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ORIGINAL PAPER

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Two-dimensional generalized thermoelastic diffusion in a half-space under axisymmetric distributions

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Abstract A two-dimensional problem for an infinite thermoelastic half-space with a permeating substance in contact with the bounding plane is developed. The formulation is applied to the generalized thermoelastic diffusion based on Lord–Shulman theory. The bounding surface is traction free and subjected to a known axisymmetric temperature distribution, and the chemical potential is assumed to be a known function of time. Integral transform technique is used to find the analytic solution in the transform domain by using a direct approach. Inversion of transforms is done employing a numerical scheme. The mathematical model is prepared for copper material, and numerical results for temperature, stress, displacement, chemical potential and concentration are obtained and illustrated graphically.

1 Introduction

For the last few decades, generalized problems of dynamic thermoelasticity have been the crux of active research in the fields of engineering where the coupling between temperature and strain fields is of importance, and various studies have been carried out. The theories of generalized thermoelasticity remove the paradox of infinite speed of propagation of thermal signals inherent in the classical coupled thermoelasticity introduced by Biot [1]. Lord and Shulman [2] developed a theory modifying the Fourier law of heat conduction by introducing the heat flux rate and a relaxation time for the special case of an isotropic body. The heat equation associated with this theory is of wave type. This theory comes under the label of hyperbolic thermoelasticity, see the survey of Hetnarski and Ignaczak [3].

Diffusion is one of the several naturally occurring transport phenomena. The diffusion in thermoelastic solids is governed by Fick's laws where diffusion is the passive movement of molecules or particles along a concentration gradient. Thermoelastic diffusion in an elastic solid involves the coupling of the fields of temperature, mass diffusion and strain. Heat and mass exchange take place during thermoelastic diffusion in an elastic solid. The diffusion phenomenon has generated a great amount of interest due to its many applications in geophysics and industrial applications. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, form the source/drain regions in metal oxide semiconductor (MOS) transistors and dope poly-silicon gates in MOS transistors. The study of the phenomenon of diffusion is used

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to improve the conditions of oil extractions and is of great interest for oil extraction companies. Nowacki [4–7] developed the theory of thermoelastic diffusion within the context of classical coupled thermoelasticity and studied some dynamical problems of diffusion in solids. The theory of Nowacki uses Fick's law. It has a major drawback that it predicts infinite speed of wave propagation. Olesiak and Pyryev [8] reported the influence of cross effects while studying thermoelastic diffusion in an elastic cylinder. They observed that the thermal excitations result in an additional mass concentration and vice versa. Recently, Sherief et al. [9] introduced the theory of thermoelastic diffusion in the framework of Lord–Shulman theory by introducing a thermal relaxation time parameter and diffusion relaxation parameters governing the field equations. Sherief and Saleh [10] and El-Maghraby [11] studied problems of thermoelastic diffusion in a half-space. Aouadi [12] discussed thermoelastic diffusion in anisotropic media. Many researchers [13–16] studied various types of problems in thermoelastic diffusion. Kothari and Mukhopadhyay [17] presented the Galerkin-type representation of solutions for thermoelastic diffusion theory. Elhagary [18] discussed a two-dimensional generalized thermoelastic diffusion problem for a half-space subjected to harmonically varying heating. Recently, El-Sayed [19] studied a two-dimensional problem in generalized thermoelastic diffusion for a half-space under thermal shock.

This work is aimed at studying the thermoelastic diffusion interactions in a half-space under axisymmetric distributions within the context of Lord–Shulman theory of generalized thermoelastic diffusion (TEDLS). The classical coupled thermoelastic diffusion theory (TEDCT) is recovered as a special case. Analytic solutions for temperature, concentration, chemical potential, displacement and stresses are obtained in the Laplace transform domain using a direct approach. Numerical inversion of Laplace transform is performed using the Gaver–Stehfast algorithm [20–22], which is considerably more stable and computationally efficient than inversion using the discrete Fourier transform [23,24]. Convergence of the Gaver–Stehfast algorithm was discussed by Kuznetsov [25]. All the integrals were evaluated using Romberg's integration technique [26] with variable step size. A mathematical model is prepared for copper material, and results are discussed along with the graphical representation.

2 Governing equations

The field equations for the generalized thermoelastic diffusion in an isotropic medium in the absence of body forces and heat source are given by [9]:

1. The equation of motion is given by

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \beta_1 T,_i - \beta_2 C,_i \tag{1}$$

where T is the absolute temperature, C is the concentration of the diffusive material, ρ is the density, λ and μ are Lamé's constants, and β_1 and β_2 are material constants given by $\beta_1 = (3\lambda + 2\mu)\,\alpha_t$ and $\beta_2 = (3\lambda + 2\mu)\,\alpha_c$, where α_t is the coefficient of linear thermal expansion, and α_c is the coefficient of linear diffusion equation.

2. The energy equation is given by

$$\mathrm{kT_{,ii}} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho \mathrm{C_E T} + T_0 \beta_1 e + T_0 a C) \tag{2}$$

where k is the thermal conductivity of the medium, C_E is the specific heat at constant strain, τ_0 is the thermal relaxation time, T_0 is the reference temperature chosen such that $|(T - T_0)/T_0| << 1$, a is the measure of thermoelastic diffusion effect, and $e = u_{i,i}$ is the cubical dilatation where u_i, $i = 1, 2, 3$ are the components of the displacement vector.

3. The equation of mass diffusion is given by

$$D\beta_2 e,_{ii} + DaT,_{ii} + \left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}\right) C = DbC,_{ii} \tag{3}$$

where D is the diffusion coefficient, b is a measure of the diffusive effect and τ is the diffusion relaxation time.

4. The constitutive equations are

$$\sigma_{\mathrm{ij}} = 2\mu e_{ij} + \delta_{ij}\,(\lambda e - \beta_1\,(T - T_0) - \beta_2 C)\,, \tag{4}$$

$$P = -\beta_2 e + bC - a(T - T_0) \tag{5}$$

where σ_{ij}, $i, j = 1, 2, 3$ are the components of stress tensor, P is the chemical potential of the material per unit mass, and e_{ij}, $i, j = 1, 2, 3$ are the components of the strain tensor, given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad i, j = 1, 2, 3. \tag{6}$$

3 Formulation of the problem

We shall consider a homogeneous isotropic thermoelastic solid occupying the region $z \geq 0$. The z-axis is taken perpendicular to the bounding plane pointing inward. The problem is considered within the context of the theory of generalized thermoelastic diffusion with one relaxation time. We shall assume that the initial state of the medium is quiescent at a temperature T_0. The surface of the medium is traction free and subjected to a known axisymmetric temperature distribution, and chemical potential is a known function of time. Axisymmetric heat sources permeate the medium. Cylindrical polar coordinates (r, φ, z) are used.

The problem is thus two-dimensional with all functions considered depending on the spatial variables r and z as well as on the time variable t.

The displacement vector, thus, has the form $\vec{u} = (u, 0, w)$.

For the two-dimensional problem, the components of the strain tensor in Eq. (6) can be written in the form

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\varphi\varphi} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \tag{7}$$

and e is the cubical dilatation given by

$$e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}.$$

The Laplacian operator is given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

In addition, the governing Eqs. (1)–(5) will take the form

$$\mu\nabla^2 u - \frac{\mu}{r^2}u + (\lambda + \mu)\frac{\partial e}{\partial r} - \beta_1\frac{\partial T}{\partial r} - \beta_2\frac{\partial C}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{8}$$

$$\mu\nabla^2 w + (\lambda + \mu)\frac{\partial e}{\partial z} - \beta_1\frac{\partial T}{\partial z} - \beta_2\frac{\partial C}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}, \tag{9}$$

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \beta_1 T_0 div\vec{u} + aT_0 C\right), \tag{10}$$

$$D\beta_2\nabla^2(div\vec{u}) + Da\nabla^2 T + \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)C - Db\nabla^2 C = 0, \tag{11}$$

$$\sigma_{\varphi\varphi} = 2\mu e_{\phi\phi} + \lambda e - \beta_1(T - T_0) - \beta_2 C, \tag{12.1}$$

$$\sigma_{rr} = 2\mu e_{rr} + \lambda e - \beta_1(T - T_0) - \beta_2 C, \tag{12.2}$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda e - \beta_1(T - T_0) - \beta_2 C, \tag{12.3}$$

$$\sigma_{rz} = \mu e_{rz}, \tag{12.4}$$

$$\sigma_{r\varphi} = \sigma_{z\varphi} = 0, \tag{12.5}$$

$$P = -\beta_2 e + bC - a(T - T_0). \tag{13}$$

We shall use the following nondimensional variables:

$$\mathrm{r}' = \mathrm{c}_1\eta\mathrm{r}, \quad \mathrm{z}' = \mathrm{c}_1\eta\mathrm{z}, \quad \mathrm{u}' = \mathrm{c}_1\eta\mathrm{u}, \quad \mathrm{w}' = \mathrm{c}_1\eta\mathrm{w}, \quad \tau_0' = \mathrm{c}_1^2\eta\tau_0, \quad \mathrm{t}' = \mathrm{c}_1^2\eta\mathrm{t}, \quad \tau_1' = \mathrm{c}_1^2\eta\tau,$$
$$\mathrm{P}' = \frac{P}{\beta_2}, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{\lambda + 2\mu}, \quad \theta' = \frac{\beta_1(T - T_0)}{(\lambda + 2\mu)}, \quad C' = \frac{\beta_2 C}{\lambda + 2\mu}$$

where $\eta = \frac{\rho \mathrm{C_E}}{k}$ is the dimensionless characteristic length, and $c_1 = \sqrt{\lambda + 2\mu/\rho}$ is the speed of propagation of isothermal elastic waves.

Using the above nondimensional variables, the governing Eqs. (8)–(13) take the form (dropping the primes for convenience)

$$\nabla^2 u - \frac{1}{r^2}u + \left(\beta^2 - 1\right)\frac{\partial e}{\partial r} - \beta^2\frac{\partial T}{\partial r} - \beta^2\frac{\partial C}{\partial r} = \beta^2\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

$$\nabla^2 w + \left(\beta^2 - 1\right)\frac{\partial e}{\partial z} - \beta^2\frac{\partial T}{\partial z} - \beta^2\frac{\partial C}{\partial z} = \beta^2\frac{\partial^2 w}{\partial t^2}, \tag{15}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)(\theta + \varepsilon e + \varepsilon\alpha_1 C), \tag{16}$$

$$\nabla^2 e + \alpha_1\nabla^2\theta + \alpha_2\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)C - \alpha_3\nabla^2 C = 0, \tag{17}$$

$$\sigma_{\varphi\varphi} = \frac{2}{\beta^2}\frac{u}{r} + \frac{\left(\beta^2 - 2\right)}{\beta^2}e - \theta - C, \tag{18.1}$$

$$\sigma_{rr} = \frac{2}{\beta^2}\frac{\partial u}{\partial r} + \frac{\left(\beta^2 - 2\right)}{\beta^2}e - \theta - C, \tag{18.2}$$

$$\sigma_{zz} = \frac{2}{\beta^2}\frac{\partial w}{\partial z} + \frac{\left(\beta^2 - 2\right)}{\beta^2}e - \theta - C, \tag{18.3}$$

$$\sigma_{rz} = \frac{1}{\beta^2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \tag{18.4}$$

$$P = -e + \alpha_3 C - \alpha_1\theta \tag{19}$$

where $\alpha_1 = \frac{a(\lambda+2\mu)}{\beta_1\beta_2}$, $\alpha_2 = \frac{(\lambda+2\mu)}{D\eta\beta_2^2}$, $\alpha_3 = \frac{b(\lambda+2\mu)}{\beta_2^2}$, $\varepsilon = \frac{\beta_1^2 T_0}{\rho C_E(\lambda+2\mu)}$.

The boundary conditions of the problem at $z = 0$ are taken as

$$\theta(r, 0, t) = f_1(r, t), \quad 0 < r < \infty, \tag{20}$$

$$\sigma_{zz}(r, 0, t) = 0, \quad 0 < r < \infty, \tag{21}$$

$$\sigma_{rz}(r, 0, t) = 0, \quad 0 < r < \infty, \tag{22}$$

$$P(r, 0, t) = H(t) f_2(r), \quad 0 < r < \infty \tag{23}$$

where $f_1(r, t)$and$f_2(r)$ are known functions and $H(t)$ is the Heaviside unit step function.

4 Analytical solution

We use the Laplace transform defined by the relation

$$\bar{f}(r, z, s) = L\left[f(r, z, t)\right] = \int_0^\infty e^{-st} f(r, z, t)dt$$

where s is the Laplace transform parameter.

The Hankel transform of order zero with respect to r of a function $\bar{f}(r, z, s)$ is defined by the relation

$$\bar{f}^*(\alpha, z, s) = H\left[\bar{f}(r, z, s)\right] = \int_0^\infty \bar{f}(r, z, s) r J_0(\alpha r) dr$$

where α is the Hankel transform parameter and J_0 is the Bessel's function of the first kind of order zero.

The inverse Hankel transform is given by the relation

$$\bar{f}(r, z, s) = H^{-1}\left[\bar{f}^*(\alpha, z, s)\right] = \int_0^\infty \bar{f}^*(\alpha, z, s)\alpha J_0(\alpha r) d\alpha.$$

Applying the Laplace transform to Eqs. (14)–(19) and using the homogeneous initial conditions, we get

$$\nabla^2\bar{u} - \frac{1}{r^2}\bar{u} + \left(\beta^2 - 1\right)\frac{\partial\bar{e}}{\partial r} - \beta^2\frac{\partial\bar{\theta}}{\partial r} - \beta^2\frac{\partial\bar{C}}{\partial r} = \beta^2 s^2\bar{u}, \tag{24}$$

$$\nabla^2\bar{w} + \left(\beta^2 - 1\right)\frac{\partial\bar{e}}{\partial z} - \beta^2\frac{\partial\bar{\theta}}{\partial z} - \beta^2\frac{\partial\bar{C}}{\partial z} = \beta^2 s^2\bar{w}, \tag{25}$$

$$\nabla^2\bar{\theta} = \left(s + \tau_0 s^2\right)\left(\bar{\theta} + \varepsilon\bar{e} + \varepsilon\alpha_1\bar{C}\right), \tag{26}$$

$$\nabla^2\bar{e} + \alpha_1\nabla^2\bar{\theta} + \alpha_2\left(s + \tau_0 s^2\right)\bar{C} - \alpha_3\nabla^2\bar{C} = 0, \tag{27}$$

$$\bar{\sigma}_{\varphi\varphi} = \frac{2}{\beta^2}\frac{\bar{u}}{r} + \frac{\left(\beta^2 - 2\right)}{\beta^2}\bar{e} - \bar{\theta} - \bar{C}, \tag{28.1}$$

$$\bar{\sigma}_{rr} = \frac{2}{\beta^2}\frac{\partial\bar{u}}{\partial r} + \frac{\left(\beta^2 - 2\right)}{\beta^2}\bar{e} - \bar{\theta} - \bar{C}, \tag{28.2}$$

$$\bar{\sigma}_{zz} = \frac{2}{\beta^2}\frac{\partial\bar{w}}{\partial z} + \frac{\left(\beta^2 - 2\right)}{\beta^2}\bar{e} - \bar{\theta} - \bar{C}, \tag{28.3}$$

$$\bar{\sigma}_{rz} = \frac{1}{\beta^2}\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial r}\right), \tag{28.4}$$

$$\bar{P} = -\bar{e} + \alpha_3\bar{C} - \alpha_1\bar{\theta}. \tag{29}$$

On simplifying Eqs. (24) and (25), we get

$$\left(\nabla^2 - s^2\right)\bar{e} = \nabla^2\bar{\theta} + \nabla^2\bar{C}. \tag{30}$$

On applying the Hankel transform to Eqs. (26), (27) and (30) we get

$$\left(D^2 - \alpha^2\right)\bar{\theta}^* = \left(s + \tau_0 s^2\right)\left(\bar{\theta}^* + \varepsilon\bar{e}^* + \varepsilon\alpha_1\bar{C}^*\right), \tag{31}$$

$$\left(D^2 - \alpha^2\right)\bar{e}^* + \alpha_1\left(D^2 - \alpha^2\right)\bar{\theta}^* + \alpha_2\left(s + \tau_0 s^2\right)\bar{C}^* - \alpha_3\left(D^2 - \alpha^2\right)\bar{C}^* = 0, \tag{32}$$

$$\left(D^2 - \alpha^2 - s^2\right)\bar{e}^* = \left(D^2 - \alpha^2\right)\bar{\theta}^* + \left(D^2 - \alpha^2\right)\bar{C}^* \tag{33}$$

where $D \equiv \partial/\partial z$.

Eliminating the transformed strain $\bar{e}^*$ and concentration $\bar{C}^*$ from Eqs. (31), (32) and (33), we obtain the following sixth-order differential equation for the transformed temperature $\bar{\theta}^*$:

$$\left(D^6 - a_1 D^4 + a_2 D^2 - a_3\right)\bar{\theta}^* = 0 \tag{34}$$

where the coefficients a_1, a_2, a_3 are given by

$$a_1 = -\frac{s}{(\alpha_3 - 1)}\left\{(s\tau_0 + 1)\left(\alpha_1\varepsilon\left(\alpha_1 + 2\right) + \alpha_3\left(\varepsilon + 1\right) - 1\right) + \alpha_2\left(s\tau + 1\right) + \alpha_3 s\right\},$$

$$a_2 = \frac{s^2}{(\alpha_3 - 1)}\left\{(s\tau_0 + 1)\left(\alpha_1^2\varepsilon s + \alpha_3 s + \alpha_2\left(s\tau + 1\right)\left(\varepsilon + 1\right)\right) + \alpha_2 s\left(s\tau + 1\right)\right\},$$

$$a_3 = \frac{s^4\alpha_2}{(\alpha_3 - 1)}\left(s\tau_0 + 1\right)\left(s\tau + 1\right).$$

Similarly, we can show that the transformed strain $\bar{e}^*$ and concentration $\bar{C}^*$ satisfy the ordinary differential equations

$$\left(D^6 - a_1 D^4 + a_2 D^2 - a_3\right)\bar{e}^* = 0, \tag{35}$$

$$\left(D^6 - a_1 D^4 + a_2 D^2 - a_3\right)\bar{C}^* = 0. \tag{36}$$

Equation (34) can also be written as

$$\left(D^2 - k_1^2\right)\left(D^2 - k_2^2\right)\left(D^2 - k_3^2\right)\bar{\theta}^* = 0 \tag{37}$$

where $\pm k_1$, $\pm k_2$ and $\pm k_3$ are the roots of the characteristic equation given by

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0. \tag{38}$$

The roots k_1, k_2 and k_3 are given by

$$k_1 = \sqrt{\frac{1}{3}\left(2p_1 \sin(p_2) + a_1\right)},$$
$$k_2 = \sqrt{\frac{1}{3}\left[a_1 - p_1\left(\sqrt{3}\cos(p_2) + \sin(p_2)\right)\right]},$$
$$k_3 = \sqrt{\frac{1}{3}\left[a_1 + p_1\left(\sqrt{3}\cos(p_2) - \sin(p_2)\right)\right]}$$

where $p_1 = \sqrt{a_1^2 - 3a_2}$, $\quad p_2 = \frac{\sin^{-1}(\gamma)}{3}$, and $\gamma = -\left(\frac{2a_1^3 - 9a_1a_2 + 27a_3}{2p^3}\right)$

The solution of Eq. (37) is of the form

$$\bar{\theta}^* = \sum_{i=1}^{3} A_i(\alpha, s) e^{-q_i z} \tag{39}$$

where $A_i(\alpha, s)$, $i = 1, 2, 3$ are parameters depending on α and s.

Similarly, the transformed strain $\bar{e}^*$ and concentration $\bar{C}^*$ can be obtained from Eqs. (35) and (36) as

$$\bar{e}^* = \sum_{i=1}^{3} A_i'(\alpha, s) e^{-q_i z}, \tag{40}$$

$$\bar{C}^* = \sum_{i=1}^{3} A_i''(\alpha, s) e^{-q_i z} \tag{41}$$

where A_i, A_i' and A_i'', $i = 1, 2, 3$ are parameters depending on α and s. Substituting from Eqs. (39), (40) and (41) into Eqs. (32) and (33), the parameters $A_i'(\alpha, s)$ and $A_i''(\alpha, s)$, $i = 1, 2, 3$ can be expressed in terms of $A_i(\alpha, s)$ as

$$A_i'(\alpha, s) = f_i A_i(\alpha, s),\ A_i''(\alpha, s) = d_i A_i(\alpha, s) \tag{42}$$

where $f_i = \frac{\left\{k_i^4 - \left[(s+\tau_0 s^2)(1-\alpha_1\varepsilon)\right]k_i^2\right\}}{\varepsilon(s+\tau_0 s^2)\left\{k_i^2(1+\alpha_1) - \alpha_1 s^2\right\}}$, $\quad d_i = \frac{\left\{k_i^4 - \left[(s+\tau_0 s^2)(1+\varepsilon) + s^2\right]k_i^2 + (s+\tau_0 s^2)s^2\right\}}{\varepsilon(s+\tau_0 s^2)\left\{k_i^2(1+\alpha_1) - \alpha_1 s^2\right\}}$.

Applying the inverse Hankel transform to Eqs. (39), (40) and (41), we get

$$\bar{\theta} = \int_0^\infty \left\{\sum_{i=1}^{3} A_i(\alpha, s) e^{-q_i z}\right\} \alpha J_0(\alpha r) d\alpha, \tag{43}$$

$$\bar{e} = \int_0^\infty \left\{\sum_{i=1}^{3} A_i'(\alpha, s) e^{-q_i z}\right\} \alpha J_0(\alpha r) d\alpha, \tag{44}$$

$$\bar{C} = \int_0^\infty \left\{\sum_{i=1}^{3} A_i''(\alpha, s) e^{-q_i z}\right\} \alpha J_0(\alpha r) d\alpha. \tag{45}$$

Inserting Eqs. (43)–(45) into Eqs. (24)–(25), solutions for the displacement components in the Laplace transform domain are obtained as

$$\bar{u}(r,z,s) = \int_0^\infty -\alpha^2 J_1(\alpha r)\left[B(\alpha,s)e^{-qz} + \sum_{i=1}^{3}\frac{\lambda_i}{\left(q_i^2-q^2\right)}e^{-q_i z}\right]d\alpha, \tag{46}$$

$$\bar{w}(r,z,s) = \int_0^\infty \alpha J_0(\alpha r)\left[C(\alpha,s)e^{-qz} + \sum_{i=1}^{3}\frac{\lambda_i q_i}{\left(q_i^2-q^2\right)}e^{-q_i z}\right]d\alpha \tag{47}$$

where the parameters $B(\alpha,s)$ and $C(\alpha,s)$ depend on α and s only.

Also $q^2=\alpha^2+\beta^2 s^2$, $\quad C(\alpha,s)=\frac{\alpha^2 B(\alpha,s)}{q}$, $\quad \lambda_i=\left\{\left(1-\beta^2\right)f_i+\beta^2\left(1+d_i\right)\right\}A_i$.

Using Eqs. (28.1)–(29) and the solutions given in Eqs. (46)–(47), we obtain the stress components and the chemical potential in the Laplace transform domain,

$$\bar{\sigma}_{\varphi\varphi} = -\frac{2}{\beta^2 r}\int_0^\infty \alpha^2 J_1(\alpha r)\left[B(\alpha,s)e^{-qz} + \sum_{i=1}^{3}\frac{\lambda_i}{\left(q_i^2-q^2\right)}e^{-q_i z}\right]d\alpha + \bar{G}, \tag{48}$$

$$\bar{\sigma}_{rr} = -\frac{2}{\beta^2}\int_0^\infty \alpha^3\left[\frac{1}{\alpha r}J_1(\alpha r) - J_0(\alpha r)\right]\left[\begin{array}{l} B(\alpha,s)e^{-qz}, \\ +\sum_{i=1}^{3}\frac{\lambda_i}{\left(q_i^2-q^2\right)}e^{-q_i z}, \end{array}\right]d\alpha + \bar{G}, \tag{49}$$

$$\bar{\sigma}_{zz} = \frac{2}{\beta^2}\int_0^\infty \left[-qC(\alpha,s)e^{-qz} - \sum_{i=1}^{3}\frac{\lambda_i q_i^2}{\left(q_i^2-q^2\right)}e^{-q_i z}\right]\alpha J_0(\alpha r)\,d\alpha + \bar{G}, \tag{50}$$

$$\bar{\sigma}_{rz} = \frac{1}{\beta^2}\int_0^\infty \left[\left(\frac{\alpha^2+q^2}{q}\right)B(\alpha,s)e^{-qz} + \sum_{i=1}^{3}\frac{\lambda_i q_i\left(1+q_i\right)}{\left(q_i^2-q^2\right)}e^{-q_i z}\right]\alpha^2 J_1(\alpha r)\,d\alpha, \tag{51}$$

$$\bar{P}(r,z,s) = \frac{2}{\beta^2}\int_0^\infty \left[\sum_{i=1}^{3}\mu_i A_i e^{-q_i z}\right]\alpha J_0(\alpha r)\,d\alpha,$$

$$\bar{G} = \int_0^\infty \alpha J_0(\alpha r)\left(\sum_{i=1}^{3}\xi_i e^{-q_i z}\right)d\alpha \tag{52}$$

where $\mu_i=\left(-f_i+\alpha_3 d_i-\alpha_1\right)$, $\xi_i=\left(\frac{(\beta^2-2)}{\beta^2}f_i-d_i-1\right)A_i$.

Applying Laplace and Hankel transform to the boundary conditions given in Eqs. (20)–(23) and making use of Eqs. (43)–(52), we get

$$\sum_{i=1}^{3}A_i(\alpha,s) - \bar{f}_1^*(\alpha,s) = 0, \tag{53}$$

$$\frac{2}{\beta^2}\left[\alpha^2 B(\alpha,s) + \sum_{i=1}^{3}\frac{\lambda_i q_i^2}{\left(q_i^2-q^2\right)}\right] = 0, \tag{54}$$

$$\left(\frac{\alpha^2+q^2}{q}\right)B(\alpha,s) + 2\sum_{i=1}^{3}\frac{\lambda_i q_i(1+q_i)}{\left(q_i^2-q^2\right)} = 0, \tag{55}$$

$$\frac{2}{\beta^2}\left[\sum_{i=1}^{3}\mu_i A_i\right] = f_2^*(\alpha). \tag{56}$$

Equations (53)–(56) are a system of linear equations with A_1, A_2, A_3 and B as unknown parameters. Solving the above system of linear equations, the complete solution of the problem is obtained in the Laplace transform domain.

5 Inversion of double transforms

The Laplace transform of a continuous function $f(t)$ is given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt \tag{57}$$

for $t > 0$ and $s = x + iy$.

If the solution is given in the Laplace domain, the inversion integral is used to find the original function $f(t)$,

$$f(t) = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{-st} \bar{f}(s)dt, \tag{58}$$

where the contour must be taken to the right of any singularities of $\bar{f}(s)$. The direct integration of Eq. (58) is usually difficult and in many cases analytically not possible.

A detailed discussion on application of Gaver–Stehfast algorithm can be found in [24,25]. Only the final formula for the inverse of the Laplace transform as obtained by Gaver [20] and Stehfast [21,22] is given below.

By this method, the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t) = \frac{\ln 2}{t} \sum_{j=1}^{K} D(j, K) F\left(j \frac{\ln 2}{t}\right) \tag{59}$$

with

$$D(j, K) = (-1)^{j+M} \sum_{n=m}^{\min(j,M)} \frac{n^M (2n)!}{(M-n)!n!(n-1)!(j-n)!(2n-j)!} \tag{60}$$

where K is an even integer, whose value depends on the word length of the computer used. $M = K/2$ and m is the integer part of $(j+1)/2$. The optimal value of K was chosen as described in Gaver–Stehfast algorithm, for the fast convergence of results with the desired accuracy. This method is easy to implement and very accurate for functions of the type $e^{-\alpha t}$. The Romberg numerical integration technique [26] with variable step size was used to evaluate the integrals involved. All the programs were made in mathematical software MATLAB.

6 Numerical results and discussion

The function $f_1(r, t)$ considered in the problem is

$$f_1(r, t) = \theta_0 H(a - r) H(t) \tag{61}$$

where θ_0 is a constant. This means that the surface of the half-space is suddenly heated to the temperature θ_0 at the start inside a circle of radius "a" and center at the origin. The rest of the surface is kept at zero temperature.

Thus on applying Laplace and Hankel transforms to Eq. (61) we get

$$\bar{f}_1^*(\alpha, s) = \frac{a\theta_0 J_1(\alpha a)}{s\alpha}.$$

The chemical potential is taken as

$$f_2(r) = P_0 H(a - r) \tag{62}$$

where P_0 is constant.

On applying Hankel transform to Eq. (62), we get

$$f_2^*(\alpha) = \frac{a P_0}{\alpha} J_1(\alpha).$$

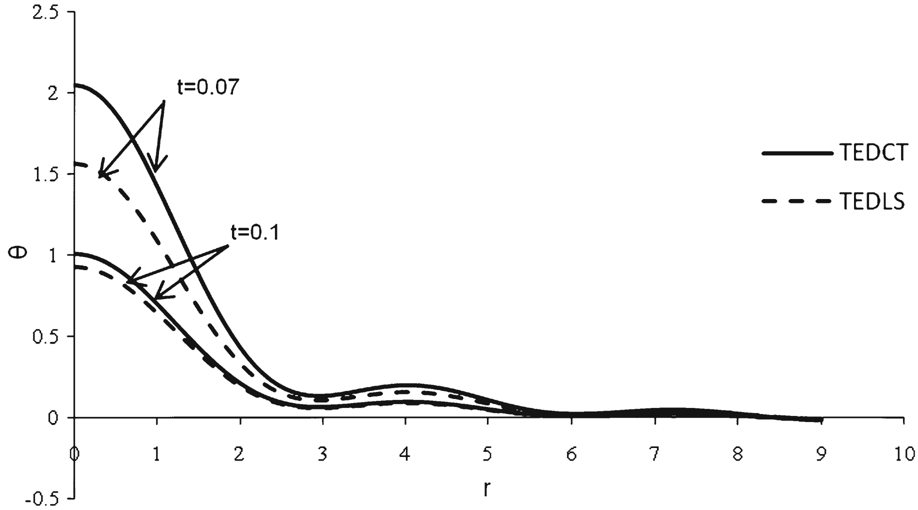

Fig. 1 Temperature θ distribution along the radial direction

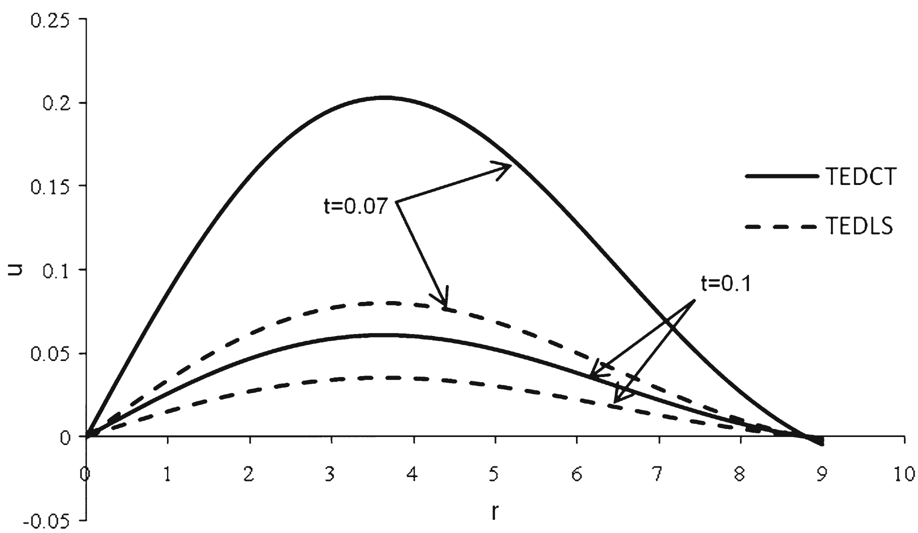

Fig. 2 Radial displacement component u distribution along the radial direction

The mathematical model is prepared with copper material for purposes of numerical computations. The material constants of the problem are thus given in S.I. units [10]:

$$T_0 = 293\,\mathrm{K},\quad \rho = 8954\,\mathrm{kg\,m^{-3}},\quad \tau_0 = 0.02,\quad \tau = 0.2,\quad k = 386\,\mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}},\quad \alpha_t = 1.78\times 10^{-5}\,\mathrm{K^{-1}}$$
$$\alpha_C = 1.78\times 10^{-5}\,\mathrm{K^{-1}},\quad c_E = 383.1\,\mathrm{J\,Kg^{-1}\,K^{-1}},\quad \mu = 3.86\times 10^{10}\,\mathrm{Nm^{-2}},\quad \lambda = 7.76\times 10^{10}\,\mathrm{Nm^{-2}}$$
$$c_1 = 4.158\times 10^{3}\,\mathrm{ms^{-1}},\quad a = 1.2\times 10^{4}\,\mathrm{m^2/s^2k},\quad b = 0.9\times 10^{6}\mathrm{m^5/kg\,s^2},\quad D = 0.88\times 10^{-8}\,\mathrm{kg\,s/m^3}.$$

Using these values, it was found that $\eta = 8886.73\,\mathrm{s\,m^{-2}}$, $\varepsilon = 0.0168\,\mathrm{N\,m\,J^{-1}}$, $\beta^2 = 4$, $\alpha_1 = 5.43$, $\alpha_2 = 0.533$ and $\alpha_3 = 36.24$. It should be noted that a unit of nondimensional time corresponds to $6.5\times 10^{-12}s$, while a unit of nondimensional length corresponds to $2.7\times 10^{-8}m$. The computations were carried out for time $t = 0.07, 0.1$.

Figures 1, 2, 3, 4, 5 and 6 exhibit the variations of temperature θ, radial displacement component u, axial displacement component w, chemical potential P, concentration C and axial stress component σ_{zz} with distance r. The variations of the various components with distance r are shown as (a) solid line for TEDCT theory and (b) dotted line for TEDLS theory. The numerical simulations are done at the bounding plane, i.e., $z = 0$.

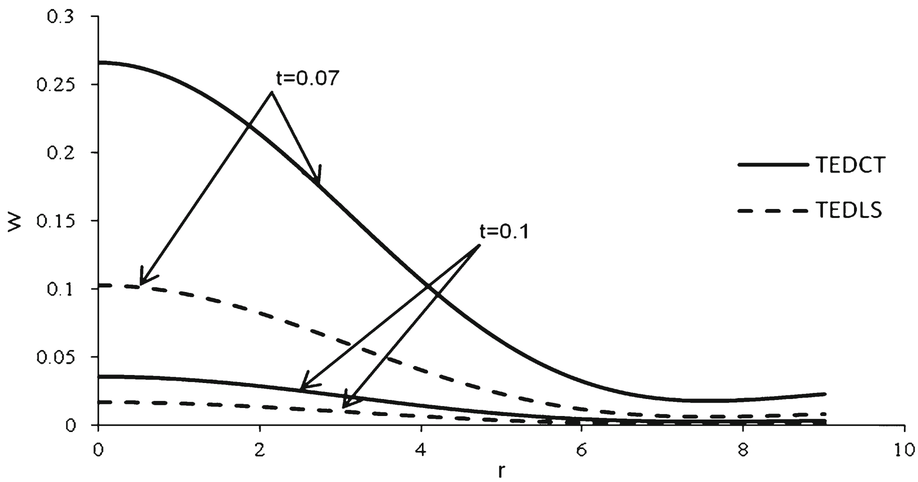

Fig. 3 Axial displacement component w distribution along the radial direction

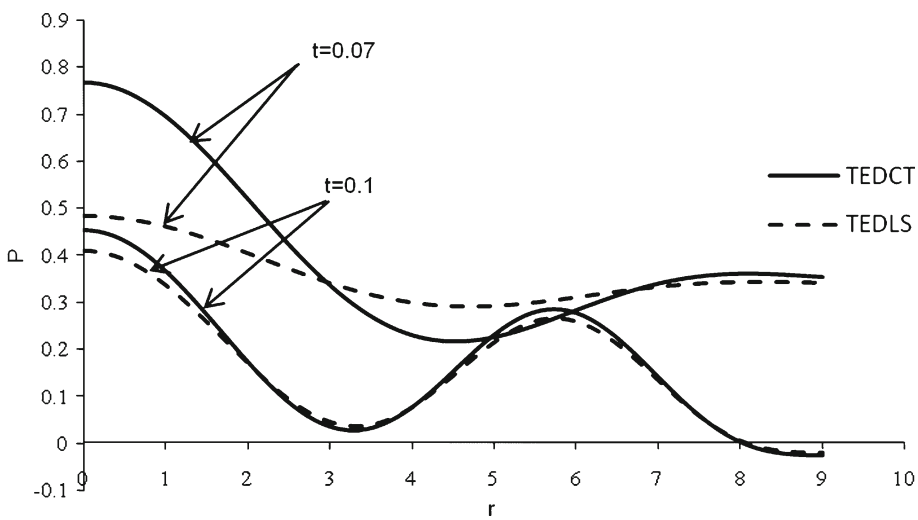

Fig. 4 Chemical potential P distribution along the radial direction

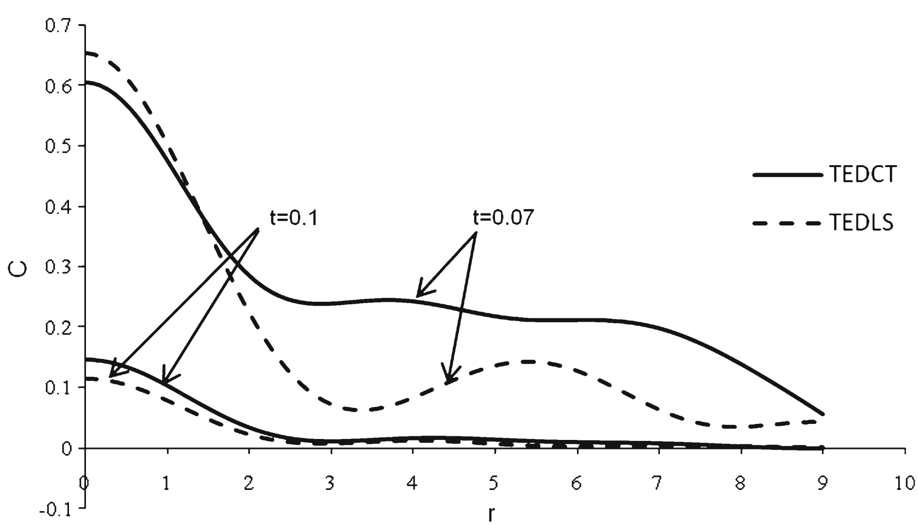

Fig. 5 Concentration C distribution along the radial direction

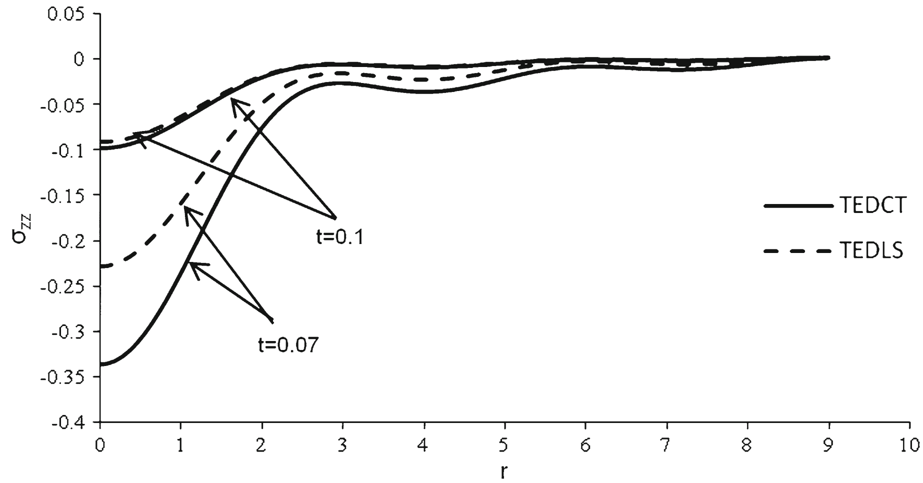

Fig. 6 Axial stress σ_{zz} distribution along the radial direction

Figure 1 exhibits the variation of temperature θ as a function of radius. It is observed that the temperature decays to zero as distance r increases. At large value of time $t = 0.1$, TEDLS and TEDCT theories show similar results; however, large variation is seen in the values of temperature for TEDLS and TEDCT theories at small time $t = 0.07$. It is observed that the behavior and trend of variations in values of temperature for TEDCT and TEDLS are almost similar except for the difference in magnitudes. In the region $3 \leq r \leq 6$ there is an increase in the values of temperature, and it follows a sinusoidal pattern afterward. As the disturbance travels through the medium, it encounters sudden changes, resulting in a nonuniform pattern of the curves which shows the effect of coupling of the fields of temperature, diffusion and strain.

In Fig. 2, the radial displacement component u increases and attains a maximum value near $r = 4$, and then, it decreases to zero. It is observed that the behavior and trend of variations in values of displacement for TEDCT and TEDLS are almost similar except that the magnitude of radial displacement in TEDCT is more as compared to TEDLS.

From Fig. 3, we observe that the axial displacement component w decreases with the increase in the radial distance. The axial displacement under TEDCT theory shows large variations as compared to TEDLS theory for small time $t = 0.07$, whereas for large time $t = 0.1$, the axial displacement shows small variations in magnitudes for both the theories.

From Fig. 4, we observe that the chemical potential P distribution along the radial direction shows oscillatory behavior throughout the medium. At time $t = 0.07$, the values of chemical potential for TEDCT theory is more than TEDLS theory in the regions $0 \leq r \leq 3$ and $7 \leq r \leq 9$, whereas for the region $3 \leq r \leq 7$ the magnitude of chemical potential for TEDLS theory is more than TEDCT theory. It is also observed that the values of chemical potential for TEDCT and TEDLS are similar in magnitudes for time $t = 0.1$.

In Fig. 5, the concentration C shows an oscillatory behavior throughout the medium. For time $t = 0.07$, the magnitude of concentration for TEDLS theory is more than TEDCT theory till $r \leq 1.5$, and then onward, the magnitude of concentration for TEDCT theory is more than TEDLS. For large time $t = 0.1$, the magnitudes of TEDCT are more than TEDLS throughout the medium.

Figure 6 exhibits the variation of axial stress σ_{zz} along the radial direction. One can observe that the variation in values of axial stress is more for small time $t = 0.07$ as compared to the values for large time $t = 0.1$. The axial stress values are compressive in the medium and gradually increase to zero. It is also observed that the behavior and trend of variations in values of axial stress for TEDCT and TEDLS are almost similar except for the magnitudes.

7 Conclusions

In this work, a two-dimensional problem for a half-space with a permeating substance was investigated. The transform method was used to obtain the analytical solution for the temperature, displacement, stress, concentration and chemical potential of the diffusive material. The method used in this study provides quite a successful approach in dealing with thermoelastic diffusion problems without any assumed restriction on the field variables. Coupling of the diffusion field, temperature and strain plays an important role in the deformation of an elastic body. As the disturbance travels through the medium, it encounters sudden changes, resulting in a nonuniform pattern of the curves. It was observed that the temperature and concentration of the diffusive material converge to a steady state with the passage of time. The results of this problem are very useful in the two-dimensional problems in axisymmetric half-space, which have various geophysical and industrial applications.

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