Acta Mech 226, 2435–2439 (2015)
DOI 10.1007/s00707-015-1315-5

NOTE

Leonardo Casetta · Celso P. Pesce

A brief note on the analytical solution of Meshchersky's equation within the inverse problem of Lagrangian mechanics

Received: 14 October 2014 / Revised: 24 December 2014 / Published online: 19 February 2015

Abstract Meshchersky's equation is a basic differential equation in the mechanics of variable-mass particles. This note particularly considers the case in which a one-dimensional and position-dependent mass particle is under the action of a potential force. The absolute velocity of mass ejection (or accretion) is supposed to be a linear function of the particle velocity. Within the formulation of the inverse problem of Lagrangian mechanics, an analytical solution of Meshchersky's equation is here derived. The solution method follows from applying the concept of constant of motion of an extremum problem, which is a fundamental ground in the theory of invariant variational principles.

1 Introduction

It is well known that the fundamental principles of classical mechanics were originally conceived for constant mass systems (see, e.g., the review article [1]). Therefore, an appropriate mathematical formalism is required to the treatment and analysis of the so-called (non-relativistic) variable-mass systems. Investigations in this sense have been ongoing since the eighteenth century and date back to Euler and Bernoulli. Other remarkable names, as Poisson [2] and Cayley [3], have also fundamentally contributed to the field (see, e.g., the historical review [4]). Nowadays, the mechanics of variable-mass systems is recognized as a particular branch of research in theoretical and applied mechanics, being also focus of investigations in the domain of applied mathematics and physics (see, e.g., [5,6]). The unfamiliar reader can find an interesting overview in the context of mechanics in the recently published book [7].

The study of variable-mass systems has been connected to the famous inverse problem of Lagrangian mechanics (see [8,9]), which is a very traditional topic of mathematical physics. Such a problem essentially means the inverse construction of a Lagrangian function that, when inserted into the principle of least action, properly yields the differential equation at hand. The first investigations on the subject are due to Helmholtz [10]. For other important contributions, see for instance [11] and [12].

Casetta and Pesce [8] have discussed that the fundamental differential equation of a variable-mass particle, namely Meshchersky's equation, is such that it results from a principle of least action. The present note thus aims at demonstrating that, seen from the perspective of the inverse problem of Lagrangian mechanics, Meshchersky's equation can be analytically solved, at least for some particular cases. To the authors' best knowledge, this particular issue has not yet been addressed.

L. Casetta (✉) · C. P. Pesce
Offshore Mechanics Laboratory, Department of Mechanical Engineering, Escola Politécnica,
University of São Paulo, São Paulo, Brazil
E-mail: lecasetta@gmail.com

C. P. Pesce
E-mail: ceppesce@usp.br

2 Theoretical background

Written for the case of a one-dimensional problem regarding a single particle, the famous Meshchersky's equation is usually presented as the following second-order differential equation (see, e.g., [13]):

$$m\ddot{q} - Q - (w - \dot{q})\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \tag{1}$$

where m, $\dot{q}$ and $\ddot{q}$ are, respectively, mass, velocity and acceleration of the particle; Q is the acting force, and w is the absolute velocity at which mass is expelled (or accreted). As discussed by Pesce [13], in the realm of analytical mechanics, the mass of a particle can be assumed to be a function of position, velocity and time.

Our analysis will be restricted to the following assumptions:

$$m = m(q), \tag{2}$$

$$Q = -\mathrm{d}V(q)/\mathrm{d}q \tag{3}$$

where V is the potential energy, and

$$w = k\dot{q} \tag{4}$$

where $k = \text{const.}$

Note, for example, that the particular value of $k = 0$ recovers Levi-Civita's case, and that the particular value of $k = 1$ refers to the isotropic loss of mass (see the discussion in [1]).

Under the mentioned assumptions, Eq. (1) becomes

$$m(q)\ddot{q} + \frac{\mathrm{d}V(q)}{\mathrm{d}q} - \alpha\dot{q}^2\frac{\mathrm{d}m(q)}{\mathrm{d}q} = 0 \tag{5}$$

where

$$\alpha = k - 1 = \text{const.} \tag{6}$$

According to [8], the principle of least action which yields Eq. (5) is

$$\delta\int_{t_1}^{t_2}\widetilde{L}\mathrm{d}t = 0 \tag{7}$$

for

$$\widetilde{L} = \frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 - \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q. \tag{8}$$

The symbol '$\sim$' is being used to avoid confusion with respect to the traditional Lagrangian L, which is given as the difference between kinetic energy and potential energy.

In fact, note that, evoking the identity

$$\delta\widetilde{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{L}}{\partial\dot{q}}\delta q\right) + \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{q}} + \frac{\partial\widetilde{L}}{\partial q}\right)\delta q, \tag{9}$$

where δq is an arbitrary virtual change required to vanish at the limiting instants t_1 and t_2, Eq. (7) is able to recover (5) via the fully equivalent expression:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{q}} - \frac{\partial\widetilde{L}}{\partial q} = m(q)^{-2\alpha}\left(\ddot{q} + \frac{\frac{\mathrm{d}V(q)}{\mathrm{d}q} - \alpha\dot{q}^2\frac{\mathrm{d}m(q)}{\mathrm{d}q}}{m(q)}\right) = 0. \tag{10}$$

By virtue of that $\partial\widetilde{L}/\partial t = 0$ [see Eq. (8)], the identity $(\mathrm{d}/\mathrm{d}t)((\partial\widetilde{L}/\partial\dot{q})\dot{q} - \widetilde{L}) = -\partial\widetilde{L}/\partial t$, which is found to be derived, for instance, in [14, p. 61], implies that

$$\frac{\partial\widetilde{L}}{\partial\dot{q}}\dot{q} - \widetilde{L} = \text{const.}, \tag{11}$$

that is, using Eq. (8) in (11):

$$\frac{1}{2}m(q)^{-2\alpha}\dot{q}^2 + \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q = \widetilde{E} \tag{12}$$

where $\widetilde{E} = \text{const}$.

Equation (12) defines a conservation law of the extremum problem (7) and, in the sense discussed by Whittaker [15, p. 62], corresponds to an energy theorem within analytical mechanics.

3 Analytical solution of Meshchersky's equation for the position-dependent case

Now we address the problem of deriving an analytical solution of Meshchersky's differential equation (5), which is our intended contribution. For that, we consider the theory of invariant variational principles, which is a noticeable chapter of mathematical physics.

Following the classical textbook of Logan [16, Chap. 1], Eq. (12) is seen as defining a 'constant on a extremal,' namely the left-hand side of Eq. (12) means a mathematical function that, along the path of the motion of the extremum problem (7), is such that it remains constant. This concept properly comes out, for example, when one attempts to demonstrate Noether's theorem using the so-called Rund–Trautman's identities (see, e.g., [17, Chap. 5], [18]).

The point in question is that, in fact, the left-hand side of Eq. (12) does not equal a constant everywhere, but along the solution of Eq. (5). This offers us the possibility of considering Eq. (12) to solve (5).

Thus, calling upon the identity

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}, \tag{13}$$

Equation (12) can be algebraically manipulated to the following expression:

$$\mathrm{d}t = \frac{\sqrt{2}}{2}\sqrt{\left(\frac{m(q)^{-2\alpha}}{\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q}\right)}\mathrm{d}q. \tag{14}$$

Equation (14) is then integrated to furnish

$$t = \frac{\sqrt{2}}{2}\int_{q(t_0)}^{q}\sqrt{\left(\frac{m(q)^{-2\alpha}}{\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q}\right)}\mathrm{d}q + t_0. \tag{15}$$

For simplicity, and with no loss of generality, we take $t_0 = 0$ and $q(t_0) = 0$, which leads to

$$t = \frac{\sqrt{2}}{2}\int\sqrt{\left(\frac{m(q)^{-2\alpha}}{\widetilde{E} - \int m(q)^{-2\alpha-1}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\mathrm{d}q}\right)}\mathrm{d}q. \tag{16}$$

According to the theory of invariant variational principles of mathematical physics, Eq. (16) is such that it rules the curve $q \to t(q)$ which describes the path of the motion of the extremum problem (7). Consequently, the function $t = t(q)$ as in Eq. (16) is a general analytical solution of the differential equation (5).

The usage of the energy theorem to provide an analytical solution of the equation of motion is a fundamental procedure of classical mechanics. This is found to be discussed, for example, in [14, Chap. 3.2] and [15, Chap. III]. In the present note, considering the formulation of the inverse problem of Lagrangian mechanics, we have generalized this method to the context of variable-mass systems.

3.1 An illustrative example: the famous Cayley's falling-chain problem

In order to demonstrate the practical applicability of Eq. (16), we address the classical Cayley's falling-chain problem (see [3]). This is a very traditional problem in the context of variable-mass systems. Within the inverse problem of Lagrangian mechanics for Meshchersky's equation, the formulation of Cayley's falling-chain problem can be found to be discussed in our previous article [8, Sect. 5.1]. In the words of Cayley [3, p. 506], the statement of his falling-chain problem is: '(…) a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system (…).' Considering that such particles to be accreted by the moving system are originally at rest, one has that $w = 0$. Thus, seeing Eqs. (4) and (6), it renders

$$\alpha = -1. \tag{17}$$

The coordinate z of the lower extremity of the chain, which is supposed to be measured vertically downward, can be conveniently used as the generalized coordinate $z = q$ of the problem. In accordance, the mass $m = m(q)$ of this moving part of the chain can be written as

$$m(z) = \rho z, \tag{18}$$

where ρ is the linear mass density. Therefore, the associated potential energy is defined as $V = -\int m(z) g \mathrm{d}z$, that is,

$$V(z) = -\frac{1}{2} g \rho z^2. \tag{19}$$

Note that, inserting Eqs. (17), (18) and (19) into (5), we obtain the equation of motion of the problem:

$$\ddot{z} - g + \frac{\dot{z}^2}{z} = 0. \tag{20}$$

To derive the Lagrangian $\widetilde{L}$ which is in correspondence with Eq. (20), we substitute Eqs. (17), (18) and (19) into (8):

$$\widetilde{L} = \frac{1}{2}(\rho z)^2 \dot{z}^2 + \frac{1}{3} g \rho^2 z^3. \tag{21}$$

Using Eqs. (21) in (11) [or, equivalently, Eqs. (17), (18) and (19) in (12)], we so obtain the following conservation law:

$$\widetilde{E} = \frac{1}{2}(\rho z)^2 \dot{z}^2 - \frac{1}{3} g \rho^2 z^3 = \text{const.} \tag{22}$$

Now we are ready to test the practical applicability of our result, that is, Eq. (16) [or (15)]. In Cayley's falling-chain problem, one has that, at $t = 0$, $z = 0^+$ and $\dot{z} = 0$. This means that we are able to use Eq. (16).

Let us notice that such initial conditions imply that

$$\widetilde{E} = 0 \tag{23}$$

[see Eq. (22)]. Finally, substituting Eqs. (17), (18), (19) and (23) into (16), and then developing the integral in $q = z$ in the right-hand side, we find:

$$t = \left(\sqrt{\frac{6}{g}}\right) z^{\frac{1}{2}}. \tag{24}$$

To verify that $t = t(z)$ as in Eq. (24) is the solution of Eq. (20), we simply invert it, that is,

$$z = \frac{g}{6} t^2. \tag{25}$$

In fact, $z = z(t)$ as in Eq. (25) solves Eq. (20) and therefore confirms the original and intriguing result of Cayley [3, p. 511] which asserts that the moving part of the chain falls with constant acceleration $g/3$.

4 Conclusions

This note has addressed the problem of analytically solving Meshchersky's differential equation. The particular and important case of a position-dependent mass particle under the action of a potential force was considered.

We have also assumed that the absolute velocity of ejection (or accretion) of mass is a linear function of the particle velocity. This has brought out a second-order ordinary differential equation, here interpreted as the famous Meshchersky's equation of variable-mass mechanics.

Aiming at solving the issue, we have used the formulation of the inverse problem of Lagrangian mechanics for Meshchersky's equation, as previously discussed by Casetta and Pesce [8]. The energy theorem, which naturally appears when treating such a position-dependent case in this formulation, was integrated to furnish the required analytical solution. As a simple example, the traditional Cayley's falling-chain problem was considered to test the practical applicability of this result. It was verified that the formulation presented in this note was then able to recover the analytical solution of Cayley's falling-chain problem.

Seeing Eq. (15) and following Whittaker's [15, Chap. III] terminology, we have demonstrated a *principle available for the integration* in the context of the analytical mechanics of variable-mass systems.

Acknowledgements The authors acknowledge FAPESP, the State of São Paulo Research Foundation, for the Postdoctoral Research Grant No. 2012/10848-4. The second author acknowledges CNPq, The National Council for Scientific and Technological Development, for the Research Grant No. 303838/2008-6. The authors thank the reviewer of this article for their important suggestion on the consideration of a practical example to demonstrate the applicability of the aimed result. The authors thank Prof. Hans Irschik and Prof. Helmut Holl (Institute of Technical Mechanics, Johannes Kepler University of Linz, Austria) for fruitful and valuable discussions on mechanics of variable-mass systems. The first author particularly thanks his friend, Maria Regina Castro, who has been providing an essential assistance for the progress of his research activities. The first author dedicates this paper to his grandmother, Maria R. C. Rossi, who passed away on October 13, 2014.

References

1. Irschik, H., Holl, H.J.: Mechanics of variable-mass systems—part 1: balance of mass and linear momentum. Appl. Mech. Rev. **57**, 145–160 (2004)
2. Poisson, S.D.: Sur le mouvement d'un système de corps, en supposant les masses variables. Bull. Sci. Soc. Philomat. Paris 60–62 (1819)
3. Cayley, A.: On a class of dynamical problems. Proc. R. Soc. Lond. **8**, 506–511 (1857)
4. Mikhailov, G.K.: On the history of variable-mass system dynamics. Mech. Solids **10**, 32–40 (1975)
5. Leach, P.G.L.: Harmonic oscillator with variable mass. J. Phys. A Math. Gen. **16**, 3261–3269 (1983)
6. Cveticanin, L.: Approximate solution of a time-dependent differential equation. Meccanica **30**, 665–671 (1995)
7. Irschik, H., Belyaev, A.K. (eds.): Dynamics of mechanical systems with variable mass. In: Series: CISM International Centre for Mechanical Sciences, vol. 557, 266 p. Springer, Berlin (2014)
8. Casetta, L., Pesce, C.P.: The inverse problem of Lagrangian mechanics for Meshchersky's equation. Acta Mech. **225**, 1607–1623 (2014)
9. Casetta, L.: The inverse problem of Lagrangian mechanics for a non-material volume. Acta Mech. **226**, 1–15 (2015)
10. Helmholtz, H.: Über die physikalische Bedeutung des Princips der kleinsten Wirkung. J. Reine Angew. Math. **100**, 137–166 (1887)
11. Havas, P.: The range of application of the Lagrange formalism—I. Suppl. Nuovo Cim. **V**(X), 363–388 (1957)
12. Santilli, R.M.: Foundations of Theoretical Mechanics I. The Inverse Problem in Newtonian Mechanics. Springer, New York (1978)
13. Pesce, C.P.: The application of Lagrange equations to mechanical systems with mass explicitly dependent on position. J. Appl. Mech. **70**, 751–756 (2003)
14. Goldstein, H., Poole, C.P., Safko, J.L.: Classical Mechanics. Addison-Wesley, San Francisco (2002)
15. Whittaker, E.T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Cambridge University Press, Cambridge (1988)
16. Logan, J.D.: Invariant Variational Principles. Academic Press, New York (1977)
17. Neuenschwander, D.E.: Emmy Noether's Wonderful Theorem. The Johns Hopkins University Press, Baltimore (2011)
18. Sarlet, W., Cantrijn, F.: Generalizations of Noether's theorem in classical mechanics. SIAM Rev. **23**, 467–494 (1981)