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The inverse problem of Lagrangian mechanics for a non-material volume

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Abstract The appropriate consideration of non-material volumes at the level of analytical mechanics is an ongoing research field. In the present paper, we aim at demonstrating the principle of stationary action that is able to yield the proper form of Lagrange's equation in the context, namely the Lagrange's equation in the form derived by Irschik and Holl (Acta Mech 153(3–4):231–248, 2002). Such issue will here be interpreted as being the inverse problem of Lagrangian mechanics for a non-material volume. The classical method of Darboux (Leçons sur la Théorie Générale des Surfaces. Gauthier-Villars, Paris, 1891) will be used as the solution technique. This means that our discussion will be restricted to the case of a single degree of freedom. Having such principle of stationary action at hand, the corresponding Hamiltonian formalism will be written in accordance with the classical theory. Furthermore, a conservation law will be demonstrated for the time-independent case. At last, two simple examples will be addressed in order to illustrate the applicability of the proposed formulation. The reader may find some mathematical analogies between the upcoming content and that discussed by Casetta and Pesce (Acta Mech, 2013. doi:10.1007/s00707-013-1004-1) in considering the inverse problem of Lagrangian mechanics for Meshchersky's equation. The mathematical formulation which will be outlined in the present paper is thus expected to consistently situate non-material volumes within the classical variational approach of mechanics.

1 Motivation

The inverse problem of Lagrangian mechanics, also named as the inverse problem of calculus of variations, is a very traditional issue of mathematical physics. Solving this problem means to find a principle of stationary action that is able to yield a previously given equation of motion. Fundamental investigations on this matter are due to Helmholtz [1], Darboux [2], Havas [3], Santilli [4] and to many other authors not cited here.

Within the original domain of the variational methods of mechanics, which comprises conservative systems with constant mass (see, e.g., comments in [5, Sect. 1]), the solution of such inverse problem is given by Hamilton's principle, with the Lagrangian equating kinetic energy minus potential energy. However, in a broader context which can involve non-conservative systems and variable-mass systems, this is in general no longer valid.

Of course that, when dealing with more general situations, one can have a variational principle by extending the classical form of Hamilton's principle in order to include non-potential terms, or, 'external terms' as in Santilli's [4] terminology. This refers to the variational principle which naturally arises from time integration, between limiting instants, of the corresponding principle of virtual work. In this type of variational formulation with external terms, one has the important advantage of preserving the original physical significance of the

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involved entities. On the other hand, one has the inconvenience of losing the bond with the classical variational approach.

The inverse problem of Lagrangian mechanics goes in the opposite direction. Its solution offers the possibility of writing a principle of stationary action for more general systems, but one has to deal with the inconvenience that the involved entities do not exhibit direct physical meanings. A motivation for having a system formulated from this latter perspective is that, as argued in [4, p. 8], 'the methodological profile is basically that for systems with forces derivable from a potential, in the sense that the analytic equations, the time evolution law, the underlying algebraic and geometric structure, etc. remain formally unchanged.' This signifies, to a good extent of the term, the prompt use of the mathematical theory which is grounded on the variational methods of mechanics. See also [5, Sect. 1] for other motivational quotations in this sense.

The question is that both formulations, namely that with external terms and that following from the solution of the inverse problem, are to be seen as complementary tools. According to Santilli [4, p. 9], 'it is hoped that a judicious interplay between these two complementary approaches to the same systems will be effective on methodological as well as physical grounds. On the former grounds, certain aspects which are difficult to treat within the context of one approach could be more manageable within the context of the other approach, and vice versa. On the latter grounds, the two complementary approaches could be useful for the identification of the physical significance of the algorithms at hand (...).'

The objective of the present paper is to address the inverse problem of Lagrangian mechanics within the context of the so-called non-material volumes. This kind of system refers to a control volume whose surface moves with a different velocity from the velocity of the material particles found to be instantaneously located at the control surface. Due to such relative motion, flux of mass may occur through the control surface, and then mass is not conserved within the non-material volume.

It is important to be mentioned that the consideration of non-material volumes at the level of analytical mechanics is an ongoing research field. In this sense, the article of Irschik and Holl [6], concerning the derivation of the proper form of Lagrange's equation in the context, has been playing a fundamental role. A noticeable point in the analysis of Irschik and Holl [6] is that, in changing from the material point of view to the non-material point of view, two new terms of surface flux come out in Lagrange's equation. This result has motivated next investigations in the field like, for example, the demonstration of the generalized form of Hamilton's principle which accommodates such new terms in a variational framework (see [7]). With these two fundamental results at hand, one so has the basis of a formulation (with external terms) for the treatment of non-material volumes.

In the present paper, we aim at contributing to the field by outlining the formulation that originates from the solution of the inverse problem. Within the context of non-material volumes, this means the proposition of a formulation which mathematically agrees with the classical variational approach of mechanics.

2 The statement of the problem

Let us consider a continuum set of material particles defining a material body. In using Ritz-type approximations (see [6, Sect. 2, 3] and [8, Chap. 11]), we can express the position vector $\boldsymbol{p}$ of the material particle as a function of its position $\boldsymbol{P}$ in the reference configuration of the body, of time t and of a finite number of generalized coordinates $q_k = q_k(t)$, that is,

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{P}, q_k, t), \tag{1}$$

where the one-to-one relation between $\boldsymbol{p}$ and $\boldsymbol{P}$ is supposed to hold. In consequence, the actual position of the material volume V turns out to be a function depending on such generalized coordinates and on time, that is, $V = V(q_k, t)$. Now, consider the non-material volume V_u. Such a volume can be defined by the continuum set of fictitious particles in the sense of Truesdell and Toupin [9], and it is considered to be instantaneously coincident with some material volume of the continuous body. The position vector $\boldsymbol{r}$ of the fictitious particle is assumed to be dependent on its position $\boldsymbol{R}$ in the reference configuration, on time and on the same set of generalized coordinates, namely

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R}, q_k, t), \tag{2}$$

where the one-to-one relation between $\boldsymbol{r}$ and $\boldsymbol{R}$ is also ensured. In the same manner, one has that $V_u = V_u(q_k, t)$.

Fictitious particles are allowed to move at an arbitrary velocity

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \tag{3}$$

which, in general, is different from the velocity

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{4}$$

of the material particles. Thus, at the bounding surface ∂V_u of V_u, there exists a relative motion occurring between the original material particles and the corresponding fictitious particles. Therefore, V_u can be understood as a non-material volume.

From such a consideration, Irschik and Holl [6] have demonstrated that the proper form of Lagrange's equation for a non-material volume is given by the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k} - \frac{\partial T_u}{\partial q_k} - \int\limits_{\partial V_u} \frac{1}{2}\rho v^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right) \cdot \boldsymbol{n}\mathrm{d}\partial V_u + \int\limits_{\partial V_u} \rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n}\mathrm{d}\partial V_u - Q_k = 0, \tag{5}$$

where $\mathrm{d}/\mathrm{d}t$ is the total time derivative which is considered to be material with respect to the velocity $\boldsymbol{u}$, $v^2 \equiv \boldsymbol{v} \cdot \boldsymbol{v}$, ρ is the volumetric mass density, Q_k is the k-th generalized force instantaneously acting on the material body in V_u, $\boldsymbol{n}$ is the outer normal unit vector at the surface, and $T_u = T_u(\dot{q}_k, q_k, t)$ is the total kinetic energy of the material particles instantaneously included in V_u, that is,

$$T_u = \int\limits_{V_u} \frac{1}{2}\rho v^2 \mathrm{d}V_u. \tag{6}$$

The article of Irschik and Holl [6] contains, for example, a discussion on the involved Ritz-type formulation as well as on following generalizations of Gauβ-Green's divergence theorem and of Reynold's transport theorem.

In the present paper, the inverse problem of Lagrangian mechanics for a non-material volume is defined as the problem of finding the principle of stationary action that leads to Eq. (5). This principle of stationary action is written as

$$\delta \int\limits_{t_1}^{t_2} \tilde{L}_u \mathrm{d}t = 0, \tag{7}$$

where t_1 and t_2 are definite limits, and $\tilde{L}_u$ represents the Lagrangian function to be properly constructed to solve such a problem, that is, $\tilde{L}_u$ has to be such that Eq. (7) yields the Lagrange's equation in the form of Eq. (5). We clarify that the symbol $\sim$ is being used to distinguishably label the $\tilde{L}_u$-Lagrangian with respect to the conventional Lagrangian L, which is normally used in the classical definition (see, e.g., [10, p. 112, Eq. (51.7)]). The subscript u indicates that the function $\tilde{L}_u$ is to be defined in accordance with the velocity $\boldsymbol{u}$ of the non-material control surface.

The crux of the matter is that, within this context of non-material volumes, three external terms may appear in Lagrange's equation (see Eq. (5)): the non-potential force, which occurs when Q_k is non-derivable from a potential, and two terms regarding surface flux, which are stated by the third term and by the fourth term of Eq. (5). These surface flux terms explicitly depend on the velocity $\boldsymbol{u}$ of the control surface and, in general, they are not derivable from a potential.

In the following, we exemplarily restrict our analysis to the case of a system described by a single generalized coordinate. We thus propose to solve the inverse problem in agreement with the classical method of Darboux [2], which means to mathematically construct the $\tilde{L}_u$-Lagrangian from the knowledge of Eq. (5)—written for a single degree of freedom.

3 An analytical and variational formulation for a single degree of freedom non-material volume: the solution via the method of Darboux

From the analytic fundamental theorems of the inverse problem of calculus of variations (see [4, Chap. 3]), one has that the mathematical procedure to have a Lagrangian constructed from the equation of motion is greatly simplified in the case in which this has the form of

$$\ddot{q} + G(q, \dot{q}, t) = 0, \tag{8}$$

where q is the generalized coordinate of a single degree of freedom system. The function $G = G(q, \dot{q}, t)$ appears from a convenient algebraic manipulation of the equation of motion.[1]

It is known that all second-order differential equations in the form of Eq. (8) admit a variational formulation. Under certain continuity and regularity conditions, namely by assuming that G is differentiable and integrable to the degree required, the theory of partial differential equations guarantees the existence of solution of the associated inverse problem (see, e.g., discussion in [3, p. 367, footnote (*)], [4, p. 12 and 139], [11] and [12, p. 72]).

A mathematical procedure that can so be used to find the Lagrangian which leads to Eq. (8) is the method of Darboux [2]. In taking the following indirect representation of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}} - \frac{\partial \tilde{L}}{\partial q} = \Lambda\left(\ddot{q} + G(q, \dot{q}, t)\right), \tag{9}$$

where Λ is the so-called Jacobi last multiplier of Eq. (8) (see, e.g., [3, p. 371], [12, p. 72] and [13]), the method of Darboux asserts that the $\tilde{L}$-Lagrangian which satisfies Eq. (9) is given by

$$\tilde{L} = \int_0^{\dot{q}} (\dot{q} - \omega)\Lambda(q, \omega, t)\mathrm{d}\omega - \int_0^{q} G(\xi, 0, t)\Lambda(\xi, 0, t)\mathrm{d}\xi, \tag{10}$$

with Λ calculated as

$$\Lambda = \exp \int \frac{\partial G(q, \dot{q}, t)}{\partial \dot{q}}\mathrm{d}t. \tag{11}$$

This means that, by inserting Eq. (10) into $\delta \int_{t_1}^{t_2} \tilde{L}\mathrm{d}t = 0$, one properly recovers the right-hand side of Eq. (9).

The derivation of Eqs. (10) and (11) can be found, for example, in [14] and [15].

The possibility of extending this technique to solve the inverse problem within the context of non-material volumes then appears as a consequence of using Ritz-type approximations (see Eqs. (1) and (2)) and of considering only single degree of freedom systems. In order to proceed, we need to transform Eq. (5) into the form of Eq. (8). Let us assume the single degree of freedom case, in which Eqs. (1) and (2) are simplified as

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{P}, q, t), \tag{12}$$

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R}, q, t). \tag{13}$$

From Eqs. (12) and (13), Eqs. (3) and (4) can be written as

$$\boldsymbol{u} = \frac{\partial \boldsymbol{r}}{\partial q}\dot{q} + \frac{\partial \boldsymbol{r}}{\partial t}, \tag{14}$$

$$\boldsymbol{v} = \frac{\partial \boldsymbol{p}}{\partial q}\dot{q} + \frac{\partial \boldsymbol{p}}{\partial t}. \tag{15}$$

We can make use of the identities

$$\frac{\partial \boldsymbol{r}}{\partial q} = \frac{\partial \boldsymbol{u}}{\partial \dot{q}}, \tag{16}$$

$$\frac{\partial \boldsymbol{p}}{\partial q} = \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \tag{17}$$

(see, e.g., [6, Eq. (2.8)]) to alternatively write Eqs. (14) and (15) as

$$\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\dot{q} + \frac{\partial \boldsymbol{r}}{\partial t}, \tag{18}$$

$$\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \dot{q}}\dot{q} + \frac{\partial \boldsymbol{p}}{\partial t}. \tag{19}$$

Substituting Eq. (19) into Eq. (6), the total of kinetic energy of the material particles instantaneously enclosed by V_u becomes

[1] Note that in the simple case of a constant-mass single particle the function G equals the negative of force divided by mass.

$$T_u = \dot{q}^2 \int\limits_{V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u + \dot{q} \int\limits_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \mathrm{d}V_u + \int\limits_{V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \mathrm{d}V_u; \tag{20}$$

and, analogously, from looking at Eqs. (18) and (19), the third term and the fourth term of Eq. (5) turn out to be

$$\begin{aligned} \int\limits_{\partial V_u} \frac{1}{2}\rho v^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u &= \dot{q}^2 \int\limits_{\partial V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u \\ &+ \dot{q} \int\limits_{\partial V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u + \int\limits_{\partial V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u, \end{aligned} \tag{21}$$

$$\begin{aligned} \int\limits_{\partial V_u} \rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} (\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}\partial V_u &= \dot{q}^2 \int\limits_{\partial V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u \\ &+ \dot{q} \int\limits_{\partial V_u} \rho \left[\frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial \boldsymbol{r}}{\partial t}\right) + \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right)\right] \cdot \boldsymbol{n} \mathrm{d}\partial V_u \\ &+ \int\limits_{\partial V_u} \rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial \boldsymbol{r}}{\partial t}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u. \end{aligned} \tag{22}$$

Now, inserting Eqs. (20), (21) and (22) into (5) and assuming that $Q = Q(q, \dot{q}, t)$, we find

$$\ddot{q}\mathcal{A}_u(q,t) + \dot{q}\mathcal{B}_u(q,t) + \dot{q}^2\mathcal{C}_u(q,t) + \mathcal{D}_u(q,t) - Q(q,\dot{q},t) = 0, \tag{23}$$

where the auxiliary functions $\mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u$ and $\mathcal{D}_u$, which are being used only for the sake of a shortened representation of Eq. (23), correspond to

$$\mathcal{A}_u(q,t) = \int\limits_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u, \tag{24}$$

$$\mathcal{B}_u(q,t) = \frac{\partial}{\partial t} \int\limits_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u + \int\limits_{\partial V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial \boldsymbol{r}}{\partial t}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u, \tag{25}$$

$$\mathcal{C}_u(q,t) = \frac{\partial}{\partial q} \int\limits_{V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u + \int\limits_{\partial V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u, \tag{26}$$

$$\begin{aligned} \mathcal{D}_u(q,t) = \frac{\partial}{\partial t} \int\limits_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \mathrm{d}V_u - \frac{\partial}{\partial q} \int\limits_{V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \mathrm{d}V_u - \int\limits_{\partial V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{p}}{\partial t} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u \\ + \int\limits_{\partial V_u} \rho \frac{\partial \boldsymbol{p}}{\partial t} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{p}}{\partial t} - \frac{\partial \boldsymbol{r}}{\partial t}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u. \end{aligned} \tag{27}$$

Note that such functional dependence of $\mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u$ and $\mathcal{D}_u$ is assured by the set of equations (12)–(19). The subscript u is attached to these functions to clearly indicate that they are defined in terms of integrals over the chosen non-material volume and also in terms of integrals over the corresponding (non-material) control surface. In the following, for the sake of accordance with such notion, the subscript u will be attached to any entity that is defined in terms of one or more of $\mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u$ and/or $\mathcal{D}_u$.

Dividing Eq. (23) by $\mathcal{A}_u$, which by hypothesis is a non-vanishing function, the Lagrange's equation for a non-material volume so arises in the required form:

$$\ddot{q} + \frac{\dot{q}\mathcal{B}_u(q,t) + \dot{q}^2\mathcal{C}_u(q,t) + \mathcal{D}_u(q,t) - Q(q,\dot{q},t)}{\mathcal{A}_u(q,t)} = 0. \tag{28}$$

Comparing Eqs. (28) and (8), the function G_u is promptly recognized as

$$G_u = \frac{\dot{q}\mathcal{B}_u(q,t) + \dot{q}^2\mathcal{C}_u(q,t) + \mathcal{D}_u(q,t) - Q(q,\dot{q},t)}{\mathcal{A}_u(q,t)}. \tag{29}$$

With Eqs. (28) and (29) at hand, we are able to follow the method of Darboux [2]. The starting point is to write the indirect representation of the problem from Eq. (28), and in analogy with Eq. (9), viz.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}_u}{\partial \dot{q}} - \frac{\partial \tilde{L}_u}{\partial q} = \Lambda_u\left(\ddot{q} + \frac{\dot{q}\mathcal{B}_u(q,t) + \dot{q}^2\mathcal{C}_u(q,t) + \mathcal{D}_u(q,t) - Q(q,\dot{q},t)}{\mathcal{A}_u(q,t)}\right). \tag{30}$$

Calling upon the definition of the Jacobi last multiplier (see Eq. (11)), we have from Eq. (29) that

$$\Lambda_u = \exp\int\left(\frac{\mathcal{B}_u(q,t) + 2\dot{q}\mathcal{C}_u(q,t) - \partial Q(q,\dot{q},t)/\partial\dot{q}}{\mathcal{A}_u(q,t)}\right)\mathrm{d}t. \tag{31}$$

Then, inserting Eqs. (29) and (31) into (10), one has that the $\tilde{L}_u$-Lagrangian which satisfies Eq. (30) is

$$\begin{aligned}\tilde{L}_u = &\int_0^{\dot{q}} (\dot{q} - \omega)\exp\int\left(\frac{\mathcal{B}_u(q,t) + 2\omega\mathcal{C}_u(q,t) - \partial Q(q,\omega,t)/\partial\omega}{\mathcal{A}_u(q,t)}\right)\mathrm{d}t\,\mathrm{d}\omega \\ &- \int_0^{q}\left(\frac{\mathcal{D}_u(\xi,t) - Q(\xi,0,t)}{\mathcal{A}_u(\xi,t)}\right)\exp\int\left(\frac{\mathcal{B}_u(\xi,t) - [\partial Q(\xi,\dot{q},t)/\partial\dot{q}]_{\dot{q}=0}}{\mathcal{A}_u(\xi,t)}\right)\mathrm{d}t\,\mathrm{d}\xi.\end{aligned} \tag{32}$$

This is the solution of the inverse problem of Lagrangian mechanics for a non-material volume which is in agreement with the method of Darboux. Therefore, inserting Eq. (32) into (7), one has at their disposal a principle of stationary action properly connected to the Lagrange's equation for a non-material volume, namely Eq. (5) written for a single degree of freedom.

3.1 The following Hamiltonian formalism and a conservation law

Having such a principle of stationary action at our disposal, we can pursue other fundamental results of the classical theory of mechanics. From the solution of the inverse problem, we define the corresponding canonical momentum as

$$\tilde{p}_u = \frac{\partial \tilde{L}_u}{\partial \dot{q}}, \tag{33}$$

and so, using Eq. (32) in (33), one has that

$$\tilde{p}_u = \int_0^{\dot{q}} \exp\int\left(\frac{\mathcal{B}_u(q,t) + 2\omega\mathcal{C}_u(q,t) - \partial Q(q,\omega,t)/\partial\omega}{\mathcal{A}_u(q,t)}\right)\mathrm{d}t\,\mathrm{d}\omega. \tag{34}$$

Applying the usual Legendre's transformation to $\tilde{L}_u = \tilde{L}_u(q,\dot{q},t)$, that is,

$$\tilde{H}_u = \tilde{p}_u\dot{q} - \tilde{L}_u, \tag{35}$$

and substituting Eqs. (32) and (34) into (35), one finds the $\tilde{H}_u$-Hamiltonian:

$$\begin{aligned}\tilde{H}_u = &\int_0^{\dot{q}} \omega\exp\int\left(\frac{\mathcal{B}_u(q,t) + 2\omega\mathcal{C}_u(q,t) - \partial Q(q,\omega,t)/\partial\omega}{\mathcal{A}_u(q,t)}\right)\mathrm{d}t\,\mathrm{d}\omega \\ &+ \int_0^{q}\left(\frac{\mathcal{D}_u(\xi,t) - Q(\xi,0,t)}{\mathcal{A}_u(\xi,t)}\right)\exp\int\left(\frac{\mathcal{B}_u(\xi,t) - [\partial Q(\xi,\dot{q},t)/\partial\dot{q}]_{\dot{q}=0}}{\mathcal{A}_u(\xi,t)}\right)\mathrm{d}t\,\mathrm{d}\xi.\end{aligned} \tag{36}$$

Given that $\tilde{L}_u$ is defined such that is the Lagrangian of a variational problem, the conventional set of canonical equations accordingly follows from Eq. (35) (see, e.g., [10, Chap. VI]):[2]

$$\dot{q} = \frac{\partial \tilde{H}_u}{\partial \tilde{p}_u}, \tag{37}$$

$$\dot{\tilde{p}}_u = -\frac{\partial \tilde{H}_u}{\partial q}. \tag{38}$$

Equations (37) and (38) then state a 'Hamiltonization' for non-material volumes.

Another aspect of the classical theory that can be pursued is the derivation of conservation laws. As discussed, for instance, in [16, p. 61], if Eq. (7) holds; hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}_u}{\partial \dot{q}}\dot{q} - \tilde{L}_u\right) = -\frac{\partial \tilde{L}_u}{\partial t}. \tag{39}$$

Thus, if $\partial \tilde{L}_u/\partial t = 0$, time integration of Eq. (39) renders

$$\frac{\partial \tilde{L}_u}{\partial \dot{q}}\dot{q} - \tilde{L}_u = \text{const.}, \tag{40}$$

which, seen Eqs. (33) and (35), shows to be mathematically equivalent to

$$\tilde{H}_u = \text{const.} \tag{41}$$

This means that, from Eq. (40) (or 41), one is able to evaluate conservation laws within the context.

4 The general single degree of freedom time-independent case

Let us assume a time-independent situation in which

$$\frac{\partial \boldsymbol{p}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial V_u}{\partial t} = \frac{\partial \rho}{\partial t} = 0, \tag{42}$$

$$Q = -\frac{\partial U(q)}{\partial q}, \tag{43}$$

where $U = U(q)$ is the 'potential energy' in the sense discussed in [10, p. 27-31].[3]

Considering Eq. (42), one obtains the following simplifications of Eqs. (24)–(27):

$$\mathcal{A}_u(q) = \int_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u, \tag{44}$$

$$\mathcal{B}_u = \mathcal{D}_u = 0, \tag{45}$$

$$\mathcal{C}_u(q) = \frac{\partial}{\partial q}\int_{V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u + \int_{\partial V_u} \frac{1}{2}\rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u. \tag{46}$$

Equations (43)–(46) simplify Eqs. (28) and (29) as

$$\ddot{q} + \frac{\dot{q}^2 \mathcal{C}_u(q) + \partial U(q)/\partial q}{A_u(q)} = 0, \tag{47}$$

$$G_u = \frac{\dot{q}^2 \mathcal{C}_u(q) + \partial U(q)/\partial q}{\mathcal{A}_u(q)}. \tag{48}$$

[2] One should read $\dot{\tilde{p}}_u \equiv \mathrm{d}\tilde{p}_u(t)/\mathrm{d}t$.

[3] Notice that our symbol U corresponds to the symbol V in [10, p. 29, Eq. (17.10)]. In the present work, the symbol V is used to define volume.

The indirect representation of the inverse problem is written from substituting Eq. (48) into (9):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}_u}{\partial \dot{q}} - \frac{\partial \tilde{L}_u}{\partial q} = \Lambda_u \left(\ddot{q} + \frac{\dot{q}^2 \mathcal{C}_u(q) + \partial U(q)/\partial q}{\mathcal{A}_u(q)} \right), \tag{49}$$

where, in consonance with the method of Darboux, namely using Eq. (48) in (11), one reads

$$\Lambda_u = \exp \int \left(2\frac{\dot{q}\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}t. \tag{50}$$

Notice that such particular form of Λ_u is a function solely dependent on the generalized coordinate:[4]

$$\Lambda_u(q) = \exp \int \left(2\frac{\dot{q}\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}t = \exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \frac{\mathrm{d}q}{\mathrm{d}t}\mathrm{d}t = \exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q. \tag{51}$$

Making use of the right-hand side of Eqs. (44) and (46) in Eq. (51), Λ_u assumes the form of

$$\Lambda_u(q) = \left(\int\limits_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u \right) \exp \int \left[\frac{\int_{\partial V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}} \right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u}{\int_{V_u} \rho \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}} \mathrm{d}V_u} \right] \mathrm{d}q. \tag{52}$$

Aiming at a shortened representation of the upcoming equations, Eq. (51) will be preferred.

The solution of the problem—Eq. (49) follows from the consideration of Eqs. (48) and (51) in (10), that is,

$$\tilde{L}_u = \frac{1}{2}\left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \dot{q}^2 - \int \left\{ \frac{\partial U(q)/\partial q}{\mathcal{A}_u(q)} \left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \right\} \mathrm{d}q. \tag{53}$$

This signifies that, in the case in which Eqs. (42) and (43) hold, Eq. (7)—with the $\tilde{L}_u$-Lagrangian given by Eq. (53) recovers the right-hand side of Eq. (49).

The associated $\tilde{p}_u$-canonical momentum shows up when introducing Eq. (53) into (33):

$$\tilde{p}_u = \left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \dot{q}. \tag{54}$$

The $\tilde{H}_u$-Hamiltonian results from the usage of Eqs. (53) and (54) in the identity given by Eq. (35):

$$\tilde{H}_u = \frac{1}{2}\left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \dot{q}^2 + \int \left\{ \frac{\partial U(q)/\partial q}{\mathcal{A}_u(q)} \left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \right\} \mathrm{d}q. \tag{55}$$

Having Eqs. (54) and (55), we are able to put the $\tilde{H}_u$-Hamiltonian in terms of the canonical variables $(q, \tilde{p}_u)$:

$$\tilde{H}_u(q, \tilde{p}_u) = \frac{\tilde{p}_u^2}{2\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q} + \int \left\{ \frac{\partial U(q)/\partial q}{\mathcal{A}_u(q)} \left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \right\} \mathrm{d}q. \tag{56}$$

Thus, substituting Eq. (56) into (37) and into (38), one finds the corresponding set of canonical equations:[5,6]

$$\dot{q} = \frac{\tilde{p}_u}{\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q}, \tag{57}$$

$$\dot{\tilde{p}}_u = \frac{\tilde{p}_u^2 \mathcal{C}_u(q)}{\mathcal{A}_u(q) \exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q} - \frac{\partial U(q)/\partial q}{\mathcal{A}_u(q)} \exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q. \tag{58}$$

At last, since $\partial \tilde{L}_u/\partial t = 0$ (see Eq. (53)), we have from Eqs. (39), (40), (41) and (55) that

$$\frac{1}{2}\left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \dot{q}^2 + \int \left\{ \frac{\partial U(q)/\partial q}{\mathcal{A}_u(q)} \left[\exp \int \left(2\frac{\mathcal{C}_u(q)}{\mathcal{A}_u(q)} \right) \mathrm{d}q \right] \right\} \mathrm{d}q = \text{const.}, \tag{59}$$

which appears as a conservation law in the case in which Eqs. (42) and (43) hold.

[4] In fact, $\dot{q} \equiv \mathrm{d}q(t)/\mathrm{d}t$.

[5] Note that, in such a time-independent case, this equation algebraically coincides with the definition of $\tilde{p}_u$.

[6] See footnote 3.

5 Illustrative problems

In order to illustrate the applicability of the proposed variational formulation, two simple problems will be studied in the following.

5.1 Flow of liquid from an open tube

Consider the two-dimensional problem of a straight and vertical tube of constant cross-sectional area which is initially filled with an ideal and incompressible liquid. Suppose that, at a certain instant of time, the full bottom of the tube is suddenly withdrawn, so allowing flux of mass through it. This is a typical open-reservoir problem (see, e.g., [8, p. 506]).[7] In this problem, the control volume has a bounding surface such that it is material with respect to the downward moving level of the liquid and also with respect to the solid wall of the tube, being then closed by a fixed control surface spatially coincident with the open bottom area of the tube. The height of the amount of liquid inside the tube is chosen to be the generalized coordinate $q = q(t)$ of the problem. Given that the liquid is ideal and incompressible, only the gravitational force instantaneously acting on the liquid included in the non-material volume is to be taken into account:

$$Q = -\rho' q g, \tag{60}$$

that is, in considering Eq. (43), one has

$$U = \frac{1}{2}\rho' q^2 g, \tag{61}$$

with ρ' being the (constant) linear mass density of the liquid.

Since the position of the liquid particles inside the tube as well as the position of the control volume do not explicitly depend on time, but on the generalized coordinate q, Eq. (42) also holds. Therefore, we are able to apply the formulation of Sect. 4.

Seen that the velocity $v = \dot{q}$ is common to any of the liquid particles in the non-material volume, Eqs. (44) and (46) are simplified as

$$\mathcal{A}_u(q) = \int\limits_{V_u} \rho \mathrm{d}V_u, \tag{62}$$

$$\mathcal{C}_u(q) = \frac{\partial}{\partial q}\int\limits_{V_u} \frac{1}{2}\rho \mathrm{d}V_u + \int\limits_{\partial V_u} \frac{1}{2}\rho \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right)\cdot \boldsymbol{n}\mathrm{d}\partial V_u. \tag{63}$$

One recognizes that the right-hand side of Eq. (62) is the total of liquid mass in the non-material volume, which can alternatively be given by

$$\mathcal{A}_u(q) = \rho' q. \tag{64}$$

Moreover, evoking Reynolds' transport theorem properly formulated for partial derivatives (see [6, p. 242, Eq. (4.15)])[8] together with the law of conservation of mass in the corresponding material volume V, one has that

$$\frac{\partial}{\partial q}\int\limits_{V} \rho \mathrm{d}V = \frac{\partial}{\partial q}\int\limits_{V_u} \rho \mathrm{d}V_u + \int\limits_{\partial V_u} \rho \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}}\right)\cdot \boldsymbol{n}\mathrm{d}\partial V_u = 0. \tag{65}$$

Looking at Eq. (65), Eq. (63) is immediately reduced to

$$C_u = 0. \tag{66}$$

The associated equation of motion is then obtained by substituting Eqs. (61), (64) and (66) into (47):

[7] In the original problem described in [8, p. 506], the area through which the liquid flows is different from the cross-sectional area of the reservoir. In the present paper, we adopt a simplified version of this problem in which such an area is assumed to be equal to the cross-sectional area of the tube.

[8] In [6, p. 242, Eq. (4.15)], the reader finds Reynolds' transport theorem properly formulated for partial derivatives—written regarding the transport of kinetic energy. Here, in total analogy, we consider the transport of mass.

$$\ddot{q} + g = 0. \tag{67}$$

Note that Eq. (67) agrees with the solution presented in [8, p. 506].[9] In fact, it is the expected result in view of the fact that the liquid mass falls freely.

The indirect representation of the inverse problem is written from inserting Eqs. (61), (64) and (66) into (49), that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}_u}{\partial \dot{q}} - \frac{\partial \tilde{L}_u}{\partial q} = \Lambda_u(\ddot{q} + g), \tag{68}$$

where

$$\Lambda_u = 1 \tag{69}$$

[10]is obtained from the substitution of Eqs. (64) and (66) into (51).

The $\tilde{L}_u$-Lagrangian which solves Eq. (68) is derived by using Eqs. (61), (64) and (66) in (53):

$$\tilde{L}_u = \frac{1}{2}\dot{q}^2 - gq. \tag{70}$$

This implies that, by inserting Eq. (70) into (7), one has a variational formulation for the problem of a liquid flowing from an open tube. Note that such a Lagrangian equation (70), which is derived within the context of non-material volumes, is mathematically equal to the Lagrangian describing the free fall motion of a single point of unitary mass—a typical conservative problem. However, we emphasize that Eq. (70) was demonstrated within the different context of (non-material) control volumes, within which one has to deal with non-conservative terms (surface flux terms) in Lagrange's equation (5).

Substituting Eqs. (61), (64) and (66) into (54), (56)–(59), we accordingly derive the $\tilde{p}_u$-canonical momentum, the $\tilde{H}_u$-Hamiltonian, the set of canonical equations, and the conservation law:[11]

$$\tilde{p}_u = \dot{q}; \tag{71}$$

$$\tilde{H}_u = \frac{1}{2}\tilde{p}_u^2 + gq; \tag{72}$$

$$\dot{q} = \tilde{p}_u, \tag{73}$$

$$\dot{\tilde{p}}_u = -g; \tag{74}$$

$$\frac{1}{2}\dot{q}^2 + gq = \text{const.} \tag{75}$$

The analogy between Eq. (75) and the energy conservation theorem of the free fall motion of a single point of unitary mass is also immediate, as expected.

5.2 The rocket motion

Now we study the problem of the rectilinear motion of a rocket. As demonstrated in [6, Sect. 6], the equation of the rocket motion can be derived from the proper consideration of Lagrange's equation for a non-material volume (see Eq. (5)). In this section, we will briefly recover such a result by following the simplifying assumptions adopted in [6, Sect. 6]. Next, we will address the inverse problem.

Let the velocity $v = v(t)$ of the solid part of the rocket be the generalized velocity of the problem $\dot{q} = \dot{q}(t)$, that is,

$$v = \dot{q}. \tag{76}$$

The control volume is conveniently chosen such that its surface is material with respect to the solid outer surface of the rocket. This surface is closed by the exhaust plane of the rocket, at which propellant is expelled backwards with relative velocity $v_{\mathrm{rel}} = v_{\mathrm{rel}}(t)$. This implies a varying-mass condition, and we consider that the total of mass instantaneously included in the control volume is given by the time-dependent function $m_u = m_u(t)$. Assuming that the material particles instantaneously included in the control volume have the same velocity $v = \dot{q}$, one has that the total of kinetic energy of such particles can be expressed as (see Eq. (6))

[9] See footnote 8.

[10] We emphasize that, in more general situations, Λ_u may not equal unity.

[11] See footnote 6.

$$T_u = \frac{1}{2} m_u(t) \dot{q}^2. \tag{77}$$

Following [6, Sect. 6], one has that

$$\int_{\partial V_u} \frac{1}{2} \rho v^2 \left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} - \frac{\partial \boldsymbol{u}}{\partial \dot{q}_k} \right) \cdot \boldsymbol{n} \mathrm{d}\partial V_u = 0, \tag{78}$$

since the flow of mass through the control surface is assumed to be independent from the generalized velocity $\dot{q}$; and also that

$$\int_{\partial V_u} \rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial \dot{q}_k} (\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n} \mathrm{d}\partial V_u = -\left(\dot{q} - v_{\mathrm{rel}}(t) \right) \dot{m}_u(t), \tag{79}$$

where $\dot{m}_u(t) \equiv \mathrm{d}m_u(t)/\mathrm{d}t$.

Inserting Eqs. (77), (78) and (79) into (5), the equation of the rocket motion shows up:

$$m_u(t)\ddot{q} + \dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t) = 0, \tag{80}$$

where

$$Q = Q(t) \tag{81}$$

is the overall external force acting instantaneously upon the control volume.

Comparing Eqs. (23) and (80), we immediately find that

$$\mathcal{A}_u = m_u(t), \tag{82}$$
$$\mathcal{B}_u = \mathcal{C}_u = 0, \tag{83}$$
$$\mathcal{D}_u = \dot{m}_u(t) v_{\mathrm{rel}}(t). \tag{84}$$

In fact, Eqs. (82), (83) and (84) follow from the consistent simplification of the general expressions for $\mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u$ and $\mathcal{D}_u$ (see Eqs. (24)–(27)). Note that

$$v(\dot{q}) = \dot{q} \tag{85}$$

in V_u.

Looking at Eqs. (19) and (85), we have that in V_u

$$\partial v/\partial \dot{q} = 1, \quad \partial p/\partial t = 0. \tag{86}$$

Furthermore,

$$v(\dot{q}, t) = \dot{q} - v_{\mathrm{rel}}(t) \tag{87}$$

at the exhaust plane.

Comparing Eqs. (19) and (87), we find that at the exhaust plane

$$\partial v/\partial \dot{q} = 1, \quad \partial p/\partial t = -v_{\mathrm{rel}}. \tag{88}$$

Since the rocket moves toward and the burned fuel is backwards expelled, one has

$$(\partial \boldsymbol{v}/\partial \dot{q}) \cdot (\partial \boldsymbol{p}/\partial t) = -v_{\mathrm{rel}}, \quad (\partial \boldsymbol{p}/\partial t) \cdot \boldsymbol{n} = v_{\mathrm{rel}}. \tag{89}$$

Also note that at the exhaust plane

$$\int_{\partial V_u} \rho (\partial \boldsymbol{p}/\partial t) \cdot \boldsymbol{n} \mathrm{d}\partial V_u = \rho v_{\mathrm{rel}} A = -\dot{m}_u(t), \tag{90}$$

where A is the constant exhaust area. We also suppose that $\rho = \mathrm{const}$.

At the material portion of ∂V_u, we have that

$$(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{n} = 0. \tag{91}$$

At last, at any point of ∂V_u, one writes

$$u(\dot{q}) = \dot{q}, \tag{92}$$

which leads to

$$\partial u/\partial \dot{q} = 1 \quad \text{and} \quad \partial r/\partial t = 0 \tag{93}$$

at any point of ∂V_u (see Eq. (18)).

The reader can now verify that Eqs. (82), (83) and (84) follow from using Eqs. (85)–(93) in (24)–(27).

Thus, substituting Eqs. (81)–(84) into (30) and into (31), one writes the indirect representation of the inverse problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}_u}{\partial \dot{q}} - \frac{\partial \tilde{L}_u}{\partial q} = \Lambda_u \left(\ddot{q} + \frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right), \tag{94}$$

with[12]

$$\Lambda_u = 1. \tag{95}$$

According to the formulation of Sect. 3, we insert Eqs. (81)–(84) into (32) in order to find the $\tilde{L}_u$-Lagrangian which solves Eq. (94), that is,

$$\tilde{L}_u = \frac{1}{2}\dot{q}^2 - \left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right) q. \tag{96}$$

The variational formulation of the rocket problem is obtained when considering Eq. (96) in (7). In harmony, the following $\tilde{p}_u$-canonical momentum and $\tilde{H}_u$-Hamiltonian are derived in substituting Eqs. (81)–(84) into (34) and into (36):

$$\tilde{p}_u = \dot{q}, \tag{97}$$

$$\tilde{H}_u = \frac{1}{2}\dot{q}^2 + \left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right) q. \tag{98}$$

The $\tilde{H}_u$-Hamiltonian can be put in terms of canonical variables by substituting Eq. (97) into (98):

$$\tilde{H}_u = \frac{1}{2}\tilde{p}_u^2 + \left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right) q. \tag{99}$$

The set of canonical equations may be achieved from inserting Eq. (99) into Eqs. (37) and (38):[13,14]

$$\dot{q} = \tilde{p}_u, \quad \dot{\tilde{p}}_u = -\left(\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t) - Q(t)}{m_u(t)} \right). \tag{100}$$

Once the $\tilde{L}_u$-Lagrangian (see Eq. (96)) explicitly depends on time, we have a situation in which $\partial \tilde{L}_u/\partial t \neq 0$; hence, a conservation law cannot be stated in the form of Eq. (40) (or (41)), at least not considering the general functions $m_u = m_u(t)$, $v_{\mathrm{rel}} = v_{\mathrm{rel}}(t)$ and $Q = Q(t)$. However, we can attempt to find particular cases of $m_u = m_u(t)$, $v_{\mathrm{rel}} = v_{\mathrm{rel}}(t)$ and $Q = Q(t)$ in which one is able to write a conservation law using Eq. (40). For the sake of this purpose, let us suppose that

$$Q = 0, \tag{101}$$

which means absence of external forces, and also that

$$\frac{\dot{m}_u(t) v_{\mathrm{rel}}(t)}{m_u(t)} = -c, \tag{102}$$

where, conveniently, $c = \text{const.}$, $c > 0$.

[12] See footnote 11.
[13] See footnote 6.
[14] See footnote 3.

Notice that in such a particular case, Eq. (96) is simplified as

$$\tilde{L}_u = \frac{1}{2}\dot{q}^2 + cq \tag{103}$$

and therefore, $\partial \tilde{L}_u / \partial t = 0$. This signifies that, by using Eq. (103) in (40), we find the following conservation law:

$$\frac{1}{2}\dot{q}^2 - cq = \text{const.} \tag{104}$$

Now, we need only to discuss under which conditions Eq. (102) can be satisfied. Let us note that time integration of Eq. (90) yields

$$\int_{m_u(t=0)}^{m_u(t)} \mathrm{d}m_u = -\rho A \int_{t=0}^{t} v_{\text{rel}}(t)\mathrm{d}t, \tag{105}$$

that is,

$$m_u(t) = m_u(t=0) - \rho A \int_{t=0}^{t} v_{\text{rel}}(t)\mathrm{d}t. \tag{106}$$

Next, substituting Eq. (106) into (102), we write

$$\rho A v_{\text{rel}}^2(t) = c m_u(t=0) - \rho A c \int_{t=0}^{t} v_{\text{rel}}(t)\mathrm{d}t, \tag{107}$$

which, from time differentiation, furnishes

$$\frac{\mathrm{d}v_{\text{rel}}}{\mathrm{d}t} = -\frac{1}{2}c, \tag{108}$$

that is,

$$v_{\text{rel}}(t) = v_{\text{rel}}(t=0) - \frac{1}{2}ct. \tag{109}$$

Thus, inserting Eq. (109) into (106), we find that

$$m_u(t) = m_u(t=0) - \rho A v_{\text{rel}}(t=0)t + \frac{1}{4}\rho A c t^2. \tag{110}$$

Additionally, from using Eq. (109) in (107), one obtains

$$c = \frac{\rho A v_{\text{rel}}^2(t=0)}{m_u(t=0)}. \tag{111}$$

In view of Eq. (111), we can rewrite Eqs. (104), (109) and (110) as

$$\frac{1}{2}\dot{q}^2 - \frac{\rho A v_{\text{rel}}^2(t=0)}{m_u(t=0)}q = \text{const.}, \tag{112}$$

$$v_{\text{rel}}(t) = v_{\text{rel}}(t=0) - \frac{1}{2}\frac{\rho A v_{\text{rel}}^2(t=0)}{m_u(t=0)}t, \tag{113}$$

$$m_u(t) = m_u(t=0) - \rho A v_{\text{rel}}(t=0)t + \frac{1}{4}\frac{\rho^2 A^2 v_{\text{rel}}^2(t=0)}{m_u(t=0)}t^2. \tag{114}$$

Equation (112) then states a conservation law for the rocket motion in the particular case in which Eqs. (101) and (102) hold. Note also that time differentiation of Eq. (112) gives

$$\ddot{q} - \frac{\rho A v_{\text{rel}}^2(t=0)}{m_u(t=0)} = 0, \tag{115}$$

which indicates a constant acceleration situation. In fact, Eq. (115) results from Eq. (80), by assuming Eqs. (101), (102) and (111).

As it was demonstrated, the condition given by Eq. (102) can be achieved by imposing Eq. (113) to describe the time evolution law of the relative velocity $v_{\text{rel}} = v_{\text{rel}}(t)$ at which propellant is expelled backwards. Consequently, from the consideration of the continuity equation (see Eq. (90)), the instantaneous mass of the rocket turns out to be given by Eq. (114). We emphasize that such particular results are valid under the condition of $\dot{m}_u(t) < 0$, namely for $t < (2m_u(t=0))/(\rho A v_{\text{rel}}(t=0))$ (see Eq. (114)).

6 Discussion and conclusions

This paper has addressed the inverse problem of Lagrangian mechanics within the context of non-material control volumes. It means that we have addressed the problem of finding a Lagrangian function that, via a principle of stationary action, is able to recover the proper form of Lagrange's equation in the context. Such a form of Lagrange's equation, as demonstrated in [6] within the framework of Ritz's method, contains terms of surface flux. From the perspective of the inverse problem, these terms are mathematically considered as non-potential terms.

We have exemplarily restricted our analysis to single degree of freedom systems, and so the classical method of Darboux could be properly applied to analytically solve the inverse problem. Thus, considering single degree of freedom systems, we could connect the Lagrange's equation in the form derived by Irschik and Holl [6] with a principle of stationary action. Consequently, in following the fundamentals of analytical mechanics, a canonical momentum and a Hamiltonian function were demonstrated in accordance. This has allowed us to write a Hamiltonian formalism for non-material volumes, in which the corresponding set of canonical equations has arisen as in the conventional form.

The case of a non-material volume in which both the position vector of the material particles and the position vector of the fictitious particles do not depend explicitly on time, but only on a single generalized coordinate, has shown up as an important case of the proposed formulation. In this case, it was shown that, when solely a time-independent potential force is assumed to act upon the non-material volume, the Hamiltonian can be put in terms of canonical variables, and a conservation law can also be demonstrated.

Within the classical context of constant-mass potential systems, it is known that, if the kinetic energy and the potential energy are both time-independent, a fundamental quantity, which is interpreted as total energy, remains constant during the motion (see, e.g., [10, p. 31–34]). The inverse problem provides a generalization of this concept. According to the particular formulation of the time-independent case, one so has that: if the total kinetic energy of the material particles instantaneously included in the non-material volume as well as the fluxes of linear momentum and of kinetic energy across the control surface are all time-independent, and if the acting force upon the corresponding material body follows from a time-independent potential, then the motion of the non-material volume, which can in general occur by translation, rotation and/or deformation, is such that it satisfies a conservation law.

At last, two simple problems were considered aiming at illustrating the applicability of the formulation.

We expect to have contributed to the construction of a proper formalism for non-material volumes, in the sense that we have established the initial grounds of a formulation which accommodates such systems within the classical variational approach of mechanics.

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