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Eshelby's inclusion and dislocation problems for an isotropic circular domain bonded to an anisotropic medium

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Abstract This paper mainly investigates a two-dimensional Eshelby's problem of an inclusion of arbitrary shape embedded within an isotropic elastic circular domain which is perfectly bonded to the surrounding infinite anisotropic elastic medium. The Muskhelishvili's complex variable formulation in isotropic elasticity and the Stroh formalism in anisotropic elasticity are employed to derive a very simple and explicit analytical solution. The coefficients in the derived six analytic functions within the isotropic circular domain only contain the Barnett-Lothe tensors for the surrounding anisotropic medium and the shear modulus and Poisson's ratio for the isotropic circular domain. Several examples are discussed in detail to demonstrate and validate the obtained analytical solution. By using a similar method, we also investigate a line dislocation located in an isotropic circular cylinder which is perfectly bonded to the surrounding anisotropic medium and derive the image force acting on the line dislocation.

1 Introduction

The celebrated Eshelby's inclusion problem is concerned with the stress and strain analysis of an unbounded homogeneous elastic body containing a subdomain undergoing uniform eigenstrains [1–3]. The Eshelby's solution is valid only when the size of the inclusion is small compared to that of the surrounding elastic body (or the representative volume element, RVE) and when the inclusion is not very close to the boundary of the RVE. In a series of papers [4–7], Li and co-workers derived Eshelby tensors for a two-dimensional circular or a three-dimensional spherical inclusion concentrically embedded in a finite circular or spherical RVE with Dirichlet and Neumann boundary conditions. Sauer et al. [8] further obtained the Composite Eshelby tensors for a spherical inclusion concentrically embedded in a spherical RVE which is in turn perfectly bonded to the surrounding infinite isotropic medium. Mejak [9] considered the case in which the inclusion can be axisymmetrically placed within a finite spherical RVE. By using a superposition method, Zou et al. [10] investigated a two-dimensional inclusion of arbitrary shape embedded in a finite elastic domain with Dirichlet and Neumann boundary conditions. The boundary value problem is finally reduced to the solution of a Fredholm-type integral equation.

In the present work, we investigate an inclusion of arbitrary shape located in an isotropic elastic circular RVE which is perfectly bonded to the surrounding generally anisotropic elastic medium using Muskhelishvili's complex variable formulation in isotropic elasticity [11], the Stroh formalism in anisotropic elasticity [12], and the method of conformal mapping and analytical continuation [13–15]. The present three-phase composite model can be used to account for the practical situation in which the outer unbounded composite phase is generally anisotropic, whereas previous studies have assumed that the composite phase is transversely isotropic

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[16–18]. The obtained expressions of the six analytic functions in the isotropic circular domain are valid for any mathematically degenerate anisotropic material surrounding the circular RVE because the coefficients in these six analytic functions only contain the Barnett-Lothe tensors for the surrounding anisotropic medium and the shear modulus and Poisson's ratio for the isotropic circular domain. The Eshelby tensors in the three phases can then be arrived at by using the obtained analytic functions. Some typical cases are presented to demonstrate and validate the obtained analytic solution. In order to demonstrate the generality of the solution method, the problem of a line dislocation located in an isotropic elastic circular cylinder which is perfectly bonded to the surrounding anisotropic elastic infinite medium is also discussed in detail.

2 Basic formulations

2.1 The Stroh formalism for anisotropic elastic materials

In a fixed Cartesian coordinate system $x_i,\ i=1,\ 2,\ 3$, the basic equations for an anisotropic elastic material are

$$\sigma_{ij,j}=0,\ \varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \sigma_{ij}=C_{ijkl}\varepsilon_{kl}=C_{ijkl}u_{k,l}, \tag{1}$$

where $\sigma_{ij},\ u_i$ and ε_{ij} are the stresses, displacements, and strains; C_{ijkl} are the elastic stiffnesses.

For two-dimensional problems in which all the physical quantities depend only on the plane coordinates x_1 and x_2, the general solution can be expressed as [12]

$$\begin{aligned}\mathbf{u}&=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}^T=\mathbf{A}\mathbf{f}(z)+\bar{\mathbf{A}}\overline{\mathbf{f}(z)},\\ \boldsymbol{\varphi}&=\begin{bmatrix}\varphi_1 & \varphi_2 & \varphi_3\end{bmatrix}^T=\mathbf{B}\mathbf{f}(z)+\bar{\mathbf{B}}\overline{\mathbf{f}(z)}\end{aligned} \tag{2}$$

where

$$\begin{aligned}&\mathbf{A}=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix},\\ &\mathbf{f}(z)=\begin{bmatrix}f_1(z_1) & f_2(z_2) & f_3(z_3)\end{bmatrix}^T,\\ &z_i=x_1+p_ix_2,\ \mathrm{Im}\{p_i\}>0,\ (i=1,2,3),\end{aligned} \tag{3}$$

with

$$\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^T\end{bmatrix}\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix}=p_i\begin{bmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{bmatrix},\ (i=1,2,3), \tag{4}$$

$$\mathbf{N}_1=-\mathbf{T}^{-1}\mathbf{R}^T,\ \mathbf{N}_2=\mathbf{T}^{-1},\ \mathbf{N}_3=\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T-\mathbf{Q}, \tag{5}$$

and

$$Q_{ik}=C_{i1k1},\ R_{ik}=C_{i1k2},\ T_{ik}=C_{i2k2}. \tag{6}$$

The stress function vector $\boldsymbol{\varphi}$ is defined, in terms of the stresses, as follows:

$$\sigma_{i1}=-\varphi_{i,2},\ \sigma_{i2}=\varphi_{i,1},\ (i=1,2,3). \tag{7}$$

Due to the fact that the two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following orthogonality relations [12]:

$$\begin{aligned}\mathbf{B}^T\mathbf{A}+\mathbf{A}^T\mathbf{B}&=\mathbf{I}=\bar{\mathbf{B}}^T\bar{\mathbf{A}}+\bar{\mathbf{A}}^T\bar{\mathbf{B}},\\ \mathbf{B}^T\bar{\mathbf{A}}+\mathbf{A}^T\bar{\mathbf{B}}&=\mathbf{0}=\bar{\mathbf{B}}^T\mathbf{A}+\bar{\mathbf{A}}^T\mathbf{B},\end{aligned} \tag{8}$$

the following three real Barnett-Lothe tensors $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$ can be introduced [12]:

$$\mathbf{S}=\mathrm{i}(2\mathbf{A}\mathbf{B}^T-\mathbf{I}),\ \mathbf{H}=2\mathrm{i}\mathbf{A}\mathbf{A}^T,\ \mathbf{L}=-2\mathrm{i}\mathbf{B}\mathbf{B}^T. \tag{9}$$

Furthermore, the two matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite, while $\mathbf{SH},\ \mathbf{LS},\ \mathbf{H}^{-1}\mathbf{S},\ \mathbf{SL}^{-1}$ are anti-symmetric.

2.2 Muskhelishvili's formulation for isotropic elastic materials

For plane strain deformation of an isotropic elastic material, the nontrivial stresses, displacements, and stress functions can be expressed in terms of two analytic functions $\phi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [11]

$$\sigma_{11} + \sigma_{22} = 2\left[\phi'(z) + \overline{\phi'(z)}\right], \tag{10}$$

$$\begin{aligned} \sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} &= 2\left[\bar{z}\phi''(z) + \psi'(z)\right], \\ 2\mu(u_1 + \mathrm{i}u_2) &= \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \\ \varphi_1 + \mathrm{i}\varphi_2 &= \mathrm{i}\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right] \end{aligned} \tag{11}$$

where $\kappa = 3 - 4\nu$; μ and ν, where $\mu > 0$ and $0 \le \nu \le 0.5$, are the shear modulus and Poisson's ratio, respectively.

For the anti-plane shear deformation of an isotropic elastic material, the shear stresses, out-of-plane displacement and stress function φ_3 can be expressed in terms of a single analytic function $h(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as

$$\sigma_{32} + \mathrm{i}\sigma_{31} = h'(z), \; \varphi_3 + \mathrm{i}\mu u_3 = h(z). \tag{12}$$

3 The Eshelby problem

We consider a generally anisotropic elastic infinite medium containing an isotropic elastic circular cylinder of radius a with its center at origin. Contained within the circular cylinder is a two-dimensional Eshelby's inclusion of arbitrary shape undergoing uniform in-plane eigenstrains $(\varepsilon_{11}^*, \varepsilon_{12}^*, \varepsilon_{22}^*)$ and anti-plane eigenstrains $(\varepsilon_{13}^*, \varepsilon_{23}^*)$, as shown in Fig. 1. Let S_0 and S_1 denote the Eshelby's inclusion and its supplement to the circular cylinder, Γ the perfect interface between S_0 and S_1, S_2 the surrounding anisotropic medium. Throughout the paper, the subscripts 0, 1, and 2 will be used to identify the associated quantities in S_0, S_1 and S_2.

The following mapping functions are considered for the surrounding anisotropic medium [19]:

$$z_\alpha = x_1 + p_\alpha x_2 = \omega_\alpha(\xi_\alpha) = \frac{1}{2}(1 - \mathrm{i}p_\alpha)\xi_\alpha + \frac{a^2}{2}(1 + \mathrm{i}p_\alpha)\frac{1}{\xi_\alpha}, \; (\alpha = 1, 2, 3) \tag{13}$$

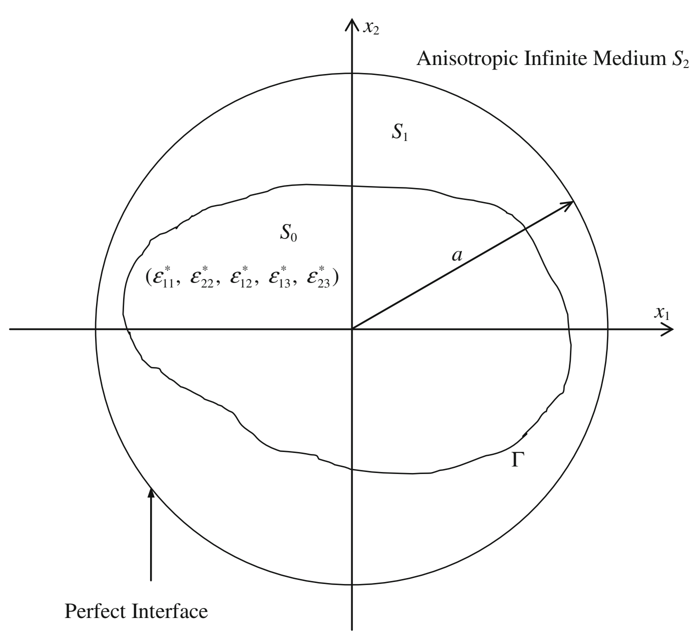

Fig. 1 An inclusion of arbitrary shape in an isotropic circular cylinder perfectly bonded to an infinite anisotropic medium

which can map the exterior of an elliptical region in the z_α-plane onto the exterior of the circle $|\xi_\alpha| > a$ in the ξ_α-plane (the circle $|z| = a$ maps to the boundary of the elliptical region in the z_α-plane, where $z_\alpha = x_1^\alpha + \mathrm{i}x_2^\alpha = x_1 + p'_\alpha x_2 + \mathrm{i}p''_\alpha x_2$ with p'_α and p''_α being the real and imaginary parts of p_α). In view of the fact that $\xi_1 = \xi_2 = \xi_3 = z$ on $|z| = a$ [19], we can first replace ξ_α by the common variable z. When the analysis is finished, the complex variable z shall be changed back to the corresponding complex variables ξ_α.

The continuity conditions of displacements and tractions across the interface Γ between S_0 and S_1 can be expressed as

$$\begin{aligned}
\kappa\phi_1(z) - z\overline{\phi_1'(z)} - \overline{\psi_1(z)} &= \kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)} + 2\mu\left[\delta_1 z + (\delta_2 + \mathrm{i}\delta_3)\bar{z}\right], \\
\phi_1(z) + z\overline{\phi_1'(z)} + \overline{\psi_1(z)} &= \phi_0(z) + z\overline{\phi_0'(z)} + \overline{\psi_0(z)}, \\
h_1(z) - \overline{h_1(z)} &= h_0(z) - \overline{h_0(z)} + 2\mathrm{i}\mu\left[(\delta_4 - \mathrm{i}\delta_5)z + (\delta_4 + \mathrm{i}\delta_5)\bar{z}\right], \\
h_1(z) + \overline{h_1(z)} &= h_0(z) + \overline{h_0(z)}, \quad z \in \Gamma
\end{aligned} \tag{14}$$

where

$$\delta_1 = \frac{\varepsilon_{11}^* + \varepsilon_{22}^*}{2},\ \delta_2 = \frac{\varepsilon_{11}^* - \varepsilon_{22}^*}{2},\ \delta_3 = \varepsilon_{12}^*,\ \delta_4 = \varepsilon_{13}^*,\ \delta_5 = \varepsilon_{23}^*. \tag{15}$$

Equation (14) can be recast into the following equivalent form:

$$\begin{aligned}
\phi_1(z) &= \phi_0(z) + \frac{2\mu}{\kappa+1}\left[\delta_1 z + (\delta_2 + \mathrm{i}\delta_3)\bar{z}\right], \\
\psi_1(z) + \bar{z}\left[\phi_1'(z) - \phi_0'(z)\right] &= \psi_0(z) - \frac{2\mu}{\kappa+1}\left[\delta_1\bar{z} + (\delta_2 - \mathrm{i}\delta_3)z\right] \\
h_1(z) &= h_0(z) + \mathrm{i}\mu\left[(\delta_4 - \mathrm{i}\delta_5)z + (\delta_4 + \mathrm{i}\delta_5)\bar{z}\right], \quad z \in \Gamma.
\end{aligned} \tag{16}$$

According to Ru [13], $\bar{z} = D(z)$ on the interface Γ. In addition, $D(z)$ is analytic in the exterior of the inclusion except at infinity where it behaves as

$$D(z) \to P(z) + O\left(\frac{1}{z}\right),\ D(z)D'(z) \to Q(z) + O\left(\frac{1}{z^2}\right), \quad |z| \to \infty, \tag{17}$$

where $P(z)$ and $Q(z)$ are polynomials of orders N and $N(N-1)$ in z if the conformal mapping $z = \omega(\xi)$, which maps the inclusion onto the exterior of the unit circle in the ξ-plane, is a polynomial of order N in $1/\xi$.

As a result, the interface conditions in Eq. (16) can be rewritten into

$$\begin{aligned}
&\phi_1(z) - \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D(z) - P(z)\right] = \phi_0(z) + \tfrac{2\mu}{\kappa+1}\left[\delta_1 z + (\delta_2 + \mathrm{i}\delta_3)P(z)\right], \\
&\psi_1(z) + \tfrac{4\mu\delta_1}{\kappa+1}\left[D(z) - P(z)\right] + \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D(z)D'(z) - Q(z)\right] \\
&= \psi_0(z) - \tfrac{2\mu}{\kappa+1}(\delta_2 - \mathrm{i}\delta_3)z - \tfrac{4\mu\delta_1}{\kappa+1}P(z) - \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)Q(z), \\
&h_1(z) - \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)\left[D(z) - P(z)\right] = h_0(z) + \mathrm{i}\mu(\delta_4 - \mathrm{i}\delta_5)z + \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)P(z), \quad z \in \Gamma.
\end{aligned} \tag{18}$$

In view of Eq. (18), we construct the following three auxiliary functions:

$$\Phi(z) = \begin{cases} \phi_1(z) - \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D(z) - P(z)\right], & z \in S_1 \\ \phi_0(z) + \tfrac{2\mu}{\kappa+1}\left[\delta_1 z + (\delta_2 + \mathrm{i}\delta_3)P(z)\right], & z \in S_0 \end{cases} \tag{19}$$

$$\Omega(z) = \begin{cases} \psi_1(z) + \tfrac{a^2}{z}\phi_1'(z) + \tfrac{4\mu\delta_1}{\kappa+1}\left[D(z) - P(z)\right] \\ \quad + \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D(z)D'(z) - \tfrac{a^2}{z}D'(z) + \tfrac{a^2}{z}P'(z) - Q(z)\right] - \tfrac{\lambda}{z}, & z \in S_1 \\ \psi_0(z) + \tfrac{a^2}{z}\phi_0'(z) - \tfrac{2\mu}{\kappa+1}(\delta_2 - \mathrm{i}\delta_3)z \\ \quad - \tfrac{4\mu\delta_1}{\kappa+1}P(z) + \tfrac{2\mu}{\kappa+1}\left[\tfrac{a^2}{z}\delta_1 + (\delta_2 + \mathrm{i}\delta_3)\left[\tfrac{a^2}{z}P'(z) - Q(z)\right]\right] - \tfrac{\lambda}{z}, & z \in S_0 \end{cases} \tag{20}$$

$$\Theta(z) = \begin{cases} h_1(z) - \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)\left[D(z) - P(z)\right], & z \in S_1 \\ h_0(z) + \mathrm{i}\mu(\delta_4 - \mathrm{i}\delta_5)z + \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)P(z), & z \in S_0 \end{cases} \tag{21}$$

where the complex constant λ is given by

$$\lambda = \begin{cases} a^2\phi_1'(0) - \frac{2a^2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D'(0) - P'(0)\right], & \text{if } z = 0 \in S_1 \\ a^2\phi_0'(0) + \frac{2a^2\mu}{\kappa+1}\left[\delta_1 + (\delta_2 + \mathrm{i}\delta_3)P'(0)\right], & \text{if } z = 0 \in S_0. \end{cases} \quad (22)$$

It is found that the introduced three auxiliary functions $\Phi(z)$, $\Omega(z)$ and $\Theta(z)$ in Eqs. (19)–(21) are continuous and analytic everywhere in the circular region $|z| \le a$. Furthermore, it is deduced from Eqs. (19)–(21) that

$$\phi_1(z) = \Phi(z) + g_1(z),\ \psi_1(z) + \frac{a^2}{z}\phi_1'(z) = \Omega(z) + g_2(z),\ h_1(z) = \Theta(z) + g_3(z) \quad (23)$$

where

$$\begin{aligned} g_1(z) &= \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[D(z) - P(z)\right], \\ g_2(z) &= -\tfrac{4\mu\delta_1}{\kappa+1}\left[D(z) - P(z)\right] + \tfrac{2\mu}{\kappa+1}(\delta_2 + \mathrm{i}\delta_3)\left[\tfrac{a^2}{z}D'(z) - D(z)D'(z) - \tfrac{a^2}{z}P'(z) + Q(z)\right] + \tfrac{\lambda}{z}, \\ g_3(z) &= \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)\left[D(z) - P(z)\right]. \end{aligned} \quad (24)$$

Expression (23) can be interpreted as follows: the regular parts of $\phi_1(z)$, $\psi_1(z) + \frac{a^2}{z}\phi_1'(z)$ and $h_1(z)$ are, respectively, $\Phi(z)$, $\Omega(z)$ and $\Theta(z)$; while their singular parts are, respectively, $g_1(z)$, $g_2(z)$ and $g_3(z)$. This fact will facilitate the ensuring analysis.

We can then present the expressions of $\Phi(z)$, $\Omega(z)$ and $\Theta(z)$ as follows:

$$\begin{aligned} \Phi(z) &= l_1\bar{g}_1(a^2/z) + l_2\bar{g}_2(a^2/z) + l_3\bar{g}_3(a^2/z), \\ \Psi(z) &= m_1\bar{g}_1(a^2/z) + m_2\bar{g}_2(a^2/z) + m_3\bar{g}_3(a^2/z), \\ \Theta(z) &= n_1\bar{g}_1(a^2/z) + n_2\bar{g}_2(a^2/z) + n_3\bar{g}_3(a^2/z),\ |z| \le a, \end{aligned} \quad (25)$$

where $l_k,\ m_k,\ n_k,\ (k = 1,\ 2,\ 3)$ are unknown complex constants to be determined. Apparently $\Phi(z)$, $\Omega(z)$ and $\Theta(z)$ in Eq. (25) are indeed analytic within $|z| \le a$. Our task below is to determine $l_k,\ m_k,\ n_k,\ (k = 1,\ 2,\ 3)$ by satisfaction of the continuity conditions of displacements and tractions across the circular interface $|z| = a$ between S_1 and S_2.

It is deduced from Eqs. (19)–(21) and (25) that the three analytic functions defined in S_0 are

$$\begin{aligned} \phi_0(z) &= l_1\bar{g}_1(a^2/z) + l_2\bar{g}_2(a^2/z) + l_3\bar{g}_3(a^2/z) - \frac{2\mu}{\kappa+1}\left[\delta_1 z + (\delta_2 + \mathrm{i}\delta_3)P(z)\right], \\ \psi_0(z) + \frac{a^2}{z}\phi_0'(z) &= m_1\bar{g}_1(a^2/z) + m_2\bar{g}_2(a^2/z) + m_3\bar{g}_3(a^2/z) + \frac{2\mu}{\kappa+1}(\delta_2 - \mathrm{i}\delta_3)z \\ &\quad + \frac{4\mu\delta_1}{\kappa+1}P(z) - \frac{2\mu}{\kappa+1}\left[\frac{a^2}{z}\delta_1 + (\delta_2 + \mathrm{i}\delta_3)\left[\frac{a^2}{z}P'(z) - Q(z)\right]\right] + \frac{\lambda}{z}, \\ h_0(z) &= n_1\bar{g}_1(a^2/z) + n_2\bar{g}_2(a^2/z) + n_3\bar{g}_3(a^2/z) - \mathrm{i}\mu(\delta_4 - \mathrm{i}\delta_5)z - \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)P(z),\ z \in S_0. \end{aligned} \quad (26)$$

It is deduced from Eqs. (23) and (25) that the three analytic functions defined in S_1 are

$$\begin{aligned} \phi_1(z) &= l_1\bar{g}_1(a^2/z) + l_2\bar{g}_2(a^2/z) + l_3\bar{g}_3(a^2/z) + g_1(z), \\ \psi_1(z) + \frac{a^2}{z}\phi_1'(z) &= m_1\bar{g}_1(a^2/z) + m_2\bar{g}_2(a^2/z) + m_3\bar{g}_3(a^2/z) + g_2(z), \\ h_1(z) &= n_1\bar{g}_1(a^2/z) + n_2\bar{g}_2(a^2/z) + n_3\bar{g}_3(a^2/z) + g_3(z),\ z \in S_1. \end{aligned} \quad (27)$$

$\phi_0'(0)$ or $\phi_0'(0)$ appearing in λ can be determined from Eqs. (26) and (27) as

$$\phi_0'(0) = \frac{\mu A}{\pi a^2(1-|l_2|^2)}\left[\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right]$$
$$-\frac{2\mu}{\kappa+1}\left[\delta_1+(\delta_2+\mathrm{i}\delta_3)P'(0)\right], \text{ if } z=0\in S_0, \tag{28}$$

$$\phi_1'(0) = \frac{\mu A}{\pi a^2(1-|l_2|^2)}\left[\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right]$$
$$+\frac{2\mu}{\kappa+1}(\delta_2+\mathrm{i}\delta_3)\left[D'(0)-P'(0)\right], \text{ if } z=0\in S_1, \tag{29}$$

where A is the area of the Eshelby inclusion of arbitrary shape.

Consequently, λ in Eq. (22) can be obtained as

$$\lambda = \frac{\mu A}{\pi(1-|l_2|^2)}\left[\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right]. \tag{30}$$

The above expression of λ is valid whether the origin $z=0$ is located in S_0 or S_1. The displacement and stress function vectors along the circular interface $|z|=a$ on the S_1 side can be obtained from Eqs. (11), (12) and (27) as

$$\hat{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu} \\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu} \\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu} \end{bmatrix} \begin{bmatrix} g_1(z) \\ g_2(z) \\ g_3(z) \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{\kappa l_1-m_1+\kappa}{4\mu} & \frac{\kappa l_2-m_2-1}{4\mu} & \frac{\kappa l_3-m_3}{4\mu} \\ \frac{\kappa l_1+m_1-\kappa}{4\mathrm{i}\mu} & \frac{\kappa l_2+m_2-1}{4\mathrm{i}\mu} & \frac{\kappa l_3+m_3}{4\mathrm{i}\mu} \\ \frac{n_1}{2\mathrm{i}\mu} & \frac{n_2}{2\mathrm{i}\mu} & \frac{n_3-1}{2\mathrm{i}\mu} \end{bmatrix} \begin{bmatrix} \bar{g}_1(a^2/z) \\ \bar{g}_2(a^2/z) \\ \bar{g}_3(a^2/z) \end{bmatrix}, \quad |z|=a, \tag{31}$$

$$\hat{\boldsymbol{\varphi}} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} \frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}} \\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2} \\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2} \end{bmatrix} \begin{bmatrix} g_1(z) \\ g_2(z) \\ g_3(z) \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{-l_1+m_1+1}{2\mathrm{i}} & \frac{-l_2+m_2-1}{2\mathrm{i}} & \frac{-l_3+m_3}{2\mathrm{i}} \\ \frac{l_1+m_1+1}{2} & \frac{l_2+m_2+1}{2} & \frac{l_3+m_3}{2} \\ \frac{n_1}{2} & \frac{n_2}{2} & \frac{n_3+1}{2} \end{bmatrix} \begin{bmatrix} \bar{g}_1(a^2/z) \\ \bar{g}_2(a^2/z) \\ \bar{g}_3(a^2/z) \end{bmatrix}, \quad |z|=a. \tag{32}$$

Consequently, the continuity conditions of displacements and tractions across the circular interface $|z|=a$ between S_1 and S_2 can be concisely expressed by

$$\begin{aligned} &\mathbf{Af}(z)+\bar{\mathbf{A}}\overline{\mathbf{f}(z)}=\hat{\mathbf{u}}, \\ &\mathbf{Bf}(z)+\bar{\mathbf{B}}\overline{\mathbf{f}(z)}=\hat{\boldsymbol{\varphi}},\ |z|=a. \end{aligned} \tag{33}$$

Pre-multiplying the two conditions in Eq. (33) by $\mathbf{B}^T$ and $\mathbf{A}^T$, respectively, adding the resulting conditions, and utilizing the orthogonality relations in Eq. (8), we arrive at the following expression for $\mathbf{f}(z)$:

$$\mathbf{f}(z)=\mathbf{B}^T\hat{\mathbf{u}}+\mathbf{A}^T\hat{\boldsymbol{\varphi}}$$
$$=\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu}\\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu}\\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}}\\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2}\\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2}\end{bmatrix}\right\}\begin{bmatrix}g_1(z)\\ g_2(z)\\ g_3(z)\end{bmatrix}$$
$$+\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa l_1-m_1+\kappa}{4\mu} & \frac{\kappa l_2-m_2-1}{4\mu} & \frac{\kappa l_3-m_3}{4\mu}\\ \frac{\kappa l_1+m_1-\kappa}{4\mathrm{i}\mu} & \frac{\kappa l_2+m_2-1}{4\mathrm{i}\mu} & \frac{\kappa l_3+m_3}{4\mathrm{i}\mu}\\ \frac{n_1}{2\mathrm{i}\mu} & \frac{n_2}{2\mathrm{i}\mu} & \frac{n_3-1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{-l_1+m_1+1}{2\mathrm{i}} & \frac{-l_2+m_2-1}{2\mathrm{i}} & \frac{-l_3+m_3}{2\mathrm{i}}\\ \frac{l_1+m_1+1}{2} & \frac{l_2+m_2+1}{2} & \frac{l_3+m_3}{2}\\ \frac{n_1}{2} & \frac{n_2}{2} & \frac{n_3+1}{2}\end{bmatrix}\right\}\begin{bmatrix}\bar{g}_1(a^2/z)\\ \bar{g}_2(a^2/z)\\ \bar{g}_3(a^2/z)\end{bmatrix},$$
$$z\in S_2. \tag{34}$$

The expression for $\mathbf{f}(z)$ cannot contain the terms $\bar{g}_k(a^2/z)$, $(k=1,2,3)$ since the fact that $\mathbf{f}(z)$ is analytic in S_2. Thus, the following condition can be obtained:

$$\mathbf{B}^T\begin{bmatrix}\frac{\kappa l_1-m_1+\kappa}{4\mu} & \frac{\kappa l_2-m_2-1}{4\mu} & \frac{\kappa l_3-m_3}{4\mu}\\ \frac{\kappa l_1+m_1-\kappa}{4\mathrm{i}\mu} & \frac{\kappa l_2+m_2-1}{4\mathrm{i}\mu} & \frac{\kappa l_3+m_3}{4\mathrm{i}\mu}\\ \frac{n_1}{2\mathrm{i}\mu} & \frac{n_2}{2\mathrm{i}\mu} & \frac{n_3-1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{-l_1+m_1+1}{2\mathrm{i}} & \frac{-l_2+m_2-1}{2\mathrm{i}} & \frac{-l_3+m_3}{2\mathrm{i}}\\ \frac{l_1+m_1+1}{2} & \frac{l_2+m_2+1}{2} & \frac{l_3+m_3}{2}\\ \frac{n_1}{2} & \frac{n_2}{2} & \frac{n_3+1}{2}\end{bmatrix}=\mathbf{0}. \tag{35}$$

Pre-multiplying Eq. (35) by $\mathbf{B}^T$ and utilizing the identities in Eq. (9), we can finally obtain the following relationships:

$$\mathbf{L}\begin{bmatrix}\mathrm{i}(\kappa l_1-m_1+\kappa) & \mathrm{i}(\kappa l_2-m_2-1) & \mathrm{i}(\kappa l_3-m_3)\\ \kappa l_1+m_1-\kappa & \kappa l_2+m_2-1 & \kappa l_3+m_3\\ 2n_1 & 2n_2 & 2(n_3-1)\end{bmatrix} \tag{36}$$
$$-2\mu(\mathbf{S}^T+\mathrm{i}\mathbf{I})\begin{bmatrix}-l_1+m_1+1 & -l_2+m_2-1 & -l_3+m_3\\ \mathrm{i}(l_1+m_1+1) & \mathrm{i}(l_2+m_2+1) & \mathrm{i}(l_3+m_3)\\ \mathrm{i}n_1 & \mathrm{i}n_2 & \mathrm{i}(n_3+1)\end{bmatrix}=\mathbf{0}.$$

The nine complex constants $l_k,\ m_k,\ n_k,\ (k=1,\ 2,\ 3)$ can be uniquely determined by solving Eq. (36) as

$$\begin{bmatrix}l_1\\ m_1\\ n_1\end{bmatrix}=\left[\mathbf{L}\begin{bmatrix}\mathrm{i}\kappa & -\mathrm{i} & 0\\ \kappa & 1 & 0\\ 0 & 0 & 2\end{bmatrix}-2\mu(\mathbf{S}^T+\mathrm{i}\mathbf{I})\begin{bmatrix}-1 & 1 & 0\\ \mathrm{i} & \mathrm{i} & 0\\ 0 & 0 & \mathrm{i}\end{bmatrix}\right]^{-1}\left[\kappa\mathbf{L}+2\mu(\mathrm{i}\mathbf{S}^T-\mathbf{I})\right]\begin{bmatrix}-\mathrm{i}\\ 1\\ 0\end{bmatrix},$$
$$\begin{bmatrix}l_2\\ m_2\\ n_2\end{bmatrix}=\left[\mathbf{L}\begin{bmatrix}\mathrm{i}\kappa & -\mathrm{i} & 0\\ \kappa & 1 & 0\\ 0 & 0 & 2\end{bmatrix}-2\mu(\mathbf{S}^T+\mathrm{i}\mathbf{I})\begin{bmatrix}-1 & 1 & 0\\ \mathrm{i} & \mathrm{i} & 0\\ 0 & 0 & \mathrm{i}\end{bmatrix}\right]^{-1}\left[\mathbf{L}+2\mu(\mathrm{i}\mathbf{S}^T-\mathbf{I})\right]\begin{bmatrix}\mathrm{i}\\ 1\\ 0\end{bmatrix}, \tag{37}$$
$$\begin{bmatrix}l_3\\ m_3\\ n_3\end{bmatrix}=\left[\mathbf{L}\begin{bmatrix}\mathrm{i}\kappa & -\mathrm{i} & 0\\ \kappa & 1 & 0\\ 0 & 0 & 2\end{bmatrix}-2\mu(\mathbf{S}^T+\mathrm{i}\mathbf{I})\begin{bmatrix}-1 & 1 & 0\\ \mathrm{i} & \mathrm{i} & 0\\ 0 & 0 & \mathrm{i}\end{bmatrix}\right]^{-1}\left[2\mathbf{L}+2\mu(\mathrm{i}\mathbf{S}^T-\mathbf{I})\right]\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},$$

which only contain the Barnett-Lothe tensors for the anisotropic elastic material in S_2 and the shear modulus and Poisson's ratio for the isotropic elastic material in $S_0\cup S_1$.

Once $l_k,\ m_k,\ n_k,\ (k=1,\ 2,\ 3)$ have been given by the above expression, $\mathbf{f}(z)$ becomes

$$\mathbf{f}(z)=\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu}\\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu}\\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}}\\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2}\\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2}\end{bmatrix}\right\}\begin{bmatrix}g_1(z)\\ g_2(z)\\ g_3(z)\end{bmatrix},\quad z\in S_2, \tag{38}$$

the full-field expression of which can be conveniently derived as

$$\mathbf{f}(z)=\sum_{j=1}^{3}\langle g_j(\xi_\alpha)\rangle \times\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu}\\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu}\\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}}\\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2}\\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2}\end{bmatrix}\right\}\mathbf{i}_j,\quad z\in S_2 \tag{39}$$

where $\langle * \rangle$ is a 3×3 diagonal matrix in which each component is varied according to the Greek index α, and

$$\xi_\alpha=\omega_\alpha^{-1}(z_\alpha)=\frac{z_\alpha+\sqrt{z_\alpha^2-a^2(1+p_\alpha^2)}}{1-\mathrm{i}p_\alpha}, \tag{40}$$

$$\mathbf{i}_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T,\ \mathbf{i}_2=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T,\ \mathbf{i}_3=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T. \tag{41}$$

In addition, the induced total in-plane strains ($\varepsilon_{11},\ \varepsilon_{12},\ \varepsilon_{22}$), anti-plane strains ($\varepsilon_{13},\ \varepsilon_{23}$) and rigid body rotation $\varpi_{21}=\frac{1}{2}(u_{2,1}-u_{1,2})$ in $S_0,\ S_1$ and S_2 can be derived from the obtained analytic functions in the three phases as follows:

$$\varepsilon_{11}+\varepsilon_{22}+2\mathrm{i}\varpi_{21}=\frac{1}{\mu}\left[\kappa\phi_0'(z)-\overline{\phi_0'(z)}\right]+\varepsilon_{11}^*+\varepsilon_{22}^*, \tag{42.1}$$

$$\varepsilon_{22}-\varepsilon_{11}+2\mathrm{i}\varepsilon_{12}=\frac{1}{\mu}\left[\bar{z}\phi_0''(z)+\psi_0'(z)\right]+\varepsilon_{22}^*-\varepsilon_{11}^*+2\mathrm{i}\varepsilon_{12}^*, \tag{42.2}$$

$$\varepsilon_{23}+\mathrm{i}\varepsilon_{13}=\frac{h_0'(z)}{2\mu}+\varepsilon_{23}^*+\mathrm{i}\varepsilon_{13}^*,\ z\in S_0, \tag{42.3}$$

$$\varepsilon_{11}+\varepsilon_{22}+2\mathrm{i}\varpi_{21}=\frac{1}{\mu}\left[\kappa\phi_1'(z)-\overline{\phi_1'(z)}\right], \tag{43.1}$$

$$\varepsilon_{22}-\varepsilon_{11}+2\mathrm{i}\varepsilon_{12}=\frac{1}{\mu}\left[\bar{z}\phi_1''(z)+\psi_1'(z)\right], \tag{43.2}$$

$$\varepsilon_{23}+\mathrm{i}\varepsilon_{13}=\frac{h_1'(z)}{2\mu},\ z\in S_1, \tag{44}$$

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12}+\varpi_{21} & 2\varepsilon_{13}\end{bmatrix}^T=\mathbf{A}\mathbf{f}'(z)+\bar{\mathbf{A}}\overline{\mathbf{f}'(z)},$$
$$\begin{bmatrix}\varepsilon_{12}-\varpi_{21} & \varepsilon_{22} & 2\varepsilon_{23}\end{bmatrix}^T=\mathbf{A}\langle p_\alpha\rangle\mathbf{f}'(z)+\bar{\mathbf{A}}<\bar{p}_\alpha>\overline{\mathbf{f}'(z)},\ z\in S_2.$$

Consequently, the Eshelby tensors in the three phases $S_0,\ S_1$, and S_2 can be arrived at by substituting Eqs. (26), (27), and (39) into Eqs. (42)–(44).

4 Discussions

4.1 The elastic material occupying S_2 is orthotropic

The Barnett-Lothe tensors for an orthotropic material with its principal axes along the $x_1,\ x_2$, and x_3 directions take the following form [12], [20]:

$$\mathbf{H}=\begin{bmatrix}H_{11} & 0 & 0\\ 0 & H_{22} & 0\\ 0 & 0 & H_{33}\end{bmatrix},\ \mathbf{L}=\begin{bmatrix}L_{11} & 0 & 0\\ 0 & L_{22} & 0\\ 0 & 0 & L_{33}\end{bmatrix},\ \mathbf{S}=\begin{bmatrix}0 & S_{12} & 0\\ S_{21} & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{45}$$

where the components are

$$\begin{array}{l}\sqrt{\gamma}H_{11}=\frac{H_{22}}{\sqrt{\gamma}}=\sqrt{2s_{11}s_{22}(1+\eta)}-\frac{(\sqrt{s_{11}s_{22}}+s_{12})^2}{\sqrt{2s_{11}s_{22}(1+\eta)}},\ H_{33}=\sqrt{s_{44}s_{55}},\\ \frac{L_{11}}{\sqrt{\gamma}}=\sqrt{\gamma}L_{22}=\frac{1}{\sqrt{2s_{11}s_{22}(1+\eta)}},\ L_{33}=\frac{1}{\sqrt{s_{44}s_{55}}},\ \frac{S_{21}}{\sqrt{\gamma}}=-\sqrt{\gamma}S_{12}=\frac{\sqrt{s_{11}s_{22}}+s_{12}}{\sqrt{2s_{11}s_{22}(1+\eta)}}>0,\end{array} \tag{46}$$

with s_{ij} being the reduced elastic compliances, γ and η being two dimensionless parameters defined as

$$\gamma=\sqrt{\frac{s_{22}}{s_{11}}},\ \eta=\frac{2s_{12}+s_{66}}{2\sqrt{s_{11}s_{22}}}>-1. \tag{47}$$

In this case, $l_k,\ m_k,\ n_k,\ (k=1,\ 2,\ 3)$ are explicitly determined as

$$\begin{array}{l}l_1=\frac{2\mu(\kappa+1)(L_{22}-L_{11})}{\Delta},\ l_2=\frac{2L_{11}L_{22}-8\mu L_{11}S_{12}-8\mu^2(S_{21}S_{12}+1)}{\Delta},\quad l_3=0,\\ m_1=\frac{2\kappa^2L_{11}L_{22}+8\kappa\mu L_{11}S_{12}-8\mu^2(S_{21}S_{12}+1)}{\Delta},\ m_2=\frac{2\mu(\kappa+1)(L_{22}-L_{11})}{\Delta},\quad m_3=0,\\ n_1=0,\ n_2=0,\ n_3=\frac{L_{33}-\mu}{L_{33}+\mu}\end{array} \tag{48}$$

where

$$\Delta=[\kappa L_{11}-2\mu(S_{21}-1)][L_{22}-2\mu(S_{12}-1)]+[L_{11}+2\mu(S_{21}+1)][\kappa L_{22}+2\mu(S_{12}+1)]. \tag{49}$$

The above result indicates that $l_k,\ m_k,\ n_k,\ (k=1,\ 2,\ 3)$ are all real numbers when the elastic material in S_2 is orthotropic. In addition, there exist only four independent nonzero coefficients $l_2,\ m_1,\ l_1\ (=m_2)$ and n_3 for an orthotropic medium. This fact implies that the in-plane stresses in S_0 and S_1 depend only on the three real parameters $l_2,\ m_1,\ l_1$, and the anti-plane stresses in S_0 and S_1 depend only on the single real parameter n_3.

Furthermore, if the elastic material occupying S_2 is isotropic, we will have from Eq. (46) that $s_{11}=\frac{1-\nu_2}{2\mu_2},\ s_{12}=\frac{-\nu_2}{2\mu_2},\ L_{11}=L_{22}=\frac{\mu_2}{1-\nu_2},\ L_{33}=\mu_2,\ S_{21}=-S_{12}=\frac{1-2\nu_2}{2(1-\nu_2)}$. As a result, Eq. (45) simplifies to

$$\begin{array}{l}l_1=0,\ l_2=\frac{\mu_2-\mu}{\kappa\mu_2+\mu},\ l_3=0,\\ m_1=\frac{\kappa\mu_2-\kappa_2\mu}{\kappa_2\mu+\mu_2},\ m_2=0,\ m_3=0,\\ n_1=0,\ n_2=0,\ n_3=\frac{\mu_2-\mu}{\mu_2+\mu}.\end{array} \tag{50}$$

l_2 and m_1 in Eq. (50) can also be equivalently expressed in terms of the Dundurs' parameters α and β as

$$\begin{array}{l}m_1=\Lambda=-\frac{\alpha+\beta}{1+\beta},\ l_2=\prod=\frac{\beta-\alpha}{1-\beta},\\ \alpha=\frac{\Gamma(\kappa_2+1)-(\kappa+1)}{\Gamma(\kappa_2+1)+(\kappa+1)},\ \beta=\frac{\Gamma(\kappa_2-1)-(\kappa-1)}{\Gamma(\kappa_2+1)+(\kappa+1)},\\ \Gamma=\frac{\mu}{\mu_2}.\end{array} \tag{51}$$

Λ and Π are just the two mismatch parameters introduced by Suo [21].

4.2 $(\sigma_{11}+\sigma_{22})$ and $(\varepsilon_{11}+\varepsilon_{22})$ at origin

It follows from Eqs. (10), (28), (29), and (42) that the mean stress $(\sigma_{11}+\sigma_{22})$ and $(\varepsilon_{11}+\varepsilon_{22})$ at the origin induced by a thermal inclusion $(\varepsilon_{11}^*=\varepsilon_{22}^*=\varepsilon^*,\ \varepsilon_{12}^*=\varepsilon_{13}^*=\varepsilon_{23}^*=0)$ can be simply given by

$$(\sigma_{11}+\sigma_{22})|_{z=0}=\begin{cases}-\frac{8\mu\varepsilon^*}{\kappa+1}\left[\frac{2(\mathrm{Re}\{l_2\}+|l_2|^2)}{1-|l_2|^2}\frac{A}{\pi a^2}+1\right], & \text{if } z=0\in S_0\\ -\frac{16\mu\varepsilon^*(\mathrm{Re}\{l_2\}+|l_2|^2)}{(\kappa+1)(1-|l_2|^2)}\frac{A}{\pi a^2}, & \text{if } z=0\in S_1,\end{cases} \tag{52}$$

$$(\varepsilon_{11}+\varepsilon_{22})|_{z=0}=\begin{cases}\frac{4\varepsilon^*}{\kappa+1}\left[1-\frac{(\kappa-1)(\mathrm{Re}\{l_2\}+|l_2|^2)}{1-|l_2|^2}\frac{A}{\pi a^2}\right] & \text{if } z=0\in S_0\\ -\frac{4\varepsilon^*(\kappa-1)(\mathrm{Re}\{l_2\}+|l_2|^2)}{(\kappa+1)(1-|l_2|^2)}\frac{A}{\pi a^2} & \text{if } z=0\in S_1.\end{cases} \tag{53}$$

In addition, the induced rigid body rotation at the origin is

$$\varpi_{21}|_{z=0} = -\frac{2A\mathrm{Im}\{l_2\}\,\varepsilon^*}{\pi a^2(1-|l_2|^2)}, \tag{54}$$

which is valid whether $z=0$ is located in S_0 or S_1.

Equations (52)–(54) indicate that $(\sigma_{11}+\sigma_{22})$, $(\varepsilon_{11}+\varepsilon_{22})$ and ϖ_{21} at the origin can be obtained once the area A of the thermal inclusion of any shape and the parameter l_2 are known.

On the other hand, if the origin is located within an Eshelby inclusion undergoing arbitrary eigenstrains, then $(\sigma_{11}+\sigma_{22})$, $(\varepsilon_{11}+\varepsilon_{22})$ and ϖ_{21} at the origin can be determined by using Eqs. (10), (28) and (42) as

$$\begin{aligned}(\sigma_{11}+\sigma_{22})|_{z=0} &= -\frac{8\mu}{\kappa+1}\left[\delta_1+\mathrm{Re}\left\{(\delta_2+\mathrm{i}\delta_3)P'(0)\right\}\right]\\ &+\frac{4\mu A}{\pi a^2(1-|l_2|^2)}\mathrm{Re}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\},\end{aligned} \tag{55}$$

$$\begin{aligned}(\varepsilon_{11}+\varepsilon_{22})|_{z=0} &= \frac{4}{\kappa+1}\delta_1-\frac{2(\kappa-1)}{\kappa+1}\mathrm{Re}\left\{(\delta_2+\mathrm{i}\delta_3)P'(0)\right\}\\ &+\frac{A(\kappa-1)}{\pi a^2(1-|l_2|^2)}\mathrm{Re}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\},\end{aligned} \tag{56}$$

$$\begin{aligned}\varpi_{21}|_{z=0} &= -\mathrm{Im}\left\{(\delta_2+\mathrm{i}\delta_3)P'(0)\right\}\\ &+\frac{A(\kappa+1)}{2\pi a^2(1-|l_2|^2)}\mathrm{Im}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\}.\end{aligned} \tag{57}$$

For example, we consider a hypotrochoidal inclusion described by

$$z=\omega(\xi)=R\left(\xi+\frac{m}{\xi^n}\right)+c,\ R>0,\ 0\le m\le\frac{1}{n}, \tag{58}$$

with $|\xi|=1$ and c a complex constant. $P(z)$ and A can be simply determined as

$$\begin{aligned}P(z) &= \frac{m(z-c)^n}{R^{n-1}}+\bar{c},\\ A &= \pi R^2(1-nm^2).\end{aligned} \tag{59}$$

Thus $(\sigma_{11}+\sigma_{22})$, $(\varepsilon_{11}+\varepsilon_{22})$ and ϖ_{21} at the origin within the hypotrochoidal inclusion are explicitly given by

$$\begin{aligned}(\sigma_{11}+\sigma_{22})|_{z=0} &= -\frac{8\mu}{\kappa+1}\left[\delta_1+\frac{mn}{R^{n-1}}\mathrm{Re}\left\{(-c)^{n-1}(\delta_2+\mathrm{i}\delta_3)\right\}\right]\\ &+\frac{4\mu R^2(1-nm^2)}{a^2(1-|l_2|^2)}\mathrm{Re}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}\right.\\ &\left.-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\},\end{aligned} \tag{60}$$

$$\begin{aligned}(\varepsilon_{11}+\varepsilon_{22})|_{z=0} &= \frac{4}{\kappa+1}\delta_1-\frac{2(\kappa-1)}{\kappa+1}\frac{mn}{R^{n-1}}\mathrm{Re}\left\{(-c)^{n-1}(\delta_2+\mathrm{i}\delta_3)\right\}\\ &+\frac{R^2(1-nm^2)(\kappa-1)}{a^2(1-|l_2|^2)}\mathrm{Re}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1}\right.\\ &\left.-(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\},\end{aligned} \tag{61}$$

$$\varpi_{21}|_{z=0} = -\frac{mn}{R^{n-1}}\mathrm{Im}\left\{(-c)^{n-1}(\delta_2+\mathrm{i}\delta_3)\right\} + \frac{R^2(1-nm^2)(\kappa+1)}{2a^2(1-|l_2|^2)}\mathrm{Im}\left\{\frac{2(l_1+l_2\bar{l}_1)\delta_2+2\mathrm{i}(l_2\bar{l}_1-l_1)\delta_3-4(l_2+|l_2|^2)\delta_1}{\kappa+1} -(l_3+l_2\bar{l}_3)\delta_5+\mathrm{i}(l_2\bar{l}_3-l_3)\delta_4\right\}. \tag{62}$$

4.3 An elliptical thermal inclusion

We now consider an elliptical thermal inclusion with the foci $2d$ described by

$$z=\omega(\xi)=d\left(R\xi+\frac{1}{R\xi}\right)+c \tag{63}$$

with $|\xi|=1,\ R>1$ and c a complex constant.

In this case, $g_1(z),\ g_2(z),\ g_3(z),\ P(z)$ and λ can be explicitly given by

$$\begin{aligned}
&g_1(z)=0,\\
&g_2(z)=\tfrac{4\mu\delta_1}{\kappa+1}\tfrac{R^4-1}{2R^2}\left[\sqrt{(z-c)^2-4d^2}-(z-c)\right]-\tfrac{4\mu A\delta_1(l_2+|l_2|^2)}{\pi(\kappa+1)(1-|l_2|^2)}\tfrac{1}{z},\\
&g_3(z)=0,\\
&P(z)=\tfrac{z-c}{R^2}+\bar{c},\ \lambda=-\tfrac{4\mu A\delta_1(l_2+|l_2|^2)}{\pi(\kappa+1)(1-|l_2|^2)}.
\end{aligned} \tag{64}$$

Substituting the above into Eqs. (26), (27), and (39), we can finally obtain the explicit solutions of the analytic functions defined in the three phases $S_0,\ S_1$ and S_2 as follows:

$$\begin{aligned}
\phi_0(z)&=\frac{2l_2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]-\frac{2\mu\delta_1}{\kappa+1}\left[\frac{2l_2(\bar{l}_2+|l_2|^2)}{1-|l_2|^2}\frac{A}{\pi a^2}+1\right]z,\\
\psi_0(z)+\frac{a^2}{z}\phi_0'(z)&=\frac{2m_2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]\\
&+\frac{4\mu\delta_1}{\kappa+1}\left[\frac{1}{R^2}-\frac{m_2(\bar{l}_2+|l_2|^2)}{1-|l_2|^2}\frac{A}{\pi a^2}\right]z-\frac{2\mu\delta_1}{\kappa+1}\left[\frac{2(l_2+|l_2|^2)}{1-|l_2|^2}\frac{A}{\pi a^2}+1\right]\frac{a^2}{z}+\frac{4\mu\delta_1}{\kappa+1}\left(\bar{c}-\frac{c}{R^2}\right),\\
h_0(z)&=\frac{2n_2\mu\delta_1}{\kappa+1}\frac{R^4-1}{R^2}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]-\frac{4n_2\mu A\delta_1(\bar{l}_2+|l_2|^2)}{\pi a^2(\kappa+1)(1-|l_2|^2)}z,\ z\in S_0,
\end{aligned} \tag{65}$$

$$\begin{aligned}
\phi_1(z)&=\frac{2l_2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]-\frac{4l_2\mu A\delta_1(\bar{l}_2+|l_2|^2)}{\pi a^2(\kappa+1)(1-|l_2|^2)}z,\\
\psi_1(z)+\frac{a^2}{z}\phi_1'(z)&=\frac{2m_2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]-\frac{4m_2\mu A\delta_1(\bar{l}_2+|l_2|^2)}{\pi a^2(\kappa+1)(1-|l_2|^2)}z\\
&+\frac{2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{(z-c)^2-4d^2}-(z-c)\right]-\frac{4\mu A\delta_1(l_2+|l_2|^2)}{\pi(\kappa+1)(1-|l_2|^2)}\frac{1}{z},\\
h_1(z)&=\frac{2n_2\mu\delta_1(R^4-1)}{R^2(\kappa+1)}\left[\sqrt{\left(\frac{a^2}{z}-\bar{c}\right)^2-4d^2}-\left(\frac{a^2}{z}-\bar{c}\right)\right]-\frac{4n_2\mu A\delta_1(\bar{l}_2+|l_2|^2)}{\pi a^2(\kappa+1)(1-|l_2|^2)}z,\ z\in S_1,
\end{aligned} \tag{66}$$

$$\mathbf{f}(z)=\left[\frac{\delta_1(R^4-1)}{2R^2(\kappa+1)}\left\langle\sqrt{(\xi_\alpha-c)^2-4d^2}-(\xi_\alpha-c)\right\rangle-\frac{A\delta_1(l_2+|l_2|^2)}{\pi(\kappa+1)(1-|l_2|^2)}\left\langle\frac{1}{\xi_\alpha}\right\rangle\right]$$

$$\times \left\{ \mathbf{B}^T \begin{bmatrix} \kappa\bar{l}_2 - \bar{m}_2 - 1 \\ \mathrm{i}(\kappa\bar{l}_2 + \bar{m}_2 - 1) \\ 2\mathrm{i}\bar{n}_2 \end{bmatrix} + 2\mu\mathbf{A}^T \begin{bmatrix} -\mathrm{i}(\bar{l}_2 - \bar{m}_2 + 1) \\ \bar{l}_2 + \bar{m}_2 + 1 \\ \bar{n}_2 \end{bmatrix} \right\}, \ z \in S_2 \tag{67}$$

where $A = \pi d^2(R^2 - R^{-2})$.

4.4 A circular inclusion

Finally, we consider a circular inclusion of radius b with its center at $z = e$ on the real axis. In this case, $g_1(z),\ g_2(z),\ g_3(z),\ P(z)$ and $Q(z)$ can be explicitly given by:

(i) If $e \neq 0$,

$$\begin{aligned} g_1(z) &= \frac{2b^2\mu(\delta_2 + \mathrm{i}\delta_3)}{\kappa + 1}\frac{1}{z - e}, \\ g_2(z) &= \frac{2b^2\mu\left[a^2(\delta_2 + \mathrm{i}\delta_3) - 2e^2\delta_1\right]}{e^2(\kappa + 1)}\frac{1}{z - e} + \frac{2\mu(\delta_2 + \mathrm{i}\delta_3)}{\kappa + 1}\left[\frac{b^2(e^2 - a^2)}{e(z - e)^2} + \frac{b^4}{(z - e)^3}\right] \\ &\quad + \left[\lambda - \frac{2a^2b^2\mu(\delta_2 + \mathrm{i}\delta_3)}{e^2(\kappa + 1)}\right]\frac{1}{z}, \\ g_3(z) &= \frac{\mathrm{i}b^2\mu(\delta_4 + \mathrm{i}\delta_5)}{z - e}, \\ P(z) &= e,\ Q(z) = 0. \end{aligned} \tag{68}$$

(ii) If $e = 0$,

$$\begin{aligned} g_1(z) &= \frac{2\mu(\delta_2 + \mathrm{i}\delta_3)}{\kappa + 1}\frac{b^2}{z}, \\ g_2(z) &= \left(\lambda - \frac{4b^2\mu\delta_1}{\kappa + 1}\right)\frac{1}{z} + \frac{2\mu(\delta_2 + \mathrm{i}\delta_3)}{\kappa + 1}\frac{b^2(b^2 - a^2)}{z^3}, \\ g_3(z) &= \mathrm{i}\mu(\delta_4 + \mathrm{i}\delta_5)\frac{b^2}{z}, \\ P(z) &= Q(z) = 0. \end{aligned} \tag{69}$$

Substituting Eq. (68) for the case of $e \neq 0$ or Eq. (69) for the case of $e = 0$ into Eqs. (26), (27), and (39), we can finally obtain the explicit solutions of the analytic functions defined in the three phases $S_0,\ S_1$, and S_2. For example, when $e = 0$, the analytic functions in the three phas es are given by

$$\begin{aligned} \phi_0(z) &= \left[\frac{2l_1b^2\mu(\delta_2 - \mathrm{i}\delta_3) - 4l_2b^2\mu\delta_1}{a^2(\kappa + 1)} + \frac{l_2\bar{\lambda} - \mathrm{i}l_3b^2\mu(\delta_4 - \mathrm{i}\delta_5)}{a^2} - \frac{2\mu\delta_1}{\kappa + 1}\right]z \\ &\quad + \frac{2l_2\mu b^2(b^2 - a^2)(\delta_2 - \mathrm{i}\delta_3)}{a^6(\kappa + 1)}z^3, \end{aligned} \tag{70.1}$$

$$\begin{aligned} \psi_0(z) &= \left[\frac{2b^2\mu\left[m_1a^2 - 3l_2(b^2 - a^2)\right](\delta_2 - \mathrm{i}\delta_3) - 4m_2a^2b^2\mu\delta_1}{a^4(\kappa + 1)}\right. \\ &\qquad \left. + \frac{m_2\bar{\lambda} - \mathrm{i}m_3b^2\mu(\delta_4 - \mathrm{i}\delta_5)}{a^2} + \frac{2\mu(\delta_2 - \mathrm{i}\delta_3)}{\kappa + 1}\right]z \\ &\quad + \frac{2m_2\mu b^2(b^2 - a^2)(\delta_2 - \mathrm{i}\delta_3)}{a^6(\kappa + 1)}z^3, \end{aligned} \tag{70.2}$$

$$h_0(z) = \left[\frac{2n_1b^2\mu(\delta_2 - \mathrm{i}\delta_3) - 4n_2b^2\mu\delta_1}{a^2(\kappa + 1)} + \frac{n_2\bar{\lambda} - \mathrm{i}n_3b^2\mu(\delta_4 - \mathrm{i}\delta_5)}{a^2} - \mathrm{i}\mu(\delta_4 - \mathrm{i}\delta_5)\right]z$$

$$+\frac{2n_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^3,\ z\in S_0, \tag{71}$$

$$\phi_1(z)=\left[\frac{2l_1b^2\mu(\delta_2-\mathrm{i}\delta_3)-4l_2b^2\mu\delta_1}{a^2(\kappa+1)}+\frac{l_2\bar{\lambda}-\mathrm{i}l_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}\right]z$$
$$+\frac{2l_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^3+\frac{2\mu(\delta_2+\mathrm{i}\delta_3)}{\kappa+1}\frac{b^2}{z}, \tag{72.1}$$

$$\psi_1(z)+\frac{a^2}{z}\phi_1'(z)=\left[\frac{2m_1b^2\mu(\delta_2-\mathrm{i}\delta_3)-4m_2b^2\mu\delta_1}{a^2(\kappa+1)}+\frac{m_2\bar{\lambda}-\mathrm{i}m_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}\right]z$$
$$+\frac{2m_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^3+\left(\lambda-\frac{4b^2\mu\delta_1}{\kappa+1}\right)\frac{1}{z}+\frac{2\mu(\delta_2+\mathrm{i}\delta_3)}{\kappa+1}\frac{b^2(b^2-a^2)}{z^3}, \tag{72.2}$$

$$h_1(z)=\left[\frac{2n_1b^2\mu(\delta_2-\mathrm{i}\delta_3)-4n_2b^2\mu\delta_1}{a^2(\kappa+1)}+\frac{n_2\bar{\lambda}-\mathrm{i}n_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}\right]z$$
$$+\frac{2n_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^3+\mathrm{i}\mu(\delta_4+\mathrm{i}\delta_5)\frac{b^2}{z},\ z\in S_1, \tag{72.3}$$

$$\mathbf{f}(z)=\left\langle\frac{1}{\xi_\alpha}\right\rangle\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu}\\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu}\\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}}\\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2}\\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2}\end{bmatrix}\right\}$$
$$\times\begin{bmatrix}\frac{2b^2\mu(\delta_2+\mathrm{i}\delta_3)}{\kappa+1}\\ \lambda-\frac{4b^2\mu\delta_1}{\kappa+1}\\ \mathrm{i}b^2\mu(\delta_4+\mathrm{i}\delta_5)\end{bmatrix}+\frac{b^2(b^2-a^2)(\delta_2+\mathrm{i}\delta_3)}{2(\kappa+1)}\left\langle\frac{1}{\xi_\alpha^3}\right\rangle\left\{\mathbf{B}^T\begin{bmatrix}\kappa\bar{l}_2-\bar{m}_2-1\\ \mathrm{i}(\kappa\bar{l}_2+\bar{m}_2-1)\\ 2\mathrm{i}\bar{n}_2\end{bmatrix}\right.$$
$$\left.+2\mu\mathbf{A}^T\begin{bmatrix}-\mathrm{i}(\bar{l}_2-\bar{m}_2+1)\\ \bar{l}_2+\bar{m}_2+1\\ \bar{n}_2\end{bmatrix}\right\},\quad z\in S_2. \tag{72.4}$$

It is deduced from Eqs. (10), (12), and (70) that the internal stress field inside the circular inclusion $|z|<a$ is

$$\sigma_{11}+\sigma_{22}$$
$$=4\mathrm{Re}\left\{\frac{2l_1b^2\mu(\delta_2-\mathrm{i}\delta_3)-2\mu(a^2+2l_2b^2)\delta_1}{a^2(\kappa+1)}+\frac{l_2\bar{\lambda}-\mathrm{i}l_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}+\frac{6l_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^2\right\}, \tag{73.1}$$

$$\sigma_{22}-\sigma_{11}+2\mathrm{i}\sigma_{12}=\frac{4b^2\mu\left[m_1a^2-3l_2(b^2-a^2)\right](\delta_2-\mathrm{i}\delta_3)-8m_2a^2b^2\mu\delta_1}{a^4(\kappa+1)}+\frac{2m_2\bar{\lambda}-2\mathrm{i}m_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}$$
$$+\frac{4\mu(\delta_2-\mathrm{i}\delta_3)}{\kappa+1}+\frac{12m_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^2+\frac{24l_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}|z|^2, \tag{73.2}$$

$$\sigma_{32}+\mathrm{i}\sigma_{31}=\left[\frac{2n_1b^2\mu(\delta_2-\mathrm{i}\delta_3)-4n_2b^2\mu\delta_1}{a^2(\kappa+1)}+\frac{n_2\bar{\lambda}-\mathrm{i}n_3b^2\mu(\delta_4-\mathrm{i}\delta_5)}{a^2}-\mathrm{i}\mu(\delta_4-\mathrm{i}\delta_5)\right]$$
$$+\frac{6n_2\mu b^2(b^2-a^2)(\delta_2-\mathrm{i}\delta_3)}{a^6(\kappa+1)}z^2,\ z\in S_0. \tag{73.3}$$

It is observed from the above expression that the internal field is non-uniform within the circular inclusion when $b<a$ and $\delta_2-\mathrm{i}\delta_3\neq 0$, and it is uniform either when $b=a$ or $\delta_2=\delta_3=0$. If the material in S_2 is monoclinic with the symmetry plane at $x_3=0$, $n_1=n_2=0$ and $n_3=(L_{33}-\mu)/(L_{33}+\mu)$ with $L_{33}=(s_{44}s_{55}-s_{45}^2)^{-1/2}$. In this case, Eq. (73.3) reads

$$\frac{\sigma_{31}}{\delta_4}=\frac{\sigma_{32}}{\delta_5}=-\mu\left(1+\frac{n_3b^2}{a^2}\right),\ z\in S_0, \tag{74}$$

which indicates that the internal anti-plane stresses inside the inclusion are uniform.

It is deduced from Eqs. (42.1), (43.1), (70.1) and (72.1) that if the imposed eigenstrains are dilatational ($\varepsilon_{11}^* = \varepsilon_{22}^* = \varepsilon^*,\ \varepsilon_{12}^* = \varepsilon_{13}^* = \varepsilon_{23}^* = 0$),

$$\varepsilon_{11} + \varepsilon_{22} = \begin{cases} \frac{4\varepsilon^*}{\kappa+1}\left[1 - \frac{(\kappa-1)(\mathrm{Re}\{l_2\}+|l_2|^2)}{1-|l_2|^2}\frac{b^2}{a^2}\right], & z \in S_0, \\ -\frac{4\varepsilon^*(\kappa-1)(\mathrm{Re}\{l_2\}+|l_2|^2)}{(\kappa+1)(1-|l_2|^2)}\frac{b^2}{a^2}, & z \in S_1, \end{cases} \tag{75}$$

which implies that the sum $(\varepsilon_{11} + \varepsilon_{22})$ is uniformly distributed both in S_0 and S_1. If the elastic stiffnesses in S_2 approach infinity and zero, we respectively have $l_2 = 1/\kappa$ and $l_2 = -1$. In the two extreme cases, Eq. (75) reduces to those by Li et al. [4] for a Dirichlet boundary and Wang et al. [5] for a Neumann boundary.

If S_2 is orthotropic, the Eshelby tensors in S_0 and S_1 can be finally derived from Eqs. (42), (43), (70), and (72) as

$$\begin{aligned} 2(\kappa+1)S^0_{ijmn} &= \left[2 - \kappa + \rho\left(m_1 - \frac{2l_2(\kappa-1)}{1-l_2}\right) + 3l_2\rho(1-\rho)\left[1-(3-\kappa)t^2\right]\right]\delta_{ij}\delta_{mn} \\ &\quad + \left[\kappa - m_1\rho + 3l_2\rho(1-\rho)(2t^2-1)\right](\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) - 6l_2(\kappa-1)\rho(1-\rho)t^2\delta_{ij}r_m r_n \\ &\quad + \rho l_1 Y_{ijmn}, \\ S^0_{i3j3} &= \frac{1-n_3\rho}{4}\delta_{ij},\ z \in S_0, \end{aligned} \tag{76}$$

$$\begin{aligned} \frac{2(\kappa+1)}{\rho}S^1_{ijmn} &= \left[\frac{\kappa-5}{t^2} + m_1 - \frac{2l_2(\kappa-1)}{1-l_2} + \frac{9\rho}{t^4} + 3l_2(1-\rho)\left[1-(3-\kappa)t^2\right]\right]\delta_{ij}\delta_{mn} \\ &\quad + \left[\frac{2}{t^2} - m_1 - \frac{3\rho}{t^4} + 3l_2(1-\rho)(2t^2-1)\right](\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\ &\quad + \left[\frac{10-2\kappa}{t^2} - \frac{12\rho}{t^4} + 6l_2(\kappa-1)(\rho-1)t^2\right]\delta_{ij}r_m r_n \\ &\quad + \frac{4}{t^2}\left(1 - \frac{3\rho}{t^2}\right)\delta_{mn}r_i r_j + \frac{8}{t^2}\left(\frac{3\rho}{t^2} - 2\right)r_i r_j r_m r_n + l_1 Y_{ijmn}, \\ S^1_{i3j3} &= \frac{\rho}{4}\left(\frac{1}{t^2} - n_3\right)\delta_{ij} - \frac{\rho}{2t^2}r_i r_j,\ z \in S_1, \end{aligned} \tag{77}$$

where l_2 and m_1 have been given by Eq. (50), δ_{ij} is the Kronecker delta, and

$$\begin{aligned} Y_{1111} &= \frac{\kappa+1-l_1}{1-l_2} + 3(1-\rho)t^2(r_1^2 - r_2^2),\ Y_{2211} = \frac{\kappa-3+l_1}{1-l_2} - 3(1-\rho)t^2(r_1^2 - r_2^2), \\ Y_{1122} &= -\frac{\kappa-3-l_1}{1-l_2} - 3(1-\rho)t^2(r_1^2 - r_2^2),\ Y_{2222} = -\frac{\kappa+1+l_1}{1-l_2} + 3(1-\rho)t^2(r_1^2 - r_2^2), \end{aligned} \tag{78}$$

$$Y_{1212} = \frac{l_1}{1+l_2} + 3(1-\rho)t^2(r_1^2 - r_2^2),\ Y_{1112} = -Y_{2212} = -Y_{1211} = Y_{1222} = 6(1-\rho)t^2 r_1 r_2, \tag{79}$$

$$\rho = \frac{b^2}{a^2} \le 1,\ t = \frac{|z|}{a},\ r_i = \frac{x_i}{|z|}.$$

It is observed from Eqs. (50), (76), and (77) that both S^0_{ijmn} and S^1_{ijmn} are fourth-order radial isotropic tensors [4], [5] when the material in S_2 is isotropic with $l_1 = 0$; S^0_{i3j3} and S^1_{i3j3} are second-order radial isotropic tensors even when S_2 is orthotropic. It is easily checked that when the elastic stiffnesses in S_2 approach infinity (i.e., $l_2 = 1/\kappa,\ m_1 = \kappa,\ l_1 = 0$) and approach zero (i.e., $l_2 = m_1 = -1,\ l_1 = 0$), the in-plane components in Eqs. (76) and (77), respectively, reduce to the Dirichlet-Eshelby tensors in Li et al. [4] and Neumann-Eshelby tensors in Wang et al. [5].

In addition, the average interior Eshelby tensor can be expediently obtained from Eq. (76) as

$$\left\langle S^0_{ijmn}\right\rangle_{S_0} = \frac{1}{\kappa+1}\left[1-\frac{l_2(\kappa-1)\rho}{1-l_2}\right]\delta_{ij}\delta_{mn} \quad (80.1)$$

$$+\frac{\kappa-m_1\rho-3l_2\rho(1-\rho)^2}{2(\kappa+1)}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\delta_{ij}\delta_{mn})+\frac{\rho l_1}{2(\kappa+1)}\left\langle Y_{ijmn}\right\rangle_{S_0},$$

$$\left\langle S^0_{i3j3}\right\rangle_{S_0} = \frac{1-n_3\rho}{4}\delta_{ij}, \quad (80.2)$$

with

$$\langle Y_{1111}\rangle_{S_0} = \frac{\kappa+1-l_1}{1-l_2},\ \langle Y_{2211}\rangle_{S_0} = \frac{\kappa-3+l_1}{1-l_2},$$
$$\langle Y_{1122}\rangle_{S_0} = -\frac{\kappa-3-l_1}{1-l_2},\ \langle Y_{2222}\rangle_{S_0} = -\frac{\kappa+1+l_1}{1-l_2},\ \langle Y_{1212}\rangle_{S_0} = \frac{l_1}{1+l_2}, \quad (81)$$
$$\langle Y_{1112}\rangle_{S_0} = \langle Y_{2212}\rangle_{S_0} = \langle Y_{1211}\rangle_{S_0} = \langle Y_{1222}\rangle_{S_0} = 0.$$

Interestingly, $\left\langle Y_{ijmn}\right\rangle_{S_0}$ is just the value of Y_{ijmn} at the origin z=0 and is independent of ρ. It is observed from Eq. (80.1) that the average $\left\langle S^0_{ijmn}\right\rangle_{S_0}$ is no longer an isotropic tensor if $l_1 \neq 0$ for an orthotropic material in S_2, and it becomes an isotropic tensor if the material in S_2 is isotropic. By letting the elastic stiffnesses in S_2 approach infinity and zero, Eq. (80.1) just reduces to the corresponding results in Li et al. [4] and Wang et al. [5].

5 The dislocation problem

The solution method presented in Sect. 3 can also be used to solve the problem of a line dislocation with Burgers vector $\mathbf{b} = \left[b_1\ b_2\ b_3\right]^T$ located at $z = z_0 = x_0 + iy_0$ within an isotropic circular cylinder of radius a perfectly bonded to a generally anisotropic infinite medium (see Fig. 2). In this case, the three analytic

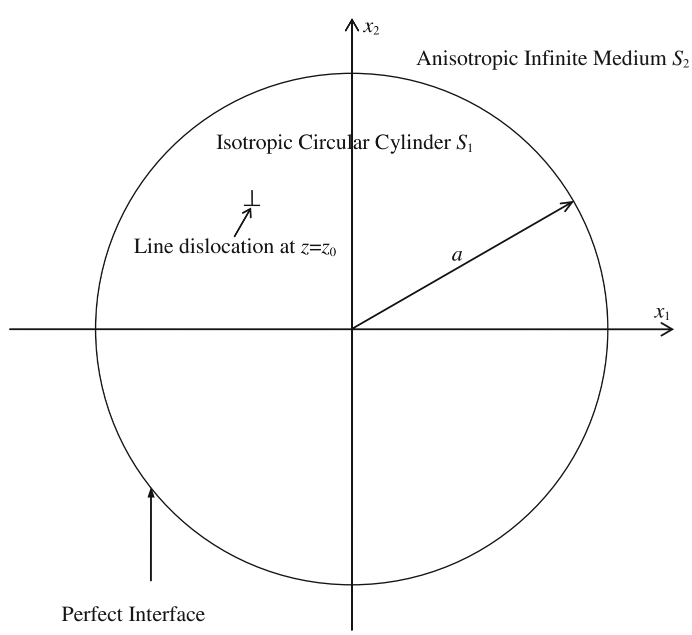

Fig. 2 A line dislocation located at $z = z_0$ in an isotropic circular cylinder perfectly bonded to an infinite anisotropic medium

functions $\phi(z),\ \psi(z)$, and $h(z)$ defined in the isotropic circular cylinder, and $\mathbf{f}(z)$ defined in the anisotropic medium can be expediently given by

$$\begin{aligned}
\phi(z) &= l_1\bar{g}_1(a^2/z) + l_2\bar{g}_2(a^2/z) + l_3\bar{g}_3(a^2/z) \\
&\quad + g_1(z) + \frac{\mu\,[2b_2(l_1+l_2) + 2\mathrm{i}b_1(l_1-l_2) + b_3 l_3(\kappa+1)]}{2\pi(\kappa+1)}\ln z, \\
\psi(z) + \frac{a^2}{z}\phi'(z) &= m_1\bar{g}_1(a^2/z) + m_2\bar{g}_2(a^2/z) + m_3\bar{g}_3(a^2/z) + g_2(z) \\
&\quad + \frac{\mu\,[2b_2(m_1+m_2) + 2\mathrm{i}b_1(m_1-m_2) + b_3 m_3(\kappa+1)]}{2\pi(\kappa+1)}\ln z, \\
h(z) &= n_1\bar{g}_1(a^2/z) + n_2\bar{g}_2(a^2/z) + n_3\bar{g}_3(a^2/z) + g_3(z) \\
&\quad + \frac{\mu\,[2b_2(n_1+n_2) + 2\mathrm{i}b_1(n_1-n_2) + b_3 n_3(\kappa+1)]}{2\pi(\kappa+1)}\ln z,\ z\in S_1,
\end{aligned} \tag{82}$$

$$\begin{aligned}
\mathbf{f}(z) &= \frac{\mathbf{B}^T\mathbf{q}}{2\pi\mathrm{i}}\ln z \\
&\quad + \left\{\mathbf{B}^T\begin{bmatrix} \frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu} \\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu} \\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu} \end{bmatrix} + \mathbf{A}^T\begin{bmatrix} \frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}} \\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2} \\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2} \end{bmatrix}\right\}\begin{bmatrix} g_1(z) \\ g_2(z) \\ g_3(z) \end{bmatrix}, \quad z\in S_2
\end{aligned} \tag{83}$$

where $l_k,\ m_k,\ n_k,\ (k=1,\ 2,\ 3)$ have been given by Eq. (37); $g_1(z),\ g_2(z)$ and $g_3(z)$ are the singular parts of $\phi(z),\ \psi(z)+\frac{a^2}{z}\phi'(z)$ and $h(z)$, and are specifically given by

$$\begin{aligned}
g_1(z) &= \tfrac{\mu(b_2-\mathrm{i}b_1)}{\pi(\kappa+1)}\ln(z-z_0), \\
g_2(z) &= \tfrac{\mu(b_2+\mathrm{i}b_1)}{\pi(\kappa+1)}\ln(z-z_0) + \tfrac{\mu(b_2-\mathrm{i}b_1)}{\pi(\kappa+1)}\tfrac{a^2-|z_0|^2}{z_0}\tfrac{1}{z-z_0} + \tfrac{a^2\phi'(0)}{z}, \\
g_3(z) &= \tfrac{\mu b_3}{2\pi}\ln(z-z_0).
\end{aligned} \tag{84}$$

By using the asymptotic condition at infinity: $\mathbf{f}(z) = \frac{\mathbf{B}^T\mathbf{b}}{2\pi\mathrm{i}}\ln z$ as $|z|\to\infty$, the complex constant vector $\mathbf{q}$ in Eq. (83) can be uniquely determined as

$$\begin{aligned}
\mathbf{q} &= \begin{bmatrix} \frac{\kappa(\bar{l}_2-\bar{l}_1)+\bar{m}_1-\bar{m}_2+\kappa+1}{2(\kappa+1)} & \frac{\mathrm{i}[-\kappa(\bar{l}_1+\bar{l}_2)+\bar{m}_1+\bar{m}_2-\kappa+1]}{2(\kappa+1)} & \frac{\mathrm{i}(\bar{m}_3-\kappa\bar{l}_3)}{4} \\ \frac{\mathrm{i}[\kappa(\bar{l}_2-\bar{l}_1)-\bar{m}_1+\bar{m}_2+\kappa-1]}{2(\kappa+1)} & \frac{\kappa(\bar{l}_1+\bar{l}_2)+\bar{m}_1+\bar{m}_2+\kappa+1}{2(\kappa+1)} & \frac{\kappa\bar{l}_3+\bar{m}_3}{4} \\ \frac{\mathrm{i}(\bar{n}_2-\bar{n}_1)}{\kappa+1} & \frac{\bar{n}_1+\bar{n}_2}{\kappa+1} & \frac{\bar{n}_3+1}{2} \end{bmatrix}\mathbf{b} \\
&\quad -\mu(\mathbf{L}^{-1}+\mathrm{i}\mathbf{S}\mathbf{L}^{-1})\begin{bmatrix} \frac{\bar{l}_2-\bar{l}_1+\bar{m}_1-\bar{m}_2+2}{\kappa+1} & \frac{\mathrm{i}(-\bar{l}_1-\bar{l}_2+\bar{m}_1+\bar{m}_2)}{\kappa+1} & \frac{\mathrm{i}(\bar{m}_3-\bar{l}_3)}{2} \\ \frac{\mathrm{i}(\bar{l}_2-\bar{l}_1-\bar{m}_1+\bar{m}_2)}{\kappa+1} & \frac{\bar{l}_1+\bar{l}_2+\bar{m}_1+\bar{m}_2+2}{\kappa+1} & \frac{\bar{l}_3+\bar{m}_3}{2} \\ \frac{\mathrm{i}(\bar{n}_2-\bar{n}_1)}{\kappa+1} & \frac{\bar{n}_1+\bar{n}_2}{\kappa+1} & \frac{\bar{n}_3+1}{2} \end{bmatrix}\mathbf{b}.
\end{aligned} \tag{85}$$

The appearance of the additional logarithmic terms in Eq. (82) is to ensure that $\phi(z),\ \psi(z)$ and $h(z)$ are analytic at $z=0$. By satisfaction of the consistency condition of $\phi'(0)$, we obtain the following equation:

$$\begin{aligned}
\phi'(0) - l_2\overline{\phi'(0)} &= -\frac{\mu\,[2b_2(l_1+l_2) + 2\mathrm{i}b_1(l_1-l_2) + l_3 b_3(\kappa+1)]}{2\pi(\kappa+1)}\frac{\bar{z}_0}{a^2} \\
&\quad + \frac{\mu l_2(b_2+\mathrm{i}b_1)}{\pi(\kappa+1)}\frac{a^2-|z_0|^2}{a^2\bar{z}_0} - \frac{\mu(b_2-\mathrm{i}b_1)}{\pi(\kappa+1)}\frac{1}{z_0},
\end{aligned} \tag{86}$$

from which $\phi'(0)$ can be uniquely determined.

The full-field expression of $\mathbf{f}(z)$ can be obtained from Eq. (83) as

$$\mathbf{f}(z)=\frac{1}{2\pi \mathrm{i}}\langle \ln \xi_\alpha\rangle \mathbf{B}^T\mathbf{q}+\sum_{j=1}^{3}\langle g_j(\xi_\alpha)\rangle \\ \times\left\{\mathbf{B}^T\begin{bmatrix}\frac{\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mu} & \frac{\kappa\bar{l}_2-\bar{m}_2-1}{4\mu} & \frac{\kappa\bar{l}_3-\bar{m}_3}{4\mu}\\ \frac{-\kappa\bar{l}_1-\bar{m}_1+\kappa}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_2-\bar{m}_2+1}{4\mathrm{i}\mu} & \frac{-\kappa\bar{l}_3-\bar{m}_3}{4\mathrm{i}\mu}\\ \frac{-\bar{n}_1}{2\mathrm{i}\mu} & \frac{-\bar{n}_2}{2\mathrm{i}\mu} & \frac{-\bar{n}_3+1}{2\mathrm{i}\mu}\end{bmatrix}+\mathbf{A}^T\begin{bmatrix}\frac{\bar{l}_1-\bar{m}_1-1}{2\mathrm{i}} & \frac{\bar{l}_2-\bar{m}_2+1}{2\mathrm{i}} & \frac{\bar{l}_3-\bar{m}_3}{2\mathrm{i}}\\ \frac{\bar{l}_1+\bar{m}_1+1}{2} & \frac{\bar{l}_2+\bar{m}_2+1}{2} & \frac{\bar{l}_3+\bar{m}_3}{2}\\ \frac{\bar{n}_1}{2} & \frac{\bar{n}_2}{2} & \frac{\bar{n}_3+1}{2}\end{bmatrix}\right\}\mathbf{i}_j,\quad z\in S_2. \tag{87}$$

If the dislocation is located on the real axis $z_0=x_0,\ (|x_0|<a)$ and the surrounding medium is orthotropic, the stress components along the real axis in the circular cylinder can be finally derived as

$$\frac{\pi(\kappa+1)}{\mu b_1}\sigma_{12}=\frac{2}{x-x_0}+\frac{x_0(2l_1-l_2-m_1)}{a^2-x_0x}+\frac{(a^2-x_0^2)\left[l_1(a^2-x_0^2)-2a^2l_2\right]}{x_0(a^2-x_0x)^2} \\ +\frac{2l_2a^2(a^2-x_0^2)^2}{x_0(a^2-x_0x)^3}-\frac{l_1}{x_0}+\frac{x_0l_1^2}{a^2(1+l_2)}, \tag{88}$$

$$\frac{\pi(\kappa+1)}{\mu b_2}\sigma_{22}=\frac{2}{x-x_0}-\frac{x_0(l_2+m_1+2l_1)}{a^2-x_0x}+\frac{(a^2-x_0^2)\left[l_1(a^2-x_0^2)-2l_2x_0^2\right]}{x_0(a^2-x_0x)^2} \\ +\frac{2l_2a^2(a^2-x_0^2)^2}{x_0(a^2-x_0x)^3}-\frac{l_1+2l_2}{x_0}-\frac{x_0(l_1+2l_2)^2}{a^2(1-l_2)}, \tag{89}$$

$$\frac{\pi(\kappa+1)}{\mu b_2}\sigma_{11}=\frac{2}{x-x_0}+\frac{x_0(m_1-3l_2-2l_1)}{a^2-x_0x}+\frac{(a^2-x_0^2)\left[-l_1(a^2-x_0^2)+2l_2(2a^2+x_0^2)\right]}{x_0(a^2-x_0x)^2} \\ -\frac{2l_2a^2(a^2-x_0^2)^2}{x_0(a^2-x_0x)^3}+\frac{l_1-2l_2}{x_0}+\frac{x_0(l_1^2-4l_2^2)}{a^2(1-l_2)}, \tag{90}$$

$$\sigma_{32}=\frac{\mu b_3}{2\pi}\left(\frac{1}{x-x_0}-\frac{n_3x_0}{a^2-x_0x}\right),\ \sigma_{31}=0, \tag{91}$$

where $x=x_1,\ (|x|<a)$, and the four real coefficients $l_1,\ l_2,\ m_1$ and n_3 are given by Eq. (48).

Thus, the image force acting on the line dislocation is obtained as [22]

$$F_1=\frac{\mu b_1^2x_0}{\pi(\kappa+1)}\left[\frac{2l_1-l_2-m_1}{a^2-x_0^2}+\frac{l_1^2}{a^2(1+l_2)}\right] \\ -\frac{\mu b_2^2x_0}{\pi(\kappa+1)}\left[\frac{l_2+m_1+2l_1}{a^2-x_0^2}+\frac{(2l_2+l_1)^2}{a^2(1-l_2)}\right]-\frac{\mu b_3^2n_3x_0}{2\pi(a^2-x_0^2)}, \tag{92.1}$$

$$F_2=\frac{2\mu b_1b_2x_0}{\pi(\kappa+1)}\frac{2l_2^2(1+l_2)-l_1^2}{a^2(1-l_2^2)} \tag{92.2}$$

where the force components F_1 and F_2 are along the x_1 and x_2 directions. F_2 is nonzero only when $b_1b_2x_0\neq 0$ and is independent of the parameter m_1. If the dislocation is very close to the interface (i.e., $x_0\to a$), Eq. (92.1) reads

$$F_1=-\frac{\mu(l_2+m_1)(b_1^2+b_2^2)}{2\pi\delta(\kappa+1)}+\frac{\mu l_1(b_1^2-b_2^2)}{\pi\delta(\kappa+1)}-\frac{\mu b_3^2n_3}{4\pi\delta} \tag{93}$$

where $\delta=a-x_0$. Equation (93) recovers Eq. (8.9–9) in Ting [12] if the upper half-space is isotropic while the lower half-space is orthotropic. The orthotropic property can be used to make $F_1\equiv 0$ in Eq. (93). For example, if the Burgers vector only contains the b_1 component and $2l_1=l_2+m_1$, then $F_1\equiv 0$. The condition $2l_1=l_2+m_1$ can be more specifically expressed by

$$\frac{1}{\sqrt{\gamma}}-\sqrt{\gamma}=\frac{\left[1+2\mu(\sqrt{s_{11}s_{22}}+s_{12})\right]^2+\left[\kappa-2\mu(\sqrt{s_{11}s_{22}}+s_{12})\right]^2}{2(\kappa+1)\mu\sqrt{2s_{11}s_{22}(1+\eta)}}-\frac{4\mu\sqrt{2s_{11}s_{22}(1+\eta)}}{\kappa+1}, \quad (94)$$

which can be solved to arrive at the ratio $\gamma=\sqrt{s_{22}/s_{11}}$ for given values of $s_{11}s_{22},\ s_{12},\ \eta,\ \kappa$ and μ.

If the dislocation is located on the imaginary axis $z_0=\mathrm{i}y_0,\ (|y_0|<a)$ and the surrounding medium is orthotropic, the image force is

$$F_2=\frac{\mu b_2^2 y_0}{\pi(\kappa+1)}\left[-\frac{l_2+m_1+2l_1}{a^2-y_0^2}+\frac{l_1^2}{a^2(1+l_2)}\right]-\frac{\mu b_1^2 y_0}{\pi(\kappa+1)}\left[\frac{l_2+m_1-2l_1}{a^2-y_0^2}+\frac{(2l_2-l_1)^2}{a^2(1-l_2)}\right]-\frac{\mu b_3^2 n_3 y_0}{2\pi(a^2-y_0^2)}, \quad (95.1)$$

$$F_1=\frac{2\mu b_1 b_2 y_0}{\pi(\kappa+1)}\frac{2l_2^2(1+l_2)-l_1^2}{a^2(1-l_2^2)}. \quad (95.2)$$

Equation (95) is obtained from Eq. (92) by using the following substitutions:

$$x_0\to y_0,\ l_1\to -l_1,\ b_1\to b_2,\ b_2\to b_1,\ F_1\to F_2,\ F_2\to F_1. \quad (96)$$

It is easily checked that the above expressions (92) and (95) will reduce to those by Dundurs [22] if the surrounding medium is elastically isotropic with $l_1=0$ and $l_2,\ m_1$ being given by Eq. (51). It is observed that an additional real number l_1 will enter the image force expression if the surrounding medium is orthotropic. Considering the values of F_1 in Eq. (92.1) and F_2 in Eq. (95.1) for a dislocation with Burgers vector $\mathbf{b}=\left[b_1\ 0\ 0\right]^T$ close to the circular interface and in the vicinity of center of the cylinder, it is seen that there exist eight different types of the behavior due to the existence of the additional real number l_1. These types are listed below:

(i) $2l_1-l_2-m_1<0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}<0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}<0:$
Stable equilibrium position at center.

(ii) $2l_1-l_2-m_1>0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}<0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}<0:$
Stable equilibrium position at center, unstable equilibrium positions on x_1 and x_2 axes.

(iii) $2l_1-l_2-m_1<0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}>0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}<0:$
Saddle point at center, stable equilibrium position on x_1 axis.

(iv) $2l_1-l_2-m_1<0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}<0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}>0:$
Saddle point at center, stable equilibrium position on x_2 axis.

(v) $2l_1-l_2-m_1>0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}>0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}<0:$
Saddle point at center, unstable equilibrium position on x_2 axis.

(vi) $2l_1-l_2-m_1<0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}>0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}>0:$
Unstable equilibrium position at center, stable equilibrium positions on x_1 and x_2 axes.

(vii) $2l_1-l_2-m_1>0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}<0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}>0:$
Saddle point at center, unstable equilibrium position on x_1 axis.

(viii) $2l_1-l_2-m_1>0,\ 2l_1-l_2-m_1+\frac{l_1^2}{1+l_2}>0,\ 2l_1-l_2-m_1-\frac{(2l_2-l_1)^2}{1-l_2}>0:$
Unstable equilibrium position at center.

Finally, the stress components $\sigma_{12},\ \sigma_{22}$ and σ_{32} given by Eqs. (88), (89), and (91) can be further applied as Green's functions to investigate mode I, II, and III radial cracks in an isotropic circular cylinder bonded to an orthotropic medium.

6 Conclusions

We have derived the elastic field of an inclusion of arbitrary shape embedded in a circular isotropic elastic domain perfectly bonded to the surrounding anisotropic medium. The analytic functions in all the three phases characterizing the elastic field are given by Eqs. (26), (27), and (39) with the appearing coefficients $l_k,\ m_k,\ n_k,\ (k = 1,\ 2,\ 3)$ determined by Eq. (37). When the surrounding medium is orthotropic, these coefficients are all real numbers and are explicitly given by Eq. (48). We also present in Eqs. (76) and (77) the Eshelby tensors in S_0 and S_1 for a circular inclusion concentrically embedded in an isotropic circular domain perfectly bonded to an infinite orthotropic medium. By letting the elastic stiffnesses in S_2 approach infinity and zero, these Eshelby tensors just recover the Dirichlet-Eshelby tensors in [4] and Neumann-Eshelby tensors in [5]. The derived Eshelby tensors can be further applied to the homogenization of composite materials, with the effective elastic property being generally anisotropic.

The problem of a line dislocation located in an isotropic circular cylinder which is perfectly bonded to an infinite anisotropic medium is also investigated by using a similar method. The additional logarithmic terms in the solution structure for a line dislocation are introduced to account for the behavior of the analytic functions at $z = 0$ and $z = \infty$.

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References

1. Eshelby, J.D.: The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. Lond. **A241**, 376–396 (1957)
2. Eshelby, J.D.: The elastic field outside an ellipsoidal inclusion. Proc. R. Soc. Lond. **A252**, 561–569 (1959)
3. Eshelby, J.D.: Elastic inclusions and inhomogeneities. In: Sneddon, N.I., Hill, R. (eds.) Progress in Solid Mechanics, vol. 2, pp. 89–104. North-Holland, Amsterdam (1962)
4. Li, S., Sauer, R., Wang, G.: A circular inclusion in a finite domain I. The Dirichlet-Eshelby problem. Acta Mech. **179**, 67–90 (2005)
5. Wang, G., Li, S., Sauer, R.: A circular inclusion in a finite domain II. The Neumann-Eshelby problem. Acta Mech. **179**, 91–110 (2005)
6. Li, S., Sauer, R.A., Wang, G.: The Eshelby tensors in a finite spherical domain: I. Theoretical formulations. ASME J. Appl. Mech. **74**, 770–783 (2007)
7. Li, S., Wang, G., Sauer, R.A.: The Eshelby tensors in a finite spherical domain: II. Applications in homogenization. ASME J. Appl. Mech. **74**, 784–797 (2007)
8. Sauer, R.A., Wang, G., Li, S.: The composite Eshelby tensors and their applications to homogenization. Acta Mech. **197**, 63–96 (2008)
9. Mejak, G.: Eshelby tensors for a finite spherical domain with an axisymmetric inclusion. Eur. J. Mech. A Solids **30**, 477–490 (2011)
10. Zou, W.N., He, Q.C., Zheng, Q.S.: Inclusions in a finite elastic body. Int. J. Solids Struct. **49**, 1627–1636 (2012)
11. Muskhelishvili, N.I.: Some Basic Problems of the Mathematical Theory of Elasticity. Noordhoff, Groningen (1953)
12. Ting, T.C.T.: Anisotropic Elasticity-Theory and Applications. Oxford University Press, New York (1996)
13. Ru, C.Q.: Analytic solution for Eshelby's problem of an inclusion of arbitrary shape in a plane or half-plane. ASME J. Appl. Mech. **66**, 315–322 (1999)
14. Ru, C.Q., Schiavone, P., Mioduchowski, A.: Elastic fields in two jointed half-planes with an inclusion of arbitrary shape. Z. Angew. Math. Phys. **52**, 18–32 (2001)
15. Wang, X., Sudak, L.J., Ru, C.Q.: Elastic fields in two imperfectly bonded half-planes with a thermal inclusion of arbitrary shape. Z. Angew. Math. Phys. **58**, 488–509 (2007)
16. Christensen, R.M., Lo, K.H.: Solutions for the effective shear properties in three phase sphere and cylinder models. J. Mech. Phys. Solids **27**, 315–330 (1979)
17. Luo, H.A., Weng, G.J.: On Eshelby's inclusion problem in a three-phase spherically concentric solid, and a modification of Mori-Tanaka's method. Mech. Mater. **6**, 347–361 (1987)
18. Luo, H.A., Weng, G.J.: On Eshelby's S-tensor in a three-phase cylindrical concentric solid and the elastic moduli of fibre-reinforced composites. Mech. Mater. **8**, 77–88 (1989)
19. Ting, T.C.T.: Common errors on mapping of nonelliptic curves in anisotropic elasticity. ASME J. Appl. Mech. **67**, 655–657 (2000)
20. Suo, Z.: Singularities, interfaces and cracks in dissimilar anisotropic media. Proc. R. Soc. Lond. **A427**, 331–358 (1990)
21. Suo, Z.: Singularities interacting with singularities and cracks. Int. J. Solids Struct. **25**, 1133–1142 (1989)
22. Dundurs, J.: Elastic interaction of dislocations with inhomogeneities. In: Mura, T. (ed.) Mathematical Theory of Dislocations, pp. 70–115. ASME, New York (1969)