Acta Mech 226, 3011–3029 (2015)
DOI 10.1007/s00707-015-1332-4

ORIGINAL PAPER

N. D. Anh · L. X. Hung · L. D. Viet · N. C. Thang

Global–local mean square error criterion for equivalent linearization of nonlinear systems under random excitation

Received: 18 August 2014 / Revised: 11 December 2014 / Published online: 1 May 2015

Abstract The paper presents a dual approach with respect to the local mean square error criterion to multi-degree-of-freedom nonlinear systems under stationary Gaussian random excitation. It results in new values of linearization coefficients that are obtained as global averaged values of all local linearization coefficients. Two examples of typical two-degree-of-freedom nonlinear systems under zero-mean stationary Gaussian random excitation are demonstrated. The results show that the accuracy of solutions by the proposed criterion is significantly improved in comparison with the one by the classical equivalent linearization method, especially when the nonlinearity is strong.

1 Introduction

Stochastic equivalent linearization or Gaussian equivalent linearization (GEL) proposed by Caughey [1] is a popular method used for analysis of stochastic nonlinear systems. The method is based on the replacement of a nonlinear oscillator under Gaussian excitation by a linear one under the same excitation. The standard way of implementing this technique is that the coefficients of linearization are to be found from the classical mean square error criterion which minimizes the equation error. Although the method is very efficient, its accuracy decreases as nonlinearity is increasing and in many cases it results in unacceptable errors. For this reason, a good deal of research has been conducted in recent decades on improving GEL, for example [2–6]. The method has also been investigated by many criteria [7–12]. In 2006, Crandall's work [13] described a number of interesting episodes in the history of the linearization technique that have arisen in the past half

N. D. Anh · L. D. Viet · N. C. Thang
Institute of Mechanics, VAST, 264, Doi Can, Hanoi, Vietnam
E-mail: ndanh@imech.ac.vn

L. D. Viet
E-mail: ldviet@imech.ac.vn

N. C. Thang
E-mail: caothang2002us@yahoo.com

N. D. Anh · L. D. Viet · N. C. Thang
University of Engineering and Technology, VNU, Hanoi, Vietnam

L. X. Hung (✉)
Hanoi Metropolitan Rail Board, 8 Ho Xuan Huong, Hanoi, Vietnam
E-mail: hunglx_hatd@fpt.vn
Tel.: 84-4-39445146
Fax: 84-4-39435126

century. In 1995, based on the assumption that the global integration domain taken in the mean square error criterion should be reduced to a local one where the response would be concentrated, Anh and Di Paola [14] proposed a local mean square error criterion (LOMSEC). Further investigation [15–17] has showed an improved accuracy of this criterion; however, the local domain in question was unknown and it has resulted in the main disadvantage of LOMSEC. Recently a dual conception was proposed in the study of responses to nonlinear systems [18] and has been developed in [19–21]. One significant advantage of the dual conception is its consideration of two different aspects of a problem in question, which allows the investigation to be more appropriate. In 2012, in a new development based on the dual conception to LOMSEC by Anh, Hung and Viet [22], the authors proposed a global–local mean square error criterion (GLOMSEC) to single-degree-of-freedom (SDOF) stochastic nonlinear systems, considering local and global levels, and thus new values of linearization coefficients are obtained as global averaged values of all local linearization coefficients. Although SDOF systems may provide an adequate mathematical representation of a considerable number of physical problems, dynamic systems with multi-degrees-of-freedom (MDOF) must be used in most cases of engineering. Thus, it is needed to address the issue of application of GEL to MDOF systems subjected to random excitation. The transition from the SDOF to the MDOF problem has been gradual over a half-century, and some researches can be mentioned for example: Atalic and Utku [23] presented the GEL procedure for MDOF systems with state space nonlinear equation of motion that are linearized according to the iterative scheme; Spanos [24] developed GEL for symmetric or asymmetric MDOF nonlinear systems; Faravelli, Casciati and Singh [25] provided two new procedures to GEL and their applicability to hysteretic systems; Casciati and Faravelli [26] developed GEL for three-dimensional frames; Casciati, Faravelli and Venini [27] investigated complex and plane structures under random excitation by means of the frequency domain method; Paola, Loppolo and Muscolino [28] analysed MDOF systems under stochastic seismic; Falsone [30] developed GEL for MDOF systems under parametric excitation; and others [31–33]. The above-mentioned researches are useful references to this paper's subject. This paper presents an extension of GLOMSEC to MDOF stochastic nonlinear systems and implements two examples for demonstration. The same as results obtained by the analysis of SDOF nonlinear systems [22], the numerical results for the two-degree-of-freedom nonlinear systems under white noise excitation demonstrate a significant improvement in the accuracy of solutions, especially when the nonlinearity is strong.

2 Formulation

We consider a MDOF nonlinear stochastic system in the following form:

$$M\ddot{q} + C\dot{q} + Kq + \Phi(q, \dot{q}) = Q(t), \tag{1}$$

where $M = [m_{ij}]_{n\times n}, C = [c_{ij}]_{n\times n}, K = [k_{ij}]_{n\times n}$ are $n \times n$ constant matrices, defined as the inertia, damping and stiffness matrices, respectively. $\Phi(q, \dot{q}) = [\Phi_1 \Phi_2 \cdots \Phi_n]^T$ is a nonlinear n-vector function of the generalized coordinate vector $q = [q_1 q_2 \cdots q_n]^T$ and its derivative $\dot{q} = [\dot{q}_1 \dot{q}_2 \cdots \dot{q}_n]^T$. The symbol (T) denotes the transpose of a matrix. The excitation $Q(t)$ is a zero-mean stationary Gaussian random vector process with the spectral density matrix $S_Q(\omega) = [S_{ij}(\omega)]_{n\times n}$, where $S_{ij}(\omega)$ is the spectral density function of elements Q_i and Q_j.

An equivalent linear system to the original nonlinear system (1) can be defined as

$$M\ddot{q} + (C + C^e)\dot{q} + (K + K^e)q = Q(t), \tag{2}$$

where $C^e = [c^e_{ij}]_{n\times n}, K^e = [k^e_{ij}]_{n\times n}$ are deterministic matrices. They are to be determined so that the n-vector difference $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]^T$ between the original and the equivalent system is minimized. For the conventional linearization method, as shown in [29] by Roberts and Spanos, the matrices C^e, K^e and are determined by the following criterion

$$E\left\{\varepsilon^T \varepsilon\right\} \to \min_{c^e_{ij}, k^e_{ij}}; \quad (i, j = 1, 2, \ldots, n), \tag{3}$$

where $E\{.\}$ denotes the mathematical expectation operation and c^e_{ij}, k^e_{ij} are the (i, j) elements of the matrices C^e, K^e and

$$\varepsilon = \Phi(q, \dot{q}) - C^e\dot{q} - K^e q. \tag{4}$$

Upon utilizing the linearity property of the expectation operator $E\{.\}$, criterion (3) can be written in the following form [29]

$$E\left\{\varepsilon_\alpha^2\right\} \to \min_{c_{ij}^e, k_{ij}^e}; \quad (\alpha = 1, 2, \ldots, n). \tag{5}$$

The necessary conditions for the criterion (5) to be true are

$$\frac{\partial}{\partial c_{ij}^e} E\left\{\varepsilon_\alpha^2\right\} = 0, \quad \frac{\partial}{\partial k_{ij}^e} E\left\{\varepsilon_\alpha^2\right\} = 0; \quad (i, j = 1, 2, \ldots, n). \tag{6}$$

Utilizing equations (4) and (6), and implementing some necessary algebraic analyses, one obtains the expressions to determine c_{ij}^e, k_{ij}^e as follows

$$c_{ij}^e = E\left\{\frac{\partial \Phi_i}{\partial \dot{q}_j}\right\}, \quad k_{ij}^e = E\left\{\frac{\partial \Phi_i}{\partial q_j}\right\}, \tag{7}$$

where Φ_i is the (i) element of $\Phi(q, \dot{q})$.

The spectral density matrix of the response process $q(t)$ is of the form

$$S_q(\omega) = [S_{q_i q_j}(\omega)], \quad (i, j = 1, 2, \ldots, n), \tag{8}$$

where $S_{q_i q_j}(\omega)$ is the $(i, j)th$ element of $S_q(\omega)$.

Using the matrix spectral input–output relationship to the linear system (2), one gets

$$S_q(\omega) = \alpha(\omega) S_Q(\omega) \alpha^T(\omega), \tag{9}$$

where $\alpha(\omega)$ is the matrix of frequency response functions. It is known as

$$\alpha(\omega) = \left[-\omega^2 M + i\omega(C + C^e) + (K + K^e)\right]^{-1}. \tag{10}$$

The mean values of the response can be calculated by the following equations:

$$E\left\{q_i q_j\right\} = \int_{-\infty}^{\infty} S_{q_i q_j}(\omega) \mathrm{d}\omega, \quad E\left\{q q^T\right\} = \int_{-\infty}^{\infty} \alpha(-\omega) S_Q(\omega) \alpha^T(\omega) \mathrm{d}\omega,$$

$$E\left\{\dot{q} \dot{q}^T\right\} = \int_{-\infty}^{\infty} \omega^2 \alpha(-\omega) S_Q(\omega) \alpha^T(\omega) \mathrm{d}\omega. \tag{11}$$

Equations (2), (7), (9–11) establish a set of nonlinear algebraic equations to determine the mean values of the response.

Denoting by $p(q)$ the stationary joint probability density function (PDF) of the vector $q = [q_1, q_2, \ldots, q_n]^T$, we can write criterion (5) in the following form

$$E\left\{\varepsilon_\alpha^2\right\} = \int_{-\infty}^{+\infty} (n) \int_{-\infty}^{+\infty} \varepsilon_\alpha^2 p(q) \mathrm{d}q \to \min_{c_{ij}^e, k_{ij}^e}; (\alpha, i, j = 1, 2, \ldots, n). \tag{12}$$

The square error criterion (the classical criterion) is very efficient, however, its accuracy decreases as nonlinearity is increasing and in many cases it leads to unacceptable errors. Thus, the problem of modifying the classical criterion in order to improve the accuracy has attracted the attention of many researches [2–12]. Since the integration is taken over the entire coordinate space $(-\infty, +\infty)$, criterion (12) may be called global mean square error criterion. In 1995, based on the assumption that the global integration domain taken in the classical criterion should be reduced to a local one where the response would be concentrated, Anh and Di Paola [14] proposed a local mean square error criterion (LOMSEC) and established an algorithm for SDOF stochastic nonlinear systems. The further investigations [15–17] have developed LOMSEC to MDOF stochastic nonlinear systems, in which criterion (12) is modified by

$$\int_{-q_{01}}^{+q_{01}} (n) \int_{-q_{0n}}^{+q_{0n}} \varepsilon_\alpha^2 p(q) \mathrm{d}q \to \underset{c_{ij}^e, k_{ij}^e}{\text{minimum}}, \tag{13}$$

where $q_{01}, q_{02}, \ldots, q_{0n}$ are given positive values. The expected integrations in (13) can be transformed to non-dimensional variables by $q_{01} = r\sigma_{q1}, q_{02} = r\sigma_{q2}, \ldots, q_{0n} = r\sigma_{qn}$ with r a given positive value; $\sigma_{q1}, \sigma_{q2}, \ldots, \sigma_{qn}$ are the normal deviations of random variables of $q_1, q_2, \ldots, q_n$, respectively. Thus, criterion (13) leads to

$$E\left[\varepsilon_\alpha^2\right] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} (n) \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} \varepsilon_\alpha^2 p(q)\mathrm{d}q \to \underset{c_{ij}^e, k_{ij}^e}{\text{minimum}}, \tag{14}$$

where $E[.]$ denotes the local mean values of random variables, which are taken as follows:

$$E[.] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} (n) \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} (.)\, p(q)\mathrm{d}q \underset{\text{For example}}{\to} E\left[q_i q_j\right] = \int_{-r\sigma_{q1}}^{+r\sigma_{q1}} (n) \int_{-r\sigma_{qn}}^{+r\sigma_{qn}} q_i q_j\, p(q)\mathrm{d}q. \tag{15}$$

In GEL, the values $\sigma_{q1}, \sigma_{q2}, \ldots, \sigma_{qn}$ are considered to be independent from c_{ij}^e, k_{ij}^e in the process of minimizing (14). Criterion (14) results in the necessary conditions for determining c_{ij}^e, k_{ij}^e as follows:

$$\frac{\partial}{\partial c_{ij}^e} E\left[\varepsilon_\alpha^2\right] = 0, \quad \frac{\partial}{\partial k_{ij}^e} E\left[\varepsilon_\alpha^2\right] = 0 \tag{16}$$

It is seen from (14) to (16) that the elements of c_{ij}^e, k_{ij}^e obtained by LOMSEC are functions depending on the local mean values of random variables and also depending on r (i.e. $c_{ij}^e = c_{ij}^e(r), k_{ij}^e = k_{ij}^e(r)$), which is not explicitly expressed here.

Since the linearization coefficients $c_{ij}^e(r), k_{ij}^e(r)$ determined by (16) are functions depending on the parameter r, they become constant values when r is determined. In this sense, the linearization coefficients $c_{ij}^e(r), k_{ij}^e(r)$ can be called local linearization coefficients. Equations (2), (15) and (16) allow to determine the unknowns $c_{ij}^e(r), k_{ij}^e(r)$ and the vector $q(t)$ when r is given. Some advantages of LOMSEC can easily be seen as follows [15–17]: First, by changing values of r, LOMSEC can create a series of various approximate solutions, and as $r = \infty$, LOMSEC gives the same solution as the classical criterion does; LOMSEC also implies the existence of a value (r_{exact}) that in principle allows to obtain the exact solution, while this is impossible for the classical criterion; the most important advantage of LOMSEC is that it enables to obtain much more accurate solutions than the ones of the classical criterion [15–17]. The main disadvantage of LOMSEC, however, is that the local domain of integration, namely in our case the value of r, is unknown and the open question is how to find it. Recently a dual conception was proposed in the study of responses to nonlinear systems [18] and has been developed in [19–21]. One of the significant advantages of the dual conception is that its consideration of two different aspects of a problem in question allows the investigation to be more appropriate. Using the dual approach to LOMSEC, it is suggested that instead of finding a special value of r one may consider its variation in all the global domain of integration. Thus, the linearization coefficients $c_{ij}^e(r), k_{ij}^e(r)$ can be suggested as global mean values of all local linearization coefficients as follows:

$$c_{ij}^e = \left\langle c_{ij}^e(r) \right\rangle = \lim_{s\to\infty} \frac{1}{s} \int_0^s c_{ij}^e(r)\mathrm{d}r, \quad k_{ij}^e = \left\langle k_{ij}^e(r) \right\rangle = \lim_{s\to\infty} \frac{1}{s} \int_0^s k_{ij}^e(r)\mathrm{d}r, \tag{17}$$

where $\langle . \rangle$ denotes conventionally the average of operators of deterministic functions.

Now Eqs. (2), (15), (16), (17) allow to determine the unknowns without specifying any value of r. The dual approach to LOMSEC leads to a new criterion that may be called global–local mean square error criterion (GLOMSEC).

3 Two-degree-of-freedom oscillator with nonlinear stiffness

For the purpose of proving the great advantage of GLOMSEC to the solution accuracy, this part investigates a two-degree-of-freedom system with nonlinear stiffness under white noise excitation [32] which is described by

$$\ddot{x}_i + h\dot{x}_i + \frac{\partial}{\partial x_i} U(x_1, x_2) = w_i(t), \quad i = 1, 2, \tag{18}$$

where $U(x_1, x_2)$ is the potential energy and is given by

$$U(x_1, x_2) = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + \gamma_1 x_1^4 + \gamma_3 x_1^2 x_2^2 + \gamma_5 x_2^4 \tag{19}$$

and $h, \omega_1, \omega_2, \gamma_1, \gamma_3, \gamma_5$ are positive constants; $w_i(t)$ with $i = 1, 2$ are the components of the vector $w(t)$, which is a zero-mean Gaussian white noise stationary random vector process with the following correlation function:

$$K_{ij}(\tau) = E\left\{w_i(t)w_j(t+\tau)\right\} = 2\pi S_i \delta_{ij}\delta(\tau), \quad (i, j = 1, 2), \tag{20}$$

where $\delta(\tau)$ is the Dirac delta function, δ_{ij} is the Kronecker symbol, S_1, S_2 are constant values of the spectral density of the random excitations $w_1(t)$, $w_2(t)$, respectively.

3.1 Exact solution

In the case of the same spectral density of random excitations $S_1 = S_2 = S_0$, the Fokker–Planck equation gives an exact stationary PDF to system (18) as follows:

$$p(x_1, x_2) = C \exp\left\{-\frac{h}{\pi S_0}U(x_1, x_2)\right\}, \tag{21}$$

$$C = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{h}{\pi S_0}U(x_1, x_2)\right\} \mathrm{d}x_1 \mathrm{d}x_2\right)^{-1}. \tag{22}$$

Here C is the normalization constant.

The exact mean square responses are defined by

$$E\left\{x_i^2\right\}_{ex} = C\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_i^2 \exp\left\{-\frac{h}{\pi S_0}U(x_1, x_2)\right\} dx_1 dx_2, \quad i = 1, 2. \tag{23}$$

The exact solution (23) is used for evaluating relative errors of approximate solutions.

3.2 LOMSEC

By utilizing (19), Eq. (18) can be rewritten in matrix form as

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}h & 0\\ 0 & h\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}4\gamma_1 x_1^3 + 2\gamma_3 x_1 x_2^2\\ 4\gamma_5 x_2^3 + 2\gamma_3 x_1^2 x_2\end{bmatrix} = \begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}. \tag{24}$$

Following Eq. (1), we denote

$$M = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \mathrm{C} = \begin{bmatrix}h & 0\\ 0 & h\end{bmatrix}, \quad \mathrm{K} = \begin{bmatrix}\omega_1^2 & 0\\ 0 & \omega_2^2\end{bmatrix}, \quad \Phi = \begin{bmatrix}4\gamma_1 x_1^3 + 2\gamma_3 x_1 x_2^2\\ 4\gamma_5 x_2^3 + 2\gamma_3 x_1^2 x_2\end{bmatrix}, \quad x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{25}$$

The linear system to (24) is taken in the following form:

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}h & 0\\ 0 & h\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}\omega_1^2 + k_{11}^e & k_{12}^e\\ k_{21}^e & \omega_2^2 + k_{22}^e\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}, \tag{26}$$

where $k_{ij}^e (i, j = 1, 2)$ are the linearization coefficients.

According to formula (4), the difference ε between (24) and (26) is

$$\varepsilon = \Phi(x) - K^e x, \tag{27}$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 4\gamma_1 x_1^3 + 2\gamma_3 x_1 x_2^2 \\ 4\gamma_5 x_2^3 + 2\gamma_3 x_1^2 x_2 \end{bmatrix}, \quad K^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{28}$$

Utilizing (16) for determining the components of K^e yields

$$\begin{aligned}
\frac{\partial}{\partial k_{11}^e} E\left[\varepsilon_1^2\right] &= \frac{\partial}{\partial k_{11}^e} E\left[\left(4\gamma_1 x_1^3 + 2\gamma_3 x_1 x_2^2 - k_{11}^e x_1 - k_{12}^e x_2\right)^2\right] \\
&= -4\gamma_1 E\left[x_1^4\right] - 2\gamma_3 E\left[x_1^2 x_2^2\right] + k_{11}^e E\left[x_1^2\right] + k_{12}^e E\left[x_1 x_2\right] = 0, \\
\frac{\partial}{\partial k_{12}^e} E\left[\varepsilon_1^2\right] &= \frac{\partial}{\partial k_{12}^e} E\left[\left(4\gamma_1 x_1^3 + 2\gamma_3 x_1 x_2^2 - k_{11}^e x_1 - k_{12}^e x_2\right)^2\right] \\
&= -4\gamma_1 E\left[x_1^3 x_2\right] - 2\gamma_3 E\left[x_1 x_2^3\right] + k_{11}^e E\left[x_1 x_2\right] + k_{12}^e E\left[x_2^2\right] = 0, \\
\frac{\partial}{\partial k_{21}^e} E\left[\varepsilon_2^2\right] &= \frac{\partial}{\partial k_{21}^e} E\left[\left(4\gamma_5 x_2^3 + 2\gamma_3 x_1^2 x_2 - k_{21}^e x_1 - k_{22}^e x_2\right)^2\right] \\
&= -4\gamma_5 E\left[x_1 x_2^3\right] - 2\gamma_3 E\left[x_1^3 x_2\right] + k_{21}^e E\left[x_1^2\right] + k_{22}^e E\left[x_1 x_2\right] = 0, \\
\frac{\partial}{\partial k_{22}^e} E\left[\varepsilon_2^2\right] &= \frac{\partial}{\partial k_{22}^e} E\left[\left(4\gamma_5 x_2^3 + 2\gamma_3 x_1^2 x_2 - k_{21}^e x_1 - k_{22}^e x_2\right)^2\right] \\
&= -4\gamma_5 E\left[x_2^4\right] - 2\gamma_3 E\left[x_1^2 x_2^2\right] + k_{21}^e E\left[x_1 x_2\right] + k_{22}^e E\left[x_2^2\right] = 0.
\end{aligned} \tag{29}$$

Since x_1, x_2 are independent as assumed, $E\left[x_1^{2n+1}\, x_2^{2m+1}\right] = 0$, and $E\left[x_1^{2n}\, x_2^{2m}\right]$ is expressed by (A.10) and (A.11) in the Appendix. Thus, equation system (29) gives $k_{12}^e(r) = k_{21}^e(r) = 0$ and the equations for determining $k_{11}^e(r), k_{22}^e(r)$ as follows:

$$\begin{aligned}
k_{11}^e(r) &= \frac{4\gamma_1 E\left[x_1^4\right] + 2\gamma_3 E\left[x_1^2 x_2^2\right]}{E\left[x_1^2\right]} = \frac{4\gamma_1 2T_{2,r}\left(E\left\{x_1^2\right\}\right)^2 2T_{0,r} + 2\gamma_3 2T_{1,r} E\left\{x_1^2\right\} 2T_{1,r} E\left\{x_2^2\right\}}{2T_{1,r} E\left\{x_1^2\right\} 2T_{0,r}} \\
&= 4\gamma_1 E\left\{x_1^2\right\} \frac{T_{2,r}}{T_{1,r}} + 2\gamma_3 E\left\{x_2^2\right\} \frac{T_{1,r}}{T_{0,r}}, \\
k_{22}^e(r) &= \frac{4\gamma_5 E\left[x_2^4\right] + 2\gamma_3 E\left[x_1^2 x_2^2\right]}{E\left[x_2^2\right]} = \frac{4\gamma_5 2T_{0,r} 2T_{2,r}\left(E\left\{x_2^2\right\}\right)^2 + 2\gamma_3 2T_{1,r} E\left\{x_1^2\right\} 2T_{1,r} E\left\{x_2^2\right\}}{2T_{0,r} 2T_{1,r} E\left\{x_2^2\right\}} \\
&= 4\gamma_5 E\left\{x_2^2\right\} \frac{T_{2,r}}{T_{1,r}} + 2\gamma_3 E\left\{x_1^2\right\} \frac{T_{1,r}}{T_{0,r}},
\end{aligned} \tag{30}$$

where, $T_{0,r}, T_{1,r}, T_{2,r}$ are defined by (A.9) in the Appendix as follows:

$$T_{0,r} = \int_0^r \eta(t)\mathrm{d}t, \quad T_{1,r} = \int_0^r t^2 \eta(t)\mathrm{d}t, \quad T_{2,r} = \int_0^r t^4 \eta(t)\mathrm{d}t, \quad \eta(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}. \tag{31}$$

3.3 Conventional linearization method

Utilizing (31) with $r \to \infty$, the following factors in (30) are defined:

$$\frac{T_{2,\infty}}{T_{1,\infty}} = \frac{\int_0^\infty t^4 \eta(t)\mathrm{d}t}{\int_0^\infty t^2 \eta(t)\mathrm{d}t} = 3, \quad \frac{T_{1,\infty}}{T_{0,\infty}} = \frac{\int_0^\infty t^2 \eta(t)\mathrm{d}t}{\int_0^\infty \eta(t)\mathrm{d}t} = 1. \tag{32}$$

Equation system (30) and the values given by (32) allow determining k_{11}^e, k_{22}^e by the conventional linearization method:

$$k_{11}^e = 12\gamma_1 E\left\{x_1^2\right\} + 2\gamma_3 E\left\{x_2^2\right\}, \quad k_{22}^e = 12\gamma_5 E\left\{x_2^2\right\} + 2\gamma_3 E\left\{x_1^2\right\}. \tag{33}$$

3.4 GLOMSEC

Utilizing (17) and (30), one obtains the equations for determining k_{11}^e, k_{22}^e by GLOMSEC:

$$k_{11}^e = \left\langle k_{11}^e(r)\right\rangle = \operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s k_{11}^e(r)\mathrm{d}r = 4\gamma_1 E\left\{x_1^2\right\}\operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{2,r}}{T_{1,r}}\mathrm{d}r + 2\gamma_3 E\left\{x_2^2\right\}\operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{1,r}}{T_{0,r}}\mathrm{d}r,$$

$$k_{22}^e = \left\langle k_{22}^e(r)\right\rangle = \operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s k_{22}^e(r)\mathrm{d}r = 4\gamma_5 E\left\{x_2^2\right\}\operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{2,r}}{T_{1,r}}\mathrm{d}r + 2\gamma_3 E\left\{x_1^2\right\}\operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{1,r}}{T_{0,r}}\mathrm{d}r. \quad (34)$$

The limitation expressions in (34) can be approximately computed, and their outputs are

$$\operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{2,r}}{T_{1,r}}\mathrm{d}r \approx 2.41189, \quad \operatorname*{Lim}_{s\to\infty}\frac{1}{s}\int_0^s\frac{T_{1,r}}{T_{0,r}}\mathrm{d}r \approx 0.83706. \quad (35)$$

3.5 Approximate mean square responses

Under the assumption (20) and $S_1 = S_2 = S_0$, the spectral density matrix $S_w(\omega)$ of $w(t)$ is defined by

$$S_w(\omega) = \begin{bmatrix} S_0 & 0 \\ 0 & S_0 \end{bmatrix}. \quad (36)$$

The matrix of frequency response function to linear system (26) is

$$\alpha(\omega) = \left[-\omega^2 M + i\omega C + (K + K^e)\right]^{-1}, \quad (37)$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}, \quad K = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}, \quad K^{\mathrm{e}} = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^{\mathrm{e}} \end{bmatrix}, \quad (38)$$

and $k_{12}^e = k_{21}^e = 0$, which are the outputs from the above analyses since the given assumption. Utilizing (38), the formula (37) can be expanded as follows:

$$\begin{aligned}\alpha(\omega) &= \left[\begin{bmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{bmatrix} + \begin{bmatrix} i\omega h & 0 \\ 0 & i\omega h \end{bmatrix} + \begin{bmatrix} \omega_1^2 + k_{11}^e & 0 \\ 0 & \omega_2^2 + k_{22}^e \end{bmatrix}\right]^{-1} \\ &= \begin{bmatrix} -\omega^2 + i\omega h + \omega_1^2 + k_{11}^e & 0 \\ 0 & -\omega^2 + i\omega h + \omega_2^2 + k_{22}^e \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (-\omega^2 + i\omega h + \omega_1^2 + k_{11}^e)^{-1} & 0 \\ 0 & (-\omega^2 + i\omega h + \omega_2^2 + k_{22}^e)^{-1} \end{bmatrix}. \end{aligned} \quad (39)$$

Introduce the following notations:

$$\omega_{e11}^2 = \omega_1^2 + k_{11}^e, \quad \omega_{e22}^2 = \omega_2^2 + k_{22}^e, \quad L(\omega) = \left(-\omega^2 + i\omega h + \omega_{e11}^2\right)\left(-\omega^2 + i\omega h + \omega_{e22}^2\right). \quad (40)$$

Utilizing (40), formula (39) can be replaced by

$$\begin{aligned}\alpha(\omega) &= \frac{L(\omega)}{L(\omega)}\begin{bmatrix} (-\omega^2 + i\omega h + \omega_{e11}^2)^{-1} & 0 \\ 0 & (-\omega^2 + i\omega h + \omega_{e22}^2)^{-1} \end{bmatrix} \\ &= \frac{1}{L(\omega)}\begin{bmatrix} -\omega^2 + i\omega h + \omega_{e22}^2 & 0 \\ 0 & -\omega^2 + i\omega h + \omega_{e11}^2 \end{bmatrix}. \end{aligned} \quad (41)$$

Table 1 Mean square of x_1 versus $S_0 = 1, h = 0.5, \omega_1 = 2, \omega_2 = 4, \gamma_1 = \gamma_3 = \gamma_5 = \gamma(0.1 - 100)$

γ	$E\left\{x_1^2\right\}_{ex}$	$E\left\{x_1^2\right\}_{co}$	Error (%)	$E\left\{x_1^2\right\}_{gl}$	Error (%)
0.1	1.1782	1.1514	2.275	1.2028	2.088
1	0.6038	0.5567	7.801	0.6059	0.348
10	0.2252	0.2008	10.835	0.2220	1.421
100	0.0746	0.0659	11.662	0.0732	1.877

Utilizing (9), (11) together with (36), (41), one obtains a matrix of mean square elements as follows:

$$E\left\{xx^T\right\} = \int_{-\infty}^{\infty} \alpha(-\omega) S_w(\omega) \alpha^T(\omega) \mathrm{d}\omega = \int_{-\infty}^{\infty} \frac{1}{L(\omega)L(-\omega)} \begin{bmatrix} r_{11}(\omega) & 0 \\ 0 & r_{22}(\omega) \end{bmatrix} \mathrm{d}\omega, \tag{42}$$

where

$$r_{11}(\omega) = S_0\left(\left(-\omega^2 + \omega_{e22}^2\right)^2 + h^2\omega^2\right), \quad r_{22}(\omega) = S_0\left(\left(-\omega^2 + \omega_{e11}^2\right)^2 + h^2\omega^2\right). \tag{43}$$

By expanding $L(\omega)$ in (40), one obtains a polynomial function of argument $i\omega$ as follows:

$$L(\omega) = R(i\omega) = \lambda_4(i\omega)^4 + \lambda_3(i\omega)^3 + \lambda_2(i\omega)^2 + \lambda_1(i\omega) + \lambda_0. \tag{44}$$

The coefficients in (44) can be easily found by comparison with the respective coefficients of the expanded $L(\omega)$, the results are

$$\lambda_4 = 1, \quad \lambda_3 = 2h, \quad \lambda_2 = \omega_{e11}^2 + \omega_{e22}^2 + h^2, \quad \lambda_1 = h\omega_{e11}^2 + h\omega_{e22}^2, \quad \lambda_0 = \omega_{e11}^2\omega_{e22}^2. \tag{45}$$

By applying formula of integrals presented in [29], the integrals in (42) can be computed as follows:

$$E\left\{x_1^2\right\} = \int_{-\infty}^{\infty} \frac{r_{11}(\omega)\mathrm{d}\omega}{R(i\omega)R(-i\omega)}, \quad E\left\{x_2^2\right\} = \int_{-\infty}^{\infty} \frac{r_{22}(\omega)\mathrm{d}\omega}{R(i\omega)R(-i\omega)}. \tag{46}$$

Then one gets

$$\begin{aligned} E\left\{x_1^2\right\} &= \frac{\pi}{\lambda_4} \begin{vmatrix} 0 & \xi_2 & \xi_1 & \xi_0 \\ -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & -\lambda_3 & \lambda_1 & 0 \\ 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix} \begin{vmatrix} \lambda_3 & -\lambda_1 & 0 & 0 \\ -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & -\lambda_3 & \lambda_1 & 0 \\ 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix}^{-1}, \\ E\left\{x_2^2\right\} &= \frac{\pi}{\lambda_4} \begin{vmatrix} 0 & \mu_2 & \mu_1 & \mu_0 \\ -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & -\lambda_3 & \lambda_1 & 0 \\ 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix} \begin{vmatrix} \lambda_3 & -\lambda_1 & 0 & 0 \\ -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & -\lambda_3 & \lambda_1 & 0 \\ 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix}^{-1}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\xi_2 = S_0, \quad \xi_1 = S_0(h^2 - 2\omega_{e22}^2), \quad \xi_0 = S_0\omega_{e22}^4; \\ &\mu_2 = S_0, \quad \mu_1 = S_0(h^2 - 2\omega_{e11}^2), \quad \mu_0 = S_0\omega_{e11}^4; \end{aligned} \tag{48}$$

Combining (47) with either (33) or (34), we get close algebraic equation systems for determining the unknowns $k_{11}^e, k_{22}^e, E\left\{x_1^2\right\}, E\left\{x_2^2\right\}$ that are given by conventional method and GLOMSEC, respectively. Such close algebraic equation systems are in general nonlinear and can be computed by numerical approaches.

Consider the system parameters to be $S_0 = 1, h = 0.5, \omega_1 = 2, \omega_2 = 4, \gamma_1 = \gamma_3 = \gamma_5 = \gamma(0.1 - 100)$. Denote by $E\left\{x_i^2\right\}_{ex}, E\left\{x_i^2\right\}_{co}, E\left\{x_i^2\right\}_{gl}$ the mean square responses of $x_i (i = 1, 2)$ given by the exact solution, conventional method, and GLOMSEC, respectively. Tables 1 and 2 give the numerical results including the relative errors.

Based on the relative errors of the approximate solutions with respect to the exact solution, it can be seen that for the considered case, GLOMSEC gives the significant improvement in the accuracy of the solution

Table 2 Mean square of x_2 versus $S_0 = 1$, $h = 0.5$, $\omega_1 = 2$, $\omega_2 = 4$, $\gamma_1 = \gamma_3 = \gamma_5 = \gamma\,(0.1 - 100)$

γ	$E\left\{x_2^2\right\}_{ex}$	$E\left\{x_2^2\right\}_{co}$	Error (%)	$E\left\{x_2^2\right\}_{gl}$	Error (%)
0.1	0.3768	0.3766	0.053	0.3793	0.663
1	0.3064	0.3028	1.175	0.3135	2.317
10	0.1699	0.1602	5.709	0.1727	1.648
100	0.0677	0.0612	9.600	0.0674	0.443

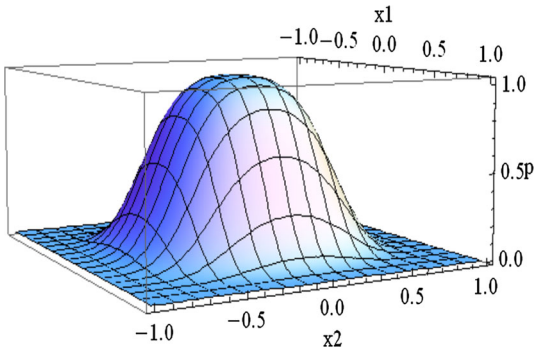

Fig. 1 PDF (21), with $U = 0.5\omega_1^2 x_1^2 + 0.5\omega_2^2 x_2^2 + \gamma_1 x_1^4 + \gamma_3 x_1^2 x_2^2 + \gamma_5 x_2^4$

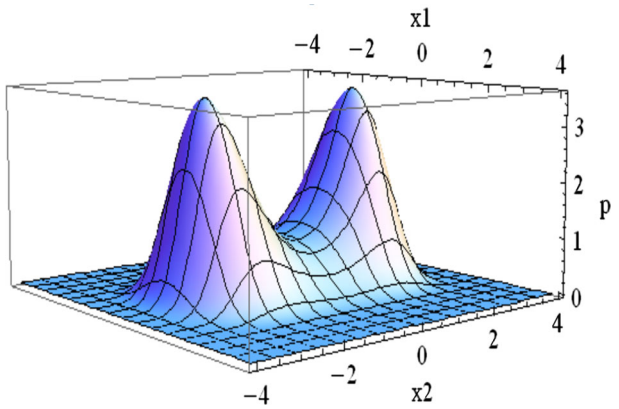

Fig. 2 PDF (21), with $U = 0.5\omega_1^2 x_1^2 - 0.5\omega_2^2 x_2^2 + \gamma_1 x_1^4 + \gamma_3 x_1^2 x_2^2 + \gamma_5 x_2^4$

in comparison with the conventional GEL method, especially when the nonlinearity is strong. The authors investigated lots of various values of the parameters, greater S_0 included, the obtained outputs results in the same remark as above.

PDF (21) of the considered case is symmetric as shown in Fig. 1 to arbitrary value of the parameters. However, in the potential energy function (19) if the first term or the second one is negative, then PDF will have two peaks, as shown in Fig. 2. Anh and Hung [16] applied LOMSEC for a Duffing system with two peaks, which happen when the first-order stiffness term is negative ($\ddot{x} + 2h\dot{x} - \beta x + \varepsilon x^3 = \sigma\xi(t)$). The result indicates that the GEL methods are not efficient (the solution errors are unacceptable) when the parameter ε is small. However, when ε is larger, the solution accuracy by LOMSEC is significantly improved in comparison with the conventional GEL.

Return to the considered system when the potential energy function is taken as $U = 0.5\omega_1^2 x_1^2 - 0.5\omega_2^2 x_2^2 + \gamma_1 x_1^4 + \gamma_3 x_1^2 x_2^2 + \gamma_5 x_2^4$ and PDF shown in Fig. 2. The mean square responses of $x_i\,(i = 1, 2)$ given in Tables 3 and 4.

The results indicate that when γ is small ($\gamma \leq 1$), the GEL methods are not efficient. However, when γ is larger ($\gamma >> 1$), the solution accuracy by GLOMSEC is significantly improved in comparison with the conventional GEL.

Table 3 Mean square of x_1 versus $S_0 = 1, h = 0.5, \omega_1 = 2, \omega_2 = 4, \gamma_1 = \gamma_3 = \gamma_5 = \gamma(0.1 - 100)$

γ	$E\{x_1^2\}_{ex}$	$E\{x_1^2\}_{co}$	Error (%)	$E\{x_1^2\}_{gl}$	Error (%)
0.1	0.5082	1.1776	131.720	1.2274	141.519
1	0.4346	0.5980	37.600	0.6489	49.310
10	0.2078	0.1919	7.652	0.2107	1.396
100	0.0728	0.0650	10.714	0.0720	1.100

Table 4 Mean square of x_2 versus $S_0 = 1, h = 0.5, \omega_1 = 2, \omega_2 = 4, \gamma_1 = \gamma_3 = \gamma_5 = \gamma(0.1 - 100)$

γ	$E\{x_2^2\}_{ex}$	$E\{x_2^2\}_{co}$	Error (%)	$E\{x_2^2\}_{gl}$	Error (%)
0.1	39.3378	0.3872	9016	0.3886	9012
1	3.2700	0.3340	898	0.3445	895
10	0.3722	0.2850	23.428	0.3279	11.902
100	0.0867	0.0736	15.110	0.0828	4.498

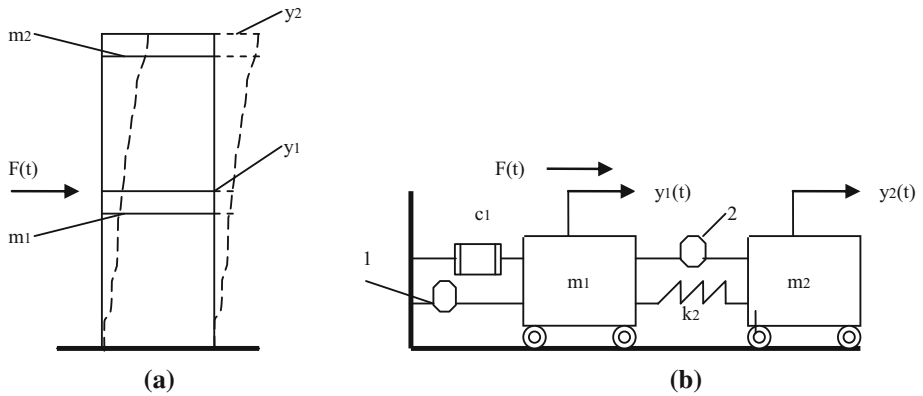

Fig. 3 Two-degree-of-freedom system with nonlinear damping and stiffness. **a** Two-storey building structure. **b** Equivalent mass-spring-damper model

4 Two-degree-of-freedom oscillator with nonlinear damping and stiffness

4.1 Equation of motion

Consider a two-storey building structure shown in Fig. 3a [29]. Here, the structure may be idealized as two rigid masses m_1 and m_2, which move horizontally as the result of applied fluctuating force $F(t)$ acting on mass m_1. An equivalent two-degree-of-freedom model of the structure is then shown in Fig. 3b. The absolute displacements of m_1 and m_2 measured from the static equilibrium position are denoted by y_1 and y_2, respectively. Mass m_1 is connected to the foundation by a linear damper with coefficient c_1 and a nonlinear spring of the linear-plus-cubic type with coefficient k_1, whereas m_1 and m_2 are connected by a linear spring of stiffness k_2 and a nonlinear damper of the linear-plus-quadratic type with coefficient c_2. The force of the nonlinear spring is given by $k_1 y_1(1+\delta_1 y_1^2)$, whereas the force in the nonlinear damper is given by $c_2(\dot{y}_2 - \dot{y}_1)(1+\delta_2|\dot{y}_2 - \dot{y}_1|)$. Assume that $F(t) = m_1 p(t)$, where $p(t)$ is obtained by passing white noise through a linear first-order filter. The spectral density function $S_p(\omega)$ of $p(t)$ is assumed to be a first-order spectrum with form $S_p(\omega) = S_0/(\alpha^2 + \omega^2)$, where α, S_0 are constants. The equation of motion of the equivalent model in terms of the coordinates y_1 and y_2 may be written as

$$\begin{aligned} &m_1\ddot{y}_1 + c_1\dot{y}_1 - k_2(y_2 - y_1) + k_1 y_1(1 + \delta_1 y_1^2) - c_2(\dot{y}_2 - \dot{y}_1)(1 + \delta_2|\dot{y}_2 - \dot{y}_1|) = m_1 p(t), \\ &m_2\ddot{y}_2 + k_2(y_2 - y_1) + c_2(\dot{y}_2 - \dot{y}_1)(1 + \delta_2|\dot{y}_2 - \dot{y}_1|) = 0. \end{aligned} \tag{49}$$

Since the nonlinear damping element force and the linear spring force depend on relative velocity and relative displacement, respectively, it is convenient to introduce the following transformation:

$$q_1 = y_1, \quad q_2 = y_2 - y_1. \tag{50}$$

Denote

$$\omega_1^2 = \frac{k_1}{m_1}, \quad \omega_2^2 = \frac{k_2}{m_2}, \quad \mu = \frac{m_2}{m_1}, \quad \beta_1 = \frac{\delta_1 k_1}{m_1}, \quad \beta_2 = \frac{\delta_2 c_2}{m_2}, \quad \zeta_1 = \frac{c_1}{2\sqrt{k_1 m_1}}, \quad \zeta_2 = \frac{c_2}{2\sqrt{k_2 m_2}}. \tag{51}$$

Utilizing (50) and (51), system (49) can be rewritten as

$$\begin{aligned} &\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2 q_1 - 2\mu\zeta_2\omega_2\dot{q}_2 - \mu\omega_2^2 q_2 + \beta_1 q_1^3 - \mu\beta_2\dot{q}_2\,|\dot{q}_2| = p(t), \\ &\ddot{q}_1 + \ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2 q_2 + \beta_2\dot{q}_2\,|\dot{q}_2| = 0. \end{aligned} \tag{52}$$

Apply the standard form (1) to system (52), in which the matrices are defined as

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2\zeta_1\omega_1 & -2\mu\zeta_2\omega_2 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix}, K = \begin{bmatrix} \omega_1^2 & -\mu\omega_2^2 \\ 0 & \omega_2^2 \end{bmatrix}, \Phi = \begin{bmatrix} \beta_1 q_1^3 - \mu\beta_2\dot{q}_2\,|\dot{q}_2| \\ \beta_2\dot{q}_2\,|\dot{q}_2| \end{bmatrix},$$
$$Q(t) = \begin{bmatrix} p(t) \\ 0 \end{bmatrix}. \tag{53}$$

4.2 LOMSEC

The equivalent linear system to (52) is taken in standard form as in Eq. (2). The difference ε between (52) and the equivalent linear equation is defined by (4) to be

$$\varepsilon = \Phi(q, \dot{q}) - C^e\dot{q} - K^e q, \tag{54}$$

where

$$\begin{aligned} &\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \quad \Phi(q,\dot{q}) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} \beta_1 q_1^3 - \mu\beta_2\dot{q}_2\,|\dot{q}_2| \\ \beta_2\dot{q}_2\,|\dot{q}_2| \end{bmatrix}, \quad C^e = \begin{bmatrix} c_{11}^e & c_{12}^e \\ c_{21}^e & c_{22}^e \end{bmatrix}, \\ &\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad K^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \end{aligned} \tag{55}$$

Equation (54) and the matrices given by (55) result in the following equation:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \beta_1 q_1^3 - \mu\beta_2\dot{q}_2\,|\dot{q}_2| - c_{11}^e\dot{q}_1 - c_{12}^e\dot{q}_2 - k_{11}^e q_1 - k_{12}^e q_2 \\ \beta_2\dot{q}_2\,|\dot{q}_2| - c_{21}^e\dot{q}_1 - c_{22}^e\dot{q}_2 - k_{21}^e q_1 - k_{22}^e q_2 \end{bmatrix}. \tag{56}$$

Utilizing (16) for determining the components of C^e and K^e yields

$$\begin{aligned} &\frac{\partial}{\partial c_{11}^e}E\left[\varepsilon_1^2\right] = -\beta_1 E\left[q_1^3\dot{q}_1\right] + \mu\beta_2 E\left[\dot{q}_1\dot{q}_2\,|\dot{q}_2|\right] + c_{11}^e E\left[\dot{q}_1^2\right] + c_{12}^e E\left[\dot{q}_1\dot{q}_2\right] + k_{11}^e E\left[q_1\dot{q}_1\right] + k_{12}^e E\left[\dot{q}_1 q_2\right] = 0, \\ &\frac{\partial}{\partial c_{12}^e}E\left[\varepsilon_1^2\right] = -\beta_1 E\left[q_1^3\dot{q}_2\right] + \mu\beta_2 E\left[\dot{q}_2^2\,|\dot{q}_2|\right] + c_{11}^e E\left[\dot{q}_1\dot{q}_2\right] + c_{12}^e E\left[\dot{q}_2^2\right] + k_{11}^e E\left[q_1\dot{q}_2\right] + k_{12}^e E\left[q_2\dot{q}_2\right] = 0, \\ &\frac{\partial}{\partial c_{21}^e}E\left[\varepsilon_2^2\right] = -\beta_2 E\left[\dot{q}_1\dot{q}_2\,|\dot{q}_2|\right] + c_{21}^e E\left[\dot{q}_1^2\right] + c_{22}^e E\left[\dot{q}_1\dot{q}_2\right] + k_{21}^e E\left[q_1\dot{q}_1\right] + k_{22}^e E\left[\dot{q}_1 q_2\right] = 0, \\ &\frac{\partial}{\partial c_{22}^e}E\left[\varepsilon_2^2\right] = -\beta_2 E\left[\dot{q}_2^2\,|\dot{q}_2|\right] + c_{21}^e E\left[\dot{q}_1\dot{q}_2\right] + c_{22}^e E\left[\dot{q}_2^2\right] + k_{21}^e E\left[q_1\dot{q}_2\right] + k_{22}^e E\left[q_2\dot{q}_2\right] = 0, \\ &\frac{\partial}{\partial k_{11}^e}E\left[\varepsilon_1^2\right] = -\beta_1 E\left[q_1^4\right] + \mu\beta_2 E\left[q_1\dot{q}_2\,|\dot{q}_2|\right] + c_{11}^e E\left[q_1\dot{q}_1\right] + c_{12}^e E\left[q_1\dot{q}_2\right] + k_{11}^e E\left[q_1^2\right] + k_{12}^e E\left[q_1 q_2\right] = 0, \\ &\frac{\partial}{\partial k_{12}^e}E\left[\varepsilon_1^2\right] = -\beta_1 E\left[q_1^3 q_2\right] + \mu\beta_2 E\left[q_2\dot{q}_2\,|\dot{q}_2|\right] + c_{11}^e E\left[\dot{q}_1 q_2\right] + c_{12}^e E\left[q_2\dot{q}_2\right] + k_{11}^e E\left[q_1 q_2\right] + k_{12}^e E\left[q_2^2\right] = 0, \\ &\frac{\partial}{\partial k_{21}^e}E\left[\varepsilon_2^2\right] = -\beta_2 E\left[q_1\dot{q}_2\,|\dot{q}_2|\right] + c_{21}^e E\left[q_1\dot{q}_1\right] + c_{22}^e E\left[q_1\dot{q}_2\right] + k_{21}^e E\left[q_1^2\right] + k_{22}^e E\left[q_1 q_2\right] = 0, \\ &\frac{\partial}{\partial k_{22}^e}E\left[\varepsilon_2^2\right] = -\beta_2 E\left[q_2\dot{q}_2\,|\dot{q}_2|\right] + c_{21}^e E\left[\dot{q}_1 q_2\right] + c_{22}^e E\left[q_2\dot{q}_2\right] + k_{21}^e E\left[q_1 q_2\right] + k_{22}^e E\left[q_2^2\right] = 0. \end{aligned} \tag{57}$$

In order to simplify the calculation, it is assumed that the responses q_1 and q_2 are independent. Moreover, as known that if $q(t)$ is a stationary Gaussian random process with zero mean, so is $\dot{q}(t)$. Besides, a stationary random process is orthogonal to its derivative, so q_1, q_2 are independent from $\dot{q}_1, \dot{q}_2$, respectively.

By denoting $x = (x_1, x_2, x_3, x_4)^T = (q_1, \dot{q}_1, q_2, \dot{q}_2)^T$, we can utilize formulas (A.10), (A.11) and (A.13) in the appendix to calculate the local means in (70) and note that $E\left[x_i^{2n+1} x_j^{2m+1}\right] = 0 (i \neq j)$. Thus, system (70) leads to the following result of the linearization coefficients which are considered as the functions depending on r

$$
\begin{aligned}
&c_{11}^e(r) = 0, c_{12}^e(r) = -\frac{\mu\beta_2 E\left[\dot{q}_2^2 |\dot{q}_2|\right]}{E\left[\dot{q}_2^2\right]} = -\frac{\mu\beta_2 2\left(E\left\{\dot{q}_2^2\right\}\right)^{3/2} T_{t^3,r}}{2T_{1,r} E\left\{\dot{q}_2^2\right\}} = -\mu\beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2} \frac{T_{t^3,r}}{T_{1,r}},\\
&c_{21}^e(r) = 0, c_{22}^e(r) = \frac{\beta_2 E\left[\dot{q}_2^2 |\dot{q}_2|\right]}{E\left[\dot{q}_2^2\right]} = \frac{\beta_2 2\left(E\left\{\dot{q}_2^2\right\}\right)^{3/2} T_{t^3,r}}{2T_{1,r} E\left\{\dot{q}_2^2\right\}} = \beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2} \frac{T_{t^3,r}}{T_{1,r}},\\
&k_{11}^e(r) = \frac{\beta_1 E\left[q_1^4\right]}{E\left[q_1^2\right]} = \frac{\beta_1 2T_{2,r}\left(E\left\{q_1^2\right\}\right)^2}{2T_{1,r} E\left\{q_1^2\right\}} = \beta_1 E\left\{q_1^2\right\} \frac{T_{2,r}}{T_{1,r}}, \quad k_{12}^e(r) = 0,\\
&k_{21}^e(r) = 0, \quad k_{22}^e(r) = 0.
\end{aligned} \tag{58}
$$

where $T_{1,r}$, $T_{2,r}$ are defined by (A.9) and $T_{t^3,r}$ is defined by (A.13) in the Appendix as follows:

$$
T_{1,r} = \int_0^r t^2 \eta(t) \mathrm{d}t, T_{2,r} = \int_0^r t^4 \eta(t) \mathrm{d}t, \quad T_{t^3,r} = \int_0^r t^3 \eta(t) \mathrm{d}t, \quad \text{here } \eta(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}. \tag{59}
$$

4.3 Conventional linearization method

Utilizing (59) with $r \to \infty$, the following factors in (58) are defined

$$
\frac{T_{t^3,\infty}}{T_{1,\infty}} = \frac{\int_0^\infty t^3 \eta(t) \mathrm{d}t}{\int_0^\infty t^2 \eta(t) \mathrm{d}t} = 2\sqrt{\frac{2}{\pi}}, \quad \frac{T_{2,\infty}}{T_{1,\infty}} = \frac{\int_0^\infty t^4 \eta(t) \mathrm{d}t}{\int_0^\infty t^2 \eta(t) \mathrm{d}t} = 3. \tag{60}
$$

Equation system (58) and the values given by (60) allow determining c_{ij}^e, k_{ij}^e by the conventional linearization method

$$
\begin{aligned}
&c_{11}^e = 0, \quad c_{12}^e = -2\sqrt{\frac{2}{\pi}} \mu\beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2}, \quad c_{21}^e = 0, \quad c_{22}^e = 2\sqrt{\frac{2}{\pi}} \beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2},\\
&k_{11}^e = 3\beta_1 E\left\{q_1^2\right\}, \quad k_{12}^e = 0, k_{21}^e = 0, \quad k_{22}^e = 0.
\end{aligned} \tag{61}
$$

4.4 GLOMSEC

Formula (17) and equation system (58) allow determining c_{ij}^e, k_{ij}^e by GLOMSEC:

$$
\begin{aligned}
&c_{11}^e = 0, \quad c_{12}^e = \left\langle c_{12}^e(r) \right\rangle = \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s c_{12}^e(r) \mathrm{d}r = -\mu\beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2} \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s \frac{T_{t^3,r}}{T_{1,r}} \mathrm{d}r,\\
&c_{21}^e = 0, \quad c_{22}^e = \left\langle c_{22}^e(r) \right\rangle = \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s c_{22}^e(r) \mathrm{d}r = \beta_2 \left(E\left\{\dot{q}_2^2\right\}\right)^{1/2} \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s \frac{T_{t^3,r}}{T_{1,r}} \mathrm{d}r,\\
&k_{11}^e = \left\langle k_{11}^e(r) \right\rangle = \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s k_{11}^e(r) \mathrm{d}r = \beta_1 E\left\{q_1^2\right\} \operatorname*{Lim}_{s\to\infty} \frac{1}{s} \int_0^s \frac{T_{2,r}}{T_{1,r}} \mathrm{d}r, \quad k_{12}^e = 0,\\
&k_{21}^e = 0, \quad k_{22}^e = 0.
\end{aligned} \tag{62}
$$

The limitation expressions in (62) can be approximately computed, their outputs are

$$\lim_{s\to\infty}\frac{1}{s}\int_0^s \frac{T_{t^3,r}}{T_{1,r}}\mathrm{d}r \approx 1.39831, \quad \lim_{s\to\infty}\frac{1}{s}\int_0^s \frac{T_{2,r}}{T_{1,r}}\mathrm{d}r \approx 2.41189. \tag{63}$$

4.5 Approximate mean square responses

As assumed in section (4.1) that the spectral density function $S_p(\omega)$ of $p(t)$ has the form $S_p(\omega) = S_0/(\alpha^2+\omega^2)$, so the spectral density matrix $S_Q(\omega)$ shall be

$$S_Q(\omega) = \begin{bmatrix} S_0/(\alpha^2+\omega^2) & 0 \\ 0 & 0 \end{bmatrix}. \tag{64}$$

The matrix of frequency response function to standard form (2) of the equivalent linear system is

$$\alpha(\omega) = \left[-\omega^2 M + i\omega(C+C^e) + (K+K^e)\right]^{-1} \tag{65}$$

The matrices in (65) were determined herein before in (53), (55), (61), (62), which are

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2\zeta_1\omega_1 & -2\mu\zeta_2\omega_2 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix}, \quad C^e = \begin{bmatrix} 0 & c_{12}^e \\ 0 & c_{22}^e \end{bmatrix}, \quad K = \begin{bmatrix} \omega_1^2 & -\mu\omega_2^2 \\ 0 & \omega_2^2 \end{bmatrix}, \quad K^e = \begin{bmatrix} k_{11}^e & 0 \\ 0 & 0 \end{bmatrix}. \tag{66}$$

The values of c_{ij}^e defined in (58) as well as (61) and (62) result as a consequence to be

$$c_{12}^e = -\mu c_{22}^e \tag{67}$$

Denote

$$\zeta_{2e} = \zeta_2 + \frac{c_{22}^e}{2\omega_2}, \quad \omega_e^2 = \omega_1^2 + k_{11}^e \tag{68}$$

Utilizing (66–68) for expanding (65), the matrix of the frequency response function is found as follows:

$$\alpha(\omega) = \begin{bmatrix} \alpha_{11}(\omega) & \alpha_{12}(\omega) \\ \alpha_{21}(\omega) & \alpha_{22}(\omega) \end{bmatrix} = \begin{bmatrix} -\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2 & -2i\mu\zeta_{2e}\omega_2\omega - \mu\omega_2^2 \\ -\omega^2 & -\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2 \end{bmatrix}^{-1}. \tag{69}$$

Matrix (69) gives elements $\alpha_{ij}(\omega)$, $(i, j = 1, 2)$ as follows:

$$\begin{aligned}
\alpha_{11}(\omega) &= \frac{-\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2}{\left(-\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2\right)\left(-\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2\right) - \omega^2\left(2i\mu\zeta_{2e}\omega_2\omega + \mu\omega_2^2\right)}, \\
\alpha_{12}(\omega) &= \frac{2i\mu\zeta_{2e}\omega_2\omega + \mu\omega_2^2}{\left(-\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2\right)\left(-\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2\right) - \omega^2\left(2i\mu\zeta_{2e}\omega_2\omega + \mu\omega_2^2\right)}, \\
\alpha_{21}(\omega) &= \frac{\omega^2}{\left(-\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2\right)\left(-\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2\right) - \omega^2\left(2i\mu\zeta_{2e}\omega_2\omega + \mu\omega_2^2\right)}, \\
\alpha_{22}(\omega) &= \frac{-\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2}{\left(-\omega^2 + 2i\zeta_1\omega_1\omega + \omega_e^2\right)\left(-\omega^2 + 2i\zeta_{2e}\omega_2\omega + \omega_2^2\right) - \omega^2\left(2i\mu\zeta_{2e}\omega_2\omega + \mu\omega_2^2\right)}.
\end{aligned} \tag{70}$$

In order to establish a close equation system for determining the unknowns, the mean square responses $E\left\{q_1^2\right\}, E\left\{q_2^2\right\}, E\left\{\dot{q}_2^2\right\}$ must be defined. Utilizing (11), (64) and (69), we obtain the following results:

$$
\begin{aligned}
E\left\{qq^{\mathrm{T}}\right\} &= \begin{bmatrix} E\left\{q_1^2\right\} & E\left\{q_1q_2\right\} \\ E\left\{q_1q_2\right\} & E\left\{q_2^2\right\} \end{bmatrix} \\
&= \int_{-\infty}^{\infty} \begin{bmatrix} \alpha_{11}(-\omega)\ \alpha_{12}(-\omega) \\ \alpha_{21}(-\omega)\ \alpha_{22}(-\omega) \end{bmatrix} \begin{bmatrix} S_0/(\alpha^2+\omega^2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11}(\omega)\ \alpha_{21}(\omega) \\ \alpha_{12}(\omega)\ \alpha_{22}(\omega) \end{bmatrix} \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \frac{S_0}{\alpha^2+\omega^2} \begin{bmatrix} \alpha_{11}(\omega)\alpha_{11}(-\omega)\ \alpha_{11}(-\omega)\alpha_{21}(\omega) \\ \alpha_{11}(\omega)\alpha_{21}(-\omega)\ \alpha_{21}(\omega)\alpha_{21}(-\omega) \end{bmatrix} \mathrm{d}\omega, \qquad (71) \\
E\left\{\dot{q}\dot{q}^{\mathrm{T}}\right\} &= \begin{bmatrix} E\left\{\dot{q}_1^2\right\} & E\left\{\dot{q}_1\dot{q}_2\right\} \\ E\left\{\dot{q}_1\dot{q}_2\right\} & E\left\{\dot{q}_2^2\right\} \end{bmatrix} \\
&= \int_{-\infty}^{\infty} \omega^2 \begin{bmatrix} \alpha_{11}(-\omega)\ \alpha_{12}(-\omega) \\ \alpha_{21}(-\omega)\ \alpha_{22}(-\omega) \end{bmatrix} \begin{bmatrix} S_0/(\alpha^2+\omega^2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11}(\omega)\ \alpha_{21}(\omega) \\ \alpha_{12}(\omega)\ \alpha_{22}(\omega) \end{bmatrix} \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \frac{S_0\omega^2}{\alpha^2+\omega^2} \begin{bmatrix} \alpha_{11}(\omega)\alpha_{11}(-\omega)\ \alpha_{11}(-\omega)\alpha_{21}(\omega) \\ \alpha_{11}(\omega)\alpha_{21}(-\omega)\ \alpha_{21}(\omega)\alpha_{21}(-\omega) \end{bmatrix} \mathrm{d}\omega. \qquad (72)
\end{aligned}
$$

Now $E\left\{q_1^2\right\}, E\left\{q_2^2\right\}, E\left\{\dot{q}_2^2\right\}$ can be obtained from (71) and (72) as follows:

$$
\begin{aligned}
E\left\{q_1^2\right\} &= S_0 \int_{-\infty}^{\infty} \frac{|\alpha_{11}(\omega)|^2}{\alpha^2+\omega^2}\mathrm{d}\omega = S_0 \int_{-\infty}^{\infty} \frac{r_{11}(\omega)}{R(i\omega)R(-i\omega)}\mathrm{d}\omega, \\
E\left\{q_2^2\right\} &= S_0 \int_{-\infty}^{\infty} \frac{|\alpha_{21}(\omega)|^2}{\alpha^2+\omega^2}d\omega = S_0 \int_{-\infty}^{\infty} \frac{r_{21}(\omega)}{R(i\omega)R(-i\omega)}\mathrm{d}\omega, \\
E\left\{\dot{q}_2^2\right\} &= S_0 \int_{-\infty}^{\infty} \frac{\omega^2\,|\alpha_{21}(\omega)|^2}{\alpha^2+\omega^2}\mathrm{d}\omega = S_0 \int_{-\infty}^{\infty} \frac{\omega^2 r_{21}(\omega)}{R(i\omega)R(-i\omega)}\mathrm{d}\omega, \qquad (73)
\end{aligned}
$$

where

$$
\begin{aligned}
&r_{11}(\omega) = \xi_2\omega^4 + \xi_1\omega^2 + \xi_0, r_{21}(\omega) = \omega^4, \\
&R(i\omega) = \lambda_5(i\omega)^5 + \lambda_4(i\omega)^4 + \lambda_3(i\omega)^3 + \lambda_2(i\omega)^2 + \lambda_1(i\omega) + \lambda_0. \qquad (74)
\end{aligned}
$$

The coefficients in (74) are determined as follows:

$$
\begin{aligned}
&\xi_2 = 1, \quad \xi_1 = \omega_2^2\left(4\zeta_{2e}^2 - 2\right), \quad \xi_0 = \omega_2^4, \\
&\lambda_5 = 1, \ \lambda_4 = \alpha + 2\zeta_1\omega_1 + 2\zeta_{2e}\omega_2 + 2\mu\zeta_{2e}\omega_2, \\
&\lambda_3 = \omega_2^2 + \omega_e^2 + \mu\omega_2^2 + 4\zeta_1\zeta_{2e}\omega_1\omega_2 + \alpha\left(2\zeta_{2e}\omega_2 + 2\zeta_1\omega_1 + 2\mu\zeta_{2e}\omega_2\right), \\
&\lambda_2 = 2\zeta_1\omega_1\omega_2^2 + 2\zeta_{2e}\omega_2\omega_e^2 + \alpha\left(\omega_2^2 + 4\zeta_1\zeta_{2e}\omega_1\omega_2 + \omega_e^2 + \mu\omega_2^2\right), \\
&\lambda_1 = \omega_e^2\omega_2^2 + \alpha\left(2\zeta_1\omega_1\omega_2^2 + 2\zeta_{2e}\omega_2\omega_e^2\right), \quad \lambda_0 = \alpha\omega_e^2\omega_2^2. \qquad (75)
\end{aligned}
$$

By computational calculation, the integrals in (73) give the following results:

$$
E\left\{q_1^2\right\} = \frac{\pi S_0}{\lambda_5} \begin{vmatrix} 0 & 0 & \xi_2 & \xi_1 & \xi_0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_3 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix} \Bigg| \begin{vmatrix} \lambda_4 & -\lambda_2 & \lambda_0 & 0 & 0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_5 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix}^{-1},
$$

Table 5 Mean square of q_i versus $\omega_1 = \omega_2 = 1, \zeta_1 = 0.05, \zeta_2 = 0.2, \mu = 1, \alpha = 2, S_0 = 1, \beta_2 = 0.05, \beta_1(0.01 - 5)$

β_1	$E\{q_1^2\}_{MC}$	$E\{q_1^2\}_{co}$	Error (%)	$E\{q_1^2\}_{gl}$	Error (%)	$E\{q_2^2\}_{MC}$	$E\{q_2^2\}_{co}$	Error (%)	$E\{q_2^2\}_{gl}$	Error (%)
0.01	1.9258	2.0210	4.943	2.0842	8.225	0.8856	0.8870	0.158	0.9046	2.145
0.05	1.4307	1.4336	0.203	1.5366	7.402	0.8288	0.8172	1.399	0.8435	1.774
0.10	1.1421	1.1229	1.681	1.2246	7.223	0.7743	0.7627	1.498	0.7933	2.454
0.20	0.8809	0.8360	5.097	0.9247	4.972	0.7104	0.6930	2.449	0.7273	2.379
0.50	0.5875	0.5353	8.885	0.5993	2.008	0.6042	0.5822	3.641	0.6195	2.532
1.00	0.4226	0.3733	11.666	0.4196	0.710	0.5166	0.4891	5.323	0.5270	2.013
2.00	0.3017	0.2593	14.054	0.2913	3.447	0.4289	0.3924	8.510	0.4292	0.070
5.00	0.1950	0.1619	16.974	0.1813	7.026	0.3114	0.2685	13.776	0.3008	3.404

Table 6 The mean square of q_i versus $\omega_1 = \omega_2 = 1, \zeta_1 = 0.05, \zeta_2 = 0.2, \mu = 1, \alpha = 2, S_0 = 1, \beta_2 = 2, \beta_1(0.01 - 5)$

β_1	$E\{q_1^2\}_{MC}$	$E\{q_1^2\}_{co}$	Error (%)	$E\{q_1^2\}_{gl}$	Error (%)	$E\{q_2^2\}_{MC}$	$E\{q_2^2\}_{co}$	Error (%)	$E\{q_2^2\}_{gl}$	Error (%)
0.01	1.8728	2.0255	8.154	2.0225	7.993	0.2737	0.2428	11.290	0.2648	3.252
0.05	1.4442	1.5122	4.709	1.5662	8.448	0.2464	0.2160	12.338	0.2397	2.719
0.10	1.1931	1.2178	2.070	1.2842	7.636	0.2268	0.1970	13.139	0.2208	2.645
0.20	0.9381	0.9307	0.065	0.9973	6.311	0.2025	0.1746	13.778	0.1978	2.321
0.50	0.6360	0.6110	3.931	0.6658	4.685	0.1665	0.1427	14.294	0.1638	1.622
1.00	0.4581	0.4281	6.549	0.4711	2.838	0.1397	0.1189	14.889	0.1377	1.432
2.00	0.3230	0.2930	9.288	0.3250	0.619	0.1148	0.0967	15.767	0.1128	1.742
5.00	0.1988	0.1732	12.877	0.1937	2.565	0.0854	0.0711	16.745	0.0836	2.108

$$E\{q_2^2\} = \frac{\pi S_0}{\lambda_5} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_5 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix} \begin{vmatrix} \lambda_4 & -\lambda_2 & \lambda_0 & 0 & 0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_5 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix}^{-1},$$

$$E\{\dot{q}_2^2\} = \frac{\pi S_0}{\lambda_5} \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_5 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix} \begin{vmatrix} \lambda_4 & -\lambda_2 & \lambda_0 & 0 & 0 \\ -\lambda_5 & \lambda_3 & -\lambda_1 & 0 & 0 \\ 0 & -\lambda_4 & \lambda_2 & -\lambda_0 & 0 \\ 0 & \lambda_5 & -\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & \lambda_4 & -\lambda_2 & \lambda_0 \end{vmatrix}^{-1}. \tag{76}$$

Combine (76) with either (61) or (62), one gets close nonlinear algebraic equation systems for determining the unknowns that given by the conventional method and GLOMSEC, respectively. These equation systems are solved by computationally numerical procedures.

Tables 5 and 6 present the numerical results of mean square responses of $q_i (i = 1, 2)$ versus two cases of the given parameters of system (52). For evaluating the relative error, a Monte-Carlo simulated solution to system (52) that is presented in [33] shall be used. Denote $E\{q_i^2\}_{\mathrm{MC}}, E\{q_i^2\}_{\mathrm{co}}, E\{q_i^2\}_{\mathrm{gl}}$ mean square responses of $q_i (i = 1, 2)$ given by Monte-Carlo simulation, the conventional method and GLOMSEC, respectively.

Based on the relative errors of the approximate solutions with respect to Monte-Carlo simulated solution, it is seen that generally when the nonlinearity is strong, GLOMSEC gives significant improvement on the accuracy of solution in comparison with the conventional GEL method.

5 Conclusion

GLOMSEC was first proposed in the previous study [22] where the algorithm formulated and investigations were targeted at SDOF nonlinear systems under zero-mean stationary Gaussian random excitation. This study develops the proposed technique to MDOF nonlinear systems under the same excitation. The mode of formulation of the algorithm is also mainly based on the classical GEL, in which a key problem is to define the matrix of equivalent linearization coefficients. There are two important improvements to formula GLOMSEC: Firstly, the matrix of equivalent linearization coefficients is defined by using LOMSEC, and the elements of this matrix shall

be functions depending on the non-dimensional parameter r which is the local domain of integration. Secondly, based on the dual conception, r is considered as varying in the global domain of integration. Thus, the constant linearization coefficients can be suggested as global mean values of all local linearization coefficients. In order to evaluate the accuracy of the solution given by the proposed technique, the paper presents two examples to be typical two-degree-of-freedom nonlinear systems under the excitation as mentioned, and the numerical results indicate an outstanding advantage of GLOMSEC that when the nonlinearity is strong, GLOMSEC gives significant improvement in the accuracy of solution in comparison with the classical GEL method.

However, two points should be further investigated: The first, the examples considered in the paper are just for a two-degree-of-freedom system, so the systems with higher-degree-of-freedom need investigation; The second, in some applications, for example, the reliability evaluation problem, like for the tails in the PDF, may be interesting to assume a weigh in the local value. In this way, we may have a loss of accuracy in the mean square value, but a greater accuracy in the evaluation of the PDF where the weight is greater.

Acknowledgments This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number "107.04-2013.19". The authors would like to say thank you to the reviewers for the sound comments and suggestions to the paper.

Appendix

Suppose that the components of the vector $x = (x_1, x_2, \ldots, x_n)^T$ are zero-mean stationary Gaussian random variables. Denote by $E\{.\}$ global mean values of random variables, which are taken as follows:

$$E\{.\} = \int_{-\infty}^{+\infty} (n) \int_{-\infty}^{+\infty} (.)\, p(x)\mathrm{d}x, \tag{A.1}$$

where $p(x)$ is the stationary joint probability density function. For the Gaussian random processes with zero mean ($E\{x_i\} = 0$), one has the following general expressions for expectations [3,12]:

$$E\{x_1 x_2 \ldots x_{2n+1}\} = 0, \quad E\{x_1 x_2 \ldots x_{2n}\} = \sum_{\text{all dependent pairs}} \left(\prod_{i \neq j} E\{x_i x_j\} \right), \tag{A.2}$$

where the number of independent pairs is equal to $(2n)!/(2^n n!)$. For example, some consequences from (A.2) are as follows:

$$\begin{aligned} &E\{x_1 x_2 x_3\} = 0, \\ &E\{x_1 x_2 x_3 x_4\} = E\{x_1 x_2\} E\{x_3 x_4\} + E\{x_2 x_3\} E\{x_1 x_4\} + E\{x_1 x_3\} E\{x_2 x_4\}, \\ &E\{x_1 x_2 x_3 x_4 x_5\} = 0. \end{aligned} \tag{A.3}$$

If x_i and $x_j (i \neq j)$ are uncorrelated, i.e. independent, then $E\{x_i x_j\} = 0$ and $E\{x_i^{2n+1} x_j^{2m+1}\} = 0$. Besides, formula (A.2) results in the following consequences:

$$E\left\{x_i^{2n} x_j^{2m}\right\} = E\left\{x_i^{2n}\right\} E\left\{x_j^{2m}\right\} = (2n-1)!!\left(E\left\{x_i^2\right\}\right)^n (2m-1)!!\left(E\left\{x_j^2\right\}\right)^m, \tag{A.4}$$

where n and m are natural numbers.

Denote by [.] the local mean values of random variables, which are taken as follows:

$$E[.] = \int_{-x_{01}}^{+x_{01}} (n) \int_{-x_{0n}}^{+x_{0n}} (.)\, p(x)\mathrm{d}x, \tag{A.5}$$

where $x_{01}, x_{02}, \ldots, x_{0n}$ are given positive values. The expected integrations in (A.5) can be transformed to non-dimensional variables by $x_{01} = r\sigma_{x1}, x_{02} = r\sigma_{x2}, \ldots, x_{0n} = r\sigma_{xn}$, where $\sigma_{x1}, \sigma_{x2} \ldots, \sigma_{xn}$ are the normal deviations of random variables, respectively, and r is a given positive value:

$$E[.] = \int_{-r\sigma_{x1}}^{+r\sigma_{x1}} (n) \int_{-r\sigma_{xn}}^{+r\sigma_{xn}} (.)p(x)\mathrm{d}x. \tag{A.6}$$

Due to the symmetry of the expected integrations in (A.6), hereby (A.2) and (A.3) are also applied to the local mean values. If x_i and $x_j (i \neq j)$ are uncorrelated, i.e. independent, then $E\left[x_i x_j\right] = 0$ and $E\left[x_i^{2n+1} x_j^{2m+1}\right] = 0$. Moreover, all higher even-order local means $E\left[x_i^{2n} x_j^{2m}\right]$ can be expressed in terms of second order global means $E\left\{x_i^2\right\}$ and $E\left\{x_j^2\right\}$ as presented hereafter:

The stationary joint probability density function of x_i and x_j has the following form

$$p(x_i, x_j) = p(x_i)p(x_j), \quad p(x_i) = \frac{1}{\sqrt{2\pi}\sigma_{xi}} \exp\left[-\frac{x_i^2}{2\sigma_{xi}^2}\right], \quad p(x_j) = \frac{1}{\sqrt{2\pi}\sigma_{xj}} \exp\left[-\frac{x_j^2}{2\sigma_{xj}^2}\right], \tag{A.7}$$

where σ_{xi}, σ_{xj} are the normal deviations of random variables, respectively. By replacing $x_i = t\sigma_{xi}, x_j = t\sigma_{xj}$ and using formulas (A.6), (A.7), one gets

$$\begin{aligned} E\left[x_i^{2n} x_j^{2m}\right] &= \int_{-r\sigma_{xi}}^{+r\sigma_{xi}} x^{2n} p(x_i)\mathrm{d}x_i \int_{-r\sigma_{xj}}^{+r\sigma_{xj}} x^{2m} p(x_j)\mathrm{d}x_j \\ &= \int_{-r}^{r} t^{2n}\sigma_{xi}^{2n} \frac{1}{\sqrt{2\pi}\sigma_{xi}} \mathrm{e}^{-t^2\sigma_{xi}^2/2\sigma_{xi}^2}\sigma_{xi}\mathrm{d}t \int_{-r}^{r} t^{2m}\sigma_{xj}^{2m} \frac{1}{\sqrt{2\pi}\sigma_{xj}} \mathrm{e}^{-t^2\sigma_{xj}^2/2\sigma_{xj}^2}\sigma_{xj}\mathrm{d}t \\ &= 2\sigma_{xi}^{2n} \int_0^r t^{2m} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\mathrm{d}t 2\sigma_{xj}^{2m} \int_0^r t^{2m} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\mathrm{d}t. \end{aligned} \tag{A.8}$$

Introduce $\sigma_{xi}^{2n} = \left(E\left\{x_i^2\right\}\right)^n, \sigma_{xj}^{2m} = \left(E\left\{x_j^2\right\}\right)^m$ since x_i and x_j are zero-mean Gaussian random variables and use the following replacements:

$$T_{n,r} = \int_0^r t^{2n}\eta(t)\mathrm{d}t, T_{m,r} = \int_0^r t^{2m}\eta(t)\mathrm{d}t, \eta(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-t^2/2}. \tag{A.9}$$

Thus, one gets

$$E\left[x_i^{2n} x_j^{2m}\right] = E\left[x_i^{2n}\right] E\left[x_j^{2m}\right] = 2T_{n,r}\left(E\left\{x_i^2\right\}\right)^n 2T_{m,r}\left(E\left\{x_j^2\right\}\right)^m. \tag{A.10}$$

If $n = 0, m \neq 0$ or $n \neq 0, m = 0$, then (A.10) leads to the following results, respectively:

$$E\left[x_i^0 x_j^{2m}\right] = 2T_{0,r}2T_{m,r}\left(E\left\{x_j^2\right\}\right)^m, \quad E\left[x_i^{2n} x_j^0\right] = 2T_{n,r}\left(E\left\{x_i^2\right\}\right)^n 2T_{0,r}, \quad \text{with} \quad T_{0,r} = \int_0^r \eta(t)dt. \tag{A.11}$$

In (A.10) and (A.11), if $r \to \infty$, one gets the same result as (A.4) of the classical case.

Consider the local mean of $x_i^2 |x_i|$ that arises in an example of the paper. By exactly the same way as presented in formula (A.8) and noting that $x_i^2 |x_i|$ is an even function, its local mean can be determined as follows:

$$E\left[x_i^2\,|x_i|\right] = \int_{-r\sigma_{xn}}^{+r\sigma_{xn}} x_i^2\,|x_i|\,p(x_i)\mathrm{d}x_i = 2\int_0^{+r\sigma_{xn}} x_i^3\,p(x_i)\mathrm{d}x_i$$

$$= 2\int_0^r t^3\sigma_{xi}^3\frac{1}{\sqrt{2\pi}\sigma_{xi}}\mathrm{e}^{-t^2\sigma_{xi}^2/2\sigma_{xi}^2}\sigma_{xi}\mathrm{d}t = 2\sigma_{xi}^3\int_0^r t^3\eta(t)\mathrm{d}t \tag{A.12}$$

where $\eta(t)$ is given by (A.9). Moreover, since $\sigma_{xi}^2 = E\left\{x_i^2\right\}$, formula (A.12) can be rewritten as

$$E\left[x_i^2\,|x_i|\right] = 2\left(E\left\{x_i^2\right\}\right)^{3/2}\int_0^r t^3\eta(t)\mathrm{d}t = 2\left(E\left\{x_i^2\right\}\right)^{3/2}T_{t^3,r} \quad \text{where } T_{t^3,r} = \int_0^r t^3\eta(t)\mathrm{d}t. \tag{A.13}$$

If we consider that $x = (x_1, x_2, \ldots, x_n)^T$ is the displacement vector, then $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)^T$ is the velocity vector and we also obtain exactly the same formulas, respectively, for the random variables of velocity.

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