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Two-dimensional elasticity solution of elastic strips and beams made of functionally graded materials under tension and bending

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Abstract This paper presents a theoretical approach to solve elastic problems of functionally graded materials (FGMs). For FGMs with exponential gradient, based on a two-dimensional theory of elasticity, a governing equation is derived by means of the Airy stress function method together with the strain compatibility equation. Simple uniaxial tension and bending are solved. For an FGM layer with transversely and/or vertically varying material properties, stress distribution and strain field under simple tension are determined according to two different assumptions. The obtained results indicate that for a thin elastic layer of thickness-wise gradient as a transition zone linking two dissimilar materials, there is a horizontal displacement difference across the transition zone due to mismatch of the material properties. In particular, when the thickness of the FGM layer reduces to zero, the horizontal displacement difference has a severe mismatch across the interface of two perfectly bonded dissimilar materials. An FGM beam subjected to a bending moment is also analyzed. The normal stress exhibits a nonlinear distribution and may arrive at its maximum tensile stress inside the beam, not at the surface. The obtained elasticity solution is useful for better understanding of the mechanical behaviors of FGMs subjected to different combined loads.

1 Introduction

Functionally graded materials (FGMs) with continuously varying material properties have attracted much attention of researchers because of their excellent performance. They have been widely used in many structures of civil, mechanical, space engineering owing to high strength and high stiffness. For instance, a homogeneous elastic layer of ceramic material may be bonded to the surface of a metallic structure and acts as a thermal barrier in high-temperature environment. However, due to mismatch of the mechanical properties of the ceramic and metallic materials, a distinct interface between two bonded dissimilar elastic media exists, which gives rise to a severe incompatibility of elastic fields when across the interface. This may lead to delamination or cracking of the interface owing to a sudden change in stresses and displacements. Due to this shortage, a distinct interface of two bonded dissimilar materials is preferably avoided. This can be achieved by designing a transition zone, in which the material properties continuously vary, rather than a sudden jump, from a ceramic medium to metal medium by gradually changing the volume fraction of the constituents involved. Therefore, FGMs possess noticeable advantages over homogeneous and layered materials in maintaining the integrity

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of a structure [1]. So far, there have been a very large number of researches available in theoretical analysis, numerical simulations, and experimental observations [2].

For elastic beams and plates made of FGMs, Sankar [3] solved bending of a simply supported FGM beam based on the theories of beams and two-dimensional elasticity. Zhong and Yu [4] obtained the general solution for a cantilever FGM beam subjected to different loads. Furthermore, Ying et al. [5] gave a two-dimensional elasticity solution for functionally graded beams resting on elastic foundations. Wang and Liu [6] analyzed a bi-material beam with graded intermediate layer subjected to uniform loading on the upper surface. Mian and Spencer [7] studied a class of elasticity solutions related to FGMs and laminated isotropic materials. For rectangular and circular FGM plates, three-dimensional elasticity solutions have been investigated by some researchers such as [8]. By the finite element method, Orakdogen et al. [9] treated a problem of the coupling effect of extension and bending in an FGM plate and obtained an elasticity solution when a transverse loading is applied on the FGM plate. Li et al. [10] made a stress analysis of FGM beams using effective principal axes. A recent review on progress of research on elastic plates made of FGMs can be found in [11].

For circular tubes and disks made of FGMs, considerable attention has been paid on both static and dynamic analyses. In this field, Horgan and Chan [12] analyzed a pressurized hollow cylinder or disk made of FGMs with power-law gradient. Li and Peng [13] extended the above problem to arbitrarily distributed gradient. Furthermore, thermal stress in a rotating functionally graded hollow circular disk with any gradient has been evaluated in [14]. Sburlati [15] further solved an elasticity solution for a pressurized hollow cylinder with internal functionally graded coatings. For an FGM annulus as a transition zone between two homogeneous annuli, the stress distribution of the whole composite was studied [16]. Nie et al. [17] presented a technique to design functionally graded hollow cylinders to attain a preferable stress state. Sofiyev [18,19] coped with static and dynamic buckling for truncated conical shells of FGMs and investigated the effect of changing shell characteristics and material properties on the critical loading.

Another interesting study is research on crack problems of FGMs [20]. In this field, great progress has been made in calculating stress intensity factors near the tips of a crack embedded in FGMs. Erdogan et al. treated the crack problems related to an FGM [21,22]. Gu and Asaro [23] handled a semi-infinite crack in a strip of an isotropic FGM under edge loading. Dolbow and Gosz [24] established an interaction energy integral for the computation of mixed-mode stress intensity factors at the tips of arbitrarily oriented cracks in FGMs. Li and Fan [25] compared dynamic stress intensity factors of a mode-III crack related to FGMs when impact loading is suddenly exerted at the crack surface and the material surface. Xu et al. [26] gave dynamic stress intensity factors of a semi-infinite crack in an orthotropic FGM. A boundary-domain integral equation formulation was suggested to evaluate stress intensity factors of a three-dimensional crack in FGMs [27].

Although a large number of papers on the theoretical analysis of elastic problems of FGMs have been published in the past several decades, study on elasticity solutions of FGMs is still limited. For example, an elasticity solution for a two-dimensional FGM under applied loading, even for a uniaxial tension, is not available yet, to the best of the authors' knowledge, because it is related to a solution of a partial differential equation with variable coefficients under appropriate boundary conditions. It is, in fact, a fundamental issue of the mechanical behavior and is of significance for better understanding of structural responses and the integrity analysis of FGMs.

The aim of this paper is to analyze the mechanical behavior of an FGM subjected to applied loadings. A governing equation related to the gradient index is derived. For an FGM under uniaxial tension, we obtain two different solutions for an FGM material based on two different assumptions. Furthermore, an FGM beam subject to a bending moment is solved. The obtained results indicate that for two bonded dissimilar homogeneous materials, there is an FGM transition zone existing near the interface such that elastic displacements vary continuously across the transition zone. If the FGM transition zone disappears, this leads to a severe mismatch horizontal displacement across the interface.

2 Statement of the problem

Consider a two-dimensional FGM occupying a region in a Cartesian coordinate system oxy and with Young's modulus obeying the following exponential gradient

$$E(x, y) = E_0 e^{\lambda_1 x + \lambda_2 y}, \tag{1}$$

where E_0 is a reference value of Young's modulus of the FGM at the origin of the Cartesian coordinate system $(x = 0, y = 0)$, λ_1 and λ_2 refer to the gradient indices, as shown in Fig. 1. To make the analysis tractable,

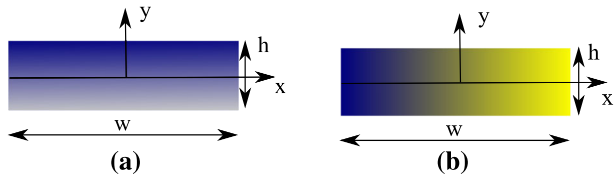

Fig. 1 Schematic of an FGM with **a** gradient index λ_2 (class I) and **b** gradient index λ_1 (class II)

we assume that Poisson's ratio of the FGM is taken as a constant, ν. This is reasonable for most situations of FGMs due to very slight variation of Poisson's ratio.

Neglecting body forces, the equilibrium equations can be written as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{2}$$

where σ_x, σ_y, and τ_{xy} are the normal and shear stress components, respectively.

Along the x- and y-directions, there are two displacement components, denoted by u and v, respectively. Using these displacement components, one has elastic strain components as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{3}$$

To link the strains with the stresses, for a linearly elastic isotropic medium made of FGMs, we have the following constitutive equations based on the two-dimensional theory of elasticity:

$$\varepsilon_x = \frac{1}{E(x, y)} \left(\sigma_x - \nu \sigma_y\right), \tag{4}$$

$$\varepsilon_y = \frac{1}{E(x, y)} \left(\sigma_y - \nu \sigma_x\right), \tag{5}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E(x, y)} \tau_{xy}, \tag{6}$$

where plane stress state is assumed. For plane strain state, it suffices to replace $E(x, y)$ and ν by $E(x, y) / \left(1 - v^2\right)$ and $v/(1-v)$, respectively, in the above constitutive equations. Therefore, in what follows, we only restrict our attention to plane stress state.

3 Governing equation

In this section, we apply the Airy stress function approach to analyze elasticity problems in two-dimensional FGMs. To this end, similar to the Airy stress function approach, we introduce an Airy stress function $\varphi(x, y)$ and express the stress components in terms of $\varphi(x, y)$ below:

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}. \tag{7}$$

Hence bearing (1) in mind one substitutes (7) into (4)–(6) and gets

$$\varepsilon_x = \frac{1}{E_0 e^{\lambda_1 x + \lambda_2 y}} \left(\frac{\partial^2 \varphi}{\partial y^2} - \nu \frac{\partial^2 \varphi}{\partial x^2}\right), \tag{8}$$

$$\varepsilon_y = \frac{1}{E_0 e^{\lambda_1 x + \lambda_2 y}} \left(\frac{\partial^2 \varphi}{\partial x^2} - \nu \frac{\partial^2 \varphi}{\partial y^2}\right), \tag{9}$$

$$\gamma_{xy} = -\frac{2(1+\nu)}{E_0 e^{\lambda_1 x + \lambda_2 y}} \frac{\partial^2 \varphi}{\partial x \partial y}. \tag{10}$$

Due to the continuity of the partial derivatives of the displacement components u and v, the strain components must satisfy the following strain compatibility equation:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{11}$$

Consequently, we insert (8)–(10) into (11) to obtain a partial differential equation as follows:

$$\nabla^2\left(\nabla^2\varphi\right) - 2\lambda_1\frac{\partial \nabla^2\varphi}{\partial x} - 2\lambda_2\frac{\partial \nabla^2\varphi}{\partial y} + \left(\lambda_1^2+\lambda_2^2\right)\nabla^2\varphi$$
$$-(1+\nu)\left(\lambda_1^2\frac{\partial^2\varphi}{\partial y^2} + \lambda_2^2\frac{\partial^2\varphi}{\partial x^2} + 2\lambda_1\lambda_2\frac{\partial^2\varphi}{\partial x\partial y}\right) = 0 \tag{12}$$

or

$$\nabla^2\left(\nabla^2\varphi\right) - 2\left(\lambda_1\frac{\partial}{\partial x} + \lambda_2\frac{\partial}{\partial y}\right)\nabla^2\varphi + \left(\lambda_1^2+\lambda_2^2\right)\nabla^2\varphi - (1+\nu)\left(\lambda_1\frac{\partial}{\partial y} + \lambda_2\frac{\partial}{\partial x}\right)^2\varphi = 0, \tag{13}$$

where ∇^2 is the two-dimensional Laplacian operator, defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{14}$$

As a result, we have obtained a governing partial differential equation (13). Once a solution to Eq. (13) is obtained under appropriate boundary conditions, all of the physical quantities of interest such as stresses and strains can be determined through (7)–(10). The remaining is to seek a solution to Eq. (13) under appropriate boundary conditions.

Consider two special cases, one corresponding to Young's modulus depending only on y and the other corresponding to Young's modulus depending only on x. For the former case, we have $\lambda_1 = 0$ and the governing equation in this case reduces to

$$\nabla^2\left(\nabla^2\varphi\right) - 2\lambda_2\frac{\partial \nabla^2\varphi}{\partial y} + \lambda_2^2\left(\frac{\partial^2\varphi}{\partial y^2} - \nu\frac{\partial^2\varphi}{\partial x^2}\right) = 0. \tag{15}$$

For the latter case, we have $\lambda_2 = 0$ and the governing equation then reduces to

$$\nabla^2\left(\nabla^2\varphi\right) - 2\lambda_1\frac{\partial \nabla^2\varphi}{\partial x} + \lambda_1^2\left(\frac{\partial^2\varphi}{\partial x^2} - \nu\frac{\partial^2\varphi}{\partial y^2}\right) = 0. \tag{16}$$

In order to keep our calculations from getting cumbersome and to make some essential features become more evident, in the following, we analyze the above two special cases under some typical loadings and make an effort to capture the nature of the influence of the gradient index on elastic fields. For convenience, we denote the cases corresponding to the governing equations (15) and (16) as class I and class II FGM, respectively, as shown in Fig. 1a, b.

4 Basic solution for class I FGMs

4.1 Simple tension

In engineering applications, structural components including FGMs under simple uniaxial tension and bending are quite common. Here, consider a rectangular elastic block made of FGMs subjected to simple uniaxial uniform tension along the y-direction, as shown in Fig. 2. For convenience of later analysis, w and h are used to express its width and height. Then, we assume $-w/2 \le x \le w/2$, $-h/2 \le y \le h/2$ and

$$E = E_0 e^{\lambda_2 y}. \tag{17}$$

In the following, we analyze two typical cases. One is a thin layer where the width is sufficiently large as compared to the height, i.e., $w \gg h$, and the other is a narrow strip where the width is sufficiently small as compared to the height, i.e., $w \ll h$. For convenience, we denote the former as case A and the latter as case B, respectively.

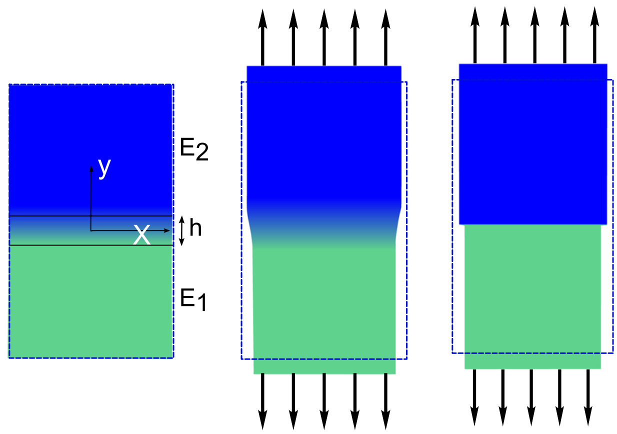

Fig. 2 Two dissimilar homogeneous isotropic media with an FGM transition zone under simple tension; **a** before deformation, **b** after deformation, **c** after deformation for a perfectly bonded bi-material without FGM transition zone, where horizontal displacements have apparent incompatibility

4.1.1 Solution for case A: thin FGM layer

For simple uniaxial uniform tension along the y-direction, appropriate boundary conditions can be stated as

$$\sigma_y = q, \qquad \tau_{xy} = 0, \quad \text{at} \quad y = \pm\frac{h}{2}. \tag{18}$$

Since simple uniaxial uniform tension is concerned, for an arbitrary cross-section $y = y_0$, $(-h/2 \le y_0 \le h/2)$, $\sigma_y = q$ can result in $\tau_{xy} = 0$. This may be explained by balance of forces, given in Appendix A. Thus, this allows us to invoke the semi-inverse solution method. In other words, using the second relationship of (7), one can express the Airy stress function $\varphi(x, y)$ as

$$\varphi(x, y) = \frac{q}{2}x^2 + xf_2(y) + f_1(y). \tag{19}$$

Making use of the last relationship of (7), one can conclude $f_2(y) = f_0$, f_0 being a constant. On the other hand, owing to xf_0 as a linear term in (19), which does not give rise to any change in the stress components and the strain components, in the following analysis, one can directly take $f_0 = 0$ without changing the distribution of elastic stresses and strains. Due to this reason, the Airy stress function $\varphi(x, y)$ is taken as

$$\varphi(x, y) = \frac{q}{2}x^2 + f_1(y), \tag{20}$$

and from (7) we get

$$\sigma_x = f_1''(y), \tag{21}$$

where the prime denotes differentiation with respect to the argument.

On the other hand, the Airy stress function $\varphi(x, y)$ for case A must satisfy the governing equation (15). Substituting Eq. (20) into Eq. (15) leads to

$$f_1^{IV}(y) - 2\lambda_2 f_1'''(y) + \lambda_2^2 f_1''(y) = \lambda_2^2 \nu q. \tag{22}$$

This is an ordinary differential equation with constant coefficient. Solving this equation, we readily find

$$f_1''(y) = (A_1 + A_2 y)\, e^{\lambda_2 y} + \nu q, \tag{23}$$

where A_1 and A_2 are unknown constants to be determined through appropriate boundary conditions. Consequently, from (21) and (7), the stress components are obtained below:

$$\sigma_x = (A_1 + A_2 y)\, e^{\lambda_2 y} + \nu q, \tag{24}$$

$$\sigma_y = q, \tag{25}$$

$$\tau_{xy} = 0. \tag{26}$$

From (24), if $\lambda_2 = 0$, we take $A_1 = -\nu q$ and $A_2 = 0$, giving

$$\sigma_x = 0, \tag{27}$$

and the well-known solution for a homogeneous isotropic medium subjected to simple uniform uniaxial tension is recovered. However, for a class I FGM with $\lambda_2 \neq 0$, it is easily found that an exact solution does not exist since it is unlikely to seek two constants A_1 and A_2 such that $\sigma_x = 0$ for any position y in the interval $[-h/2, h/2]$. To overcome this difficulty, the condition $\sigma_x = 0$ is relaxed in the subsequent analysis in this subsection. Similar to the treatment of an elastic beam in the theory of elasticity, we replace the condition $\sigma_x = 0$ with the following two conditions:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \mathrm{d}y = 0, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} y\sigma_x \mathrm{d}y = 0. \tag{28}$$

It should be pointed out that the above relaxation treatment is suitable because of Saint-Venant's principle and the distribution of stress components to be obtained is acceptable and reasonable for those positions relatively far away from the relaxed boundaries. Therefore, strictly speaking, the solutions in the present paper are still approximate, not exact.

Using both conditions in (28), we immediately determine

$$A_1 = -\alpha \nu q \frac{\alpha^2 \sinh\alpha - 2\alpha\cosh\alpha + 2\sinh\alpha}{\sinh^2\alpha - \alpha^2}, \tag{29}$$

$$A_2 = \alpha\lambda_2 \nu q \frac{\alpha\cosh\alpha - \sinh\alpha}{\sinh^2\alpha - \alpha^2}, \tag{30}$$

where

$$\alpha = \frac{\lambda_2 h}{2}. \tag{31}$$

Consequently, the normal stress σ_x is determined as

$$\sigma_x = \nu q \left\{1 + \frac{\alpha e^{2\alpha y/h}}{\sinh^2\alpha - \alpha^2}\left[2\alpha\left(1 + \frac{\alpha y}{h}\right)\cosh\alpha - \left(\alpha^2 + 2 + \frac{2\alpha y}{h}\right)\sinh\alpha\right]\right\}. \tag{32}$$

From the above, we make some observations. Firstly, setting $\lambda_2 \to 0$, one finds that

$$\sigma_x = 0. \tag{33}$$

This result no doubt confirms that when the gradient index disappears, our solution reduces to the classic elastic solution for a homogeneous isotropic medium subjected to simple uniaxial tension. Here, it is important to point out that in the presence of the gradient index, tensile loading σ_y gives rise to the appearance of the transverse normal stress σ_x. Under such stress components, from (8)–(10), the strain components are as follows:

$$\varepsilon_x = \frac{\alpha \nu q}{E_0\left(\sinh^2\alpha - \alpha^2\right)}\left[2\alpha\left(1 + \frac{\alpha y}{h}\right)\cosh\alpha - \left(\alpha^2 + 2 + \frac{2\alpha y}{h}\right)\sinh\alpha\right], \tag{34}$$

$$\varepsilon_y = \frac{q\left(1 - \nu^2\right)}{E_0 e^{2\alpha y/h}} - \frac{\alpha \nu^2 q}{E_0\left(\sinh^2\alpha - \alpha^2\right)}\left[2\alpha\left(1 + \frac{\alpha y}{h}\right)\cosh\alpha - \left(\alpha^2 + 2 + \frac{2\alpha y}{h}\right)\sinh\alpha\right], \tag{35}$$

$$\gamma_{xy} = 0. \tag{36}$$

From the above-derived strain components in connection with (3), it is easy to obtain the elastic displacement components

$$u = \frac{\alpha v q x}{E_0\left(\sinh^2\alpha - \alpha^2\right)}\left[2\alpha\left(1+\frac{\alpha y}{h}\right)\cosh\alpha - \left(\alpha^2+2+\frac{2\alpha y}{h}\right)\sinh\alpha\right], \tag{37}$$

$$v = -\frac{q\left(1-v^2\right)}{\lambda_2 E_0 e^{2\alpha y/h}} - \frac{vq\alpha}{hE_0\left(\sinh^2\alpha-\alpha^2\right)}\left\{\alpha\left[vhy\left(2+\frac{\alpha y}{h}\right)+\alpha x^2\right]\cosh\alpha - \left[vhy\left(\alpha^2+2+\frac{\alpha y}{h}\right)+\alpha x^2\right]\sinh\alpha\right\}, \tag{38}$$

where rigid translation and rotation are neglected. Note that such a relaxation treatment is suitable for a thinner FGM layer, i.e., the height h is small enough compared to its width w.

4.1.2 Solution for case B: narrow FGM strip

In the above section, we found that for rectangular FGMs under simple uniaxial uniform tension, tensile loading σ_y induces the transverse normal stress σ_x. For a narrow FGM strip, we analyze a situation of uniaxial tension which requires $\sigma_x = 0$ and $\tau_{xy} = 0$ at the surface. In addition, within the FGM strip, we assume $\sigma_x = 0$ but σ_y may be variable, not a constant. A simple derivation leads to $\tau_{xy} = 0$ within the FGM strip (see Appendix B). Thus from (7) one gets

$$\varphi = f_1(x) + yC, \tag{39}$$

where C is a constant, which does not change the distribution of the stresses and is chosen as zero for the sake of simplicity, and $f_1(x)$ is an unknown function to be determined. Substituting (39) into the governing equation (15) leads to

$$f_1^{IV}(x) - \nu\lambda_2^2 f_1''(x) = 0. \tag{40}$$

Solving the above differential equation, one has

$$f_1''(x) = A_3 e^{\lambda_2\sqrt{\nu}x} + A_4 e^{-\lambda_2\sqrt{\nu}x}, \tag{41}$$

where A_3 and A_4 are two constants. This actually gives the vertical normal stress σ_y as

$$\sigma_y = A_3\cosh\left(\lambda_2\sqrt{\nu}x\right) + A_4\sinh\left(\lambda_2\sqrt{\nu}x\right). \tag{42}$$

This suggests again that the uniaxial tensile stress σ_y is no longer a constant unless the gradient index $\lambda_2 = 0$. In the case of $\lambda_2 \neq 0$, we still employ the above relaxation treatment method. That is, we replace uniform tensile loading $\sigma_y = q$ with the following conditions:

$$\frac{1}{w}\int_{-w/2}^{w/2}\sigma_y \mathrm{d}x = q, \qquad \int_{-w/2}^{w/2} x\sigma_y \mathrm{d}x = 0. \tag{43}$$

Thus, we get

$$A_3 = \frac{qw\lambda_2\sqrt{\nu}}{2\sinh\left(w\lambda_2\sqrt{\nu}/2\right)}, \qquad A_4 = 0. \tag{44}$$

and

$$\sigma_y = \frac{qw\lambda_2\sqrt{\nu}}{2\sinh\left(w\lambda_2\sqrt{\nu}/2\right)}\cosh\left(\lambda_2\sqrt{\nu}x\right). \tag{45}$$

This indicates that for uniaxial tension of an FGM, we may require $\sigma_x = \tau_{xy} = 0$ only if applied tensile stress σ_y obeys the distribution of (45). In this case, the strain components read

$$\varepsilon_x = -\frac{vqw\lambda_2\sqrt{\nu}\cosh\left(\lambda_2\sqrt{\nu}x\right)}{2E_0 e^{\lambda_2 y}\sinh\left(w\lambda_2\sqrt{\nu}/2\right)}, \tag{46}$$

$$\varepsilon_y = \frac{qw\lambda_2\sqrt{\nu}\cosh\left(\lambda_2\sqrt{\nu}x\right)}{2E_0 e^{\lambda_2 y}\sinh\left(w\lambda_2\sqrt{\nu}/2\right)}, \tag{47}$$

$$\gamma_{xy} = 0. \tag{48}$$

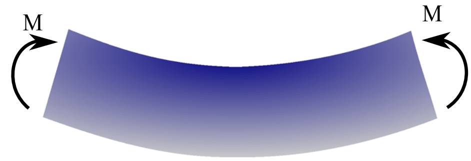

Fig. 3 Schematic of an FGM beam with thickness-wise varying Young's modulus subjected to a bending moment

If further considering a limit case of the width of the FGM close to zero, $w \to 0$, from (45) and (47), we immediately find

$$\sigma_y = q, \quad \varepsilon_y = \frac{q}{E_0 e^{\lambda_2 y}}, \quad \text{or} \quad \sigma_y = E_0 e^{\lambda_2 y} \varepsilon_y, \tag{49}$$

which is in exact agreement with Hooke's law for one-dimensional exponentially graded structures. From (46)–(48), one obtains the corresponding elastic displacement components to be

$$u = -\frac{\nu q w \sinh\left(\lambda_2 \sqrt{\nu} x\right)}{2E_0 e^{\lambda_2 y} \sinh\left(w \lambda_2 \sqrt{\nu}/2\right)}, \qquad v = -\frac{q w \sqrt{\nu} \cosh\left(\lambda_2 \sqrt{\nu} x\right)}{2E_0 e^{\lambda_2 y} \sinh\left(w \lambda_2 \sqrt{\nu}/2\right)}, \tag{50}$$

where rigid translation and rigid rotation have been neglected.

4.2 Pure bending

Next, let us turn our attention to another interesting case. Consider pure bending of an FGM beam subjected to a bending moment, as shown in Fig. 3, that is, the boundary conditions can be expressed as

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \mathrm{d}y = 0, \int_{-\frac{h}{2}}^{\frac{h}{2}} y \sigma_x \mathrm{d}y = -M, \quad \forall x \in \left(-\frac{w}{2}, \frac{w}{2}\right). \tag{51}$$

Since pure bending is focused, the shear stress vanishes, $\tau_{xy} = 0$, similar to the treatment for pure bending of a homogeneous isotropic beam. Also, another important assumption on the lateral compression is adopted. That is, for two planes parallel to the neutral surface, the lateral compression between them is reasonably small and negligible ($\sigma_y = 0$) as compared to σ_x, although it is existent. Then, the Airy stress function $\varphi(x, y)$ can be written as

$$\varphi(x, y) = g(y). \tag{52}$$

In the above, we notice that a linear function of x in the $\varphi(x, y)$ is removed since it does not give rise to a change in the distribution of the stresses. To satisfy the governing equation (15), $g(y)$ must satisfy the following homogeneous ordinary differential equation:

$$g^{IV}(y) - 2\lambda_2 g'''(y) + \lambda_2^2 g''(y) = 0. \tag{53}$$

Its solution is easily obtained to be

$$g''(y) = (A_5 + A_6 y) e^{\lambda_2 y}. \tag{54}$$

or

$$\sigma_x = (A_5 + A_6 y) e^{\lambda_2 y}, \tag{55}$$

where A_5 and A_6 are constants to be determined by appropriate boundary conditions. To gain A_5 and A_6, applying the boundary conditions (51), we can determine A_5 and A_6, and they are

$$A_5 = \frac{\lambda_2^2 M (\alpha \cosh\alpha - \sinh\alpha)}{2\left(\sinh^2\alpha - \alpha^2\right)}, \quad A_6 = -\frac{\lambda_2^3 M \sinh\alpha}{2\left(\sinh^2\alpha - \alpha^2\right)}. \tag{56}$$

We insert A_5 and A_6 back into (55) and σ_x can be derived as

$$\sigma_x = -\frac{\lambda_2^2 M \left[(1 + \lambda_2 y) \sinh\alpha - \alpha \cosh\alpha\right]}{2\left(\sinh^2\alpha - \alpha^2\right)} e^{\lambda_2 y}. \tag{57}$$

When $\lambda_2 \to 0$, we perform a limit manipulation and find $A_5 \to 0$, $A_6 \to -12M/h^3$, which implies $\sigma_x = -12My/h^3$, identical to the well-known result of a homogeneous isotropic beam subjected to a bending moment. Nevertheless, it is worth noting that for an elastic beam made of FGMs with thickness-wise varying material properties, the normal stress is no longer a linear distribution, but a nonlinear dependence. Owing to the definite explicit expression (57) for σ_x, one can judge the position of the maximum stress. Setting $\mathrm{d}\sigma_x/\mathrm{d}y = 0$ yields

$$y_0 = \frac{h}{2}\left(\coth\alpha - \frac{2}{\alpha}\right). \tag{58}$$

From the above, it is concluded that if there exists λ_2 such that $\left|\coth\alpha - \frac{2}{\alpha}\right| < 1$, then the location of maximum tensile normal stress σ_x appears within the beam, rather than at the beam surfaces. This is a great benefit to safe design of beam structures in a state of pure bending.

5 Basic solution for class II FGMs

5.1 Simple tension

Consider a rectangular elastic block made of FGMs subjected to simple uniaxial uniform tension along the y-direction, as well as shown in Fig. 1b. Here, elastic modulus is supposed to be

$$E = E_0 e^{\lambda_1 x}. \tag{59}$$

For simple uniaxial uniform tension along the y-direction, boundary conditions are the same as those given in (18).

5.1.1 Solution for case A: thin FGM layer

Similar to class I FGMs mentioned before for case A, the Airy stress function $\varphi(x, y)$ here is taken as

$$\varphi(x, y) = \frac{q}{2}x^2 + f_1(y). \tag{60}$$

Simultaneously, the Airy stress function $\varphi(x, y)$ must satisfy the governing equation (16). After substituting Eq. (60) into Eq. (16), one obtains

$$f_1^{IV}(y) - \lambda_2^2 v f_1''(y) = -\lambda_2^2 q. \tag{61}$$

Solving the differential equation (61), we have

$$f_1'' = B_1 e^{\lambda_1\sqrt{v}y} + B_2 e^{-\lambda_1\sqrt{v}y} + \frac{q}{v}, \tag{62}$$

where the coefficients B_1 and B_2 will be determined by appropriate boundary conditions. Thus the stress components are written as

$$\sigma_x = B_1 e^{\lambda_1\sqrt{v}y} + B_2 e^{-\lambda_1\sqrt{v}y} + \frac{q}{v}, \tag{63}$$

$$\sigma_y = q, \tag{64}$$

$$\tau_{xz} = 0. \tag{65}$$

Here, if $\lambda_1 = 0$, we let $B_1 = B_2 = -q/2v$, which gives

$$\sigma_x = 0. \tag{66}$$

In order to determine B_1 and B_2, in a similar manner as that for a thin class I FGM layer, we replace the condition $\sigma_x = 0$ with the following relaxation conditions:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \mathrm{d}y = 0, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} y\sigma_x \mathrm{d}y = 0. \tag{67}$$

Then we have

$$B_1 = B_2 = -\frac{\beta q}{2v \sinh \beta}, \tag{68}$$

where

$$\beta = \frac{\lambda_1 h \sqrt{v}}{2}. \tag{69}$$

Consequently, one can get the normal stress σ_x as

$$\sigma_x = \frac{q}{v}\left(1 - \frac{\beta}{\sinh \beta} \cosh \frac{2\beta y}{h}\right). \tag{70}$$

Apparently, setting $\lambda_1 \to 0$, one finds that

$$\sigma_x = 0. \tag{71}$$

It indicates that our solution reduces to the classic elastic solution for a homogeneous isotropic medium subjected to simple uniaxial tension. Noting that no matter whether the material properties of FGMs vary in the x-direction or y-direction, tensile loading σ_y causes the appearance of the transverse normal stress σ_x. According to the constitutive equations (4)–(6), the strain components are

$$\varepsilon_x = \frac{q}{E_0 e^{\lambda_1 x}}\left(\frac{1-\nu^2}{v} - \frac{\beta}{v \sinh \beta} \cosh \frac{2\beta y}{h}\right), \tag{72}$$

$$\varepsilon_y = \frac{\beta q}{E_0 e^{\lambda_1 x} \sinh \beta} \cosh \frac{2\beta y}{h}, \tag{73}$$

$$\gamma_{xy} = 0. \tag{74}$$

Using (3), we get

$$u = -\frac{q}{E_0 \lambda_1 e^{\lambda_1 x}}\left(\frac{1-\nu^2}{v} - \frac{\beta}{v \sinh \beta} \cosh \frac{2\beta y}{h}\right) + u_0(y), \tag{75}$$

$$v = \frac{hq}{2E_0 e^{\lambda_1 x} \sinh \beta} \sinh \frac{2\beta y}{h} + v_0(x), \tag{76}$$

where $u_0(y)$ and $v_0(x)$ are two unknown functions satisfying

$$\left(\frac{2\beta^2}{vh\lambda_1} - \frac{\lambda_1 h}{2}\right) \frac{q}{E_0 e^{\lambda_1 x} \sinh \beta} \sinh \frac{2\beta y}{h} + u_0'(y) + v_0'(x) = 0. \tag{77}$$

Considering (69), it is easily found that the desired elastic displacements are

$$u = \frac{q\beta}{vE_0 \lambda_1 e^{\lambda_1 x} \sinh \beta}\left[\cosh \frac{2\beta y}{h} - \frac{\left(1-\nu^2\right) \sinh \beta}{\beta}\right], \tag{78}$$

$$v = \frac{q\beta}{\lambda_1 \sqrt{v} E_0 e^{\lambda_1 x} \sinh \beta} \sinh \frac{2\beta y}{h}, \tag{79}$$

where rigid translation and rotation are neglected.

5.1.2 Solution for case B: narrow FGM strip

Similarly, for a narrow class II FGM strip, we do not take σ_y as a constant, but $\sigma_x = 0$ and $\tau_{xy} = 0$ are required. The Airy stress function is chosen as

$$\varphi = f_1(x) + yC. \tag{80}$$

Substituting (80) into the governing equation (16) leads to

$$f_1^{IV}(x) - 2f_1'''(x) + \lambda_1^2 f_1''(x) = 0. \tag{81}$$

Solving the above differential equation, one obtains

$$f_1''(x) = (B_3 + B_4 x)\, e^{\lambda_1 x}, \tag{82}$$

where B_3 and B_4 are constants to be determined by appropriate boundary conditions. In view of (7) and (82), the normal stress σ_y is

$$\sigma_y = (B_3 + B_4 x)\, e^{\lambda_1 x}. \tag{83}$$

In order to determine B_3 and B_4, we replace uniform tensile loading $\sigma_y = q$ with the following relaxation conditions:

$$\frac{1}{w}\int_{-w/2}^{w/2} \sigma_y \mathrm{d}x = q, \qquad \int_{-w/2}^{w/2} x\sigma_y \mathrm{d}x = 0. \tag{84}$$

Then we have

$$B_3 = \theta q \frac{\theta^2 \sinh\theta - 2\theta\cosh\theta + 2\sinh\theta}{\sinh^2\theta - \theta^2}, \tag{85}$$

$$B_4 = -\theta\lambda_1 q \frac{\theta\cosh\theta - \sinh\theta}{\sinh^2\theta - \theta^2}, \tag{86}$$

and

$$\sigma_y = \frac{\theta q\left[\left(\lambda_1 x + 2 + \theta^2\right)\sinh\theta - (\lambda_1 x + 2)\,\theta\cosh\theta\right]}{\sinh^2\theta - \theta^2} e^{\lambda_1 x}, \tag{87}$$

where

$$\theta = \frac{\lambda_1 w}{2}. \tag{88}$$

Thus, the strain components are

$$\varepsilon_x = -\frac{\theta\nu q\left[\left(\lambda_1 x + 2 + \theta^2\right)\sinh\theta - (\lambda_1 x + 2)\,\theta\cosh\theta\right]}{E_0\left(\sinh^2\theta - \theta^2\right)}, \tag{89}$$

$$\varepsilon_y = \frac{\theta q\left[\left(\lambda_1 x + 2 + \theta^2\right)\sinh\theta - (\lambda_1 x + 2)\,\theta\cosh\theta\right]}{E_0\left(\sinh^2\theta - \theta^2\right)}, \tag{90}$$

$$\gamma_{xy} = 0. \tag{91}$$

It is observed that the distribution of strain components ε_x and ε_y are both linear along the x-direction, and unchanged in the y-direction. Furthermore, one gets the elastic displacements as

$$u = -\frac{\theta q\left\{\left[\lambda_1\left(\nu x^2 + y^2\right) + 2\nu x\left(2 + \theta^2\right)\right]\sinh\theta - \left[\lambda_1\left(\nu x^2 + y^2\right) + 4\nu x\right]\theta\cosh\theta\right\}}{2E_0\left(\sinh^2\theta - \theta^2\right)}, \tag{92}$$

$$v = \frac{\theta q y\left[\left(\lambda_1 x + 2 + \theta^2\right)\sinh\theta - (\lambda_1 x + 2)\,\theta\cosh\theta\right]}{E_0\left(\sinh^2\theta - \theta^2\right)}, \tag{93}$$

where rigid translation and rotation are both neglected.

5.2 Pure bending

Consider a rectangular elastic beam made of class II FGMs subjected to a bending moment M at two ends. The material properties are assumed as

$$E = E_0 e^{\lambda_1 x}, \tag{94}$$

and the boundary conditions read

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \mathrm{d}y = 0, \int_{-\frac{h}{2}}^{\frac{h}{2}} y\sigma_x \mathrm{d}y = -M. \tag{95}$$

An analogous treatment allows us to take the Airy stress function in this case as

$$\varphi(x, y) = g(y). \tag{96}$$

Substituting Eq. (96) into Eq. (16) leads to

$$g^{IV}(y) - \lambda_1^2 \nu g''(y) = 0. \tag{97}$$

By solving Eq. (97), we can acquire

$$g''(y) = B_5 e^{\lambda_1 \sqrt{\nu} y} + B_6 e^{-\lambda_1 \sqrt{\nu} y} \tag{98}$$

and

$$\sigma_x = B_5 e^{\lambda_1 \sqrt{\nu} y} + B_6 e^{-\lambda_1 \sqrt{\nu} y}, \tag{99}$$

$$\sigma_y = 0, \tag{100}$$

$$\tau_{xy} = 0. \tag{101}$$

Where B_5 and B_6 are constants, which can be determined through the boundary conditions (95) as

$$B_5 = -B_6 = \frac{-\beta^2 M}{h^2(\beta \cosh \beta - \sinh \beta)}. \tag{102}$$

With these, the normal stress σ_x can be obtained to be

$$\sigma_x = -\frac{2\beta^2 M}{h^2(\beta \cosh \beta - \sinh \beta)} \sinh \frac{2\beta y}{h}. \tag{103}$$

From the above, the normal stress σ_x is an odd function with respect to y and $\mathrm{d}\sigma_x/\mathrm{d}y \neq 0$ over $[-h/2, h/2]$, implying that the maximal normal stress σ_x occurs at the beam surfaces, coinciding with the classical result. Additionally, we also check a limit case of the gradient index $\lambda_1 \to 0$. In this case, $\sigma_x = -12My/h^3$, which is identical to the classical stress distribution in a homogeneous isotropic rectangular beam subjected to a bending moment.

6 Results and discussion

In this section, the influence of inhomogeneity on the elastic fields of an FGM layer is examined. As representative examples, numerical results are given only for class I FGMs, i.e., the material properties are assumed to vary along the thickness-wise direction.

FGMs are commonly designed to connect two dissimilar materials with mismatch properties through a transition zone in which the material properties continuously vary from one material to the other by arranging a continuous change of the volume fraction of the constituents. As the first example, let us consider a thinner FGM layer occupying $-h/2 < y < h/2$ between two dissimilar homogeneous isotropic media occupying $y \leq -h/2$ and $y \geq h/2$, respectively, subject to simple tension $\sigma_y = q \quad (q > 0)$ (Fig. 2a). If denoting Young's moduli of the media in $y \leq -h/2$ and $y \geq h/2$ as E_1 and E_2, respectively, we have

$$\alpha = \frac{\lambda_2 h}{2} = \frac{1}{2} \ln \frac{E_2}{E_1}. \tag{104}$$

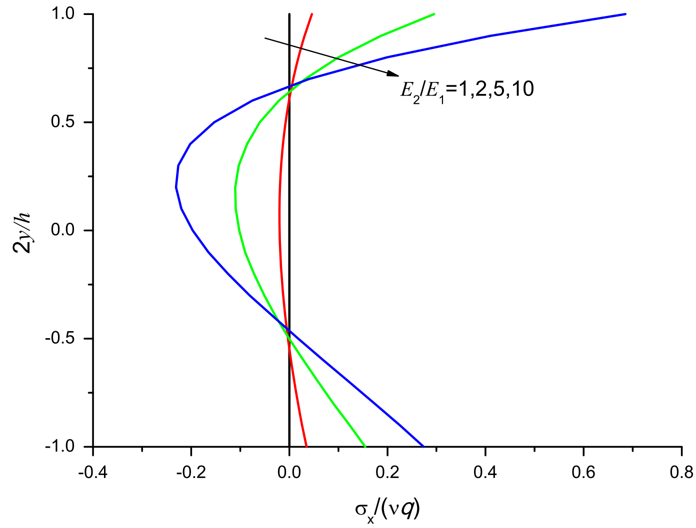

Fig. 4 Normalized normal stress $\sigma_x/(\nu q)$ in the FGM layer for various mismatch indices E_2/E_1 for case A

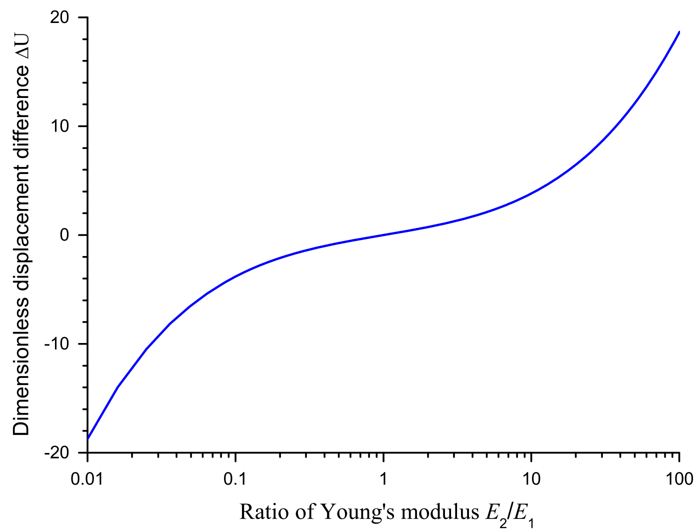

Fig. 5 Dimensionless displacement difference $\triangle U = E^* \triangle u/(x\nu q)$ between an FGM layer as a function of mismatch index E_2/E_1

For convenience, in the following, we assume that E_2 in $y \geq h/2$ is always greater than E_1 in $y \leq -h/2$, i.e., $\alpha \geq 0$.

Figure 4 shows the variation of the normalized normal stress $\sigma_x/(\nu q)$ in the FGM layer. As above mentioned, uniform tensile loading σ_y gives rise to appearance of the transverse normal stress σ_x, and the distribution of the normal stress σ_x is nonlinear along the thickness direction. The gradient index λ_2 or mismatch index E_2/E_1 of materials properties has a great influence on the distribution of normal stress σ_x. From Fig. 4, one finds that the normal stress σ_x is always tensile in the region near the interfaces $y = \pm h/2$ and compressive in certain central region inside the FGM layer. Precisely, the compressive stress mainly lies in a region close to the medium with the greater Young's modulus. In particular, if $E_2 = E_1$, we find that σ_x vanishes and is identical to the classical results, as expected. It is worth noting that although tensile or compressive stress σ_x exists in the FGM layer, the resultant and moment with respect to the $y = 0$ vanish.

In what follows, we examine the relative displacement difference between the interface $y = \pm h/2$, i.e., $\triangle u = u(x, h/2) - u(x, -h/2)$. With the aid of (37), it is quite simple to get

$$\triangle u = \frac{2\alpha^2 \nu q x e^{-\alpha}}{E_1 \left(\sinh^2 \alpha - \alpha^2\right)} (\alpha \cosh \alpha - \sinh \alpha). \tag{105}$$

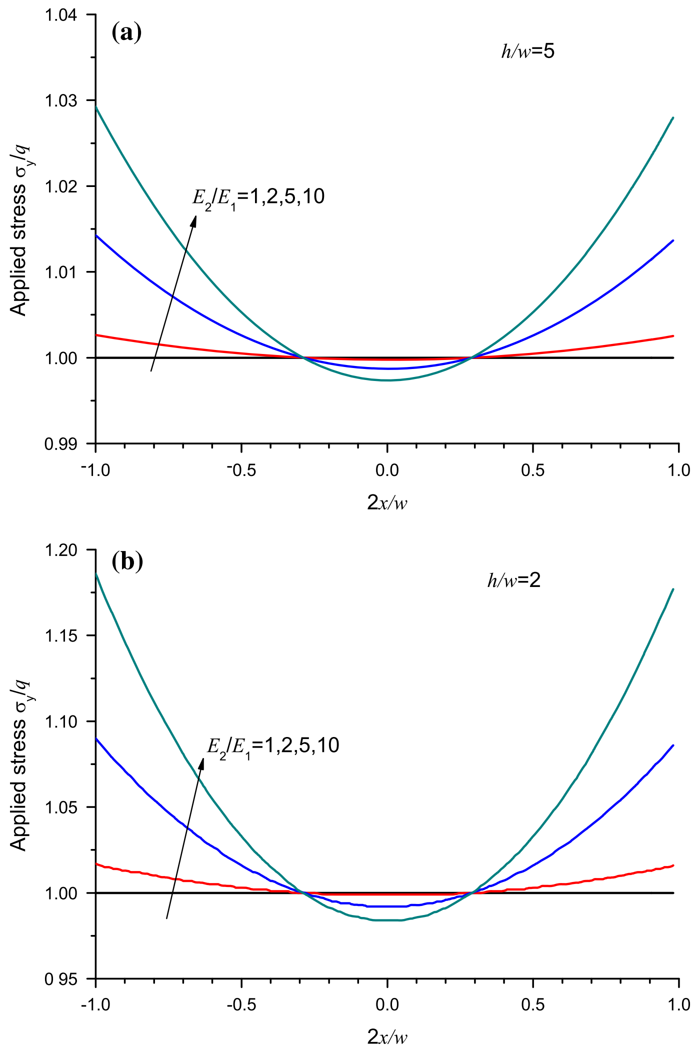

Fig. 6 Normalized normal stress σ_y/q in an FGM layer for various mismatch indices E_2/E_1 for case B with $\nu = 0.3$; **a** $h/w = 5$; **b** $h/w = 2$

Likely, for the vertical displacement v, using (38), we also calculate the relative vertical displacement difference, $\triangle v = v\,(x, h/2) - v\,(x, -h/2)$,

$$\triangle v = \frac{q\left(1 - v^2\right) he^{-\alpha}}{\alpha E_1} \sinh \alpha - \frac{v^2 q\alpha he^{-\alpha}}{E_1\left(\sinh^2 \alpha - \alpha^2\right)} \left[2\alpha \cosh \alpha - \left(\alpha^2 + 2\right) \sinh \alpha\right]. \qquad (106)$$

From the above, $\triangle v = 0$ if setting $h \to 0$. This reveals that two vertical displacements are continuous at the interface between two perfectly bonded dissimilar elastic media. However, the relative displacement difference $\triangle u$ is linearly dependent on the position variable x. Now we introduce an average modulus as $E^* = (E_1 + E_2)/2$, a dimensionless displacement difference $\triangle U = E^* \triangle u/(x\nu q)$ as a function of mismatch index E_2/E_1 is displayed in Fig. 5. From Fig. 5, we see that $\triangle u$ is monotonically increasing with the mismatch index E_2/E_1 of materials properties of two dissimilar media becoming greater. This result implies that the profile of the FGM after deformation is unsymmetrical with respect to the mid-plane, $y = 0$, unless $E_2/E_1 = 1$. An illustrative sketch of a bi-material with an FGM transition zone under simple tension is shown in Fig. 2a, b before and after deformation, respectively. In particular, it is noted that $\triangle u \neq 0$ in the case of vanishing thickness $h \to 0$, which implies a severe discontinuity of the horizontal displacement component u at the interface between two perfectly bonded dissimilar elastic media subject to uniform uniaxial tension (Fig. 2c). This incompatible relation is contradictory to the usual displacement continuity conditions at the interface

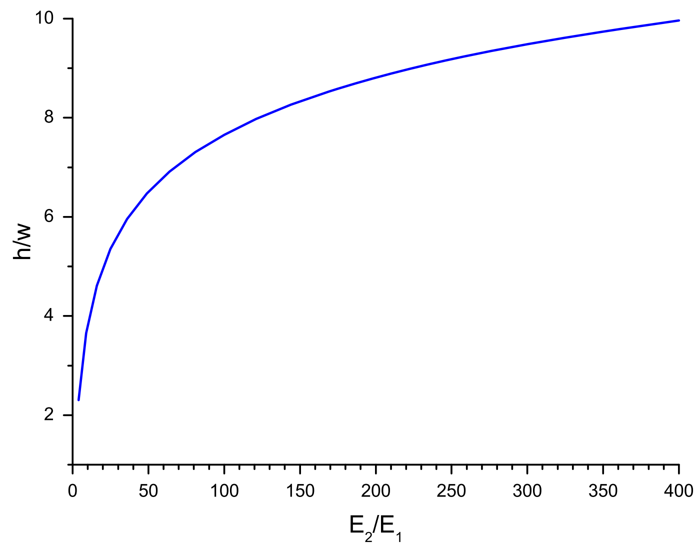

Fig. 7 Aspect ratio h/w as a function of E_2/E_1 for a narrow FGM strip in which the maximum normal stress σ_y is 5 % more than uniform stress

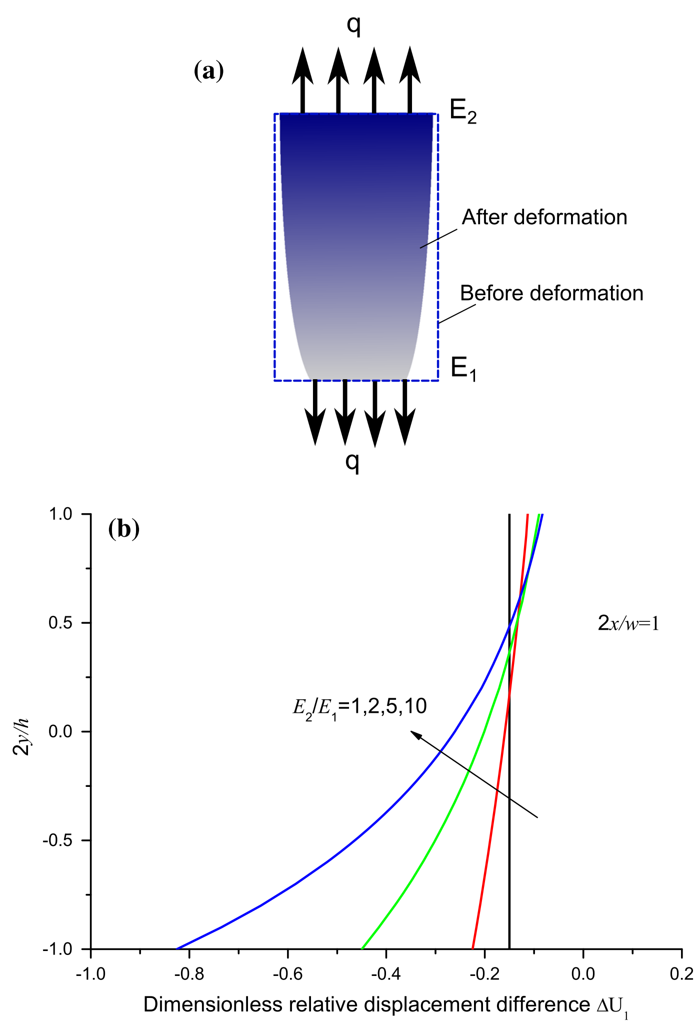

Fig. 8 **a** Profile of an FGM strip under uniaxial tension; **b** dimensionless displacement difference $\triangle U_1 = E^* \triangle u/(wq)$ between the surface and the center against $2y/h$ for various mismatch indices E_2/E_1

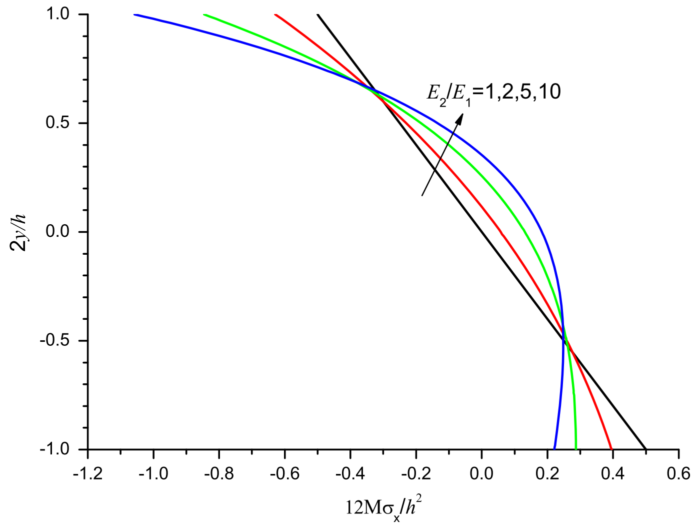

Fig. 9 Normalized normal stress $12M\sigma_x/h^2$ in an FGM beam subjected to a bending moment M for various mismatch indices E_2/E_1

between two perfectly bonded dissimilar materials. This may be a possible reason why up to date there is no exact simple elasticity solution available for two perfectly bonded dissimilar elastic half-planes subject to uniform uniaxial tension, to the best of the authors' knowledge. Of course, if singular stress fields are admissible, some elasticity solutions of two perfectly bonded dissimilar elastic strips can be determined (see e.g., [28,29]). On the other hand, due to the fact that chemical reaction and atomic diffusion in two dissimilar media near the interface take place, a distinct interface of vanishing thickness does not occur in practice. As a consequence, a narrow transition zone with continuously varying material properties from a medium to another medium is formed and the present result gives a reasonable explanation for this phenomenon, and moreover, derives an exact solution for two bonded dissimilar elastic media with an FGM layer as a transition zone subject to uniform uniaxial tension.

Example 2 is devoted to a narrow FGM strip subjected to simple tension. For this case, to ensure the conditions $\sigma_x = 0$ and $\tau_{xy} = 0$ in the FGM, applied stress σ_y must obey a distribution, which is shown in Fig. 6a,b for $\nu = 0.3$. It is interesting to point out that tensile stress is no longer a constant, but a symmetric hyperbolic function. From Fig. 6a, for an FGM strip with $h/w = 5$, the normal tensile stress may be approximately understood as a constant since the maximum normal stress is only 1.03 times as much as uniform stress q, even for $E_2/E_1 = 10$. For aspect ratio $h/w = 2$, the maximum tensile stress reaches at 20 % more than uniform stress q for $E_2/E_1 = 10$. Therefore, if the normal stress σ_y in simple tension is permitted to vary in a range of $q \pm 0.05q$ in $|x| < w/2$, we can acquire the aspect ratio h/w as a function of E_2/E_1, shown in Fig. 7.

Furthermore, we examine the horizontal displacement difference $\triangle u_1 = u\,(x, y) - u\,(0, y)$, which stands for a change of the horizontal displacement relative to the center. From (46) it is easy to obtain

$$\triangle u_1 = -\frac{\nu q w \sinh\left(\lambda_2\sqrt{\nu}x\right)}{2E_0 e^{\lambda_2 y}\sinh\left(w\lambda_2\sqrt{\nu}/2\right)}.$$

Figure 8 illustrates the horizontal displacement at the surface $x = w/2$ relative to the center $\triangle U_1 = E^*\Delta u_1/(wq)$ with $E^* = (E_1 + E_2)/2$ as a function of $2y/h$ with $\nu = 0.3$. This may be understood as a deformed profile of lateral surface of a narrow FGM strip after tension, and a sketch is displayed in Fig. 8a.

Finally, we consider an FGM beam subject to a bending moment. The stress distribution of σ_x is presented in Fig. 9. In this case, the distribution of the normal stress σ_x is no longer linear, but nonlinear along the thickness direction. Also, the position at which $\sigma_x = 0$ is not at $y = 0$, but shifts toward the surface close to the larger Young's modulus (see Fig. 10). This means that the physically neutral surface of an FGM beam moves to the stiffer side. This is an apparent discrepancy as compared to the homogeneous isotropic beam subjected to a bending moment. Due to the fact that, in practice, a material suffers a compressive stress in magnitude greater than a tensile stress, making use of this feature, one can design an FGM such that a greater compressive stress is achieved at the beam surface. Moreover, in tensile region of the beam, the normal stress

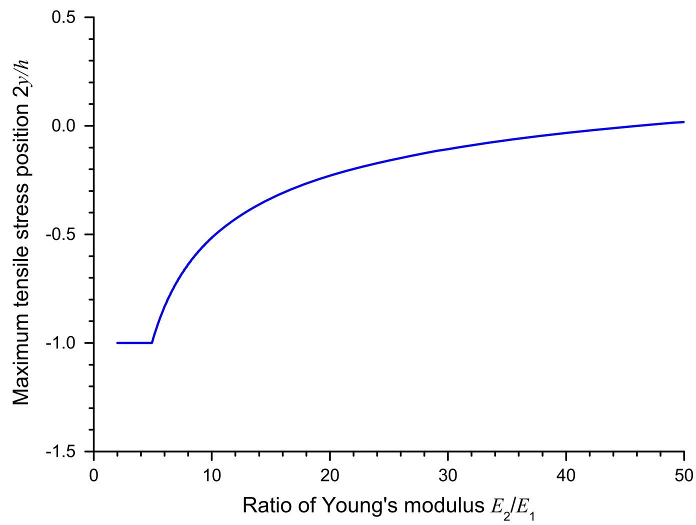

Fig. 10 Position of maximum tensile stress against the mismatch index E_2/E_1 in an FGM beam subjected to a bending moment M, where E_1 and E_2 are Young's moduli at the bottom and top surfaces of an FGM beam

σ_x only has a very slight variation. The maximum tensile stress occurs at the beam surface or inside the beam and is less than the classical maximum tensile stress at the beam surface. Owing to this reason, we can infer that the bending strength or load-carrying capability of an FGM beam is much improved as compared to a homogeneous beam.

7 Conclusions

For FGMs with the material properties dependent on the thickness-wise direction or the width-wise direction in form of an exponential function, a theoretical approach was presented to solve plane elasticity problems. Firstly, according to the two-dimensional theory of elasticity, the Airy stress function method was used to derive a governing differential equation of FGMs. Based on the resulting equation, we obtained several typical elasticity solutions for an FGM subjected to simple tension or bending moment. The main conclusions are drawn as follows:

- Elasticity solution for a thin FGM layer subjected to simple tension is obtained. The normal stress perpendicular to applied loading can be induced. It is dependent on the gradient index and Poisson's ratio.
- Elasticity solution for a narrow FGM strip subjected to simple tension is obtained. A symmetrically distributed gradient-dependent stress must be applied to ensure $\sigma_x = 0$ and $\tau_{xy} = 0$.
- For rectangular FGMs under simple tension, tensile loading σ_y induces the transverse normal stress σ_x, whereas tensile loading σ_y is required to obey a specific distribution if $\sigma_x = 0$ at the lateral surfaces.
- For an FGM beam subject to a bending moment, the normal stress exhibits a nonlinear distribution. Maximum tensile stress may occur inside the beam for thickness-wise varying gradient, and always occurs at the beam surface for length-wise varying gradient. The neutral surface shifts toward the stiffer surface for thickness-wise gradient.
- For two bonded dissimilar materials with a thin FGM transition zone subjected to simple tension, an exact elasticity solution is derived. When the thickness of the FGM reduces to zero, the transverse displacement incompatibility takes place at the interface between two perfectly bonded dissimilar materials.

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Appendix A

To explain shear stress $\tau_{xy} = 0$ at an arbitrary cross-section, we sketch a rectangle element, as shown in Fig. 11a, the top surface of which coincides with that of the elastic layer and the bottom surface corresponds

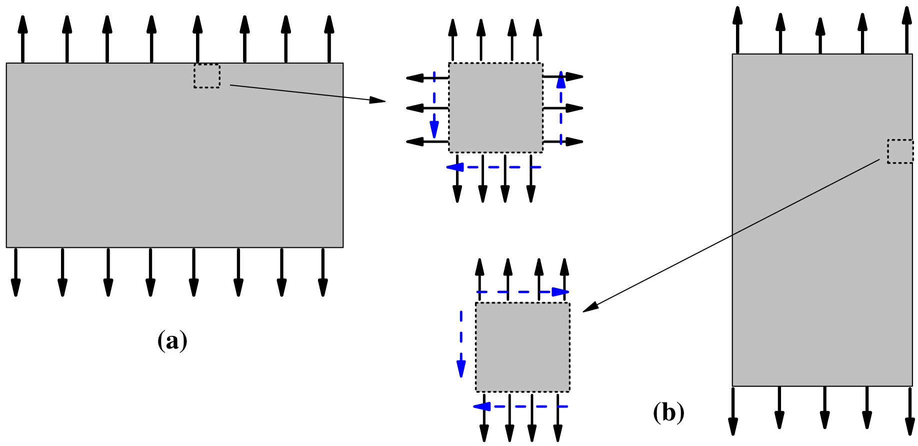

Fig. 11 A rectangle element; **a** close to the top surface of an FGM, **b** close to the right-hand side surface of an FGM, where the gradient variation is not shown here

to an arbitrary cross-section, $y = y_0$. Thus, the normal stresses $\sigma_y = q$ at these two surfaces. Since the top surface of the elastic layer is free of shear stress, the balance of force allows us to infer that the shear stress vanishes at the bottom surface of the rectangle, i.e., an arbitrary cross-section. Furthermore, the balance of moment gives that the shear stress equals to zero at the other edges of the rectangle.

Appendix B

For a narrow FGM strip, we choose a rectangle element, as shown in Fig. 11b, the right-hand side surface of which coincides with that of the FGM strip and the left-hand side surface corresponds to an arbitrary cross-section, $x = x_0$. Since $\sigma_x = 0$ at an arbitrary cross-section $x = x_0$, the balance of force allows us to obtain $\tau_{xy} = 0$ at an arbitrary cross-section, $x = x_0$, from the condition that $\tau_{xy} = 0$ at the surface of the FGM strip. Furthermore, the balance of moment gives that the shear stress equals to zero at the other edges of the rectangle.

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