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A constitutive theory for multi-functional fiber reinforced composites

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Abstract The present work considers fiber reinforced composite materials in which the fibers have more than just a stiffening function. The composite is assumed to be composed of a non-conducting matrix reinforced with electroelastic fibers that conduct both current and heat in addition to supporting an applied load. The material system is treated as equivalent homogenized material that is nonlinearly elastic and transversely isotropic with the fiber direction as the direction of transverse isotropy. General constitutive equations are developed for the stress, polarization vector, current density vector and heat flux in terms of the deformation, electric field vector and temperature gradient. From these the special constitutive equations are extracted for a non-conducting matrix with conducting reinforcing fibers.

1 Introduction

Composite materials consisting of a relatively soft matrix reinforced with stiff fibers are an important class of engineering materials. Applications involve tires, hoses and soft biological tissues. Such materials are often treated as equivalent homogenized material systems and are modeled as being nonlinearly elastic and transversely isotropic with the fiber direction as the direction of transverse isotropy. Although such models provide little insight into fiber/matrix interaction, they are useful for studying the overall structural influence of the reinforcement. There is now a large body of literature on such studies. In these models, the fibers are not capable of responding to external fields such as electromagnetic fields. Their sole function is to provide stiffness in specific directions.

Before proceeding with the articulation of our problem of interest, namely situations wherein the fiber is capable of being stimulated by external fields and not merely the applied traction, a few words concerning the status of the "homogenization" of the fiber reinforced composites are warranted. In the real composite that is made up of the matrix and the fiber, at any point belonging to the matrix, we usually have a material that is isotropic. Also, with regard to the fiber itself, each point in the fiber might be isotropic or possess some other symmetry. However, the inhomogeneous composite, when subject to deformations, globally might seem to have preferential directional response, say that in the direction of the fibers. This does not mean that at every point in the composite the symmetry is that which the global response seems to suggest; it is important to bear in mind that notions such as material symmetry refer to response at a point in the material, and one should not confuse the body's inhomogeneity that leads to its global response having directional preference as being an anisotropic body wherein at each material point the body has a certain material symmetry. Nonetheless, in

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view of the fact that the body globally responds with directional preference, in our case the direction along which the fiber is aligned, we model the body as though it is transversely isotropic, knowing full well that this is not strictly correct.

There is increasing interest in composite materials in which the fibers have more than just a stiffening function. The light weight requirements of advanced aerospace applications suggest the need for the development of composites in which fibers perform several functions. There are numerous examples of multi-functional fiber reinforced composites in biomechanics. The heart, uterus and bladder are reinforced by muscle fibers that contract in response to electrical signals, and the brain has a distribution of fiber-like axons that transmit signals. It can be expected that innovations in material science will lead to the production of reinforcing fibers that can carry load as well as perform other functions. These examples motivate this paper, which is to develop a constitutive theory for composite materials whose fibers are multi-functional.

The work presented here is based on the methods developed in [1,2], and summarized in [3], for determining the relations between tensors of various orders for different material symmetry groups. In an early application of these methods, Pipkin and Rivlin [4] developed a constitutive equation for the electric current density vector as a function of electric field and deformation assuming the material to be isotropic. They then [5] developed constitutive equations for the electric current density vector, heat flux and magnetic induction as functions of the electric field vector, magnetic induction vector and temperature gradient, again for isotropic materials, but without considering deformation. Toupin and Rivlin [6] discussed electro-magneto-optical effects for isotropic materials. Using the methods developed in [1,2], Rajagopal and Wineman [7] developed a constitutive relation for the stress in an electroactive solid in terms of the deformation and the electric field vector. Later, Dorfmann and Ogden [8] developed a constitutive theory for nonlinear electroelastic materials within the context of the thermodynamically based electromagnetic theory presented by Kovetz [9]. Recently, Bustamante and Rajagopal have developed implicit constitutive relations to describe the response of electroelastic [10–12] and magnetoelastic bodies.

Interest in fiber reinforced composite materials led to the development of stress–strain relations for transverse isotropy [13]. An important feature of these constitutive equations is that the process of imposing the material symmetry restrictions introduces terms that can be identified with the fiber stretch and the fiber tension. Other quantities also arise in the constitutive equation whose interpretation is less clear. Merodio and Ogden [14] suggested that these are related to the interaction between the fiber and the matrix. O'Neill and Spencer [15] developed a constitutive equation for the heat flux vector as a function of the temperature gradient and deformation for transversely isotropic materials. Material symmetry restrictions introduce several quantities, one of which can be identified with the temperature gradient along a fiber.

A general observation concerning material symmetry is warranted. There are different points of view with regarding the description of the material symmetry and the consequent constitutive representation of a body. The original description of anisotropy seems to be restricted to invariance of response of the body to sub-groups of rotations (see the discussion in Rajagopal [16]). However, symmetry considerations also consider invariance to the full orthogonal group, namely rotations and inversions, and also unimodular transformations. From the point of view of the experimentalist, while one can subject a body to rotations and determine the response of the body, one cannot subject a body to inversions. Of course, one can subject another body whose internal structure is the mirror image of the body of interest and consider response, but this is not the same body. Also, while one can subject a body to a unimodular transformation like shear, and consider its response, this was clearly not the intent of the original classification of anisotropy.

The present work is concerned with fiber reinforced composite materials in which the fibers are deformable and can conduct electrical current and heat. Representations for transverse isotropy are developed for constitutive equations for the stress, polarization vector, current density vector and heat flux in terms of the deformation, electric field vector and temperature gradient. It is shown that each dependent variable depends on quantities that have the physical interpretation of the components of the electric field vector and temperature gradient along the stretched fiber. The constitutive assumptions, influence of superposed rigid body motions and material symmetry restrictions are introduced in Sect. 2. Section 3 contains a description of the symmetry groups associated with transverse isotropy. This section also provides the invariants used in developing constitutive representations. The representations for the stress, polarization, electric current density and heat flux are developed in Sect. 4. Section 5 presents the special case of a non-conducting matrix with conducting reinforcing fibers. Concluding comments are provided in Sect. 6.

2 Formulation

Consider a composite material consisting of extensible elastic fibers distributed within an elastic matrix. It is assumed that both constituents can conduct electrical currents when in an electrical field as well as conduct heat when there is a temperature gradient and that both phenomena are affected by deformation. As has been done when only mechanical effects are of interest, it is assumed that the fiber-matrix system can be represented as an equivalent homogenized nonlinear elastic transversely isotropic material that conducts both electrical current and heat.

Let $\mathbf{X}$ and $\mathbf{x}$ denote, respectively, the position vectors of a particle in the reference and current configurations. They are related by the motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ for which the corresponding deformation gradient is $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$. The right and left Cauchy-Green tensors are $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, respectively.

Guided by the discussions in [4–6,8,17], it is assumed that the total stress tensor $\mathbf{T}$, the polarization vector $\mathbf{P}$, the electrical current density vector $\mathbf{J}$ and the heat flux vector $\mathbf{q}$ are functions of the deformation gradient $\mathbf{F}$, the electric field vector $\mathbf{E}$ and the spatial temperature gradient $\mathbf{g} = \partial\theta/\partial\mathbf{x}$.

In thermodynamic treatments of nonlinear electroelasticity, i.e., [8], a free energy ψ is introduced that is assumed to depend on the above list of independent variables. Considerations based on the second law of thermodynamics show that ψ is independent of the temperature gradient $\mathbf{g} = \partial\theta/\partial\mathbf{x}$. It is also shown that $\mathbf{T}$ is expressed in terms of $\partial\psi/\partial\mathbf{F}$ and $\mathbf{P}$ is expressed in terms of $\partial\psi/\partial\mathbf{E}$. Thus, $\mathbf{T}$ and $\mathbf{P}$ depend on $\mathbf{F}$ and $\mathbf{E}$ but not $\mathbf{g}$.

Although the intent here is to develop representations for constitutive equations using the methods presented in [1–3], it is useful to incorporate the above mentioned simplifications obtained from thermodynamical considerations. Thus, the assumed constitutive relations are:

$$\mathbf{T} = \mathbf{T}(\mathbf{F}, \mathbf{E}), \tag{1}$$
$$\mathbf{P} = \mathbf{P}(\mathbf{F}, \mathbf{E}), \tag{2}$$
$$\mathbf{J} = \mathbf{J}(\mathbf{F}, \mathbf{E}, \mathbf{g}), \tag{3}$$
$$\mathbf{q} = \mathbf{q}(\mathbf{F}, \mathbf{E}, \mathbf{g}). \tag{4}$$

Consideration of the influence of superposed rigid body rotations shows that (1)–(4) have the forms:

$$\mathbf{T} = \mathbf{F}\boldsymbol{\Phi}_T(\mathbf{C}, \mathbf{F}^{\mathbf{T}}\mathbf{E})\mathbf{F}^T, \tag{5}$$
$$\mathbf{P} = \mathbf{F}\boldsymbol{\Phi}_P(\mathbf{C}, \mathbf{F}^{\mathbf{T}}\mathbf{E}), \tag{6}$$
$$\mathbf{J} = \mathbf{F}\boldsymbol{\Phi}_J(\mathbf{C}, \mathbf{F}^{\mathbf{T}}\mathbf{E}, \mathbf{F}^{\mathbf{T}}\mathbf{g}), \tag{7}$$
$$\mathbf{q} = \mathbf{F}\boldsymbol{\Phi}_q(\mathbf{C}, \mathbf{F}^{\mathbf{T}}\mathbf{E}, \mathbf{F}^{\mathbf{T}}\mathbf{g}). \tag{8}$$

For notational convenience, let $\mathbf{e} = \mathbf{F}^T\mathbf{E}$ and $\tilde{\mathbf{g}} = \mathbf{F}^T\mathbf{g}$.

Let $\mathbf{M}$ denote a transformation of a material symmetry group. It is assumed that the same material symmetry group applies to all of the constitutive relations (5)–(8). Standard arguments show that the response functions in (5)–(8) are subject to these restrictions:

$$\boldsymbol{\Phi}_T(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}) = \mathbf{M}^T\boldsymbol{\Phi}_T(\mathbf{C}, \mathbf{e})\mathbf{M}, \tag{9}$$
$$\boldsymbol{\Phi}_P(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}) = \mathbf{M}^T\boldsymbol{\Phi}_P(\mathbf{C}, \mathbf{e}), \tag{10}$$
$$\boldsymbol{\Phi}_J(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\tilde{\mathbf{g}}) = \mathbf{M}^T\boldsymbol{\Phi}_J(\mathbf{C}, \mathbf{e}, \tilde{\mathbf{g}}), \tag{11}$$
$$\boldsymbol{\Phi}_q(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\tilde{\mathbf{g}}) = \mathbf{M}^T\boldsymbol{\Phi}_q(\mathbf{C}, \mathbf{e}, \tilde{\mathbf{g}}). \tag{12}$$

The method for imposing these restrictions, developed in [1,2] and summarized in [3], is briefly reviewed here. Let $\mathbf{u}$ be an arbitrary vector and define the scalar functions:

$$\hat{\Phi}^{(T)} = \mathbf{u}\cdot\boldsymbol{\Phi}_T\mathbf{u}, \tag{13}$$
$$\hat{\Phi}^{(P)} = \boldsymbol{\Phi}_P\cdot\mathbf{u}, \tag{14}$$
$$\hat{\Phi}^{(J)} = \boldsymbol{\Phi}_J\cdot\mathbf{u}, \tag{15}$$
$$\hat{\Phi}^{(q)} = \boldsymbol{\Phi}_q\cdot\mathbf{u}. \tag{16}$$

It then follows from (9) to (12) that

$$\hat{\Phi}^{(T)}(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\mathbf{u}) = \hat{\Phi}^{(T)}(\mathbf{C}, \mathbf{e}, \mathbf{u}), \tag{17}$$
$$\hat{\Phi}^{(P)}(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\mathbf{u}) = \hat{\Phi}^{(P)}(\mathbf{C}, \mathbf{e}, \mathbf{u}), \tag{18}$$
$$\hat{\Phi}^{(J)}(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\tilde{\mathbf{g}}, \mathbf{M}^T\mathbf{u}) = \hat{\Phi}^{(J)}(\mathbf{C}, \mathbf{e}, \tilde{\mathbf{g}}, \mathbf{u}), \tag{19}$$
$$\hat{\Phi}^{(q)}(\mathbf{M}^T\mathbf{C}\mathbf{M}, \mathbf{M}^T\mathbf{e}, \mathbf{M}^T\tilde{\mathbf{g}}, \mathbf{M}^T\mathbf{u}) = \hat{\Phi}^{(q)}(\mathbf{C}, \mathbf{e}, \tilde{\mathbf{g}}, \mathbf{u}). \tag{20}$$

Thus, $\hat{\Phi}^{(T)}$, $\hat{\Phi}^{(P)}$, $\hat{\Phi}^{(J)}$ and $\hat{\Phi}^{(q)}$ are scalar invariants under the transformations of the symmetry group. $\hat{\Phi}^{(T)}$ is quadratic in the components of $\mathbf{u}$, while $\hat{\Phi}^{(P)}$, $\hat{\Phi}^{(J)}$ and $\hat{\Phi}^{(q)}$ are linear in the components of $\mathbf{u}$. Each can be expressed in terms of a basic set of invariants, an integrity basis, with those depending on $\mathbf{u}$ playing a particular role. Before providing further details, it is necessary to specify certain aspects of transverse isotropy.

3 Transverse isotropy

Let $\mathbf{a}_o$ be a unit vector along the fibers in the reference configuration. This defines the preferred direction for transverse isotropy. Transverse isotropy is characterized by the following transformations [15]: $\mathbf{M}(\alpha)$—rotation about $\mathbf{a}_o$ through angle α, $\mathbf{R}_T$—reflection in a plane containing $\mathbf{a}_o$, $\mathbf{R}_L$—reflection in a plane perpendicular to $\mathbf{a}_o$, $\mathbf{D}$—rotation through π about an axis perpendicular to $\mathbf{a}_o$. These generate five distinct material symmetry groups: $T_1 : [\mathbf{M}(\alpha)]$; $T_2 : [\mathbf{M}(\alpha), \mathbf{R}_T]$; $T_3 : [\mathbf{M}(\alpha), \mathbf{R}_L]$; $T_4 : [\mathbf{M}(\alpha), \mathbf{R}_T, \mathbf{R}_L]$; $T_5 : [\mathbf{M}(\alpha), \mathbf{D}]$.

As mentioned earlier, in the laboratory, one would only be able to determine the response by rotating a sample. That is, the only symmetries that can be tested with respect to the body are T_1 and T_5. While another body whose structure corresponds to that which can be obtained by transformations belonging to T_2, T_3 and T_4 can be tested, we have to bear in mind that one cannot obtain a new body from one that is being tested by members belonging to T_2, T_3 and T_4. Here, we shall interpret material symmetry within the context of the definition given in [15]. The body whose symmetry group is given by T_1 is referred to as a rotationally transversely isotropic body. In this study, we shall be primarily interested in the constitutive representation for rotationally transversely isotropic bodies.

As is shown in [18], the invariants of $\mathbf{C}$, $\mathbf{e}$, $\tilde{\mathbf{g}}$ and $\mathbf{u}$ under any of these material symmetry groups are isotropic invariants of $\mathbf{C}$, $\mathbf{e}$, $\tilde{\mathbf{g}}$, $\mathbf{u}$ and $\mathbf{a}_o$. These invariants for group T_1 are listed here in two sets.

3.1 Invariants of $\mathbf{C}$, $\mathbf{e}$, $\tilde{\mathbf{g}}$ and $\mathbf{a}_o$

These consist of the following:

$$\begin{aligned}
&I_1' = \mathrm{tr}\mathbf{C}, \quad I_2' = \frac{1}{2}\left[\mathrm{tr}\mathbf{C}^2 - \mathrm{tr}(\mathbf{C})^2\right], \quad I_3' = \det(\mathbf{C}),\\
&I_4' = \mathbf{a}_o \cdot \mathbf{C}\mathbf{a}_o, \quad I_5' = \mathbf{a}_o \cdot \mathbf{C}^2\mathbf{a}_o,\\
&I_6' = \mathbf{e}\cdot\mathbf{e}, \quad I_7' = \mathbf{e}\cdot\mathbf{C}\mathbf{e}, \quad I_8' = \mathbf{e}\cdot\mathbf{C}^2\mathbf{e},\\
&I_9' = \tilde{\mathbf{g}}\cdot\tilde{\mathbf{g}}, \quad I_{10}' = \tilde{\mathbf{g}}\cdot\mathbf{C}\tilde{\mathbf{g}}, \quad I_{11}' = \tilde{\mathbf{g}}\cdot\mathbf{C}^2\tilde{\mathbf{g}},\\
&I_{12}' = \mathbf{e}\cdot\mathbf{a}_o, \quad I_{13}' = \mathbf{e}\cdot\mathbf{C}\mathbf{a}_o, \quad I_{14}' = \mathbf{e}\cdot\mathbf{C}^2\mathbf{a}_o,\\
&I_{15}' = \tilde{\mathbf{g}}\cdot\mathbf{a}_o, \quad I_{16}' = \tilde{\mathbf{g}}\cdot\mathbf{C}\mathbf{a}_o, \quad I_{17}' = \tilde{\mathbf{g}}\cdot\mathbf{C}^2\mathbf{a}_o\\
&I_{18}' = \mathbf{e}\cdot\tilde{\mathbf{g}}, \quad I_{19}' = \mathbf{e}\cdot\mathbf{C}\tilde{\mathbf{g}}, \quad I_{20}' = \mathbf{e}\cdot\mathbf{C}^2\tilde{\mathbf{g}},
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&\tilde{I}_1 = \left[\mathbf{e}, \mathbf{C}\mathbf{e}, \mathbf{C}^2\mathbf{e}\right], \quad \tilde{I}_2 = \left[\tilde{\mathbf{g}}, \mathbf{C}\tilde{\mathbf{g}}, \mathbf{C}^2\tilde{\mathbf{g}}\right], \quad \tilde{I}_3 = \left[\mathbf{a}_o, \mathbf{C}\mathbf{a}_o, \mathbf{C}^2\mathbf{a}_o\right],\\
&\tilde{I}_4^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\mathbf{a}_o\right], \quad \tilde{I}_5^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\mathbf{a}_o, \mathbf{C}^C\mathbf{a}_o\right],\\
&\tilde{I}_6^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\tilde{\mathbf{g}}\right], \quad \tilde{I}_7^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\tilde{\mathbf{g}}, \mathbf{C}^C\tilde{\mathbf{g}}\right],\\
&\tilde{I}_8^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\tilde{\mathbf{g}}, \mathbf{C}^C\mathbf{a}_o\right], \quad \tilde{I}_9^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\mathbf{a}_o, \mathbf{C}^C\mathbf{a}_o\right],\\
&\tilde{I}_{10}^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\mathbf{a}_o\right]
\end{aligned} \tag{22}$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ denotes the scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

3.2 Invariants of $\mathbf{C}$, $\mathbf{e}$, $\tilde{\mathbf{g}}$, $\mathbf{a}_o$ and $\mathbf{u}$

This set of invariants contains those listed in (21) and (22) along with

$$
\begin{aligned}
&J_1 = \mathbf{u}\cdot\mathbf{u}, \quad J_2 = \mathbf{e}\cdot\mathbf{u}, \quad J_3 = \tilde{\mathbf{g}}\cdot\mathbf{u}, \quad J_4 = \mathbf{a}_o\cdot\mathbf{u},\\
&J_5 = \mathbf{u}\cdot\mathbf{C}\mathbf{u}, \quad J_6 = \mathbf{u}\cdot\mathbf{C}^2\mathbf{u}, \quad J_7 = \mathbf{a}_o\cdot\mathbf{C}\mathbf{u}, \quad J_8 = \mathbf{a}_o\cdot\mathbf{C}^2\mathbf{u},\\
&J_9 = \mathbf{e}\cdot\mathbf{C}\mathbf{u}, \quad J_{10} = \mathbf{e}\cdot\mathbf{C}^2\mathbf{u}, \quad J_{11} = \tilde{\mathbf{g}}\cdot\mathbf{C}\mathbf{u}, \quad J_{12} = \tilde{\mathbf{g}}\cdot\mathbf{C}^2\mathbf{u},
\end{aligned}
\tag{23}
$$

and the scalar triple products,

$$
\begin{aligned}
&\tilde{J}_1^{(ABC)} = \left[\mathbf{C}^A\mathbf{a}_o, \mathbf{C}^B\mathbf{a}_o, \mathbf{C}^C\mathbf{u}\right], \quad \tilde{J}_2^{(ABC)} = \left[\mathbf{C}^A\mathbf{a}_o, \mathbf{C}^B\mathbf{u}, \mathbf{C}^C\mathbf{u}\right],\\
&\tilde{J}_3^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\mathbf{u}\right], \quad \tilde{J}_4^{(ABC)} = \left[\mathbf{C}^A\mathbf{e}, \mathbf{C}^B\mathbf{u}, \mathbf{C}^C\mathbf{u}\right],\\
&\tilde{J}_5^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\tilde{\mathbf{g}}, \mathbf{C}^C\mathbf{u}\right], \quad \tilde{J}_6^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\mathbf{u}, \mathbf{C}^C\mathbf{u}\right],\\
&\tilde{J}_7^{(ABC)} = \left[\mathbf{C}^A\mathbf{a}_o, \mathbf{C}^B\tilde{\mathbf{g}}, \mathbf{C}^C\mathbf{u}\right], \quad \tilde{J}_8^{(ABC)} = \left[\mathbf{C}^A\mathbf{a}_o, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\mathbf{u}\right],\\
&\tilde{J}_9^{(ABC)} = \left[\mathbf{C}^A\tilde{\mathbf{g}}, \mathbf{C}^B\mathbf{e}, \mathbf{C}^C\mathbf{u}\right].
\end{aligned}
\tag{24}
$$

For the purpose of economy of presentation, the exponents A, B, C in the scalar triple products take on the values 0, 1, 2. Some of the scalar triple products may be expressible in terms of other invariants and therefore may be eliminated from the list. In other words, the set of invariants in (22) and (24) may be reducible. Moreover, as pointed out in [15], some of the scalar triple products change sign under the transformations of groups T_2 through T_5. Their squares and products must then be considered in the list of invariants, as was done in [15]. If these can be expressed as polynomials of other invariants, then they need not be included in (22) and (24). The determination of such relations for these material symmetry groups is beyond the scope of this work. Even in the simpler case when there is only heat conduction and the electric field is not considered, Spencer [18] devoted several publications to this issue. Indeed, it will be shown that for the particular application considered later, i.e., the specialization of these results to fiber reinforced composites, it may not require the development of an irreducible set.

4 Constitutive relations

Let $\mathbf{e} = \mathbf{F}^T\mathbf{E}$ and $\tilde{\mathbf{g}} = \mathbf{F}^T\mathbf{g}$ be substituted in the invariants (21). With use of the Cayley–Hamilton theorem, this leads to an equivalent set of invariants now expressed in terms of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{g}$ and $\mathbf{a}_o$:

$$
\begin{aligned}
&I_1 = \mathrm{tr}\mathbf{B}, \quad I_2 = \frac{1}{2}\left[\mathrm{tr}\mathbf{B}^2 - \mathrm{tr}(\mathbf{B})^2\right], \quad I_3 = \det(\mathbf{B}),\\
&I_4 = \mathbf{F}\mathbf{a}_o\cdot\mathbf{F}\mathbf{a}_o, \quad I_5 = \mathbf{F}\mathbf{a}_o\cdot\mathbf{B}\mathbf{F}\mathbf{a}_o,\\
&I_6 = \mathbf{E}\cdot\mathbf{E}, \quad I_7 = \mathbf{E}\cdot\mathbf{B}\mathbf{E}, \quad I_8 = \mathbf{E}\cdot\mathbf{B}^2\mathbf{E},\\
&I_9 = \mathbf{g}\cdot\mathbf{g}, \quad I_{10} = \mathbf{g}\cdot\mathbf{B}\mathbf{g}, \quad I_{11} = \mathbf{g}\cdot\mathbf{B}^2\mathbf{g},\\
&I_{12} = \mathbf{E}\cdot\mathbf{F}\mathbf{a}_o, \quad I_{13} = \mathbf{E}\cdot\mathbf{B}\mathbf{F}\mathbf{a}_o, \quad I_{14} = \mathbf{E}\cdot\mathbf{B}^2\mathbf{F}\mathbf{a}_o,\\
&I_{15} = \mathbf{g}\cdot\mathbf{F}\mathbf{a}_o, \quad I_{16} = \mathbf{g}\cdot\mathbf{B}\mathbf{F}\mathbf{a}_o, \quad I_{17} = \mathbf{g}\cdot\mathbf{B}^2\mathbf{F}\mathbf{a}_o,\\
&I_{18} = \mathbf{E}\cdot\mathbf{g}, \quad I_{19} = \mathbf{E}\cdot\mathbf{B}\mathbf{g}, \quad I_{20} = \mathbf{E}\cdot\mathbf{B}^2\mathbf{g}.
\end{aligned}
\tag{25}
$$

The scalar triples in (22) can be expressed in terms of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{g}$ and $\mathbf{a}_o$ in a similar manner. The resulting expressions will play no role in the special theory to be presented later and therefore are omitted.

Several of the invariants in (25) represent physical quantities of interest. Invariant I_4 represents the square of the fiber stretch ratio. Invariants I_6 and I_9 represent the magnitudes of the electric field and temperature gradients, respectively. Invariants I_{12} and I_{15}, respectively, represent the components of the electric field and temperature gradient along the stretched fiber. Let $\hat{I}$ denote the set of these invariants, i.e.,

$$
\hat{I} = [I_4, \quad I_6, \quad I_9, \quad I_{12}, \quad I_{15}]\,. \tag{26}
$$

4.1 Representation for the stress

Let $\hat{\Phi}^{(T)} = \sum \phi_\alpha^{(T)} \hat{J}^{(\alpha)}$ in which $\phi_\alpha^{(T)}$ is a function of the invariants in (21) and (22) that do not depend on $\tilde{\mathbf{g}}$ and $\hat{J}^{(\alpha)}$ is a polynomial that is quadratic in the components of vector $\mathbf{u}$ and formed from the invariants in (23) and (24) that do not depend on $\tilde{\mathbf{g}}$,

$$\begin{aligned}\hat{\Phi}^{(T)} = {} & \phi_o J_1^{(T)} + \phi_1^{(T)} J_2^2 + \phi_2^{(T)} J_4^2 + \phi_3^{(T)} J_2 J_4 + \phi_4^{(T)} J_5 + \phi_5^{(T)} J_6 + \phi_6^{(T)} J_7^2 \\ & + \phi_7^{(T)} J_8^2 + \phi_8^{(T)} J_9^2 + \phi_9^{(T)} J_{10}^2 + \phi_{10}^{(T)} J_2 J_7 + \phi_{11}^{(T)} J_2 J_8 + \phi_{12}^{(T)} J_2 J_9 \\ & + \phi_{13}^{(T)} J_2 J_{10} + \phi_{14}^{(T)} J_4 J_7 + \phi_{15}^{(T)} J_4 J_8 + \phi_{16}^{(T)} J_4 J_9 + \phi_{17}^{(T)} J_4 J_{10} + \phi_{18}^{(T)} J_7 J_8 \\ & + \phi_{19}^{(T)} J_7 J_9 + \phi_{20}^{(T)} J_7 J_{10} + \phi_{21}^{(T)} J_8 J_9 + \phi_{22}^{(T)} J_8 J_{10} + \phi_{23}^{(T)} J_9 J_{10} + \hat{\Phi}_*^{(T)}. \end{aligned} \tag{27}$$

$\hat{\Phi}_*^{(T)}$ represents a summation over polynomials formed from the invariants in (24) that are quadratic in $\mathbf{u}$ as well as a summation over the exponents A, B, C in the scalar triple products as they take on the values 0, 1, 2. An explicit listing of these terms would substantially lengthen the expression for $\hat{\Phi}^{(T)}$. Some of the terms may vanish, for example, the term with $\tilde{J}_1^{ABC}$ in which $A = B$. Others may be expressible in terms of other invariants and therefore may be eliminated from the list. However, for the purposes of this paper, it is sufficient to just describe the structure of the terms represented by $\hat{\Phi}_*^{(T)}$.

The tensor $\mathbf{\Phi}_T$ in (13) is recovered by the operation

$$(\mathbf{\Phi}_T)_{ij} = \frac{\partial^2 \hat{\Phi}^{(T)}}{\partial u_i \partial u_j}. \tag{28}$$

The expression obtained by combining (5) and (28), and then setting $\mathbf{e} = \mathbf{F}^T\mathbf{E}$, is simplified with the use of the Cayley–Hamilton theorem. This leads to a constitutive equation for $\mathbf{T}$ in terms of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{Fa}_o$,

$$\begin{aligned}\mathbf{T} = {} & \alpha_o^{(T)}\mathbf{I} + \alpha_1^{(T)}\mathbf{B} + \alpha_2^{(T)}\mathbf{B}^2 \\ & + \mathrm{sym}\left(\left[\hat{\alpha}_o^{(T)}\mathbf{I} + \hat{\alpha}_1^{(T)}\mathbf{B} + \hat{\alpha}_2^{(T)}\mathbf{B}^2\right]\mathbf{Fa}_o \otimes \left[\hat{\beta}_o^{(T)}\mathbf{I} + \hat{\beta}_1^{(T)}\mathbf{B} + \hat{\beta}_2^{(T)}\mathbf{B}^2\right]\mathbf{Fa}_o\right) \\ & + \mathrm{sym}\left(\left[\hat{\alpha}_3^{(T)}\mathbf{I} + \hat{\alpha}_4^{(T)}\mathbf{B} + \hat{\alpha}_5^{(T)}\mathbf{B}^2\right]\mathbf{E} \otimes \left[\hat{\beta}_3^{(T)}\mathbf{I} + \hat{\beta}_4^{(T)}\mathbf{B} + \hat{\beta}_5^{(T)}\mathbf{B}^2\right]\mathbf{E}\right) \\ & + \mathrm{sym}\left(\mathbf{Fa}_o \otimes \left[\hat{\alpha}_6^{(T)}\mathbf{I} + \hat{\alpha}_7^{(T)}\mathbf{B} + \hat{\alpha}_8^{(T)}\mathbf{B}^2\right]\mathbf{E}\right) \\ & + \mathrm{sym}\left(\mathbf{BFa}_o \otimes \left[\hat{\alpha}_9^{(T)}\mathbf{I} + \hat{\alpha}_{10}^{(T)}\mathbf{B} + \hat{\alpha}_{11}^{(T)}\mathbf{B}^2\right]\mathbf{E}\right) \\ & + \mathrm{sym}\left(\mathbf{B}^2\mathbf{Fa}_o \otimes \left[\hat{\alpha}_{12}^{(T)}\mathbf{I} + \hat{\alpha}_{13}^{(T)}\mathbf{B} + \hat{\alpha}_{14}^{(T)}\mathbf{B}^2\right]\mathbf{E}\right) + \mathbf{F\Phi}_T^*\mathbf{F}^T, \end{aligned} \tag{29}$$

where $\mathbf{\Phi}_T^*$ represents the terms obtained from substituting $\hat{\Phi}_*^{(T)}$ in (28) and where $\mathrm{sym}\,(\mathbf{A} \otimes \mathbf{B}) = \mathbf{A}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{A}$. An example of the terms arising from $\mathbf{F\Phi}_T^*\mathbf{F}^T$ can be seen by noting that vectors whose components are given by $\mathbf{C}^A\mathbf{a}_o$ and $\mathbf{C}^B\mathbf{e}$ can be written as $\mathbf{F}^T\mathbf{B}^{A-1}\mathbf{Fa}_o$ and $\mathbf{F}^T\mathbf{B}^B\mathbf{E}$, respectively. Then, $\tilde{J}_8^{ABC}$ with $C = 0$ would introduce a term of the form

$$\begin{aligned}&\left(\mathbf{F}^T\mathbf{B}^{A_1}\mathbf{Fa}_o \times \mathbf{F}^T\mathbf{B}^{B_1}\mathbf{E}\right)_i \left(\mathbf{F}^T\mathbf{B}^{A_2}\mathbf{Fa}_o \times \mathbf{F}^T\mathbf{B}^{B_2}\mathbf{E}\right)_j \\ & \quad + \left(\mathbf{F}^T\mathbf{B}^{A_1}\mathbf{Fa}_o \times \mathbf{F}^T\mathbf{B}^{B_1}\mathbf{E}\right)_j \left(\mathbf{F}^T\mathbf{B}^{A_2}\mathbf{Fa}_o \times \mathbf{F}^T\mathbf{B}^{B_2}\mathbf{E}\right)_i. \end{aligned} \tag{30}$$

The scalar coefficients in (29) are now functions of the invariants I_1–I_8 and I_{12}–I_{14} in (25) and the invariants in (22) that do not depend on $\tilde{\mathbf{g}}$, were they to be re-expressed in terms of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{Fa}_o$.

For an incompressible material, the first term in (29) would be replaced by $-p\mathbf{I}$ with p representing an arbitrary spherical stress and invariant I_3 would be dropped from the list in (25). The tensorial structure of the first three terms in (29) is the same as for an isotropic elastic solid. When considering an isotropic matrix reinforced with fibers, these three terms are usually interpreted as representing the contribution of the matrix. The tensorial structure of the next group of terms, containing $\mathbf{B}$ and $\mathbf{Fa}_o$, is the same as in the constitutive equation for a transversely isotropic material. It has been shown in [15,18] that some of the tensor products in this group can be expressed in other tensors and invariants and thus can be eliminated from (29). In other words, there is no loss in generality if $\hat{\beta}_1^{(T)} = \hat{\beta}_2^{(T)} = 0$. The tensorial structure of the third group of terms, containing $\mathbf{B}$ and $\mathbf{E}$, appears in the constitutive equations for an isotropic electroelastic material in [7,8]. Similarly, there is no loss in generality if $\hat{\beta}_4^{(T)} = \hat{\beta}_5^{(T)} = 0$. The remaining tensorial terms in (29) are unique to the present constitutive theory and introduce effects that arise from the interaction of the electric field, the deformation and transverse isotropy. It may be possible to express some of these in terms of other tensors and invariants, thereby reducing the complexity of the constitutive equations, but such a study is beyond the scope of the present work. The scalar coefficients in (29) introduce additional interactions through their dependence on the invariants I_7, I_8 and I_{12}–I_{14} in (25). It should be noted that the material symmetry considered in [7,8] was the full orthogonal group. Phenomena arising from triple scalar invariants such as in (22) did not appear and were not studied.

4.2 Representation for the polarization

Let $\hat{\Phi}^{(P)} = \sum \phi_\alpha^{(P)} \hat{J}^{(\alpha)}$ in which $\phi_\alpha^{(P)}$ is a function of the invariants in (21) and (22) that do not depend on $\tilde{\mathbf{g}}$ and $\hat{J}^{(\alpha)}$ is an invariant of (23) or (24) that does not depend on $\tilde{\mathbf{g}}$ and is linear in the components of vector $\mathbf{u}$:

$$\hat{\Phi}^{(P)} = \phi_o^{(P)} J_2 + \phi_1^{(P)} J_4 + \phi_2^{(P)} J_7 + \phi_3^{(P)} J_8 + \phi_4^{(P)} J_9 + \phi_5^{(P)} J_{10} + \hat{\Phi}_*^{(P)}. \tag{31}$$

$\hat{\Phi}_*^{(P)}$ represents terms constructed from the scalar triple products in (24). Comments similar to those following (27) apply here also.

The vector $\mathbf{\Phi}_P$ in (14) is recovered by the operation

$$(\mathbf{\Phi}_P)_i = \frac{\partial \hat{\Phi}^{(P)}}{\partial u_i}. \tag{32}$$

The expression obtained by combining (6) and (32), setting $\mathbf{e} = \mathbf{F}^T\mathbf{E}$, and then simplifying with the use of the Cayley–Hamilton theorem leads to

$$\mathbf{P} = \left[\hat{\alpha}_0^{(P)}\mathbf{I} + \hat{\alpha}_1^{(P)}\mathbf{B} + \hat{\alpha}_2^{(P)}\mathbf{B}^2\right]\mathbf{E} + \left[\hat{\alpha}_4^{(P)}\mathbf{I} + \hat{\alpha}_5^{(P)}\mathbf{B} + \hat{\alpha}_6^{(P)}\mathbf{B}^2\right]\mathbf{F}\mathbf{a}_o + \mathbf{F}\mathbf{\Phi}_P^* \tag{33}$$

where $\mathbf{\Phi}_P^*$ represents the terms obtained from substituting $\hat{\Phi}_*^{(P)}$ in (32). The scalar coefficients in (33) are now functions of the invariants I_1–I_8 and I_{12}–I_{14} in (25) and the invariants in (22) that do not depend on $\tilde{\mathbf{g}}$, were they to be re-expressed in terms of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{F}\mathbf{a}_o$.

An example of the terms contained in $\hat{\Phi}_*^{(P)}$ can be seen by considering the invariant $\tilde{J}_8^{ABC}$ with $A = B = C = 0$ in (24). It would generate the term

$$\det(\mathbf{F})\mathbf{E} \times \mathbf{B}^{-1}\mathbf{F}\mathbf{a}_o. \tag{34}$$

The tensorial structure of the first expression in (33), with $\mathbf{B}$ and $\mathbf{E}$, appears in the polarization vector developed in [8]. The terms with $\mathbf{F}\mathbf{a}_o$ are introduced by transverse isotropy.

4.3 Representation for the electrical current density

Let $\hat{\Phi}^{(J)} = \sum \phi_\alpha^{(J)} \hat{J}^{(\alpha)}$ in which $\phi_\alpha^{(J)}$ is a function of the invariants in (21) and (22) and $\hat{J}^{(\alpha)}$ is an invariant of (23) or (24) that is linear in the components of the vector $\mathbf{u}$:

$$\begin{aligned}\hat{\Phi}^{(J)} &= \phi_o^{(J)} J_2 + \phi_1^{(J)} J_3 + \phi_2^{(J)} J_4 + \phi_3^{(J)} J_7 + \phi_4^{(J)} J_8 \\ &\quad + \phi_5^{(J)} J_9 + \phi_6^{(J)} J_{10} + \phi_7^{(J)} J_{11} + \phi_8^{(J)} J_{12} + \hat{\Phi}_*^{(J)}.\end{aligned} \tag{35}$$

$\hat{\Phi}_*^{(J)}$ represents terms constructed from the scalar triple products in (24). Comments similar to those following (27) also apply here.

The tensor $\mathbf{\Phi}_J$ in (15) is recovered by the operation

$$(\mathbf{\Phi}_J)_i = \frac{\partial \hat{\Phi}^{(J)}}{\partial u_i}. \tag{36}$$

The expression obtained by combining (7) and (36), setting $\mathbf{e} = \mathbf{F}^T\mathbf{E}$ and $\tilde{\mathbf{g}} = \mathbf{F}^T\mathbf{g}$, and then simplifying with the use of the Cayley–Hamilton theorem, leads to

$$\begin{aligned}\mathbf{J} = {} & \left[\hat{\alpha}_0^{(J)}\mathbf{I} + \hat{\alpha}_1^{(J)}\mathbf{B} + \hat{\alpha}_2^{(J)}\mathbf{B}^2\right]\mathbf{E} + \left[\hat{\alpha}_4^{(J)}\mathbf{I} + \hat{\alpha}_5^{(J)}\mathbf{B} + \hat{\alpha}_6^{(J)}\mathbf{B}^2\right]\mathbf{F}\mathbf{a}_o \\ & + \left[\hat{\alpha}_7^{(J)}\mathbf{I} + \hat{\alpha}_8^{(J)}\mathbf{B} + \hat{\alpha}_9^{(J)}\mathbf{B}^2\right]\mathbf{g} + \mathbf{F}\mathbf{\Phi}_J^*. \end{aligned} \tag{37}$$

The scalar coefficients in (37) are now functions of the invariants in (25) and the invariants in (22), were they to be re-expressed in terms of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{g}$ and $\mathbf{F}\mathbf{a}_o$.The term in $\hat{\Phi}_*^{(J)}$ generated by invariant $\tilde{J}_9^{ABC}$ with $A = B = C = 0$ is

$$\det(\mathbf{F})\mathbf{g} \times \mathbf{E}. \tag{38}$$

4.4 Representation for the heat flux

The constitutive equation for $\mathbf{q}$ has the same mathematical form as for $\mathbf{J}$. Thus,

$$\begin{aligned}\mathbf{q} = {} & \left[\hat{\alpha}_0^{(q)}\mathbf{I} + \hat{\alpha}_1^{(q)}\mathbf{B} + \hat{\alpha}_2^{(q)}\mathbf{B}^2\right]\mathbf{E} + \left[\hat{\alpha}_4^{(q)}\mathbf{I} + \hat{\alpha}_5^{(q)}\mathbf{B} + \hat{\alpha}_6^{(q)}\mathbf{B}^2\right]\mathbf{F}\mathbf{a}_o \\ & + \left[\hat{\alpha}_7^{(q)}\mathbf{I} + \hat{\alpha}_8^{(q)}\mathbf{B} + \hat{\alpha}_9^{(q)}\mathbf{B}^2\right]\mathbf{g} + \mathbf{F}\mathbf{\Phi}_q^*. \end{aligned} \tag{39}$$

The scalar coefficients in (39) are now functions of the invariants in (25) and the invariants in (22), were they to be re-expressed in terms of $\mathbf{B}$, $\mathbf{E}$, $\mathbf{g}$ and $\mathbf{F}\mathbf{a}_o$.

5 Non-conducting matrix with conducting fibers

The constitutive Eqs. (29), (33), (37) and (39) were developed for a general transversely isotropic nonlinear electroelastic solid that conducts electrical current and heat. The equations are quite complex, with many terms and scalar coefficients that are functions of many invariant arguments. In order to be useful, these equations should be simplified in accordance with a particular application. The present work considers the simplest theory for a composite material composed of a non-conducting matrix reinforced with electroelastic fibers that conduct both current and heat.

Equation (29) for the stress reduces to

$$\mathbf{T} = \alpha_o^{(T)}\mathbf{I} + \alpha_1^{(T)}\mathbf{B} + \alpha_2^{(T)}\mathbf{B}^2 + \alpha_4^{(T)}\mathbf{F}\mathbf{a}_o \otimes \mathbf{F}\mathbf{a}_o. \tag{40}$$

Scalar coefficients $\alpha_o^{(T)}$, $\alpha_1^{(T)}$ and $\alpha_2^{(T)}$ depend only on invariants I_1, I_2 and I_3 while $\alpha_4^{(T)}$ depends on invariants I_4. I_6 and I_{12}. The first three terms in (40) represent the stress carried by the matrix, and the last term represents the normal stress carried by the fibers. The stress in the matrix depends only on the deformation of the matrix, while the stress in the fibers is affected by fiber stretch and the electric field along the fibers.

Equation (33) for the polarization becomes

$$\mathbf{P} = \alpha_0^{(P)}\mathbf{F}\mathbf{a}_o. \tag{41}$$

The polarization vector is in the direction of the deformed fiber and is affected by fiber stretch and the electric field along the fibers through the dependence of $\alpha_0^{(P)}$ on invariants I_4. I_6 and I_{12}. Equation (37) for the electric current density becomes

$$\mathbf{J} = \alpha_0^{(J)}\mathbf{F}\mathbf{a}_o. \tag{42}$$

The electrical current is along the deformed fiber and is affected by fiber stretch, the electric field along the fibers and heat conduction along the fibers through the dependence of $\alpha_0^{(J)}$ on the invariants in (26). Finally, Eq. (39) for the heat flux becomes

$$\mathbf{q} = \alpha_0^{(q)}\mathbf{F}\mathbf{a}_o. \tag{43}$$

Remarks similar to those for the current apply here.

6 Concluding comments

In this paper, we have obtained representations for various physical quantities such as the stress, polarization, electric current density and heat flux vector for a composite body comprised of a matrix embedded with fibers that are aligned along a particular direction. These representations are very complex and involve material moduli which are functions of various relevant invariants and are just too cumbersome to be useful. In fact, no experimental program can be envisaged which will provide adequate information concerning these material functions. Thus, it is left to the modeler to simplify these constitutive representations based on physical insight for the specific initial-boundary value problem under consideration. Our intent was to provide the general representation which can then be simplified.

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