = \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \\ H_{33} \end{Bmatrix} = \mathrm{e}^{sx_3} \begin{Bmatrix} b_1 \cos px_1 \sin qx_2 \\ b_2 \sin px_1 \cos qx_2 \\ b_3 \sin px_1 \sin qx_2 \\ b_4 \sin px_1 \sin qx_2 \end{Bmatrix} \tag{8}$$

where

$$\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} = \left(\begin{bmatrix} 0 & 0 & C_{55}p & R_6 p \\ 0 & 0 & C_{44}q & R_5 q \\ -C_{13}p & -C_{23}q & 0 & 0 \\ -R_1 p & -R_2 q & 0 & 0 \end{bmatrix} + s \begin{bmatrix} C_{55} & 0 & 0 & 0 \\ 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{33} & R_3 \\ 0 & 0 & R_3 & K_3 \end{bmatrix} \right) \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}. \tag{9}$$

Introducing two vectors

$$\mathbf{a} = \{a_1, a_2, a_3, a_4\}^t, \quad \mathbf{b} = \{b_1, b_2, b_3, b_4\}^t, \tag{10}$$

for the coefficients in Eq. (9), it can be shown that the vector **b** is related to **a** by

$$\mathbf{b} = \left(-\mathbf{R}^t + s\mathbf{T}\right)\mathbf{a} \tag{11}$$

where the superscript t denotes matrix transpose, and

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & C_{13}p & R_1 p \\ 0 & 0 & C_{23}q & R_2 q \\ -C_{55}p & -C_{44}q & 0 & 0 \\ -R_6 p & -R_5 q & 0 & 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{55} & 0 & 0 & 0 \\ 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{33} & R_3 \\ 0 & 0 & R_3 & K_3 \end{bmatrix}. \tag{12}$$

Similarly, the other stress components in Eq. (3) are obtained as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \\ H_{31} \\ H_{32} \end{Bmatrix} = \mathrm{e}^{sx_3} \begin{Bmatrix} c_1 \sin px_1 \sin qx_2 \\ c_2 \cos px_1 \cos qx_2 \\ c_3 \sin px_1 \sin qx_2 \\ c_4 \cos px_1 \sin qx_2 \\ c_5 \sin px_1 \cos qx_2 \end{Bmatrix} \tag{13}$$

where

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{bmatrix} -C_{11}p & -C_{12}q & C_{13}s & R_1 s \\ C_{66}q & C_{66}p & 0 & 0 \\ -C_{12}p & -C_{22}q & C_{23}s & R_2 s \\ R_6 s & 0 & R_6 p & K_1 p \\ 0 & R_5 s & R_5 q & K_2 q \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}. \tag{14}$$

Substituting all stress components in Eqs. (8) and (13) in the equilibrium Eq. (2), we have

$$\begin{aligned} &\left(-C_{11}p^2a_1 - C_{66}q^2a_1 - C_{12}pqa_2 - C_{66}pqa_2\right) + s\left(C_{13}pa_3 + C_{55}pa_3 + R_1pa_4 + R_6pa_4\right) + s^2C_{55}a_1 = 0, \\ &\left(-C_{66}pqa_1 - C_{12}pqa_1 - C_{66}p^2a_2 - C_{22}q^2a_2\right) + s\left(C_{23}qa_3 + C_{44}qa_3 + R_2qa_4 + R_3qa_4\right) + s^2C_{44}a_2 = 0, \\ &\left(-C_{55}p^2a_3 - C_{44}q^2a_3 - R_6p^2a_4 - R_5q^2a_4\right) + s\left(-C_{55}pa_1 - C_{13}pa_1 - C_{44}qa_2 - C_{23}qa_2\right) \\ &\quad + s^2\left(C_{33}a_3 + R_3a_4\right) = 0, \\ &\left(-R_6p^2a_3 - R_5q^2a_3 - K_1p^2a_4 - K_2q^2a_4\right) + s\left(-R_6pa_1 - R_1pa_1 - R_5qa_2 - R_2qa_2\right) \\ &\quad + s^2\left(R_3a_3 + K_3a_4\right) = 0. \end{aligned} \tag{15}$$

In terms of vector **a**, Eq. (15) can be simplified as

$$\left[\mathbf{Q} + s\left(\mathbf{R} - \mathbf{R}^t\right) + s^2\mathbf{T}\right]\mathbf{a} = \mathbf{0} \tag{16}$$

where

$$\mathbf{Q} = \begin{bmatrix} -\left(C_{11}p^2 + C_{66}q^2\right) & -pq\left(C_{12}+C_{66}\right) & 0 & 0 \\ -pq\left(C_{12}+C_{66}\right) & -\left(C_{22}q^2 + C_{66}p^2\right) & 0 & 0 \\ 0 & 0 & -\left(C_{55}p^2 + C_{44}q^2\right) & -\left(R_6p^2 + R_5q^2\right) \\ 0 & 0 & -\left(R_6p^2 + R_5q^2\right) & -\left(K_1p^2 + K_2q^2\right) \end{bmatrix}. \tag{17}$$

Because Eq. (16) is similar to the Stroh formalism [40], this formalism was named as the pseudo-Stroh formalism [33]. From Eq. (16), it can be seen that eight eigenvalues s will form four opposite pairs rather than conjugate complex pairs in the original Stroh formalism.

Making use of Eqs. (11) and (16), we also obtain the following relation between vectors **a** and **b**:

$$\mathbf{b} = -\frac{1}{s}\left(\mathbf{Q} + s\mathbf{R}\right)\mathbf{a}. \tag{18}$$

Then using Eqs. (11) and (18), Eq. (16) can be recast into a 8×8 linear eigensystem

$$\mathbf{N}\boldsymbol{\xi} = s\boldsymbol{\xi}, \quad \boldsymbol{\xi} = \{\mathbf{a}, \mathbf{b}\}^t \tag{19}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{T}^{-1}\mathbf{R}^t & \mathbf{T}^{-1} \\ -\mathbf{Q} - \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^t & -\mathbf{R}\mathbf{T}^{-1} \end{bmatrix}. \tag{20}$$

A non-trivial solution for $\boldsymbol{\xi}$ exists if the determinant of the characteristic matrix in Eq. (19) vanishes. In other words,

$$\det\left(\mathbf{N} - s\mathbf{I}\right) = 0, \tag{21}$$

with $\mathbf{I}$ being the 4×4 unit matrix, yields eight eigenvalues s. The corresponding eigenvectors $\boldsymbol{\xi}$ are determined from Eq. (19). We point out that these eigenvalues may not be distinct. If repeated roots occur, a slight change in any material constant would result in distinct roots with negligible error in our calculations [41]. Therefore, only the case with distinct eigenvalues will be considered.

We assume that the first four eigenvalues have positive real parts (if the root is purely imaginary, we then pick up the one with positive imaginary part) and the other four have opposite signs to the first four, as do their associated eigenvectors **a** and **b**. That is,

$$s_{\alpha+4} = -s_\alpha,\ Re(s_\alpha) > 0 \quad (\alpha = 1, 2, 3, 4), \tag{22}$$

$$\mathbf{a}_{\alpha+4} = -\mathbf{a}_\alpha,\ \mathbf{b}_{\alpha+4} = -\mathbf{b}_\alpha \tag{23}$$

where Re represents the real part of the quantity, then the general solution for the displacement vector in Eq. (6) and traction vector in Eq. (8) is derived as

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \left\langle e^{s^* x_3} \right\rangle \begin{Bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{Bmatrix} \tag{24}$$

where

$$\begin{aligned} \mathbf{A}_1 &= [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4],\ \mathbf{A}_2 = [\mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8], \\ \mathbf{B}_1 &= [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4],\ \mathbf{B}_2 = [\mathbf{b}_5, \mathbf{b}_6, \mathbf{b}_7, \mathbf{b}_8], \\ \left\langle e^{s^* z} \right\rangle &= \mathrm{diag}\left[e^{s_1x_3}, e^{s_2x_3}, e^{s_3x_3}, e^{s_4x_3}, e^{-s_1x_3}, e^{-s_2x_3}, e^{-s_3x_3}, e^{-s_4x_3}\right], \end{aligned} \tag{25}$$

and $\mathbf{K}_1$ and $\mathbf{K}_2$ are two 4×1 constant column matrices to be determined.

Equation (24) is a general solution for a homogeneous and simply supported plate made of 1D orthorhombic QC. It should be noted that certain thin plate results can also be deduced from this solution by expanding the exponential term in terms of a Taylor series [42].

It is imperative to note that crystals can be seen as special QCs with all phason-field physical quantities being zero. From Eq. (3), it can be seen that if we set $R_1 \to 0$, $K_1 = K_2 = K_4 \to 0$, then $H_{33} \to 0$. It can be inferred that for multilayered plates containing both QC layers and crystal layers, the phonon stresses and strains of the QC are infinitely close to those in the corresponding purely elastic crystal. Therefore, the general solution in Eq. (24) can be used for the purely elastic crystal simply supported plates by regarding a crystal layer as “a special QC” layer with the phason-field elastic constants approaching zero. Thus, we regard a crystal layer as a “special QC” layer where the coupling constants are zero and the phason elastic constants are relatively very small (compared to the corresponding phonon elastic constants).

For a multilayered plate containing both 1D QC layers and crystal layers, the interface boundary condition in phason field satisfies [43]

$$H_{33} = 0. \tag{26}$$

By processing crystal layers as the “special QC” layers, the interface boundary condition in Eq. (26) can be very closely approximated. That is, the continuity conditions for x_3-direction phason traction forces along the interfaces in Eq. (5) can be satisfied. Therefore, by virtue of the general solution in Eq. (24), the continuity conditions along the interfaces in Eq. (5) and the boundary conditions on the top and bottom surfaces, the phonon physical quantities and phason stresses can be accurately obtained for the multilayered plate containing both QC layers and crystal layers. For QC layers, all phonon and phason stresses and displacements can be exactly obtained under the boundary condition in Eq. (4). For crystal layers, there is no physical meaning at all for phason displacement, and thus, one can simply set it to zero [35].

In summary, the general solution in Eq. (24) along with the boundary/interface conditions can be used to solve the problems of multilayered 1D QC and crystal plates. To deal with a multilayered structure with relatively large numbers of layers, the propagator matrix method can be employed instead [44]. In the next section, this method is introduced to simplify the solution procedure involving multilayered 1D QC plates.

4 Solutions of the layered system

From Eq. (24), it can be seen that the constant column matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ for layer j can be solved as follows:

$$\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}_j = \left\langle \mathrm{e}^{s^*(x_3 - x_3^{(j)})} \right\rangle^{-1} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3} \tag{27}$$

where the subscript j indicates layer j and s^* are the eigenvalues of layer j, and $x_3^{(j)} \le x_3 \le x_3^{(j+1)}$. Letting x_3 be $x_3^{(j)}$ and $x_3^{(j+1)}$, we then have

$$\begin{Bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{Bmatrix}_j = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = x_3^{(j)}} = \left\langle \mathrm{e}^{s^* h_j} \right\rangle^{-1} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = x_3^{(j+1)}} \tag{28}$$

where h_j is the thickness of layer j. From Eq. (28), the displacement $\mathbf{u}$ and traction $\mathbf{t}$ on the upper surface $x_3 = x_3^{(j+1)}$ can be expressed in terms of those on the lower surface $x_3 = x_3^{(j)}$ of layer j as

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = x_3^{(j+1)}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \left\langle \mathrm{e}^{s^* h_j} \right\rangle \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = x_3^{(j)}}. \tag{29}$$

Assuming that the displacement $\mathbf{u}$ and traction $\mathbf{t}$ are continuous across the interfaces, Eq. (29) can be applied repeatedly so that one can propagate the physical quantities from the bottom surface $x_3 = 0$ to the top surface $x_3 = H$ of the multilayered 1D QC plate. Therefore, we have

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = H} = \mathbf{P}_N(h_N)\mathbf{P}_{N-1}(h_{N-1}) \ldots \mathbf{P}_2(h_2)\mathbf{P}_1(h_1) \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3 = 0} \tag{30}$$

where

$$\mathbf{P}_j(h_j) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \left\langle \mathrm{e}^{s^* h_j} \right\rangle \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1}, (j = 1, 2, \ldots N) \tag{31}$$

is defined as the propagating matrix or propagator of layer j.

The most remarkable feature in the pseudo-Stroh formalism is that calculation of the inverse matrix in Eq. (31) can be avoided using the following simple relation [33]:

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{B}_2^t & \mathbf{A}_2^t \\ \mathbf{B}_1^t & -\mathbf{A}_1^t \end{bmatrix} \tag{32}$$

where the matrices $\mathbf{A}_l$ and $\mathbf{B}_l (l = 1, 2)$ are normalized according to

$$-\mathbf{B}_2^t\mathbf{A}_1 + \mathbf{A}_2^t\mathbf{B}_1 = \mathbf{I}. \tag{33}$$

Equation (30) is a very simple matrix propagation relation, and for given boundary conditions the unknowns involved can be directly solved. Boundary conditions prescribed on the top and bottom surfaces can be either displacement components u_i and w_3 or traction components σ_{i3} and H_{33}, or a suitable linear combination of both. Since the normal loading (such as uniform or point loading) can be expanded as a double Fourier series in x_1 and x_2, it is sufficient to consider only one term in the double Fourier series. As an example, we assume that an x_3-direction traction component is applied on the top surface of the plate as

$$\sigma_{13} = \sigma_{23} = 0, \quad \sigma_{33} = \sigma_0 \sin px_1 \sin qx_2, H_{33} = 0 \tag{34}$$

where σ_0 is the amplitude of the loading. All other traction components on the top and bottom surfaces are assumed to be zero.

For loadings given by Eq. (34), it is reasonable to assume that only one term in the series expansion of Eq. (6) will be non-vanishing. Thus, for this one-term solution, Eq. (30) is simplified to

$$\begin{pmatrix} \mathbf{u}(H) \\ \mathbf{t}(H) \end{pmatrix}_{x_3=H} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \mathbf{0} \end{pmatrix} \tag{35}$$

where $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$ are the multiplications of the propagator matrices in Eq. (30), and $\mathbf{t}(H)$ is the given traction on the boundary of the top surface, i.e.,

$$\mathbf{t}(H) = \{0, 0, \sigma_0 \sin px_1 \sin qx_2, 0\}^t . \tag{36}$$

Substituting Eq. (36) in Eq. (35), the unknown displacements at the bottom and top surfaces can be obtained as

$$\mathbf{u}(0) = \mathbf{C}_3^{-1}\mathbf{t}(H), \;\; \mathbf{u}(H) = \mathbf{C}_1\mathbf{C}_3^{-1}\mathbf{t}(H). \tag{37}$$

Thus, for the displacement and traction vectors at any depth $x_3^{(j)} \leq x_3 \leq x_3^{(j+1)}$, the solution is

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3} = \mathbf{P}_j(x_3 - x_3^{(j-1)})\mathbf{P}_{j-1}(h_{j-1}) \ldots \mathbf{P}_2(h_2)\mathbf{P}_1(h_1) \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{x_3=0} . \tag{38}$$

With the solved displacement and traction vectors at any given depth, the corresponding stress vector in Eq. (13) can be evaluated.

Similar exact closed-form solutions for other boundary conditions can also be simply obtained. Therefore, for a multilayered 1D orthorhombic QC rectangular plate, we have derived the exact closed-form solution based on the pseudo-Stroh formalism and the propagator matrix method. In the next section, we apply our solution to investigate the response of a sandwiched QC plate under surface loading.

5 Numerical studies

Here we consider a square sandwich plate made of 1D orthorhombic QC Al–Ni–Co and crystal $BaTiO_3$ with $L_1 = L_2 = 1$ m and $H = 0.3$ m. The three layers have equal thicknesses of 0.1 m. According to the material coefficients of QCs shown by Fan [18] and Sladek et al. [28], the material properties for Al–Ni–Co are given by

$$C_{11} = C_{22} = 23.433 \times 10^{10}\,\text{N/m}^2,\ C_{12} = 5.741 \times 10^{10}\,\text{N/m}^2,\ C_{13} = C_{23} = 6.663 \times 10^{10}\,\text{N/m}^2,$$
$$C_{33} = 23.222 \times 10^{10}\,\text{N/m}^2,\ C_{44} = C_{55} = 7.019 \times 10^{10}\,\text{N/m}^2,\ C_{66} = (C_{11} - C_{12})/2 = 8.846 \times 10^{10}\,\text{N/m}^2,$$
$$R_1 = R_2 = R_3 = R_5 = R_6 = 8.846 \times 10^9\,\text{N/m}^2,\ K_1 = K_2 = 12.2 \times 10^{10}\,N/m^2,\ K_3 = 2.4 \times 10^{10}\,\text{N/m}^2.$$

The material coefficients for crystal $BaTiO_3$ are obtained by Lee and Jiang [45] as

$$C_{11} = C_{22} = 16.6 \times 10^{10}\,\text{N/m}^2,\ C_{12} = 7.7 \times 10^{10}\,\text{N/m}^2,\ C_{13} = C_{23} = 7.8 \times 10^{10}\,\text{N/m}^2,\ C_{33} = 16.2 \times 10^{10}\,\text{N/m}^2,$$
$$C_{44} = C_{55} = 4.3 \times 10^{10}\,\text{N/m}^2,\ C_{66} = (C_{11} - C_{12})/2 = 4.45 \times 10^{10}\,\text{N/m}^2,$$
$$R_1 = R_2 = R_3 = R_5 = R_6 = 0,\ K_1 = K_2 = K_3 = 0.$$

It should be noted that in a crystal $BaTiO_3$ layer, a very small value for K_i $(i = 1, 2, 3)$ is assumed during the calculation so that the system matrices of each layer have the same dimension (about 10^{-10} of the corresponding K_i value in QC layer) to ensure that the system matrices are not singular.

Two stacking sequences, Al–Ni–Co/$BaTiO_3$/Al–Ni–Co (called QC/C/QC) and $BaTiO_3$/Al–Ni–Co/$BaTiO_3$ (called C/QC/C) of the layered plate are investigated. On the top surface of the layered plate ($x_3 = 0.3$ m), an x_3-direction traction is applied by Eq. (36) with $n = m = 1$ and amplitude $\sigma_0 = 1\,\text{N/m}^2$, while on the top and bottom surfaces all other traction components are zero; that is, $\sigma_{13} = \sigma_{23} = H_{33} = 0$. To show the stress and displacement responses of the plate in the thickness direction under the top surface loading, we choose the vertical line (x_3 from 0 to 0.3m) with fixed horizontal coordinates at $(x_1, x_2) = (0.75L_1, 0.75L_2)$.

Figures 2 and 3 show, respectively, the variations of the stress components in the phonon and phason fields along the x_3-direction in the sandwich plate. From Figs. 2d and 3b, it can be seen that the values of the stress

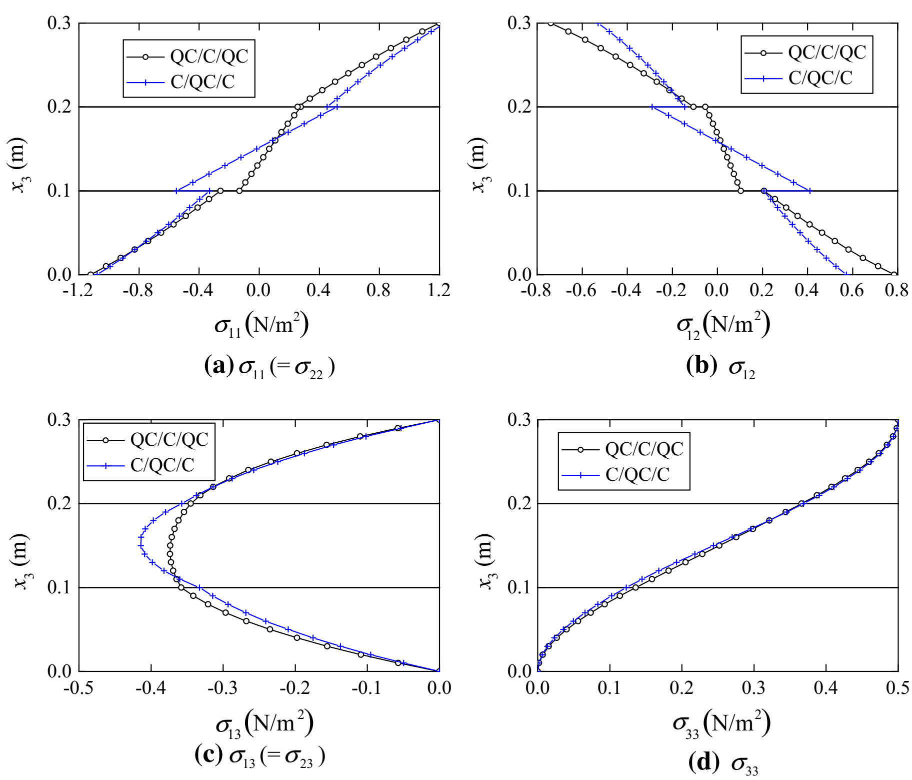

Fig. 2 Variation of the stress components in the phonon field along the thickness direction of the plate

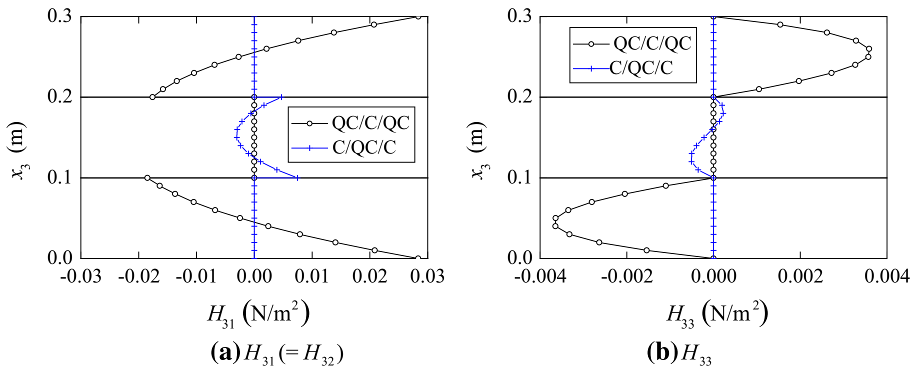

Fig. 3 Variation of the stress components in the phason field along the thickness direction of the plate

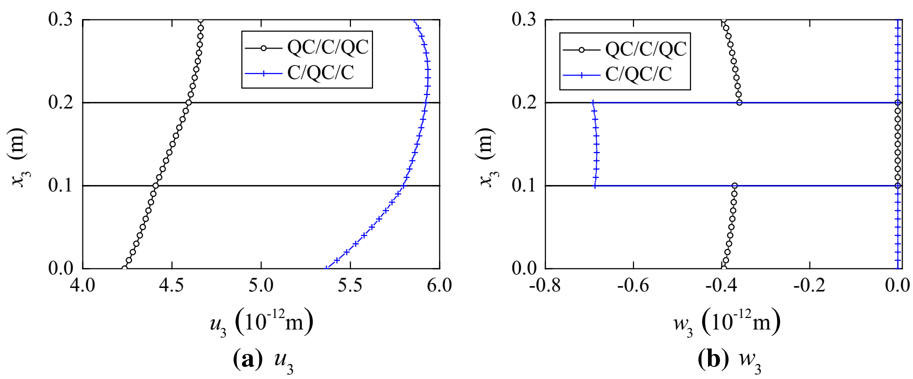

Fig. 4 Variation of displacement components u_3 and w_3 along the thickness direction of the plate

components in the x_3-direction on the top and bottom surfaces satisfy the traction boundary conditions, which also partially verifies the correctness of the derived solution. It is clear that the top surface loading produces quite different responses in these two sandwich structures, which clearly demonstrates the role played by the material stacking sequences. Figures 2 and 3 also show that the stress components in Eq. (13) are discontinuous across the interfaces and are nonzero on the bottom and top surfaces, while the traction components in Eq. (8) are continuous across the interfaces. These stress components are approximately either symmetric or antisymmetric about the middle plane. Figure 4 shows the variation of the displacement components u_3 and w_3 along the x_3-direction. It is clear that the displacement u_3 is continuous, while the displacement w_3 is not.

The model results may have potential applications in the field of laminated structures made of 1D QCs and crystals. For example, to design a QC/C/QC sandwich plate, the stress level (or distribution) within the plate under a normal surface loading on the top surface is required.

6 Conclusions

Utilizing the powerful pseudo-Stroh formalism, we have derived an exact closed-form solution for a simply supported and multilayered 1D orthorhombic plate under surface loading. The propagator matrix method is also introduced in order to treat efficiently and accurately the multilayered cases. Thus, our solution can be applied to 1D QCs with arbitrary material property layering in the layered plate.

The typical sandwich plate made of 1D QC Al–Ni–Co and crystal $BaTiO_3$ with two different stacking sequences is investigated. It is observed that the stacking sequences can substantially influence all physical quantities, which should be of interest to the design of the 1D QC laminated plates. The results can also be employed to verify the accuracy of the solutions by numerical methods, such as the finite element and difference methods, when analyzing laminated composites made of QCs.

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