Acta Mech 226, 2461–2472 (2015)
DOI 10.1007/s00707-015-1320-8

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New integrable problems in rigid body dynamics with quartic integrals

Received: 4 December 2014 / Revised: 31 December 2014 / Published online: 20 March 2015

Abstract We consider the general problem of motion of a rigid body about a fixed point under the action of an axisymmetric combination of potential and gyroscopic forces. We will construct two new conditional integrable problems. These cases are combined generalizations of several previously known ones, namely those of Chaplygin and Yehia by the introduction of additional parameters to the structure of each.

1 Preliminaries

The models of a rigid body and its generalization, the gyrostat, have found a wide range of applications in various fields of physics, in addition to their classical applications in mechanics and astronomy. For example, the gyrostat was used as a model of the Earth that takes account of some stationary transport processes on it [1], as a model of the atmosphere and of rotating fluid (e.g., [2]) and as a controlling device in satellite dynamics (e.g., [3]).

The study of the dynamics of a rigid body is one of the most interesting problems in mechanics, even in the simplest case of motion under the action of a uniform gravity field. It has been studied by Euler and Lagrange who indicated the first integrable cases (see, e.g., [4]). The interest intensified after Kovalevskaya introduced the case known under her own name [5]. It was probably the first known case of a mechanical system having an integral quartic in velocities in addition to the energy integral [5]. It was followed shortly by the case due to Chaplygin of motion of a body in liquid [6] (see also [7]). All efforts led only to few numbers of integrable cases of this dynamics under very restricted types of forces. A little fraction of those is composed of the general case, valid for all admissible initial conditions, and the rest are cases valid only on a single level of the cyclic integral, usually the zero level. Up-to-date tables of known integrable cases are available in the literature [7,8]. As there is no criterion at present to single out the forms that make the dynamics integrable, it is thus of great importance to construct, classify and tabulate new integrable problems as possible.

Consider a rigid body in motion about its fixed point O. Let $OXYZ$ and $Oxyz$ be the two Cartesian coordinate systems, fixed in space and in the body, respectively. Let also $\boldsymbol{\omega} = (p, q, r)$ be the angular velocity of the body and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vector in the direction of the Z-axis. All vectors are referred to the body system which we take as the system of principal axes of inertia.

Those variables can be expressed in terms of Euler's angles: the angle of precession ψ about the Z-axis, the angle of nutation θ (between the z- and Z-axes) and the angle of proper rotation φ about the z-axis. They have the form

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$$\boldsymbol{\gamma} = (\sin\theta \sin\varphi, \sin\theta\cos\varphi, \cos\theta) \tag{1}$$

and

$$\omega = \left(\dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \dot{\psi}\cos\theta + \dot{\varphi}\right). \tag{2}$$

In this article, we consider the general problem of motion of a rigid body about a fixed point under the action of a combination of conservative axisymmetric around the Z-axis potential and gyroscopic forces. This problem is described by a Lagrangian, e.g., [9],

$$L = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V, \tag{3}$$

where $\mathbf{I} = \mathrm{diag}(A, B, C)$ is the inertia matrix of the body. The first term represents the kinetic energy of the rigid body. The potential V and the vector $\mathbf{l}$ rely only on the Eulerian angles through $\gamma_1, \gamma_2, \gamma_3$. As shown in [9,10], potential terms can be interpreted in most cases of physical interest in terms of three classical interactions: gravitational, electric and magnetic. Gyroscopic terms appear naturally after reduction in higher dimensional systems by applying Routh's procedure for ignoring cyclic coordinates. They can be accounted for also by attaching rotors to the body and adding Lorentz forces. Explicit cases of interpretation of this type are given in [8].

Equations of motion for the Lagrangian system (3) with arbitrary $\mathbf{l}(\gamma)$ in Euler–Poisson variables can be written in the form [9]

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \boldsymbol{\mu}) = \boldsymbol{\gamma}\times\frac{\partial \mathrm{V}}{\partial\boldsymbol{\gamma}'}\,\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}, \tag{4}$$

where

$$\boldsymbol{\mu} = \frac{\partial}{\partial\boldsymbol{\gamma}}(\mathbf{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\mathbf{l}\right)\boldsymbol{\gamma}. \tag{5}$$

Equations (4) and (5) are Lagrangian equations in non-Lagrangian variables. They admit three general first integrals:

(i) Jacobi's integral

$$I_1 = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + V = h, \tag{6}$$

where h is an arbitrary constant which represents the numerical value of the Jacobi integral.

(ii) An integral linear in the components of angular velocity corresponding to the cyclic angle of precession around the axis of the field:

$$I_2 = (\boldsymbol{\omega}\mathbf{I} + \mathbf{l})\cdot\boldsymbol{\gamma} = f, \tag{7}$$

where f is the value of cyclic integral.

(iii) The geometric integral

$$I_3 = \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1. \tag{8}$$

According to Jacobi's theorem on the last integrating multiplier [11], four integrals are sufficient for the integration of (4). This means that one additional integral to the known three (6)–(8) is required.

For such problems, the angle of precession ψ around the Z-axis is a cyclic variable. Moreover, we restrict our consideration to the case when the body exhibits axial dynamical symmetry $A = B$ and the vector $\mathbf{l}$ lies along the axis of dynamical symmetry, i.e., $\mathbf{l} = (0, 0, l_3)$. Therefore, this problem reduces after ignoring the cyclic angle ψ to the Routhian

$$R = \frac{1}{2}\left[\dot{\theta}^2 + \frac{C\sin^2\theta}{A-(A-C)\cos^2\theta}\dot{\varphi}^2\right] + \frac{fC\cos\theta + Al_3\sin^2\theta}{A[A-(A-C)\cos^2\theta]}\dot{\varphi} - \frac{1}{A}\left[V + \frac{(f-l_3\cos\theta)^2}{2\,[A-(A-C)\,cos\theta]}\right]. \tag{9}$$

In [8], a method which generalizes all known integrable systems was introduced. This method is based on the invariance of equations of motion (4) under the transformation

$$\boldsymbol{\omega} = \boldsymbol{\omega}' + \rho\,(\gamma)\,\gamma. \tag{10}$$

Applying this transformation, we get the Lagrangian

$$L' = \frac{1}{2}\boldsymbol{\omega}'\mathbf{I}\cdot\boldsymbol{\omega}' + \mathbf{l}'\cdot\boldsymbol{\omega}' - V'. \tag{11}$$

The equations of motion derived from the new Lagrangian (11) are

$$\dot{\omega}'\mathbf{I} + \boldsymbol{\omega}' \times (\boldsymbol{\omega}'\mathbf{I} + \boldsymbol{\mu}') = \boldsymbol{\gamma} \times \frac{\partial V'}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}' \times \boldsymbol{\gamma} = \mathbf{0}. \tag{12}$$

The cyclic integral (7) becomes

$$\left(\boldsymbol{\omega}'\mathbf{I} + \mathbf{l}'\right)\cdot\boldsymbol{\gamma} = f, \tag{13}$$

where

$$\begin{aligned} V' &= V + (f - \mathbf{l}\cdot\boldsymbol{\gamma})\,\rho - \frac{\rho^2}{2}\gamma\mathbf{I}\cdot\gamma, \quad \mathbf{l}' = \mathbf{l} + \rho\gamma\mathbf{I}, \\ \boldsymbol{\mu}' &= \boldsymbol{\mu} - 2\rho\boldsymbol{\gamma}\bar{\mathbf{I}} + \boldsymbol{\gamma}\mathbf{I} \times (\nabla\rho \times \boldsymbol{\gamma}), \quad \bar{\mathbf{I}} = \frac{1}{2}\mathrm{tr}\,(\mathbf{I})\,\boldsymbol{\delta} - \mathbf{I}, \end{aligned} \tag{14}$$

where $\boldsymbol{\delta}$ is a unit matrix. The system described by (12), (13) and (14) is mathematically equivalent to that described by (4) and (7) but physically different (for more details see [8]). This method has been applied in many articles such as [4,9].

In this article, we aim to construct new integrable problems in rigid body dynamics. Each of these cases is identified by two scalar and vector functions V and $\boldsymbol{\mu}$. The reason is that those functions are unique for mechanical problem, while the Lagrangian is not. An expression $\frac{\partial F}{\partial \gamma} \times \gamma$, where F is an arbitrary function of γ, can be added to the vector $\mathbf{l}$ without changing the equations of motion or the integral (7). In other words, the two functions V and $\boldsymbol{\mu}$ are invariant under gauge transformation.

2 Yehia's method for constructing quartic integrals

Up to now, only a very limited number of integrable cases of a particle in the Euclidean plane with quartic integral were found, mostly in the past thirty years or so (e.g., [12–26]). Most of those cases are listed in Hietarinta's review [27]. In [28], Yehia has introduced a method for constructing integrable conservative two-dimensional mechanical systems whose second integral of motion is polynomial in velocities. This method appeared to be successful in constructing a great number of irreversible systems (involving gyroscopic forces) with a second integral quadratic (see, e.g., [29,30]), cubic [31] and quartic (see, e.g., [23] and [32–36]). In this method, the configuration space is not assumed to be the Euclidean plane. This expands the applicability of the results to various mechanical systems to include such problems as rigid body dynamics. Many new irreversible systems were obtained by using this method. Some of these systems generalize previously known ones by introducing additional parameters, and so any of the configuration manifolds and the potential of the forces acting on the system, or both, may be changed. Other systems are completely new. This method is applicable to two-dimensional mechanical systems only. To this type belongs, for example, the problem of motion of a natural mechanical system with n degrees of freedom, having $n-2$ cyclic coordinates. Another example is the problem of motion of a particle on a smooth (fixed or rotating) surface under a variety of forces. Further examples are given by the problem of motion about a fixed point of a rigid body acted upon by potential and gyroscopic forces that allow a cyclic variable [37,38]. These systems can be characterized or reduced to a mechanical system with Lagrangian

$$L = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + a_1\dot{q}_1 + a_2\dot{q}_2^2 - V, \tag{15}$$

where the six coefficients a_{ij}, a_i, V are functions of q_1, q_2 and dots denote differentiation with respect to time t. According to a theorem of Birkhoff [39], there always exists a coordinate transformation that reduces (15) to a system of the type

$$L = \frac{\Lambda}{2}\left(\dot{x}^2 + \dot{y}^2\right) + l_1\dot{x} + l_2\dot{y} - V, \tag{16}$$

where Λ, V, l_1, l_2 are certain functions of x, y. The Lagrangian system (16) admits the Jacobi integral:

$$I_1 = \frac{\Lambda}{2}\left(\dot{x}^2 + \dot{y}^2\right) + V = h. \tag{17}$$

If this system admits an additional first integral, independent of Eq. (17), then this system is integrable, i.e., the solution of the mechanical problem reduces to a number of quadratures and to the inversion of certain integrals. This is always guaranteed by the Liouville theorem for the equivalent Hamiltonian system (see, e.g., [40]). Performing the following time transformation

$$\mathrm{d}t = \Lambda \mathrm{d}\tau, \tag{18}$$

the Lagrangian (16) can be expressed as

$$L = \frac{1}{2}\left(x'^2 + y'^2\right) + l_1 x' + l_2 y' + U, \tag{19}$$

where $U = \Lambda(h - V)$ and dashes denote derivatives with respect to τ. The equations of motion take the form

$$x'' + \Omega y' = \frac{\partial U}{\partial x}, \quad y'' + \Omega x = \frac{\partial U}{\partial y}, \tag{20}$$

where $\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}$. This system admits the Jacobi integral:

$$I_1 = \frac{1}{2}\left(x'^2 + y'^2\right) - U = 0. \tag{21}$$

The Jacobi constant h for the original system (16) enters as parameters in the new potential$-U$. It is known from the results of [28] that the integral can be written in the form

$$I_2 = x'^4 + P_3 x'^3 + Q_3 x'^2 y' + P_2 x'^2 + Q_2 x' y' + P_1 x' + Q_1 y' + R = c_0, \tag{22}$$

where P_j, Q_j, R are functions in both variables x, y, and c_0 is an arbitrary constant. Differentiating (22) with respect to τ and using Jacobi's integral again as in [28], we obtain nonlinear system of partial differential equations:

$$\frac{\partial P_3}{\partial x} - \frac{\partial Q_3}{\partial y} = 0, \quad \frac{\partial P_3}{\partial y} + \frac{\partial Q_3}{\partial x} - 4\Omega = 0, \quad \frac{\partial P_2}{\partial y} + \frac{\partial Q_2}{\partial x} - 3\Omega P_3 = 0, \tag{23}$$

$$\frac{\partial P_2}{\partial x} - \frac{\partial Q_2}{\partial y} + 3\Omega Q_3 + 4U = 0, \quad P_1\frac{\partial U}{\partial x} + Q_1\frac{\partial U}{\partial y} + 2U\frac{\partial Q_1}{\partial y} - 2\Omega Q_2 U = 0, \tag{24}$$

$$\frac{\partial P_1}{\partial x} - \frac{\partial Q_1}{\partial y} + 2\Omega Q_2 + 3P_3\frac{\partial U}{\partial x} + Q_3\frac{\partial U}{\partial y} + 2U\frac{\partial Q_3}{\partial y} = 0, \tag{25}$$

$$\frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} + 2Q_3\frac{\partial U}{\partial x} - 2\Omega P_2 = 0, \tag{26}$$

$$\frac{\partial R}{\partial x} + 2P_2\frac{\partial U}{\partial x} + Q_2\frac{\partial U}{\partial y} + 2U\frac{\partial Q_2}{\partial y} + \Omega Q_1 - 4\Omega U Q_3 = 0, \quad \frac{\partial R}{\partial y} + Q_2\frac{\partial U}{\partial x} - \Omega P_1 = 0. \tag{27}$$

This is a system of nine equations in nine unknown functions. It is not known, however, whether this system is solvable, in the sense that its complete set of solutions can be found. Regarding the second equation of Eqs. (23) and the definition of Ω, one can now construct a Lagrangian (19) compatible with the integral (22) as

$$L = \frac{1}{2}\left(x'^2 + y'^2\right) + \frac{1}{4}\left(P_3 x' - Q_3 y'\right) + U. \tag{28}$$

It is evident that any solution of the system (23)–(27) can be interpreted as determining a mechanical system that admits a quartic integral on its zero level of the Jacobi integral. If it happens that in the solution the function U has the structure $U + cU_1$, where c is an arbitrary constant, then U_1 can be identified as the coefficient Λ in (18) and c as Jacobi's constant. It may even happen, as will be seen later, the constant c and U can be chosen in more than one way. In some cases, the same constant enters in one or more of the coefficients of the integral. To obtain an unrestricted integral of the motion, this constant should be eliminated in virtue of Jacobi integral. This situation is frequently utilized below.

Setting $Q_3(x, y) = 0$, Eq. (23) leads to

$$P_3 = \kappa f(y), \quad \Omega = \frac{1}{4}\frac{\mathrm{d}f}{\mathrm{d}y}, \quad Q_2 = -F_{xy}, \quad P_2 = F_{xx} + \frac{3}{8}\kappa^2 f^2, \tag{29}$$

where F is an arbitrary function in both variables x, y, $f(y)$ is an arbitrary function of y, and κ is an arbitrary constant. Taking into account all results obtained, Eq. (24) gives

$$U = \frac{1}{4}\nabla^2 F, \quad Q_1 = \kappa\left[G_y + \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}y}F_x\right], \quad P_1 = \kappa\left[-G_x + \frac{\kappa^2}{16}f^3\right], \tag{30}$$

where G is an arbitrary function in two variables x, y. From Eq. (27), one can express the function R—up to an additive constant—in the form

$$R(x, y) = -\int\left(Q_2\frac{\partial U}{\partial x} - \Omega P_1\right)\mathrm{d}y - \int\left[2P_2\frac{\partial U}{\partial x} + Q_2\frac{\partial U}{\partial y} + 2U\frac{\partial Q_2}{\partial y} + \Omega Q_1 - 4\Omega U Q_3\right]_0 \mathrm{d}x, \tag{31}$$

where $[]_0$ means that the expression in the bracket is computed for y taking an arbitrary constant value y_0 (say). It must be noted that $R(x, y)$ satisfies the compatibility condition

$$\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x}\right). \tag{32}$$

Taking all obtained results into account, Eqs. (25), (26) and (32) become

$$\kappa\left(4\nabla^2 G + 3f\frac{\partial}{\partial x}\nabla^2 F + 2\frac{d^2 f}{\mathrm{d}y^2}F_x + 4\frac{\mathrm{d}f_0}{\mathrm{d}y}F_{xy}\right) = 0, \tag{33}$$

$$\kappa\left\{\left[\kappa^2 f + 8\left(2G_y + \frac{\mathrm{d}f}{\mathrm{d}y}F_x\right)\right]\nabla^2 F_x + 8\left[\frac{\mathrm{d}f}{\mathrm{d}y} + 2G_x + 4G_{yy} - \frac{\mathrm{d}f}{\mathrm{d}y}F_{xy}\right]\nabla^2 F\right\} = 0, \tag{34}$$

$$\begin{aligned}&\kappa^2\left[\frac{\mathrm{d}^2 f}{\mathrm{d}y^2}\left(G_y + \frac{\mathrm{d}f}{\mathrm{d}y}F_x\right) + \frac{1}{2}\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2 F_{xy} - \frac{\mathrm{d}f}{\mathrm{d}y}\left(G_{xx} - G_{yy}\right) - \frac{3}{2}f\frac{\mathrm{d}f}{\mathrm{d}y}\nabla^2 F_x - \frac{3}{4}f^2\nabla^2 F_{xy}\right]\\ &-F_{xy}F_{xxxx} - 2F_{xx}F_{yxxx} + 2F_{yyyx}F_{yy} + F_{xy}F_{yyyy} - 3F_{xxy}F_{xxx} + 3F_{xyy}F_{yyy} = 0.\end{aligned} \tag{35}$$

Note that when the parameter κ vanishes, the present problem becomes reversible and Eqs. (33)–(35) reduce to a single equation

$$-F_{xy}F_{xxxx} - 2F_{xx}F_{yxxx} + 2F_{yyyx}F_{yy} + F_{xy}F_{yyyy} - 3F_{xxy}F_{xxx} + 3F_{xyy}F_{yyy} = 0. \tag{36}$$

This equation appeared for the first time in [35], and it is called *resolving equation*. Its solution was constructed with certain assumption. It is also used in [41] to construct new two-dimensional integrable problems with quartic integral. Following [35], we can set

$$x = \int\frac{\mathrm{d}\zeta}{\sqrt[4]{a_4\zeta^4 + a_3\zeta^3 + a_2\zeta^2 + a_1\zeta + a_0}}, \quad y = \int\frac{\mathrm{d}\xi}{\sqrt[4]{a_4\xi^4 + b_3\xi^3 + b_2\xi^2 + b_1\xi + b_0}}, \tag{37}$$

where $a_4, a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0$ are arbitrary constants. Thus the problem is formulated for the general case, but it is solved only for the case when the configuration space characterizes a rigid body dynamics.

3 Applications to rigid body dynamics

In the irreversible case, the solution of (33)–(35) is expressed as the solution in the reversible case plus some additional terms under conditions leading to a metric of the Kovalevskaya type of rigid body dynamics. These conditions are

$$
\begin{aligned}
& a_4 = 16, \quad a_3 = b_3 = 0, \quad a_2 = -32\beta^2, \quad a_1 = 0, \quad a_0 = 16\beta^4, \quad b_2 = 2b_0 = -96, \quad b_1 = 128, \\
& \zeta = \frac{\cos^4\theta}{1-\cos^2\theta} + 1 \quad \text{and} \quad \xi = \beta \sin 2(\varphi - \varphi_0),
\end{aligned} \tag{38}
$$

where β, φ_0 are arbitrary constants. After some calculations which are not presentable in a suitable size, we will obtain two new cases. These cases are expressed in terms of Eulerian angles as generalized coordinates.

3.1 First new integrable case

This case is characterized by the following Lagrangian:

$$
\begin{aligned}
L = {} & \frac{1}{2}\left[\varphi'^2 + \frac{2-\gamma_3^2}{\left(1-\gamma_3^2\right)^2}\gamma_3'^2\right] + \frac{1-\gamma_3^2}{2-\gamma_3^2}\left(k + \frac{a(1+(1-\gamma_3^2)\cos^2\varphi)}{(1-\gamma_3^2)\sin^2\varphi}\right)\varphi' - \frac{1-\gamma_3^2}{2\left(2-\gamma_3^2\right)}\left\{\frac{\lambda}{2\gamma_3^2}\right. \\
& + \left(1-\gamma_3^2\right)(b_1\cos 2\varphi - b_2 \sin 2\varphi) - \frac{ak\gamma_3^2}{\gamma_1^2} + \frac{a^2\gamma_3^2\left(2(1-\gamma_3^2)\cos^2\varphi - \gamma_3^2\right)}{2\left(1-\gamma_3^2\right)^2\cos^4\varphi} \\
& \left. - 2p_0 + \frac{\gamma_3^2}{2\left(2-\gamma_3^2\right)}\left(\frac{\left(k + a(1+(1-\gamma_3^2)\cos^2\varphi)\right)}{\left(1-\gamma_3^2\right)\sin^2\varphi}\right)^2\right\}.
\end{aligned} \tag{39}
$$

Its conditional Jacobi's integral becomes

$$
\begin{aligned}
I_1 = {} & \frac{1}{2}\left[\varphi'^2 + \frac{2-\gamma_3^2}{\left(1-\gamma_3^2\right)^2}\gamma_3'^2\right] + \frac{1-\gamma_3^2}{2\left(2-\gamma_3^2\right)}\left\{\frac{\lambda}{2\gamma_3^2} - \frac{ak\gamma_3^2}{\gamma_1^2} + \left(1-\gamma_3^2\right)(b_1\cos 2\varphi - b_2\sin 2\varphi)\right. \\
& - \frac{ak\gamma_3^2}{\gamma_1^2} + \frac{a^2\gamma_3^2\left(2\left(1-\gamma_3^2\right)\cos^2\varphi - \gamma_3^2\right)}{2\left(1-\gamma_3^2\right)^2\cos^4\varphi} - 2p_0 + \frac{\gamma_3^2}{2\left(2-\gamma_3^2\right)} \\
& \left. \times\left(\frac{\left(k + a(1+\left(1-\gamma_3^2\right)\cos^2\varphi)\right)}{(1-\gamma_3^2)\sin^2\varphi}\right)^2\right\} = 0.
\end{aligned} \tag{40}
$$

The conditional quartic integral is

$$
\begin{aligned}
I_2 = {} & \varphi'^4 - \frac{4k}{\gamma_3^2-2}\varphi'^3 + \left\{-p_0 + \frac{\gamma_3^4-\gamma_3^2-1}{\gamma_3^2-1}\left[b_1\cos 2\varphi - b_2\sin 2\varphi\right] - \frac{k^2\left(\gamma_3^4 - 4\gamma_3^2+1\right)}{2\left(2-\gamma_3^2\right)^2}\right\} \\
& \times\varphi'^3 + \left\{\frac{\gamma_3^3\left(\gamma_3^2-2\right)}{\left(1-\gamma_3^2\right)^2}\left[b_2\cos 2\varphi + b_1\sin 2\varphi\right]\right\}\gamma_3'\varphi' + \left\{\frac{2p_0k}{\gamma_3^2-2} + \frac{k^3\gamma_3^2\left(\gamma_3^2-4\right)}{\left(\gamma_3^2-2\right)^3}\right. \\
& \left. + \frac{2k\left(2\gamma_3^2-1\right)}{\gamma_3^2-2}(b_1\cos 2\varphi - b_2\sin 2\varphi)\right\}\varphi' - \frac{2k\gamma_3}{\gamma_3^2-1}\left[b_2\cos 2\varphi + b_1\sin 2\varphi\right]\gamma_3' \\
& - \frac{1}{8}\left(\gamma_3^4-1\right)\left[\left(b_1^2-b_2^2\right)\cos 4\varphi - 2b_1b_2\sin 4\varphi\right] + \frac{1}{4}\left[b_1\cos 2\varphi - b_2\sin 2\varphi\right] \\
& \times\left(\left(3k^2+2h-\lambda\right)\gamma_3^4 + \left(2k^2-4p_0+2\lambda\right)\gamma_3^2 - 4k^2 + 4p_0 - 2\lambda\right),
\end{aligned} \tag{41}
$$

where $k, a, b_1, b_2, \lambda, p_0$ are arbitrary constants. This problem is a conditional problem since it is valid only on the zero level of Jacobi integral (40). Setting $p_0 = h$ and preforming a time transformation

$$d\tau = \frac{2-\gamma_3^2}{1-\gamma_3^2} dt, \tag{42}$$

the Lagrangian (39) becomes

$$\begin{aligned} L &= \frac{1}{2}\left[\frac{1-\gamma_3^2}{2-\gamma_3^2}\dot{\varphi}^2 + \frac{\dot{\gamma}_3^2}{1-\gamma_3^2}\right] + \frac{1-\gamma_3^2}{2-\gamma_3^2}\left(k + \frac{\alpha(1+(1-\gamma_3^2)\cos^2\varphi)}{(1-\gamma_3^2)\sin^2\varphi}\right)\dot{\varphi} \\ &\quad - \frac{1}{2}\left\{\frac{\gamma}{2\gamma_3^2} + \left(1-\gamma_3^2\right)(b_1\cos 2\varphi - b_2 \sin 2\varphi) - \frac{ak\gamma_3^2}{\gamma_1^2} + \frac{a^2\gamma_3^2\left(2(1-\gamma_3^2)\cos^2\varphi - \gamma_3^2\right)}{2\left(1-\gamma_3^2\right)^2\cos^4\varphi}\right. \\ &\quad \left. + \frac{\gamma_3^2}{2\left(2-\gamma_3^2\right)}\left(\frac{\left(k + a(1+(1-\gamma_3^2)\cos^2\varphi\right)}{(1-\gamma_3^2)\sin^2\varphi}\right)^2\right\} + h. \end{aligned} \tag{43}$$

Note that the presence of the arbitrary parameter h in the last Lagrangian is insignificant and can be ignored. The same arbitrary constant h is now interpreted as the value of the Jacobi integral I_1:

$$\begin{aligned} I_1 &= \frac{1}{2}\left[\frac{1-\gamma_3^2}{2-\gamma_3^2}\dot{\varphi}^2 + \frac{\dot{\gamma}_3^2}{1-\gamma_3^2}\right] + \frac{1}{2}\left\{\frac{\lambda}{2\gamma_3^2} + (1-\gamma_3^2)(b_1\cos 2\varphi - b_2\sin 2\varphi) - \frac{ak\gamma_3^2}{\gamma_1^2}\right. \\ &\quad \left. + \frac{a^2\gamma_3^2\left(2(1-\gamma_3^2)\cos^2\varphi - \gamma_3^2\right)}{2\left(1-\gamma_3^2\right)^2\cos^4\varphi} + \frac{\gamma_3^2}{2\left(2-\gamma_3^2\right)}\left(\frac{\left(k + a(1+(1-\gamma_3^2)\cos^2\varphi\right)}{(1-\gamma_3^2)\sin^2\varphi}\right)^2\right\} = h. \end{aligned} \tag{44}$$

The unconditional quartic integral becomes

$$I_2 = I_2\left(\gamma_3, \varphi, \Lambda\dot{\gamma}_3, \Lambda\dot{\varphi}\right). \tag{45}$$

It is more suitable that the constant h in Eq. (45) should be replaced by its expression (44) in terms of state variables. Comparing (9), (43) and using (1) to express the results in terms of Euler–Poisson variables, we obtain

$$\begin{aligned} l_3 &= K + \frac{\nu\left(1+\gamma_2^2\right)}{\gamma_1^2}, \\ V &= C\left\{d\gamma_1\gamma_2 + c\left(\gamma_1^2 - \gamma_2^2\right) + \frac{\lambda}{2\gamma_3^2} - \frac{\nu K\gamma_3^2}{\gamma_1^2} - \frac{\nu^2\gamma_3^2}{2\gamma_1^4}\left(\gamma_3^2 + 2\gamma_2^2\right)\right\}, \end{aligned} \tag{46}$$

where K, ν, d, c are arbitrary parameters, introduced instead of the original parameters for convenience. This case is a new integrable problem in rigid body dynamics. It generalizes the case obtained by Goriachev by two free parameters ($K = \nu = 0$) [42]. When $K = \nu = \lambda = 0$, the remaining potential characterizes the Chaplygin case of a rigid body in a liquid [6]. It also involves one constant ν more than the case found by Yehia [43]. Taking into account (1) and (2), the complementary integral (45) can be written in terms of Euler–Poisson variables as:

$$\begin{aligned} I_4 &= \left\{p^2 - q^2 + c\gamma_3^2 - \frac{\lambda\left(\gamma_1^2 - \gamma_2^2\right)}{2\gamma_3^2}\right\}^2 + \left\{2pq + \frac{d}{2}\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}\right\}^2 - 2\left(p^2+q^2\right)\left\{\frac{\nu^2\gamma_3^2}{\gamma_1^4} + \left[\frac{\nu\gamma_3^2}{\gamma_1^2}\right.\right. \\ &\quad \left. + \nu - K\right][r + (K-\nu)^2\} + r\left\{\lambda\left(1 + \frac{1}{\gamma_3^2}\right)(K-\nu) + \frac{2\nu\gamma_3^2}{\gamma_1^2}\left[\frac{\nu^2\left(\gamma_1^2+\gamma_3^2\right)}{\gamma_1^4} - \frac{K\nu}{\gamma_1^2} - \frac{1}{\gamma_1^2+\gamma_2^2}\right.\right. \\ &\quad \left.\left. \times \left[c\left(2\gamma_1^4 + \gamma_3^4\left(2-\gamma_1^2\right) - \gamma_3^6 - \gamma_1^2 - \gamma_3^2\right) - d\gamma_1\gamma_2\left(\gamma_1^2 - \gamma_3^2\right)\right] + \frac{\lambda\left(\gamma_1^2+\gamma_2^2\right)}{2\gamma_3^2}\right]\right\} - \lambda\left(K-\nu\right)^2 \end{aligned}$$

$$\times\left(1+\frac{1}{\gamma_3^2}\right)-\frac{4p}{\gamma_1^2\left(\gamma_1^2+\gamma_2^2\right)}\left\{\frac{d}{2}\left(2\nu\gamma_2\gamma_1^4+\gamma_2\gamma_1^2\left[(K-\nu)\left(\gamma_3^4+1\right)-2K\gamma_3^2\right]-\left(\gamma_1^2+\gamma_2^2\right)^2\right.\right.$$
$$\left.\left.\times\nu\gamma_2\gamma_3^2\right]\right)+c\left(2\nu\gamma_1^5+\left[(K-2\nu)\left(\gamma_3^4+1\right)-2\gamma_3^2(K-\nu)\right]\gamma_1^3-2\nu\gamma_1\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)^2\right)\Bigg\}$$
$$+\frac{4\gamma_3 q}{\gamma_1\left(\gamma_1^2+\gamma_2^2\right)}\left\{c\gamma_1\gamma_2\left[K-2(K-\nu)\gamma_3^2+K\gamma_3^2-2\nu\gamma_1^2\right]+d\left[\nu\gamma_1^4-\frac{\gamma_1^2}{2}\left[(K-\nu)\gamma_3^2\left(\gamma_3^2-2\right)\right.\right.\right.$$
$$\left.\left.\left.+K+\nu\right]-\frac{\nu\gamma_3^2}{2}\left(\gamma_1^2+\gamma_2^2\right)\left(\gamma_3^2-3\right)\right]\right\}-\frac{2\nu\gamma_3^2r^2}{\gamma_1^2}\left\{K+\frac{\nu}{\gamma_1^2}(\gamma_3^2-2+2\gamma_2^2)\right\}+\frac{\nu^4\left(1-\gamma_2^2\right)}{\gamma_1^8}$$
$$\times\frac{\nu^4\left(1-\gamma_2^2\right)\left(2\gamma_1^4+\gamma_1^2\gamma_3^2+\gamma_3^2\right)}{\gamma_1^8}+\frac{2K\nu^3\gamma_3^2\left(3\gamma_1^2+\gamma_3^2-2\right)}{\gamma_1^4}+\frac{\nu^2\gamma_3^2}{\gamma_1^4}\left\{\left(6\gamma_2^2+5\gamma_3^2-4\right)K^2\right.$$
$$-\frac{2c}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[-\gamma_3^8+\left(5-4\gamma_2^2\right)\gamma_3^6-3\left(\gamma_2^4-6\gamma_2^2+2\right)\gamma_3^4+\gamma_3^2\left(12\gamma_2^4-16\gamma_2^2+3\right)+2\gamma_2^6\right.$$
$$\left.\left.-5\gamma_2^4+4\gamma_2^2-1\right]-\frac{2\mathrm{d}\gamma_1\gamma_2}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[\gamma_3^6+2\left(\gamma_2^2-4\right)\gamma_3^4-3\left(1-2\gamma_2^2\right)\gamma_3^2\right]+\frac{\lambda}{\gamma_3^2}\left(2\gamma_1^2+\gamma_3^2-1\right)\right\}$$
$$-1)\}+\frac{2\nu K\gamma_3^2}{\gamma_1^2}\left\{K^2+\frac{\lambda}{\gamma_3^2}-\frac{1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[c\gamma_3^6+\gamma_3^4\left[c\left(3\gamma_1^2-4\right)+2\mathrm{d}\gamma_1\gamma_2\right]+\gamma_3^2\left[c\left(5-8\gamma_1^2\right)\right.\right.\right.$$
$$\left.\left.\left.-5\mathrm{d}\gamma_1\gamma_2\right]+\left(2+\gamma_1^2\right)\left[\mathrm{d}\gamma_1\gamma_2+c\left(2\gamma_1^2-1\right)\right]\right]\right\}. \tag{47}$$

Further generalization can be obtained by applying the transformation (10) with

$$\rho(\gamma)=n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2, \tag{48}$$

where n, n_1 and n_2 are arbitrary constants. Then one can formulate the following theorem:

Theorem 1 *Let the moments of inertia satisfy the Kovalevskaya condition $A=B=2C$ and let the scalar and vector functions V and $\boldsymbol{\mu}$ be given by*

$$V=C\left\{\mathrm{d}\gamma_1\gamma_2+c\left(\gamma_1^2-\gamma_2^2\right)+\frac{\lambda}{2\gamma_3^2}-\frac{\nu K\gamma_3^2}{\gamma_1^2}-\frac{\nu^2\gamma_3^2}{2\gamma_1^4}\left(\gamma_3^2+2\gamma_2^2\right)-\frac{1}{2}\left(2\gamma_1^2+2\gamma_2^2+\gamma_3^2\right)\right.$$
$$\left.\times\left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)^2-\left[n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right]\gamma_3\left(K+\nu\frac{1+\gamma_2^2}{\gamma_1^2}\right)\right\} \tag{49}$$

and

$$\boldsymbol{\mu}=C\left(\gamma_1\left(9n_1+n-7\gamma_3^2-10\gamma_1^2\right)-n_2\gamma_2\left(5\gamma_1^2+\gamma_3^2-2\right)+\frac{2\nu\gamma_3}{\gamma_1^3}\left(\gamma_1^2+\gamma_3^2-2\right),\right.$$
$$\gamma_2(n_1-n-3\gamma_3^2+10\gamma_1^2)+n_2\gamma_1\left(5\gamma_1+4\gamma_3^2-3\right)+\frac{2\nu\gamma_2\gamma_3}{\gamma_1^2},$$
$$\left.K-\frac{\nu}{\gamma_1^2}\left(\gamma_1^2+\gamma_3^2-2\right)-\gamma_3\left(7n_2\gamma_1\gamma_2+3n+7n_1+2\gamma_1^2+\gamma_3^2\right)\right), \tag{50}$$

or, equivalently,

$$\mathbf{l}=C\left(2\left[n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right]\gamma_1,2\left[n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right]\gamma_2,\right.$$
$$\left.K+\frac{\nu\left(1+\gamma_2^2\right)}{\gamma_1^2}+\left[n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right]\gamma_3\right),$$

where $c, d, \lambda, \nu, K, n, n_1$ and n_2 are free parameters. Then Euler–Poisson equations (4) with (49) and (50) are integrable on the zero level of the cyclic integral

$$I_2 = 2p\gamma_1 + 2q\gamma_2 + \left(r + K + \frac{\nu\left(1+\gamma_2^2\right)}{\gamma_1^2}\right)\gamma_3 + \left[n + n_1\left(\gamma_1^2 - \gamma_2^2\right) + n_2\gamma_1\gamma_2\right] \times \left[2\gamma_1^2 + 2\gamma_2^2 + \gamma_3^2\right] = 0,$$

and the complementary integral is

$$\begin{aligned} I_4 = &\Bigg\{\left[p + \left(n + n_1\left(\gamma_1^2-\gamma_2^2\right) + n_2\gamma_1\gamma_2\right)\gamma_1\right]^2 - \left[q + \left(n + n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_2\right]^2 + c\gamma_3^2 \\ &- \frac{\lambda\left(\gamma_1^2-\gamma_2^2\right)}{2\gamma_3^2}\Bigg\}^2 + \Bigg\{2\left[p + \left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_1\right]\left[q + \left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\right. \\ &\left.\times\gamma_2\right] + \frac{d}{2}\gamma_3^2 - \frac{\lambda\gamma_1\gamma_2}{\gamma_3^2}\Bigg\}^2 - 2\Bigg\{\left[p+\left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_1\right]^2 + \left[q + \left(n + n_1\left(\gamma_1^2\right.\right.\right. \\ &\left.\left.\left.-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_2\right\}\left\{\frac{\nu^2\gamma_3^2}{\gamma_1^4} + \left[\frac{\nu\gamma_3^2}{\gamma_1^2} + \nu - K\right]\left[r + (n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2)\gamma_3\right] + (K-\nu)^2\right\} \\ &+ \left[r + \left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_3\right]\Bigg\{\lambda\left(1+\frac{1}{\gamma_3^2}\right)(K-\nu) + \frac{2\nu\gamma_3^2}{\gamma_1^2}\Bigg[\frac{\nu^2\left(\gamma_1^2+\gamma_3^2\right)}{\gamma_1^4} - \frac{K\nu}{\gamma_1^2} \\ &- \frac{1}{\gamma_1^2+\gamma_2^2}\left[c\left(2\gamma_1^4 + \gamma_3^4\left(2-\gamma_1^2\right) - \gamma_3^6 - \gamma_1^2 - \gamma_3^2\right) - \mathrm{d}\gamma_1\gamma_2\left(\gamma_1^2-\gamma_3^2\right)\right] + \frac{\lambda\left(\gamma_1^2+\gamma_2^2\right)}{2\gamma_3^2}\Bigg]\Bigg\} \\ &- \lambda(K-\nu)^2\left(1+\frac{1}{\gamma_3^2}\right) - \frac{4}{\gamma_1^2\left(\gamma_1^2+\gamma_2^2\right)}\left[p + \left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_1\right]\Bigg\{\frac{d}{2}\left(2\nu\gamma_2\gamma_1^4\right. \\ &\left.+\gamma_2\gamma_1^2\left[(K-\nu)\left(\gamma_3^4+1\right) - 2K\gamma_3^2\right] - \nu\gamma_2\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)^2\right]\Big) + c\Big(2\nu\gamma_1^5 + \left[(K-2\nu)\left(\gamma_3^4+1\right)\right. \\ &\left.-2\gamma_3^2(K-\nu)\right]\gamma_1^3 - 2\nu\gamma_1\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)^2\Big)\Bigg\} + \frac{4\gamma_3}{\gamma_1\left(\gamma_1^2+\gamma_2^2\right)}\left[q + (n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_2\gamma_1)\gamma_2\right] \\ &\times\Bigg\{c\gamma_1\gamma_2\left[K - 2(K-\nu)\gamma_3^2 + K\gamma_3^2 - 2\nu\gamma_1^2\right] + d\Bigg[\nu\gamma_1^4 - \frac{\gamma_1^2}{2}\left((K-\nu)\gamma_3^2\left(\gamma_3^2-2\right) + K + \nu\right) \\ &-\frac{\nu\gamma_3^2}{2}\left(\gamma_1^2+\gamma_2^2\right)\left(\gamma_3^2-3\right)\Bigg]\Bigg\} - \frac{2\nu\gamma_3^2}{\gamma_1^2}\left[r + \left(n+n_1\left(\gamma_1^2-\gamma_2^2\right)+n_2\gamma_1\gamma_2\right)\gamma_3\right]^2\Bigg\{K + \frac{\nu}{\gamma_1^2}\left(\gamma_3^2 - 2\right. \\ &\left.+2\gamma_2^2\right)\Bigg\} + \frac{\nu^4\left(1-\gamma_2^2\right)\left(2\gamma_1^4+\gamma_1^2\gamma_3^2+\gamma_3^2\right)}{\gamma_1^8} + \frac{2K\nu^3\gamma_3^2\left(3\gamma_1^2+\gamma_3^2-2\right)}{\gamma_1^4} + \frac{\nu^2\gamma_3^2}{\gamma_1^4}\Bigg\{\left(6\gamma_2^2+5\gamma_3^2\right. \\ &\left.-4\right)K^2 - \frac{2c}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[-\gamma_3^8 + \left(5-4\gamma_2^2\right)\gamma_3^6 - 3\left(\gamma_2^4-6\gamma_2^2+2\right)\gamma_3^4 + \gamma_3^2\left(12\gamma_2^4-16\gamma_2^2+3\right)\right. \\ &\left.+2\gamma_2^6 - 5\gamma_2^4 + 4\gamma_2^2 - 1\right] - \frac{2\mathrm{d}\gamma_1\gamma_2}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[\gamma_3^6 + 2\left(\gamma_2^2-4\right)\gamma_3^4 - 3\left(1-2\gamma_2^2\right)\gamma_3^2\right] + \frac{\lambda}{\gamma_3^2}\left(2\gamma_1^2+\gamma_3^2\right. \\ &\left.-1\right)\Big\} + \frac{2\nu K\gamma_3^2}{\gamma_1^2}\Bigg\{K^2 + \frac{\lambda}{\gamma_3^2} - \frac{1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[c\gamma_3^6 + \gamma_3^4\left[c\left(3\gamma_1^2-4\right) + 2\mathrm{d}\gamma_1\gamma_2\right] + \gamma_3^2\left[c\left(5-8\gamma_1^2\right)\right.\right. \\ &\left.\left.-5\mathrm{d}\gamma_1\gamma_2\right] + \left(2+\gamma_1^2\right)\left[\mathrm{d}\gamma_1\gamma_2 + c\left(2\gamma_1^2-1\right)\right]\right]\Bigg\}. \end{aligned}$$

This is a new integrable problem in rigid body dynamics. It contains eight free parameters. It also generalizes some problems in this field. Let us clarify that in the following table:

References	Conditions on parameters
Yehia [43]	$\nu = n = n_1 = n_2 = 0$
Goriachev [42]	$K = \nu = n = n_1 = n_2 = 0$
Chaplygin [6]	$K = \nu = n = n_1 = n_2 = \lambda = 0$

3.2 Second new case

In a similar way, one can formulate the following theorem:

Theorem 2 *For a rigid body with moments of inertia satisfying the Kovalevsky condition $A = B = 2C$ and let the scalar and vector functions V and $\boldsymbol{\mu}$ be given by*

$$V = C\left\{c\left(\gamma_1^2-\gamma_2^2\right)+2\mathrm{d}\gamma_1\gamma_2+\frac{\lambda}{\gamma_3^2}+\rho\left(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6}\right)+\frac{\gamma_3^2\left(\gamma_3^2-2\right)}{2\gamma_1^4\gamma_2^2}\left(\nu_1\gamma_1+\nu_2\gamma_2\right)^2\right.$$
$$+\frac{\gamma_3\left(\gamma_3^2-2\right)\left(\nu_1\gamma_1+\nu_2\gamma_2\right)}{\gamma_1^2\gamma_2}\left[n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2\right]$$
$$\left.-\frac{1}{2}(2\gamma_1^2+2\gamma_2^2+\gamma_3^2)(n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2)^2\right\} \tag{51}$$

and

$$\boldsymbol{\mu} = C\left(\gamma_1\left[9n+n_1\left(3-7\gamma_3^2\right)\right]+n_2\gamma_2\left(15\gamma_1^2+2\gamma_2^2+\gamma_3^2\right)-\frac{\nu_1\gamma_3}{\gamma_1^2\gamma_2^2}\left(4\gamma_1^2+2\gamma_2^2+\gamma_3^2\right)\right.$$
$$-\frac{2\nu_2\gamma_3}{\gamma_1^3}\left(3\gamma_1^2+2\gamma_2^2+\gamma_3^2\right),\gamma_2\left[9n+n_1\left(11\gamma_1^2-4\gamma_3^2-13\gamma_2^2\right)+n_2\gamma_1(2\gamma_1^2+15\gamma_2^2\right.$$
$$\left.-\gamma_3^2\right)+\frac{\nu_1\gamma_3}{\gamma_1\gamma_2^2}\left(2\gamma_1^2+3\gamma_3^2-4\right)-\frac{2\nu_2\gamma_2\gamma_3}{\gamma_1^2},\gamma_3\left[7n+n_1\left(\gamma_1^2-7\gamma_2^2-2\gamma_3^2\right)\right.$$
$$\left.+\ 11n_2\gamma_1\gamma_2\right]-\frac{5\gamma_3^2-2}{\gamma_1^2\gamma_2}(\nu_1\gamma_1+\nu_2\gamma_2)\Bigg), \tag{52}$$

or, equivalently,

$$\mathbf{l} = C\Bigg(2\gamma_1\left[n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2\right],2\gamma_2\left[n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2\right],$$
$$\gamma_3[n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2]+\frac{\left(2-\gamma_3^2\right)}{\gamma_1^2\gamma_2}\left(\nu_1\gamma_1+\nu_2\gamma_2\right)\Bigg),$$

where $c, d, \lambda, \nu_1, \nu_2, n, n_1$ and n_2 are free parameters. Then Euler–Poisson equations (4) with (51) and (52) are integrable on the zero level of the cyclic integral

$$I_2 = 2p\gamma_1+2q\gamma_2+\left(r+\frac{\left(2-\gamma_3^2\right)}{\gamma_1^2\gamma_2}\left(\nu_1\gamma_1+\nu_2\gamma_2\right)\right)\gamma_3+\left[n-n_1\left(\gamma_2^2-\gamma_1^2\right)+n_2\gamma_1\gamma_2\right]$$
$$\times\ \left(2\gamma_1^2+2\gamma_2^2+\gamma_3^2\right)=0.$$

The complementary integral can be written in the form

$$
\begin{aligned}
I_4 = \Bigg[& \left(p + \left[n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right]\gamma_1\right)^2 - \left(q + \left[n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right]\gamma_2\right)^2 + c\gamma_3^2 \\
& - \frac{\lambda\left(\gamma_1^2 - \gamma_2^2\right)}{\gamma_3^2}\Bigg]^2 + \Bigg[2\left(p + \left(n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right)\gamma_1\right)\left(q + \left(n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\right.\right. \\
& \times \gamma_2)\gamma_2) + \mathrm{d}\gamma_3^2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2}\Bigg]^2 + \Big[\left(p + \left[n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right]\gamma_1\right)^2 + \left(q + \left(n - n_1\left(\gamma_2^2\right.\right.\right. \\
& - \left.\left.\left.\gamma_1^2\right) + n_2\gamma_1\gamma_2\right)\gamma_2\right)^2\Big]\Bigg\{2\rho\left(\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6}\right) - \frac{2\gamma_3^2\left(\nu_1\gamma_1 + \nu_2\gamma_2\right)}{\gamma_1^4\gamma_2^2}\left(\nu_1\gamma_1 + \nu_2\gamma_2 + \gamma_2\gamma_1^2\ (r + (n\right. \\
& \left. - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2)\gamma_3)\right)\Bigg\} + \frac{\gamma_3^4\left(\nu_1\gamma_1 + \nu_2\gamma_2\right)^4}{\gamma_1^8\gamma_2^4} + \frac{\rho}{\gamma_3^4}\Bigg\{\frac{\left(\gamma_2^2 + \gamma_1^2\right)^2}{\gamma_3^8}\left(\rho - 2\lambda\gamma_3^4\right) \\
& + 2\left[c\left(\gamma_1^2 - \gamma_2^2\right) + 2\mathrm{d}\gamma_1\gamma_2\right]\Bigg\} + 2\frac{\left(\nu_1\gamma_1 + \nu_2\gamma_2\right)^2}{\gamma_1^4\gamma_2^2}\Bigg\{\left(\gamma_1^2 + \gamma_2^2\right)\left(\lambda + \frac{\rho}{\gamma_3^4}\right) + \frac{\gamma_3^4}{2}(r + (n \\
& + n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2)\gamma_3)^2 + \frac{\gamma_3^4\left[c\left(\gamma_1^2 - \gamma_2^2\right) + 2\mathrm{d}\gamma_1\gamma_2\right]}{\left(\gamma_2^2 + \gamma_1^2\right)^2}\left(\gamma_3^4 - 3\gamma_3^2 + 3\right)\Bigg\} \\
& + 2\frac{\nu_1\gamma_1 + \nu_2\gamma_2}{\gamma_1^2\gamma_2}\Bigg\{\Bigg[\frac{\gamma_3^4\left(c\left(\gamma_1^2 - \gamma_2^2\right) + 2\mathrm{d}\gamma_1\gamma_2\right)}{\left(\gamma_2^2 + \gamma_1^2\right)^2} + \frac{\gamma_3^4(\nu_1\gamma_1 + \nu_2\gamma_2)^2}{\gamma_1^4\gamma_2^2} + \rho\frac{\gamma_2^2 + \gamma_1^2}{\gamma_3^4} - \lambda\left(\gamma_1^2\right. \\
& \left. + \gamma_2^2\right)\Bigg]\left(r + \left(n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right)\gamma_3\right) - 2\frac{\gamma_3^3}{\left(\gamma_2^2 + \gamma_1^2\right)^2}\left[\left(c\gamma_1\left(2\gamma_2^2 + 3\gamma_3^2 - \gamma_3^4 - 2\right)\right.\right. \\
& + \mathrm{d}\gamma_2\left(2\gamma_2^2 - \gamma_3^4 + 4\gamma_3^2 - 3\right)\Big)\left(p + \left(n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right]\gamma_1\right) - (\mathrm{d}\gamma_1(\gamma_3^4 + 2\gamma_2^2 \\
& - 2\gamma_3^2 + 1) - c\gamma_2\left(\gamma_3^4 + 2\gamma_2^2 - \gamma_3^2\right)\Big)\left(q + \left(n - n_1\left(\gamma_2^2 - \gamma_1^2\right) + n_2\gamma_1\gamma_2\right)\gamma_2\right)\Big)\Bigg\}.
\end{aligned}
$$

This is a new integrable problem in rigid body dynamics. It contains nine free parameters. It generalizes some problems as shown in the following table:

Reference	Conditions on parameters
Yehia [43]	$\nu_1 = \nu_2 = n = n_1 = n_2 = 0$
Goriachev [42]	$\nu_1 = \nu_2 = n = n_1 = n_2 = \rho = 0$
Chaplygin [6]	$\nu_1 = \nu_2 = n = n_1 = n_2 = \rho = \lambda = 0$

Acknowledgments The author is very grateful to Professor Hamad M. Yehia for his help, encouragement and comments during my work on this article.

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