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Edelen's dissipation potentials and the visco-plasticity of particulate media

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Abstract Following is an elaboration on D. G. B. Edelen's (1972–1973) nonlinear generalization of the classical Rayleigh-Onsager dissipation potentials and the implications for the models of viscoplasticity. A brief derivation is given via standard vector calculus of Edelen's potentials and the associated non-dissipative or "gyroscopic" forces and fluxes. It is also shown that certain extensions of Edelen's formulae can be obtained by means of a recently proposed source-flux relation or "inverse divergence," a generalization of the classical Gauss-Maxwell construct. The Legendre–Fenchel duality of Edelen's potentials is explored, with important consequences for rate-independent friction or plasticity. The use of dissipation potentials serves to facilitate the development of viscoplastic constitutive equations, a point illustrated here by the special cases of Stokesian fluid-particle suspensions and granular media. In particular, we consider inhomogeneous systems with particle migration coupled to gradients in particle concentration, strain rate, and fabric. Employing a mixture-theoretic treatment of Stokesian suspensions, one is able to identify particle stress as the work conjugate of the global deformation of the particle phase. However, in contrast to past treatments, this stress is not assumed to be a privileged driving force for particle migration. A comparison is made with models based on extremal dissipation or entropy production. It is shown that such models yield the correct dissipative components of force or flux but generally fail to capture certain non-dissipative, but mechanically relevant components. The significance of Edelen's gyroscopic forces and their relation to reactive constraints or other reversible couplings is touched upon. When gyroscopic terms are absent, one obtains a class of *strongly dissipative* or *hyperdissipative* materials whose quasi-static mechanics are governed by variational principles based on dissipation potential. This provides an interesting analog to elastostatic variational principles based on strain energy for hyperelastic materials and to the associated material instabilities arising from loss of convexity.

1 Introduction

As an extension of the classical quadratic forms of Rayleigh and Onsager, Edelen gives a mathematical construct of a more general *dissipation potential* [8–11] which applies to any strictly dissipative process, linear or nonlinear. Familiar physical examples are provided by discrete networks of nonlinear electrical resistors, by systems of chemical reactions [3] and by various continuum models of plasticity [6,32,36,49,51], viscoplasticity [19], or fracture [5].

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Dissipation potentials are not only interesting in their own right, but they are also essential to long-standing variational principles of continuum mechanics, [16,17,33], a subject that is revisited in the present work. We recall that the same principles have found application in the field of *homogenization*, i.e., the derivation of effective continuum models for microscopically heterogeneous media. For the special case of strongly dissipative systems considered below, the dissipation potential plays a role analogous to that of strain energy in elastic systems, with variational methods providing bounds on constitutive parameters such as viscosity [29]. This same idea is of course more broadly applicable to the homogenization of plastic and viscoplastic systems.

The existence and derivation of dissipation potentials, dubbed "quasi-potentials" by Maugin [35], are sometimes predicated on extremum principles such as maximal (or extremal) dissipated power and entropy production [49,51] or on related thermodynamic or physical postulates [6,25,35,38,43]. Otherwise, they are assumed as a matter of convenience, as seems the case in the treatise of Hill [26], who cites earlier works on plasticity as motivation, or, finally, they are invoked axiomatically in mathematical treatments such as that of Moreau [37]. By contrast, Edelen's elegant mathematical derivation is virtually free of special assumptions and, as such, it seems to merit much more attention than it has apparently received, particularly in the field of continuum mechanics. Notable exceptions are the treatises of Maugin (Sect. 3) [35], and Eringen [14], and the recent paper of Ostoja-Starzewski and Zubelewicz [39], all of whom acknowledge the work of Edelen [9,10] but without elaboration on its several consequences.

The purpose of the present work is multifold: To provide a straightforward derivation of Edelen's formulae, to explore certain far-reaching consequences of the associated Legendre–Fenchel duality, and to consider selected applications to the mechanics of Stokesian fluid-particle suspensions and granular media. As a sidelight, the Appendix shows how an extension of Edelen's formulae may be derived from a recently proposed source-flux formula [21] or "inverse divergence", which also provides a foundation for the "peridynamics" of Silling and co-workers (and references therein) [30]. In one interesting form of this result, the Green's function for the Laplace operator yields the classical Gauss-Maxwell formula for a flux as gradient of a potential with given source.

By focusing on strictly dissipative systems, we circumvent a broader and long-standing thermodynamic framework [6,24,27,35,38], with conventional thermodynamic potentials (or free energies) depending on dissipative internal variables and their conjugate forces. Without attempting to establish a connection with this approach, we note that small elastic effects in stiff systems may be described by a combination of elastic and dissipative potentials [20].

As mathematical background, a brief derivation of Edelen's formulae is given in Sect. 2, with generalization deferred to the Appendix. Then, in Sect. 3, we show how convexity and duality guarantee the existence of viscoplastic moduli assumed in the previous work [19] and imply certain flow rules for rate-independent plasticity and friction. At the same time, it is shown that Edelen's gyroscopic terms, which represent a failure of generalized Onsager symmetry, lead to asymmetric moduli. A brief discussion is given in Edelen's previous work on the subject [12,10] and the possible significance of such terms.

For most of the discussion of Sects. 1, 2, 3, we suppress Edelen's notation for dependence of forces and fluxes on the local thermodynamic state, which as pointed out by Eringen [14] (Section 2.4) may also be regarded as dependence on thermomechanical history. This is illustrated in Sect. 4 of the present paper by the dependence of force-flux relations on evolutionary parameters such as particle fraction and fabric that depend on past history of deformation.

2 Edelen forms and transforms

In his treatise on exterior calculus [11] (Chapts. 4, 5 & 8), Edelen shows how his theory of dissipation potential follows from a homotopy operation, which also provides the basis for Poincaré's Lemma. A closely related result is given by Eq. (79) in the Appendix of the present paper, and a somewhat more transparent derivation based on standard vector calculus is given[1] in the paragraphs below.

In the discussion immediately following, we employ the notation $\mathbf{x}$, $\mathbf{j}$ for generalized force-flux pairs, in lieu of the Onsager notation $\mathbf{X}$, $\mathbf{J}$ employed by Edelen. As a slight variant of Edelen's work, we deal with various tensor fields $\boldsymbol{\tau}(\mathbf{x})$ over an n-dimensional linear vector space $\mathbf{x} \in \mathbb{X}$ equipped with smooth (i.e., differentiable) norm $||\mathbf{x}||$, which we denote simply by $|\mathbf{x}|$, and with dual space $\mathbb{X}^*$ of linear functions $\mathbb{X} \to \mathbb{R}$.

[1] With all due respect to Edelen's view [10] (p. 76) that the "increased generality and understanding" is worth the time required to acquire facility in exterior calculus.

For the most part. we employ the direct notation for abstract vectors and physical-space tensors, with vectors denoted by lower case Roman font and tensors of arbitrary rank denoted by lower case bold Greek. In the later applications, we denote second-rank tensors in physical space by upper case Roman. When deemed essential for clarity, we employ the notation

$$\boldsymbol{\tau} = \tau^{i_1,i_2,\ldots,i_m}\mathbf{g}_{i_1} \otimes \mathbf{g}_{i_2} \cdots \otimes \mathbf{g}_{i_n} \hat{=} [\tau^{i_1,i_2,\ldots,i_m}] \tag{1}$$

to specify for an mth rank tensor on a general basis $\mathbf{g}_k$. i.e., we suppress notation for the basis and display in square brackets the components, shown here as contravariant. The standard symbols $\otimes$ denotes tensor (or dyadic) products and $\oplus$ the direct sum of vector spaces.

As done in (1), we employ the standard tensor-summation convention. In the conventional way, we can associate contravariant and covariant components, respectively, with tangent and cotangent spaces for $\mathbb{X}$, representing vectors in $\mathbb{X}$ and co-vectors in the dual space $\mathbb{X}^*$ of linear functions. Thus, if $\mathbf{y}\hat{=}[y_i]$ refers to an element of $\mathbb{X}^*$, the dual space of $\mathbb{X}$ with elements $\mathbf{x} \hat{=} [x^i]$, we let dot-product $\mathbf{y}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{y} \hat{=} y_i x^i$ denote a pairing between $\mathbf{x}$ and linear function $\mathbf{y}$, $\mathbb{X} \to \mathbb{R}$. Depending on the setting, $\mathbf{y}$ can be interpreted as the flux of Onsager and Edelen, or, equivalently, as the generalized velocity of others works [49,37], with dissipation given by $\mathfrak{D} = \mathbf{y}\cdot\mathbf{x}$.

Given a vector field $\mathbf{y} = \mathbf{j}(\mathbf{x})$, with $\mathbf{x} \in \mathbb{X}$, $\mathbf{j} \in \mathbb{X}^*$, where $\mathbf{j}$ represents flux and $\mathbf{x}$ conjugate force, then Eqs. (2.15)–(2.16) of Edelen [9] are represented in conventional vector notation as:

$$\mathbf{j}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) + \mathbf{u}(\mathbf{x}), \quad \text{with } \mathbf{x}\cdot\mathbf{u} = 0, \tag{2}$$

a relation which is obtained by taking $m = 1$, $\boldsymbol{\varrho} = \mathbf{j}$, $\boldsymbol{\varphi} = \varphi$, $\boldsymbol{\upsilon} = \mathbf{u}$ in the more general form (79) given in the Appendix. Then, the dissipated non-negative-definite power is given by

$$\mathfrak{D} = \mathbf{j}\cdot\mathbf{x} = \mathbf{x}\cdot\nabla\varphi \geq 0, \tag{3}$$

where equality applies if and only if $\mathbf{x} = \mathbf{0}$.

The roles of $\mathbf{x}$ and $\mathbf{j}$ in the preceding formulae can be reversed, and the relevant conjugate forms are made explicit below. Also, in keeping with the analysis of the preceding section and as done (2), we employ metric-based gradients ∇ in the following, in lieu of the metric-free partial derivatives $\boldsymbol{\partial}_{\mathbf{x}} \hat{=} [\partial/\partial x^i]$ employed by Edelen [8,9].

Referring to Eqs. (2.15)–(2.16) in Edelen [9], as represented here by (2), we can obtain his Eqs. (2.17)–(2.18) for φ, $\mathbf{u}$ by means of the projective decompositions of co-vectors:

$$\begin{aligned}
&\mathbf{j} \equiv \mathbf{e}^*\mathbf{e}\cdot\mathbf{j} + (\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e})\mathbf{j} = \mathbf{e}^* j_r + \mathbf{j}_{\mathbf{e}}, \quad \text{with } j_r = \mathbf{e}\cdot\mathbf{j}, \;\; \mathbf{j}_{\mathbf{e}} = (\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e})\mathbf{j},\\
&\text{and } \nabla\varphi \equiv \mathbf{e}^*\mathbf{e}\cdot\nabla\varphi + (\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e})\nabla\varphi \equiv \mathbf{e}^*\partial_r\varphi + \frac{1}{r}(\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e})\partial_{\mathbf{e}}\varphi,\\
&\text{where } r = |\mathbf{x}|, \;\; \mathbf{e} = \frac{\mathbf{x}}{r}, \;\; \mathbf{e}^* = \nabla r, \;\; \mathbf{e}^*\cdot\mathbf{e} = 1, \;\; \nabla\mathbf{e} = \frac{1}{r}(\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e}),
\end{aligned} \tag{4}$$

with $\mathbf{e}$ defining points on $\mathbb{S} = \{\mathbf{x} : r = 1\}$.

We adopt a general norm $|\mathbf{x}|$ in lieu of the Euclidean form employed by Ziegler [49] and others, which assumes a uniform metric for various force and flux components, such that $\mathbf{e}^* \equiv \mathbf{e}$.

Then, (2) follows from the projections in (4) provided that

$$\mathbf{e}\cdot\nabla\varphi = \frac{\partial\varphi}{\partial r}(\mathbf{x}) \equiv j_r(\mathbf{x}) = \mathbf{e}\cdot\mathbf{j}, \;\; \text{and } \mathbf{u} = (\mathbf{I} - \mathbf{e}^*\otimes\mathbf{e})\cdot(\mathbf{j} - \nabla\varphi), \tag{5}$$

where $\mathbf{u}$ is unique up to the gradient of a scalar. This relation indicates explicitly that $\mathbf{u}$ involves a projection onto tangent planes of the hypersurface r = constant.

Now, the first equation in (5) is satisfied by integration with respect to r at constant $\mathbf{e}$:

$$\varphi(\mathbf{x}) = \int_0^r \mathbf{e}\cdot\mathbf{j}(r'\mathbf{e})\mathrm{d}r' + \varphi_0(\mathbf{e}) = \int_0^1 \mathbf{x}\cdot\mathbf{j}(\lambda\mathbf{x})\mathrm{d}\lambda + \varphi_0(\mathbf{e}), \tag{6}$$

which is equivalent to the second member of (79), with $\boldsymbol{\varrho} = \mathbf{j}$, and identical with Eq. (2.17) of Edelen [9], provided that $\varphi_0 \equiv 0$. Since $\mathbf{x}\cdot\nabla\varphi_0 = 0$, $\nabla\varphi_0$ could be incorporated into the orthogonal term $\mathbf{u}$. However, to render that term unique, we must take $\varphi(\mathbf{x}) = 0$ at $\mathbf{x} = \mathbf{0}$.

To obtain Edelen's formula for $\mathbf{u}$, note that for any co-vector field $\mathbf{w}(\mathbf{x})$, we have that

$$\begin{aligned}(\mathbf{I}-\mathbf{e}^*\otimes\mathbf{e})\mathbf{w} &= r(\nabla\mathbf{e})\mathbf{w} = r[\nabla(\mathbf{e}\cdot\mathbf{w})-(\nabla\mathbf{w})\mathbf{e}]\\ &= r\left\{\nabla(\mathbf{e}\cdot\mathbf{w})+[(\nabla\mathbf{w})^T-\nabla\mathbf{w}]\mathbf{e}-\mathbf{e}\cdot\nabla\mathbf{w}\right\} = r\left\{\nabla w_r+[(\nabla\mathbf{w})^T-\nabla\mathbf{w}]\mathbf{e}-\frac{\partial\mathbf{w}}{\partial r}\right\}.\end{aligned}$$

However, with $\mathbf{w}=\mathbf{j}-\nabla\varphi$, (5) gives $w_r\equiv 0$, $\mathbf{u}\equiv\mathbf{w}$ and, hence,

$$\mathbf{u}+r\frac{\partial\mathbf{u}}{\partial r}=\mathbf{x}\cdot\mathrm{Curl}\,\mathbf{j}(\mathbf{x}),\quad\text{where}\;\;\mathrm{Curl}\,\mathbf{j}(\mathbf{x})=\nabla\mathbf{j}-(\nabla\mathbf{j})^{\mathrm{T}},\tag{7}$$

where the notation "Curl" is introduced in the Appendix. Then, integrating the final relation with respect to r, we obtain skew-symmetric pseudo-linear forms:

$$\mathbf{u}=\boldsymbol{\Omega}(\mathbf{x})\mathbf{x},\;\;\text{with}\;\;\boldsymbol{\Omega}=-\boldsymbol{\Omega}^{\mathrm{T}}=-\mathrm{Curl}\,\mathfrak{g}(\mathbf{x}),\;\;\text{where}\;\;\mathfrak{g}(\mathbf{x})=\int_0^1\mathbf{j}(\lambda\mathbf{x})\mathrm{d}\lambda,\tag{8}$$

which is equivalent to Eq. (2.18) of Edelen [9] and is subsumed by the last member of (79) in the Appendix.

Finally, (3) and (6), with $\varphi_0\equiv 0$, give the relation between dissipation potential $\varphi(\mathbf{x})$ and dissipation function $\mathfrak{D}=D(\mathbf{x})=D(r\mathbf{e})$ as an elementary linear integro-differential transform:

$$\begin{aligned}\varphi(\mathbf{x})=\mathfrak{E}\{D\}:=\int_0^1 D(\lambda\mathbf{x})\frac{\mathrm{d}\lambda}{\lambda}=\int_0^r D(r'\mathbf{e})\frac{\mathrm{d}r'}{r'}\\ \text{with}\;\;D(\mathbf{x})=\mathfrak{E}^{-1}\{\varphi\}=\mathbf{x}\cdot\nabla\varphi(\mathbf{x})=r\frac{\partial\varphi}{\partial r},\end{aligned}\tag{9}$$

provided that $D(\mathbf{x})=o(r^\nu)$ with $\nu>0$, for $r\to 0$. The relations of (8) and (9) represent a special case of the general homotopy operation defined by Edelen in Eqs. (5–3.2) and (8.5.3) [11]. We note that (9) defines the so-called *perspective* of $D(\mathbf{x})$, a scaling transformation that preserves positivity and convexity, which has found application in other fields [13,34].

We henceforth assume that D is a convex function vanishing at the origin $\mathbf{x}=\mathbf{0}$, so that the first member of (9) implies the same properties for φ. We consider below the special case where $\{D,\varphi\}$ are homogeneous of degree $p\geq 1$, such that $D(\mathbf{x})\equiv p\varphi(\mathbf{x})$, i.e., such that p is an eigenvalue of $\mathfrak{E}^{-1}$. Hence, whenever a given set of homogeneous functions is complete in a properly restricted function space, it provides the corresponding spectral representation of $\mathfrak{E}^{-1}$. Prominent examples are multivariate Taylor series expansions in $\mathbf{x}$ for C^∞ functions, which involve a discrete spectrum, or the multivariate (inverse) Mellin transform with continuous spectrum. Thus, expansions in polynomial invariants, such as those employed by Ziegler [53] and Edelen [12] for Reiner-Rivlin fluids (see below), may be viewed as spectral representations in homogeneous forms of successively higher degree.

3 Convex duality, dissipative moduli and Onsager symmetry

To emphasize the duality and to provide a connection to certain past works [6,35,37,49], we employ the notation anticipated above, with $\mathbf{y}\hat{=}[y_i]$ denoting an element of $\mathbb{X}^*$, the dual space of $\mathbb{X}$. Depending on the setting, $\mathbf{y}$ can be interpreted as the flux of Onsager and Edelen, or, equivalently, as the generalized velocity of others works [37,49], with dissipation given by $\mathfrak{D}=\mathbf{y}\cdot\mathbf{x}$. In the following, we occasionally refer to $\mathbf{y}$ simply as "rate."

Thus, as an improvement on a previous treatment [21], we first invoke the Legendre–Fenchel duals or convex conjugates[2] [17,35,37,49]

$$\varphi(\mathbf{x})+\psi(\mathbf{y})=\mathbf{y}\cdot\mathbf{x},\tag{10}$$

[2] The notation φ, ψ is that employed by Ziegler [49] (p. 144) for the dissipative analogs of Gibbs and Helmholtz free energies, while [37] and [35] denote ψ, $\mathbf{y}$ by the mathematically more suggestive φ^*, $\mathbf{x}^*$. Also, our dot-product represents the conventional symbol $\langle\,,\,\rangle$.

where $\varphi : \mathbb{X} \to \mathbb{R}$, and $\psi : \mathbb{X}^* \to \mathbb{R}$ are assumed convex and differentiable, with invertible maps

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{y}) = \nabla_{\mathbf{y}}\psi, \text{ and } \mathbf{y} = \boldsymbol{\eta}(\mathbf{x}) = \nabla_{\mathbf{x}}\varphi, \text{ with } \boldsymbol{\xi} = \boldsymbol{\eta}^{-1} \tag{11}$$

arising from the dual extrema:

$$\psi(\mathbf{y}) = \max_{\mathbf{x}} \{\mathbf{y}\cdot\mathbf{x} - \varphi(\mathbf{x})\}, \text{ and } \varphi(\mathbf{x}) = \max_{\mathbf{y}} \{\mathbf{y}\cdot\mathbf{x} - \psi(\mathbf{y})\}, \tag{12}$$

discussed further below.

If φ or ψ is only piecewise differentiable, then max should be replaced by sup in (12) and the respective gradients ∇ in (11) should be interpreted as sub-gradients (set-valued sub-differentials), of the type associated with the "fan of normals" at vertices on non-smooth plastic limit surfaces [35,37].[3] At such points of discontinuity the invertibility assumed in (11) no longer applies.

Wherever (11) does apply, we may write:

$$\varphi(\mathbf{x}) = \mathbf{x}\cdot\nabla_{\mathbf{x}}\Psi(\mathbf{x}) - \Psi(\mathbf{x}) \text{ where } \Psi(\mathbf{x}) = \int_0^1 [\varphi(\lambda\mathbf{x}) - \lambda\mathbf{x}\cdot\nabla_{\mathbf{x}}\varphi(0)]\,\frac{d\lambda}{\lambda^2} + \mathbf{x}\cdot\nabla_{\mathbf{x}}\varphi(0),$$

and

$$\psi(\mathbf{y}) = \mathbf{y}\cdot\nabla_{\mathbf{y}}\Phi(\mathbf{y}) - \Phi(\mathbf{y}) \text{ where } \Phi(\mathbf{y}) = \int_0^1 \left[\psi(\lambda\mathbf{y}) - \lambda\mathbf{y}\cdot\nabla_{\mathbf{y}}\psi(0)\right]\frac{d\lambda}{\lambda^2} + \mathbf{y}\cdot\nabla_{\mathbf{y}}\psi(0). \tag{13}$$

Hence,

$$\nabla_{\mathbf{x}}\varphi = \mathbf{x}\cdot\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\Psi, \text{ and } \nabla_{\mathbf{y}}\psi = \mathbf{y}\cdot\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\Phi, \tag{14}$$

which depend on the existence of second-derivatives, a more restrictive condition than assumed by Edelen [8]. With this restriction, (3)–(14) lead to the *pseudolinear* forms:

$$\begin{aligned}
&\mathbf{y} = \boldsymbol{\eta}(\mathbf{x}) = \nabla_{\mathbf{x}}\varphi = \mathbf{L}\mathbf{x}, \text{ and } \mathbf{x} = \boldsymbol{\xi}(\mathbf{y}) = \nabla_{\mathbf{y}}\psi = \mathbf{R}\mathbf{y},\\
&\text{with } \mathbf{L}(\mathbf{x}) = \nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\Psi, \ \mathbf{R}(\mathbf{y}) = \nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\Phi = \mathbf{L}^{-1},\\
&\text{and } \mathfrak{D} = D(\mathbf{x}) = \mathbf{x}\cdot\mathbf{L}\mathbf{x} = L_{ij}x^i x^j = D^*(\mathbf{y}) = \mathbf{y}\cdot\mathbf{R}\mathbf{y} = R^{ij}y_i y_j,
\end{aligned} \tag{15}$$

with dissipative moduli $\mathbf{L}$ and $\mathbf{R} = \mathbf{L}^{-1}$ (resp., *conductance* or *mobility* and *resistance*), whose symmetry is evident and whose positivity is ensured by the convexity of the conjugate potentials. Rewriting (6), with $\mathbf{j} = \mathbf{y}$ and $\varphi_0 \equiv 0$, one obtains the dual of (9) connecting $\psi(\mathbf{y})$ and $D^*(\mathbf{y})$.

The relations (15) define *secant* moduli to be distinguished from the *tangent* moduli $\mathbf{M}$ and its inverse compliance $\mathbf{K} = \mathbf{M}^{-1}$ defined by the incremental forms:

$$\begin{aligned}
&d\mathbf{y} = d\boldsymbol{\eta}(\mathbf{x}) = \mathbf{K}d\mathbf{x}, \text{ and } d\mathbf{x} = d\boldsymbol{\xi}(\mathbf{y}) = \mathbf{M}d\mathbf{y},\\
&\text{with } \mathbf{K}(\mathbf{x}) = \nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\varphi(\mathbf{x}) \ \& \ \mathbf{M}(\mathbf{y}) = \nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\psi(\mathbf{y}).
\end{aligned} \tag{16}$$

The first two relations in (15) serve to generalize and supersede the pseudolinear viscoplastic moduli invoked previously [19].

Generalized Onsager symmetry

Without appeal to the moduli (15), the relation (11) yields the basic symmetry restrictions on forces and fluxes:

$$\frac{\partial x^i}{\partial y_j} = \frac{\partial x^j}{\partial y_i}, \text{ and } \frac{\partial y_i}{\partial x^j} = \frac{\partial y_j}{\partial x^i}, \tag{17}$$

the analogs of the Maxwell relations of equilibrium thermodynamics that may offer useful consistency tests of constitutive equations.

[3] More general derivatives of "superpotentials" have been proposed by others [40].

The failure of $\mathbf{x}(\mathbf{y})$ or $\mathbf{y}(\mathbf{x})$ to satisfy (17) does not rule out the dissipative potentials, but rather serves to define Edelen's gyroscopic terms by means of the relations

$$\mathbf{u}(\mathbf{x}) = \mathbf{y}(\mathbf{x}) - \boldsymbol{\eta}(\mathbf{x}), \text{ with } \mathbf{u}\cdot\mathbf{x} = 0, \quad \mathbf{v}(\mathbf{y}) = \mathbf{x}(\mathbf{y}) - \boldsymbol{\xi}(\mathbf{y}), \text{ with } \mathbf{y}\cdot\mathbf{v} = 0, \tag{18}$$

cf. Bataille et al. [4] and Edelen (p. 92) [10]. Given sufficient differentiability, the state of affairs can be made formally explicit in terms of *asymmetric moduli* as follows.

In light of (8) and (15), it is clear that we can set down the dual pseudo-linear forms:

$$\begin{aligned}
\mathbf{y} &= \boldsymbol{\Lambda}(\mathbf{x})\mathbf{x}, \text{ with } \boldsymbol{\Lambda} = \mathbf{L} + \boldsymbol{\Omega} \text{ and } \boldsymbol{\Omega}(\mathbf{x}) = -\boldsymbol{\Omega}^{\mathrm{T}}(\mathbf{x}) = -\int_0^1 \mathrm{Curl}_{\mathbf{x}}\, \boldsymbol{\eta}(\lambda\mathbf{x})\lambda \mathrm{d}\lambda, \\
\mathbf{x} &= \boldsymbol{\Lambda}^*(\mathbf{y})\mathbf{y}, \text{ with } \boldsymbol{\Lambda}^* = \mathbf{R} + \boldsymbol{\Omega}^* \text{ and } \boldsymbol{\Omega}^*(\mathbf{y}) = -\boldsymbol{\Omega}^{*\mathrm{T}}(\mathbf{y}) = -\int_0^1 \mathrm{Curl}_{\mathbf{y}}\, \boldsymbol{\xi}(\lambda\mathbf{y})\lambda \mathrm{d}\lambda,
\end{aligned} \tag{19}$$

where the obvious relation $\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}^{-1}$ requires the following connection between $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^*$:

$$\boldsymbol{\Omega} = \mathbf{L}^{1/2}\mathbf{W}\mathbf{L}^{1/2}, \quad \mathbf{W} = -\mathbf{W}^*(\mathbf{I} + \mathbf{W}^*)^{-1}, \text{ with } \mathbf{W}^* = \mathbf{L}^{1/2}\boldsymbol{\Omega}^*\mathbf{L}^{1/2}. \tag{20}$$

If one defines $\mathbf{L}^* := \mathbf{L}^{-1} = \mathbf{R}$, one has complete symmetry between starred and unstarred quantities. The symmetry and positivity of $\mathbf{L}$ or $\mathbf{R}$ together with the skew symmetry of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^*$ ensure the validity of the implied matrix operations, which become transparent in terms of the diagonal forms of $\mathbf{L}$ and $\mathbf{R}$ on their mutual principal axes.

The relations (19) represent dual forms of Edelen's nonlinear generalization of Onsager's theory. At the same time, they serve to define *strongly dissipative* systems, for which $\boldsymbol{\Omega} = \boldsymbol{\Omega}^* \equiv \mathbf{0}$, with the relations (17) satisfied identically. Otherwise, we encounter the failure of generalized Onsager symmetry, the possibility of which is envisaged by Edelen's postulated non-dissipative stresses in viscous fluids [12], [10, pp. 110 ff.], and by the comments of Eringen [14, Section 2.4], who questions the general validity of Onsager symmetry for nonlinear processes. It is the opinion of this author that this breakdown of symmetry may be attributed to non-dissipative coupling, generally non-holonomic, between certain kinematic degrees of freedom. Since the physical origins of such gyroscopic effects are not clear, we shall restrict attention in the balance of this paper to strongly dissipative systems.

3.1 Homogeneous potentials and singular limits

We now consider the important special case of homogeneous potentials where norms play a paramount role. Without detailed specification of the norms,[4] one obtains the counterparts in $\mathbb{X}^*$ of those indicated on the last line of (4), where $\nabla = \nabla_{\mathbf{x}}$:

$$s = |\mathbf{y}|^*, \quad \mathbf{f} = \frac{\mathbf{y}}{s}, \quad \mathbf{f}^* = \nabla_{\mathbf{y}} s, \quad \mathbf{f}^*\cdot\mathbf{f} = 1, \quad \nabla_{\mathbf{y}}\mathbf{f} = \frac{1}{s}(\mathbf{I} - \mathbf{f}^*\otimes\mathbf{f}), \tag{21}$$

where $|\mathbf{y}|^*$ denotes an arbitrary norm in $\mathbb{X}^*$. Then, (11) takes on the "radial" or normal forms:

$$\eta_r = \boldsymbol{\eta}\cdot\mathbf{e} = \frac{D(\mathbf{x})}{r} = \frac{\partial}{\partial r}\varphi(r, \mathbf{e}), \text{ and } \xi_s = \mathbf{f}\cdot\boldsymbol{\xi} = \frac{D^*(\mathbf{y})}{s} = \frac{\partial}{\partial s}\psi(s, \mathbf{e}^*). \tag{22}$$

These components of rate and force give only their dissipative contributions, which are distinguished from their magnitudes given by the norms:

$$\sigma(\mathbf{x}) = |\boldsymbol{\eta}(\mathbf{x})|^*, \text{ and } \rho(\mathbf{y}) = |\boldsymbol{\xi}(\mathbf{y})| \tag{23}$$

which figure prominently in the following.

[4] A standard example is $r = (G_{\mathrm{ij}}x^i x^j)^{1/2}$ and $s = (H^{\mathrm{ij}}y_i y_j)^{1/2}$, with positive matrices $[G_{\mathrm{ij}}]$, $[H^{\mathrm{ij}}]$.

Limit surfaces

The formulae (10)–(11) yield the textbook case [47] of homogeneous functions $D(\mathbf{x})$ and $D^*(\mathbf{y})$ of respective degrees $p > 1$ and $q > 1$:

$$\begin{aligned} &\varphi(\mathbf{x}) = \frac{1}{p} D(\mathbf{x}) = \frac{D_0}{p}\left(\frac{r}{r_0}\right)^p, \quad \psi(\mathbf{y}) = \frac{1}{q} D^*(\mathbf{y}) = \frac{D_0^*}{q}\left(\frac{s}{s_0}\right)^q, \\ &\text{with} \\ &\frac{1}{p} + \frac{1}{q} = 1, \quad \frac{\rho}{\rho_0} = \left(\frac{s}{s_0}\right)^{q-1}, \quad \frac{\sigma}{\sigma_0} = \left(\frac{r}{r_0}\right)^{p-1} \end{aligned} \tag{24}$$

where ρ, σ refer to functions defined by (23), and

$$\begin{aligned} \mathbf{f}(\mathbf{e}) &= \left[\mathbf{e}^* + \frac{1}{p}(\mathbf{I} - \mathbf{e}^* \otimes \mathbf{e})\partial_{\mathbf{e}} \ln \frac{D_0}{r_0^p}\right] \mathbf{f}\cdot\mathbf{e}, \quad \text{with } \mathbf{e}^*(\mathbf{e}) = \nabla_{\mathbf{x}} r, \\ \mathbf{e}(\mathbf{f}) &= \left[\mathbf{f}^* + \frac{1}{q}(\mathbf{I} - \mathbf{f}^* \otimes \mathbf{f})\partial_{\mathbf{f}} \ln \frac{D_0^*}{s_0^q}\right] \mathbf{f}\cdot\mathbf{e}, \quad \text{with } \mathbf{f}^*(\mathbf{f}) = \nabla_{\mathbf{y}} s, \\ &\text{where } \mathbf{e}\cdot\mathbf{f} = \frac{D_0(\mathbf{e})}{r_0(\mathbf{e}) s_0(\mathbf{f})} = \frac{D_0^*(\mathbf{f})}{r_0(\mathbf{e}) s_0(\mathbf{f})}, \quad \rho_0(\mathbf{f}) = r_0(\mathbf{e}), \quad \sigma_0(\mathbf{e}) = s_0(\mathbf{f}). \end{aligned} \tag{25}$$

Specification in (25) of D_0, r_0 and $\mathbf{f}\cdot\mathbf{e}$ as functions of $\mathbf{e}$ or of D_0^*, s_0 and $\mathbf{f}\cdot\mathbf{e}$ as functions of $\mathbf{f}$, provides parametric maps between fiducial (hyper)surfaces $s = s_0(\mathbf{f})$ in $\mathbb{X}^*$ and $r = r_0(\mathbf{e})$ in $\mathbb{X}$, maps that become simpler for iso-dissipative surfaces $D_0 = D_0^* =$ constant.

It is easy to see that the moduli in (15) must take on the form:

$$\mathbf{L} = \left(\frac{r}{r_0}\right)^{p-2} \mathbf{L}_0(\mathbf{e}), \quad \mathbf{R} = \left(\frac{s}{s_0}\right)^{q-2} \mathbf{R}_0(\mathbf{f}), \tag{26}$$

and for $p = q = 2$ in (26), one obtains quadratic forms representing the Rayleigh-Onsager dissipation with linear force-rate relations provided $\mathbf{L}_0$ and $\mathbf{R}_0$ are independent of $\mathbf{e}$ and $\mathbf{f}$. Otherwise, one obtains a more general degree-two homogeneity, with pseudolinear forms of the type mentioned above [21].

In the marginal limit $q \to 1, \;\; p \to \infty$, appropriate to rate-independent friction or plasticity, the relations (24)–(26) yield $\psi(\mathbf{y}) \equiv D^*(\mathbf{y})$, together with

$$\boldsymbol{\xi} = \rho_0(\mathbf{f})\left(\frac{s}{s_0}\right)^{q-1} \mathbf{e} \to \rho_0(\mathbf{f})\mathbf{e}(\mathbf{f}) \tag{27}$$

for $q \to 1$ with fixed $s > 0$, with $\boldsymbol{\xi}$ otherwise indeterminate. On the other hand, for bounded functions $D_0(\mathbf{e})$ with bounded derivatives, it follows from (25) that

$$\begin{aligned} &\boldsymbol{\eta} = \sigma_0\left(\frac{r}{r_0}\right)^{p-1} \mathbf{f} \to \begin{cases} \mathbf{0}, & \text{for } r < r_0(\mathbf{e}), \\ \sigma_0(\mathbf{e})\mathbf{f}(\mathbf{e}), & \text{for } r = r_0(\mathbf{e}), \end{cases} \\ &\text{with} \\ &\mathbf{f}(\mathbf{e}) \to (\mathbf{f}\cdot\mathbf{e})\left[\mathbf{e}^* - \frac{1}{r_0}(\mathbf{I} - \mathbf{e}^* \otimes \mathbf{e})\partial_{\mathbf{e}} r_0(\mathbf{e})\right] \equiv (\mathbf{f}\cdot\mathbf{e})\nabla_{\mathbf{x}}\left[r - r_0(\mathbf{e})\right]\big|_{r=r_0}, \end{aligned} \tag{28}$$

for $p \to \infty$, where, because of (25) and (27), r_0 may be taken an upper bound for r. The relation (28), which represents a multidimensional generalization of the sgn function appearing in one-dimensional versions of rigid plasticity, is depicted schematically in Fig. 1.

Thus, the assumption of rate-independent force, corresponding to the marginal limit of convexity, leads to bounding (hyper)surfaces in $\mathbb{X}$ and $\mathbb{X}^*$. Hence, the assumed yield surface of continuum plasticity arises quite naturally as a mathematical consequence of rate-independence and convex duality. While this situation is appreciated by others, [6,35,37], the present treatment makes it more palpable.

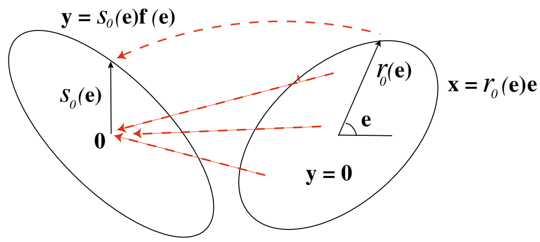

Fig. 1 Map from $\mathbb{X}$ to $\mathbb{X}^*$ for rate-independent forces

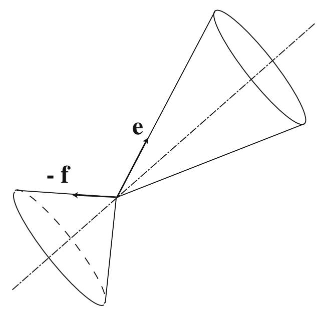

Fig. 2 Conical limit surfaces for Coulomb friction

Rate-independence also rules out a material time scale, so that the quantities s_0, σ_0 are arbitrary up to a multiplicative factor familiar in classical theories of rigid plasticity, a factor which is determined by externally imposed rates.

As a further point, we see that the rate $\boldsymbol{\eta}$ in (28) is collinear with the gradient of $r - r_0(\mathbf{e})$ and, hence, "orthogonal" to the limit surface $r = r_0(\mathbf{e})$ provided the latter is sufficiently regular.

An important exception to the above state of affairs arise in the singular case of Coulomb friction represented by a cone $\mathbb{K}$, where r_0 may be defined formally as the piecewise-constant, set-valued function:

$$r_0(\mathbf{e}) = \infty, \quad \text{for } \mathbf{e} \in \mathbb{K}, \quad \text{and } r_0(\mathbf{e}) \in [0, \infty), \quad \text{for } \mathbf{e} \in \partial\mathbb{K}. \tag{29}$$

In this case, the limit of $\mathbf{f}$ in (28) may be taken formally to be $(\mathbf{f}{\cdot}\mathbf{e})\mathbf{e}^*$. Thus, Coulomb friction can be represented by

$$s_0(\mathbf{e}) = 0, \quad \text{for } \mathbf{e} \in \mathbb{K}, \quad \text{and } \mathbf{f}(\mathbf{e}) = \mathbf{L}_0\mathbf{e}, \quad \text{with } \mathbf{L}_0 = \cos\phi_f \, \mathbf{e}^* \otimes \mathbf{e}^*, \quad \text{for } \mathbf{e} \in \partial\mathbb{K}, \tag{30}$$

which represents a non-normal flow rule with power factor $\cos\phi_f(\mathbf{e}) = \mathbf{f} \cdot \mathbf{e}$ prescribed as function of $\mathbf{e}$, where $\phi_f(\mathbf{e})$ represents the so-called angle of internal friction. The situation is depicted schematically in Fig. 2 for the case of circular (Drucker-Prager) cones, suggesting that a general cone in $\mathbb{X}$ represents a solitary plastic "vertex".

We note that the dual of (28), with

$$\{\mathbf{y}, r, q, \mathbf{L}, \mathbf{e}\} \Leftrightarrow \{\mathbf{x}, s, p, \mathbf{R}, \mathbf{f}\}, \quad \text{with } p \to 1, \quad q \to \infty, \tag{31}$$

corresponds to the peculiar notion of force-independent rate and a strictly dissipative form of "locking" behavior [17,41].

It should also be noted that the above theory covers the case of "stiff elastoplasticity" with small elastic deformation superposed on large plastic deformation. The plastic moduli invoked in a previous work [20] can now be regarded secant moduli of the form (15), with the resulting elastoplastic model described by two scalar potentials, one plastic and one elastic. With potentials that depend on parameters such as particle fraction and fabric tensor discussed below, one readily obtains models of history-dependent elastoplasticity.

4 Visco-plasticity of suspensions and granular media

We summarize briefly the application of the preceding ideas to the rheology of fluid-particle systems, ranging from suspensions of rigid particles in viscous fluids to dry granular media composed of rigid frictional particles.

Most of the background, with references to related work, is presented in a recent paper [19] dealing with fluid-particle suspensions.[5]

For the present purposes, we set aside Edelen's non-dissipative terms, assuming that our materials are strongly dissipative or else that non-dissipative terms may be added *a posteriori*, as done by Edelen [12] for viscous fluids and by the present author [20], to represent the dilatancy of granular media.

We now revert to the notation of Edelen, with $\mathbf{J}$ and $\mathbf{X}$ denoting rates and conjugate forces, and $\partial_{\mathbf{J}}$ denoting derivatives with respect to $\mathbf{J}$. We also employ $\mathbf{x}$, ∇ as conventional notation for physical space positions and gradients. Finally, we employ Sym and Skw to denote symmetric and skew parts of second-rank tensors.

Then, we are concerned with a class of visco-plastic constitutive models in which the local Cauchy stress $\mathbf{T}(\mathbf{x})\hat{=}[T^{\mathrm{ij}}]$ depends on local velocity gradient $\nabla\mathbf{v}$ though the objective deformation rate

$$\mathbf{D}(\mathbf{x}) = \mathrm{Sym}[\nabla\mathbf{v}(\mathbf{x})] \hat{=} [D_{\mathrm{ij}}] = [(v_{j;i} + v_{i;j})/2], \tag{32}$$

the local solid fraction ϕ, and an additional objective, traceless and symmetric second-rank "fabric" tensor $\mathbf{A}\hat{=}[A^{\mathrm{ij}}] = \mathbf{A}^T$ representing a restricted form of shear-induced anisotropy. The latter depends on the past history of the velocity gradient $\nabla\mathbf{v}$ at given material point through an appropriate evolution equation, which will be left unspecified in the present paper (cf. a previous paper [19]).

We shall also consider the case of mobile particles involving a local particle flux $\mathbf{j}$ which depends also on the spatial gradients of ϕ, $\nabla\mathbf{v}$ and $\mathbf{A}$.

Whereas the previous analysis [19] was predicated on the existence of dissipative moduli, these are subsumed in the above general theory of dissipation potentials, and we now focus on the rate-dependent potential $\psi(\mathbf{J})$.

4.1 Rheological models of Stokesian fluid-particle suspensions

We consider neutrally buoyant suspensions of rigid non-Brownian particles in incompressible liquids, whose microscopic dynamics are assumed to be governed by the well-known Stokes equations. The linearity of these equations should also apply to Stokesian suspensions [19], so that any nonlinear rheological effects must arise solely from the dependence of the underlying Stokesian dynamics on the past history of the kinematics, assumed for simplicity to be represented by the fabric $\mathbf{A}$.

Spatially homogeneous suspensions

Previous works [19] (and references therein), borrow the representation of Cowin [7] for fabric-dependent anisotropy in linear-elastic solids in order to deduce the corresponding constitutive model $\mathbf{T}' = \mathbf{X}(\mathbf{A}, \mathbf{D})$, with linear dependence on $\mathbf{J} = \mathbf{D}$, where $\mathbf{T}'$ denotes deviatoric stress, and tr $\mathbf{D}$=0. While this leads to a fourth-rank viscosity tensor, the analog of the elasticity tensor, this representation and the associated symmetries are subsumed by the above general theory of a dissipation potential $\psi(\mathbf{A}, \mathbf{D})$, with $\mathbf{X} = \partial_{\mathbf{J}}\psi$.

The corresponding model of flow-induced anisotropy in Stokesian suspensions is obtained by taking ψ to be a quadratic in $\mathbf{D}$, given by a linear combination of the joint isotropic scalar invariants of $\mathbf{D}$ and $\mathbf{A}$:

$$\mathrm{tr}\,\mathbf{D}^2,\ \ (\mathrm{tr}\,\mathbf{AD})^2,\ \ (\mathrm{tr}\,\mathbf{A^2D})^2,\ \ \mathrm{tr}\,\mathbf{AD}^2,\ \ \mathrm{tr}\,\mathbf{A}^2\mathbf{D}^2,\ \ (\mathrm{tr}\,\mathbf{A}^2\mathbf{D})(\mathrm{tr}\,\mathbf{AD}), \tag{33}$$

with coefficients depending on tr $\mathbf{A}^k$, $k = 1, 2, 3$. It is readily seen that derivatives $\partial_{\mathbf{D}}$ of (33) reproduce the results found previously [19] by much more lengthy arguments.

One obtains more general models, which we designate as *quasi-Stokesian*, by simply including additional non-Stokesian quadratic forms involving $|\mathbf{D}| = \sqrt{\mathrm{tr}\,\mathbf{D}^2}$ in the list (33), and the same is true for the models considered in the following Subsection.

[5] As correction of typographical errors, Eqs. (11)–(12) should respectively read

$$\mu^{\mathrm{ab}}_{i_1\ldots i_m, j_1\ldots j_n}(m,n) = \int_{\mathbf{x}\in\mathcal{A}}\int_{\mathbf{x}^*\in\mathcal{B}} r^a_{i_1}\ldots r^a_{i_m} n_k(\mathbf{x})\mu_{i_m k j_n l}(\mathbf{x},\mathbf{x}^*) r^b_{j_1}\ldots r^b_{j_n} n_l(\mathbf{x}^*)\mathrm{d}S(\mathbf{x})\mathrm{d}S(\mathbf{x}^*),$$

and $\boldsymbol{\mu}^{\mathrm{ab}}(m,n) = \partial\boldsymbol{\tau}^{a(m)}/\partial\mathbf{u}^{b(n)}$, while in Eqs. (26)–(29) and the accompanying text, the gradient $\boldsymbol{\nabla}$ should be replaced by the divergence $\boldsymbol{\nabla}\cdot$ when followed by second-rank tensors.

Inhomogeneous suspensions

Experiments reveal the existence of particle migration in the presence of spatial gradients in ϕ or $\mathbf{D}$, and various "quasi-Stokesian" models have been put forth to describe this phenomenon [15,48].

The Stokesian alternative proposed by the present author [19] involves antisymmetric stress, hyperstress (moment stress), and particle flux, together with particle fraction, deformation rate and fabric and their various spatial gradients. However, despite the ostensible generality, this model may be deemed restrictive in several respects. First, it employs the gradient $\nabla\mathbf{D}$ of deformation rate rather than the double gradient of velocity $\nabla\nabla\mathbf{v}$, which also includes objective gradients in vorticity $\mathbf{W} = \mathrm{Skw}[\nabla\mathbf{v}]$. Secondly, the development of the model proceeds from the conventional view that $\nabla\phi$ is a driving force for particle migration. While a plausible analog of the concentration gradients that drive diffusion in molecular and colloidal systems, it seems advisable to proceed without prejudice from more general dissipation principles.

A more important weakness of the above model is its restriction to single-phase suspension mechanics, with relative particle flux $\mathbf{j}$ as the only new kinematic variable. After a consideration of this restricted model, we introduce a more comprehensive mixture-theory model involving two-phase mechanics.

Single-phase models

Building on the previous analysis [19], with terms of the same order in various gradients, we assume a dissipation potential ψ that is given in terms of the joint isotropic scalar invariants of the generalized rate,

$$\mathbf{J} = \mathbf{j} \oplus \mathbf{D} \oplus \nabla\nabla\mathbf{v}, \tag{34}$$

and the parameters ϕ, $\mathbf{A}$, $\nabla\phi$, $\nabla\mathbf{A}$, invariants which are simultaneously quadratic in $\mathbf{J}$ and linear in $\nabla\phi$, $\nabla\mathbf{A}$. Here, $\mathbf{j} \mathrel{\hat{=}} [\mathrm{j}_k]$ denotes the volume-flux of particles relative to the volume-average velocity of the suspension, and $\oplus$ is standard notation for the direct sum of vector spaces. This serves to define the conjugate force as linear function of rate,

$$\mathbf{X} \;=\; \mathfrak{f} \oplus \mathbf{T}' \oplus \boldsymbol{\tau} \;=\; \partial_{\mathbf{j}}\psi \oplus \partial_{\mathbf{D}}\psi \oplus \partial_{\nabla\nabla\mathbf{v}}\psi, \tag{35}$$

where $\mathfrak{f} \mathrel{\hat{=}} [\mathfrak{f}^i]$ is the force conjugate to $\mathbf{j}$, and $\boldsymbol{\tau} \mathrel{\hat{=}} [\tau^{\mathrm{ijk}}]$ the (hyper)stress conjugate to $\nabla\nabla\mathbf{v}$. Although not done at this point, we could include additional kinematic variables, such as an hypothetical (*Cosserat*) particle rotation and its gradient (see below).

Focusing here on the particle flux $\mathbf{j}$, but not listing all the relevant scalar invariants, one can anticipate the following general form for the conjugate force (i.e., force per unit volume of suspension):

$$\begin{aligned}\mathfrak{f} \mathrel{\hat{=}} [\mathfrak{f}^i] &= \mathbf{Zj} + \mathbf{K}\nabla\phi + \boldsymbol{\zeta}\nabla\nabla\mathbf{v} + \boldsymbol{\kappa}\nabla\mathbf{A} \\ &\mathrel{\hat{=}} [Z^{ik}\mathrm{j}_k + K^{ik}\phi_{,k} + \zeta^{ijkl}v_{j;kl} + \kappa^{ijkl}A_{jk;l}],\end{aligned} \tag{36}$$

where $\mathbf{Z}$ and $\boldsymbol{\zeta}$ are isotropic functions of ϕ and $\mathbf{A}$, while $\mathbf{K}$ and $\boldsymbol{\kappa}$ are isotropic functions of ϕ, $\mathbf{A}$ and $\mathbf{D}$, which are also linear in $\mathbf{D}$.

We note that certain terms in (36) have counterparts in various phenomenological molecular and colloidal theories, where $\mathbf{Z}$ represents viscous drag and $\mathbf{K}$ is linear in a thermo-kinetic energy (or temperature). Borrowing from such theories, various quasi-Stokesian suspension models[6] assume $\mathbf{K}$ to be linear in term $|\mathbf{D}|$, while the term involving $\nabla\nabla\mathbf{v}$ is replaced by $\nabla|\mathbf{D}|$, both of which involve the distinctly non-Stokesian form $|\mathbf{D}|$. To the best of the author's knowledge, the term in $\nabla\mathbf{A}$, representing migration due to gradients in anisotropy, has been identified only recently [19], and its physical importance has yet to be established.

An alternative to (36), emphasizing the dependence on rates, is

$$\mathfrak{f} \mathrel{\hat{=}} [\mathfrak{f}^i] = \mathbf{Zj} + \boldsymbol{\nu}\mathbf{D} + \boldsymbol{\zeta}\nabla\nabla\mathbf{v}, \;\; \text{with} \;\; \boldsymbol{\nu}\mathbf{D} \;=\; \mathbf{K}\nabla\phi + \boldsymbol{\kappa}\nabla\mathbf{A} \mathrel{\hat{=}} [\nu^{ijk}D_{kl}], \tag{37}$$

where the third-rank tensor $\boldsymbol{\nu}$ is an isotropic function of ϕ, $\mathbf{A}$, $\nabla\phi$, $\nabla\mathbf{A}$, linear in the final two arguments. Further, effort would be required to set down analytic forms for all the moduli listed in (36)–(37), which at this juncture may not be fully justified. Instead, we focus on the general significance of the force $\mathfrak{f}$.

Within the confines of a single-phase model $\mathfrak{f}$ must be regarded as the sole internal force driving particle migration and, in the absence of inertial effects, the sum of this force plus any external body force $\mathfrak{b}_{\mathrm{s}}$ must be zero. Thus, (36) gives the expression for particle flux,

$$\mathbf{j} = -\mathbf{Z}^{-1}[\mathbf{K}\nabla\phi + \boldsymbol{\zeta}\nabla\nabla\mathbf{v} + \boldsymbol{\kappa}\nabla\mathbf{A} - \mathfrak{b}_{\mathrm{s}}], \tag{38}$$

[6] cf. the references in previous works [15,19,48].

which is equivalent to the form presented elsewhere [19] (Eq. (20)). The term in $\nabla\phi$ is of course the analog of colloidal (Einstein-Stokes) gradient-diffusion models. As shown elsewhere [19], there is no guarantee that the apparent diffusivity $\mathbf{Z}^{-1}\mathbf{K}$ will remain positive definite in transient deformations.

When combined with the particle balance (see below):

$$\frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = -\nabla\cdot\mathbf{j}, \tag{39}$$

(38) provides an essential particle-transport equation. Upon replacing terms in $\nabla\mathbf{v}$ and $\mathbf{D}$ by $|\mathbf{D}|$, one obtains *mutatis mutandi* a generalization of the aforementioned quasi-Stokesian models of particle migration. Since such models usually emphasize the role of various particle stresses, we consider the brief outlines of a comprehensive treatment that introduces such stresses via their kinematic conjugates.

Mixture-theoretic models

Previous works [15,48] provide reviews and elaborations upon previous research on various quasi-Stokesian models, models that are largely subsumed in the classical theory of mixtures, well summarized in at least one monograph [42].

We recall that each constituent in a mixture is endowed with its own kinematics and conjugate forces, in the present case specified by the respective velocity fields for particulate solid and continuous fluid phases, say $\mathbf{v}_\mathrm{s}(\mathbf{x})$, $\mathbf{v}_\mathrm{f}(\mathbf{x})$, and their various gradients, together with conjugate forces consisting of internal forces of interaction $\mathfrak{f}_\mathrm{s} = -\mathfrak{f}_\mathrm{f}$, plus external body forces and stresses conjugate to the gradients of velocity. For the present purposes, we employ volume-average mixture velocity and fluxes:

$$\mathbf{v} = \phi\mathbf{v}_\mathrm{s} + (1-\phi)\mathbf{v}_\mathrm{f}, \;\; \text{with } \mathbf{j} := \mathbf{j}_\mathrm{s} = -\mathbf{j}_\mathrm{f} = \phi(\mathbf{v}_\mathrm{s} - \mathbf{v}) = \phi(1-\phi)(\mathbf{v}_\mathrm{s} - \mathbf{v}_\mathrm{f}). \tag{40}$$

Then, for a suspension of rigid particles in an incompressible fluid, such that $\nabla\cdot\mathbf{v}_\mathrm{f} = 0$, balances on fluid and particles give [42] (Eq. (2.55)):

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + \mathbf{v}_\mathrm{f}\cdot\nabla\phi = 0, \;\; \text{and } \frac{\partial\phi}{\partial t} + \mathbf{v}_\mathrm{s}\cdot\nabla\phi = -\phi\nabla\cdot\mathbf{v}_\mathrm{s}, \\ &\therefore \;\; \nabla\cdot\mathbf{v} = 0, \;\; \text{and } \phi\nabla\cdot\mathbf{v}_\mathrm{s} = (\mathbf{v}_\mathrm{f} - \mathbf{v}_\mathrm{s})\cdot\nabla\phi = -\frac{\mathbf{j}\cdot\nabla\phi}{\phi(1-\phi)}, \end{aligned} \tag{41}$$

with particle-phase dilatation $\nabla\cdot\mathbf{v}_\mathrm{s}$ representing changes in particle number density.

The balance (39) still applies, requiring once more a constitutive equation for $\mathbf{j}$ and involving the augmented set of frame-indifferent rates:

$$\mathbf{J} = \mathbf{J}_0 \oplus \mathbf{D}_\mathrm{s} \oplus \boldsymbol{\Omega} \oplus \nabla\nabla\mathbf{v}_\mathrm{s} \oplus \nabla\boldsymbol{\Omega}, \tag{42}$$

where $\mathbf{J}_0$ denotes the set (34), and

$$\mathbf{D}_\mathrm{s} = \mathrm{Sym}[\nabla\mathbf{v}_\mathrm{s}], \;\; \boldsymbol{\Omega} = \mathrm{Skw}[\nabla\mathbf{v}_\mathrm{s} - \nabla\mathbf{v}]. \tag{43}$$

The skew-symmetric tensor $\boldsymbol{\Omega}$, analogous to a *Cosserat rotation*, involves contributions from the relative rotations of particle centers and from rotations of individual particles about their centroids, and the gradient $\nabla\boldsymbol{\Omega}$ defines *wryness* or "curvature". We do not bother to reduce $\boldsymbol{\Omega}$ to a conventional vector form.

Various conjugate forces are given once again as partial derivatives of ψ with respect to the rates in (42). Thus, the isotropic part of the particle stress $\mathbf{T}_\mathrm{s}$:

$$p_\mathrm{s} = -\mathrm{tr}\,\mathbf{T}_\mathrm{s}, \;\; \text{where } \mathbf{T}_\mathrm{s} = \partial_{\mathbf{D}_\mathrm{s}}\psi, \tag{44}$$

defines a particle-pressure conjugate to tr $\nabla\mathbf{v}_\mathrm{s}$ = tr $\mathbf{D}_\mathrm{s}$ whose dissipated power is due solely to changes in particle density. While the latter is obviously connected to particle migration by the last member of (41), it is not clear to this author that the gradients in this pressure should viewed as a driving force for particle migration, as proposed by others [15,48] and discussed further below.

For the Stokesian suspensions at hand, we can presumably employ the principle of minimum dissipation and standard functional analysis to obtain the various field equations for quasi-static equilibrium, exactly as done for complex elastic continua. For rigid particles suspended in an incompressible fluid in volume V, with external body force densities $\mathfrak{b}_a,\ a = f, s$ but no external body couples, this involves the variational problem

$$\min_{\mathbf{v}_f(\mathbf{x}),\mathbf{v}_s(\mathbf{x})} \int_V [\psi(\mathbf{J}) - \mathfrak{b}_s\cdot\mathbf{v}_s - \mathfrak{b}_f\cdot\mathbf{v}_f]\mathrm{d}V, \tag{45}$$

subject to the local incompressibility constraint $\nabla\cdot\mathbf{v}_f(\mathbf{x}) = 0$ and prescribed values of $\mathbf{v}_f, \mathbf{v}_s$ on ∂V or of related volume averages.

We note that (45) has the same form as a general variational principle discussed below in Sect. 5. For the problem defined by (45), it is convenient to employ the alternative decomposition

$$\begin{aligned}
&\mathbf{J} = \mathbf{v}^\sharp \oplus \mathbf{D}_s \oplus \mathbf{D}_f \oplus \mathbf{W}^\sharp \oplus \nabla\nabla\mathbf{v}_s \oplus \nabla\nabla\mathbf{v}_f \oplus \nabla\mathbf{W}^\sharp\\
&\text{where}\\
&\mathbf{v}^\sharp = \mathbf{v}_s - \mathbf{v}_f = \mathfrak{j}/\phi(1-\phi),\ \text{ and }\ \mathbf{W}^\sharp = \mathrm{Skw}[\mathbf{v}^\sharp] = \boldsymbol{\Omega} + \mathrm{Skw}[\nabla\mathfrak{j}/(1-\phi)].
\end{aligned} \tag{46}$$

Without setting down all the details here, we note that for finite V with velocity-controlled boundaries, it is easy to obtain from (45)–(46) the standard equations of quasi-static equilibrium for the partial Cauchy stresses [42]:

$$\nabla\cdot\mathbf{T}_a + \mathfrak{b}_a + \mathfrak{f}_a^\sharp = 0,\ \ a = s, f,\ \ \text{with}\ \ \mathfrak{f}_s^\sharp = -\mathfrak{f}_f^\sharp = \mathfrak{f}^\sharp := \partial_{\mathbf{v}^\sharp}\psi = \phi(1-\phi)\mathfrak{f}, \tag{47}$$

where the $\mathfrak{f}_a$ represent interaction-force densities, and the isotropic part of $\mathbf{T}_f$ is a rheologically indeterminant pressure. We do not record here various higher-order balances, which may be construed as representing angular momentum or related moments.

According to mixture theory, the relations $\mathbf{T} = \mathbf{T}_s + \mathbf{T}_f$ & $\mathfrak{b} = \mathfrak{b}_s + \mathfrak{b}_f$ for the overall stress and body force lead to the global equation of equilibrium for $\mathbf{T}$. While the partial stresses may be asymmetric, a more detailed analysis should show the overall stress $\mathbf{T}$ to be symmetric.

Without regard to higher-order stresses and kinematics, the relation $\mathbf{D} = \phi\mathbf{D}_s + (1-\phi)\mathbf{D}_f$ and the additional condition on the nominal overall dissipation

$$\mathbf{T}\cdot\mathbf{D} + \mathfrak{f}\cdot\mathfrak{j} = \mathbf{T}_s\cdot\mathbf{D}_s + \mathbf{T}_f\cdot\mathbf{D}_f + \mathfrak{f}_f\cdot\mathbf{v}_f + \mathfrak{f}_s\cdot\mathbf{v}_s \tag{48}$$

would require further that

$$(\phi\mathbf{T_f} - \{1-\phi\}\mathbf{T}_s)\cdot(\mathbf{D}_f - \mathbf{D}_s) = 0, \tag{49}$$

whose implications for suspension models do not seem to have been explored.

The balances (47) are common to the quasi-Stokesian models proposed by others [15,48], cognizant of the fact that their closure requires constitutive equations for $\mathfrak{f}$ and $\mathbf{T}_s$. However, contrary to the view expressed in many of these works, it is not evident to the present author that the particle stress $\mathbf{T}_s$ should be viewed as a privileged driving force for the flux $\mathfrak{j}$. Among other things, this implies a highly reduced form of constitutive equations like (36) or (37), even without the additional terms in (42).

The preceding point is made clearer by the consideration of (36)–(38), where replacement according to (47) of $\mathfrak{b}_s$ in (38) by $\nabla\cdot\mathbf{T}_s + \mathfrak{b}_s$ yields an expression for $\mathfrak{j}$ that involves several ostensibly independent terms. Of course, the dependence on $\mathbf{D}_s$ in certain terms may be construed roughly as dependence on $\mathbf{T}_s$. Furthermore, certain effects may be highly correlated for restricted kinematics, in particular the simple shear currently employed almost exclusively for suspension rheology.

4.2 Granular media

The particle balance for a dry granular medium is described by the second member of (41) with $\mathbf{v}_s = \mathbf{v}$, where $\mathbf{v}(\mathbf{x})$ represents the relevant material velocity in the associated continuum model. Hence, parallel to the above development for inhomogeneous suspensions, we can identify the rates as

$$\mathbf{J} = \mathbf{D} \oplus \boldsymbol{\Omega} \oplus \nabla\boldsymbol{\Omega} \oplus \nabla\nabla\mathbf{v}, \tag{50}$$

to first order in the relevant kinematic gradients. The first three variables are common to a well-known Cosserat model [32,36], whose assumed quadratic forms in $\mathbf{X}$ serve to define both norm and yield condition, as common to many plasticity models.

The standard model for the quasi-static mechanics of dry granular media provides an example of the rate-independent, pressure-sensitive (Coulomb) plasticity discussed in Sect. 3 above. In particular, the potential $\psi(\mathbf{J})$ is homogeneous degree-one in $\mathbf{J}$ and identical with the dissipation rate. Once ψ is specified in terms of an assumed set of isotropic scalar invariants of $\mathbf{J}$, the conjugate forces are given by an expression of the form

$$\mathbf{X} = \partial_{\mathbf{J}}\psi(\mathbf{J}) = |\mathbf{X}|\,\hat{\mathbf{E}}(\hat{\mathbf{F}}), \quad \text{with } \hat{\mathbf{E}} = \partial_{\mathbf{J}}\psi/|\partial_{\mathbf{J}}\psi|, \quad \text{and } \hat{\mathbf{F}} = \mathbf{J}/|\mathbf{J}|^*, \tag{51}$$

where $|\mathbf{X}|$ and $|\mathbf{J}|^*$ are respective norms for force and rate, while $\hat{\mathbf{E}}$ and $\hat{\mathbf{F}}$ are the associated directors.

In the case of non-cohesive grains $|\mathbf{X}|$ may be assumed proportional to confining pressure, which together with the first relation in (51) represents the standard frictional yield condition (cf. Mohan et al. [36]). A further accounting for the effects of density, anisotropy, and inhomogeneity would involve dependence of ψ on ϕ, $\mathbf{A}$, $\nabla\phi$, $\nabla\mathbf{A}$, all are which require evolution or "transport" equations for ϕ and $\mathbf{A}$. Recent constitutive models introduce a further internal variable representing particle-contact density or coordination number [46], which conveniently can be included in the trace of an appropriately re-defined fabric tensor $\mathbf{A}$.

Without pursuing more detailed constitutive equations, here, we note that the evolution of ϕ is governed by the particle balance:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\ln\phi = -\nabla\cdot\mathbf{v} = -\mathrm{tr}\,\mathbf{D}, \tag{52}$$

with the right-hand side representing granular dilatancy. According to one theoretical interpretation [20], the (Reynolds) dilatancy for systems of rigid grains should be regarded as a rate-independent non-holonomic constraint, with tr $\mathbf{D}$ proportional to $|\mathbf{D}'|$. Hence, the deviator $\mathbf{D}'$ replaces $\mathbf{D}$ as the relevant kinematic variable.

Consistent with the above interpretation of granular dilatancy, there should exist non-dissipative reactions to the dilatancy constraint, perhaps represented by Edelen's formulae. Although these do not contribute to dissipation, they are nevertheless relevant to the mechanics, and, to the extent that the constraints are evolutionary, they generally require further constitutive equations [20].

The preceding considerations raise interesting questions as to the general significance of Edelen's non-dissipative forces and rates, as represented by $\mathbf{u}$ and $\mathbf{v}$ in (18). At least one article [39] addresses the question via the near-equilibrium statistical mechanics of molecular systems, which does not preclude a more general continuum-mechanical interpretation. Thus, as suggested by the example of granular dilatancy, non-dissipative forces may reflect non-dissipative and possibly non-holonomic constraints of the type envisaged by the thermodynamical treatment of Green et al. [23]. On the other hand, such forces serve also to represent reversible coupling of otherwise dissipative processes, as suggested by a recent analysis of thermoelectricity [22] (which overlooks Green et al. [23]). Other examples are afforded by the reversible coupling of reactions and other transport processes in biological systems.

5 Extremum and variational principles

The preceding results serve to clarify various extremum principles pursued extensively by Ziegler and co-workers [49,52]. One version of the principle assumes extremal dissipation $\mathfrak{D}$ subject to fixed force $\mathbf{x}$ or flux $\mathbf{y}$, which yields dual relations of the form

$$\boldsymbol{\eta}(\mathbf{x}) = \frac{\nabla_{\mathbf{x}} D(\mathbf{x})}{\mathbf{x}\cdot\nabla_{\mathbf{x}} D(\mathbf{x})} D(\mathbf{x}), \quad \text{and } \boldsymbol{\xi}(\mathbf{y}) = \frac{\nabla_{\mathbf{y}} D^*(\mathbf{y})}{\mathbf{y}\cdot\nabla_{\mathbf{y}} D^*(\mathbf{y})} D^*(\mathbf{y}), \tag{53.1,2}$$

in terms of the respective dissipation functions $D(\mathbf{x})$ or $D^*(\mathbf{y})$. These represent Ziegler's *orthogonality principle*,[7] according to which rate or force are normal to surfaces of constant dissipation $\mathfrak{D}$.

The expressions for rate or force in (53.1,2) yield trivially the correct value dissipation and are invariant under replacement of D (or D^*) in the equation for $\boldsymbol{\eta}/D$ (or $\boldsymbol{\xi}/D^*$) by any differentiable scalar-valued function of D (or D^*). Considering the first expression in (53.1,2), one sees that it takes on the form $\boldsymbol{\eta} = \nabla_{\mathbf{x}}\varphi$ for the case of homogeneous dissipation but e.g., an ostensibly different form for a sum of homogeneous functions of different degrees. The same can be said of the second expression in (53.1,2), which applies to certain forms of

[7] Ziegler's orthogonality, confounded in [22] with the orthogonality represented by $\boldsymbol{v}$ and $\mathbf{u}$ in (79), requires that the latter be zero.

the Reiner-Rivlin fluid considered [53], for which our $\mathbf{y}$ represents the deformation rate tensor.[8] The general status of (53.1,2) is perhaps clarified by the following considerations.

Instead of the equation for $\boldsymbol{\eta}$ in (53.1,2), one may employ the more general relation

$$\boldsymbol{\eta}(\mathbf{x}) = \frac{\mathbf{g}(\mathbf{x})}{\mathbf{x}\cdot\mathbf{g}(\mathbf{x})} D(\mathbf{x}) = \frac{\mathbf{g}(\mathbf{x})}{g_r(\mathbf{x})}\frac{D(\mathbf{x})}{r} = \frac{\mathbf{g}(\mathbf{x})}{g_r(\mathbf{x})}\frac{\partial\varphi}{\partial r}, \text{ with } \therefore\ \mathbf{x}\cdot\boldsymbol{\eta} = D(\mathbf{x}), \tag{54}$$

in which the (co)vector $\mathbf{g}(\mathbf{x})$, $\mathbb{X} \to \mathbb{X}^*$, with $g_r = \mathbf{e}\cdot\mathbf{g} \neq 0$, is otherwise arbitrary and not necessarily the gradient of a scalar, and in which the fourth equality follows from (22). Substituting the decomposition

$$\mathbf{g} = g_r\mathbf{e}^* + (\mathbf{I} - \mathbf{e}^* \otimes \mathbf{e})\mathbf{g}$$

into (54), one finds the correct form for the dissipative component η_r of $\boldsymbol{\eta}$, independently of the form of $\mathbf{g}$.

As propitious choice for $\mathbf{g}$ in (54), one may take

$$\mathbf{g} = \nabla_{\mathbf{x}} r = \mathbf{e}^*, \text{ with } g_r = 1, \tag{55}$$

such that the flux is orthogonal to surfaces of constant r, where r is our unspecified norm. If Ziegler's Euclidean norm is replaced by an arbitrary norm r, one obtains a generalization of his dual principles [51] of maximal (or stationary) D at fixed r or minimum (or stationary) r at fixed D. Both cases involves stationarity of $r(\mathbf{x})+\lambda D(\mathbf{x})$, with Lagrange multiplier λ treated as constant but ultimately chosen to give the correct expression for local dissipation.

The extrema just discussed are to be compared with (12), of which e.g. the first can be reformulated as the minimum of $\varphi(\mathbf{x}) \geq 0$ subject to fixed dissipation D and rate $\mathbf{y}$.

This is tantamount to the stationarity of $\mathbf{y}\cdot\mathbf{x} - \lambda\varphi$ with $\mathbf{y}$ fixed, and it leads to (54), with

$$\mathbf{g} = \lambda\nabla_{\mathbf{x}}\varphi, \text{ where } \lambda = \frac{D}{\mathbf{x}\cdot\nabla_{\mathbf{x}}\varphi}.$$

That is, as with the first extremum in (12), the rate is normal to surfaces of constant potential, which are generally not coincident with surfaces of constant dissipation. Choosing λ equal to a constant serves simply to rescale φ, and the choice $\lambda = 1$ gives complete equivalence to the first members of (12),

A related treatment, motivated ostensibly by the works of Ziegler, is given by Bataille et al. [4] who propose a more general form of (2) with positive scalar $u(\mathbf{x})$ multiplying the right-hand side of (2) and with alternative functions Φ and $\mathbf{U}$ instead of φ and $\mathbf{u}$. This is summarized by their Eq. (28), and it is readily seen that the transformation of their notation

$$\{\mathbf{X}, \Phi, u\mathbf{U}\} \to \{\mathbf{x}, D, \mathbf{u}\} \text{ where } u = \frac{D}{\mathbf{x}\cdot\nabla_{\mathbf{x}} D}$$

yields a result identical with (2) and the Eq. (53.1).

As the above considerations serve to emphasize, and as evident from (22)–(23), there generally exist non-dissipative components of rates or forces that are derivable from the gradient of a scalar-valued function. That these do not dissipate energy does not imply they are mechanically ignorable. This point is partly illustrated by the Reiner-Rivlin fluid model considered above, with viscometric normal stresses that dissipate no power but nevertheless must be balanced in order to maintain simple shearing.

Thus, we conclude that the orthogonality principle, championed by Ziegler [50] and adopted by others (e.g., [6,25,43]), will generally fail to give a complete description of force or rate. In any event, it does not offer great advantage over Edelen's method which in principle provides all non-dissipative forces or rates.

Proper variational principles and consequences

Without a thorough investigation, the above considerations suggest that for strongly dissipative systems we should adopt variational principles based on dissipation potentials rather than dissipation *per se*. For exam-

[8] One may employ as norm $|\mathbf{y}|$ any positive function of the invariants denoted by $d_{(2)}, d_{(3)}$ by Ziegler [53] that is positively homogeneous degree-one in $\mathbf{y}$ and that also satisfies the triangle inequality in $\mathbb{X}^*$, the most obvious being the standard $\sqrt{d_{(2)}}$.

ple, the equations of equilibrium for the quasi-static motion of a viscoplastic material, subject to prescribed velocities on the boundary ∂V of a region V and usually to an incompressibility constraint, are given by the variational problem for the velocity field $\mathbf{v}(\mathbf{x})$,

$$\min_{\mathbf{v}(\mathbf{x})} \int_V [\psi(\mathbf{D}) - \mathbf{b}\cdot\mathbf{v}]\mathrm{d}V, \tag{56}$$

with $\mathbf{b}(\mathbf{x})$ denoting a prescribed body force density and $\mathbf{D}[\mathbf{v}]$ given by (32). Note that one obtains a more general problem of thermo-viscoplastic systems by letting ψ also depend on the temperature gradient, with stress and heat flux being regarded as the relevant fluxes. This serves *inter alia* to extend certain variational principles for non-Newtonian fluids [28,44] to more general classes of thermo-viscoplastic materials.

Such variational principles are not only relevant to the homogenization of heterogeneous media, but they also may find applications to problems involving loss of convexity, leading to material instability [18] and viscoplastic bifurcations (or thermo-viscoplastic phenomena such as adiabatic shear bands). Here, one may draw on a well-developed mathematical literature on the analogous elastostatic problems [2]. Indeed, the class of strongly dissipative or "hyperdissipative" materials may be regarded as the mirror image of the much more thoroughly studied hyperelastic materials.

6 Conclusions

The Abstract and Introduction provide generally adequate summaries of the overall goals and the principal results of the foregoing analysis.

A major objective of the present work is to highlight the importance of Edelen's work for the development of continuum models of viscoplasticity. The present analysis demonstrates that important various rigid-plastic flow rules follow quite simply from convexity and duality of dissipation potentials without further physical assumptions. The existence of such potentials also allows for a relatively easy formulation of properly invariant constitutive equations for Stokesian fluid-particle suspensions and, by extension, particle suspensions in any inelastic fluid. Moreover, the stationarity of dissipation potential yields variational methods that may be useful for the solution of boundary-value problems and for the homogenization of heterogeneous systems.

It is worth pointing out other potentially useful extensions of the above models for particulate media. First of all, as suggested by elementary phenomenological models, useful viscoplastic models of fluid-saturated granular media or dense non-Stokesian suspensions may possibly be suggested by a judicious combination of dissipation potentials for Stokesian suspensions and granular media. With viscous and plastic effects attributed, respectively, to the interstitial fluid and to frictional contacts between particles, one can envisage a possible "phase" transition from viscous to viscoplastic behavior at some critical value of solids fraction ϕ for a given fluid-particle system.

We further note that general models of particle migration and segregation in poly-disperse granular systems may be obtained by an appropriate definition of constituents in the relevant mixture theory.

In closing, it should be noted that several results and conclusions are based on the assumption of strongly dissipative systems, i.e., on generalized Onsager symmetry. As discussed above, there remain interesting and open questions as to the failure of such symmetry and the emergence of Edelen's non-dissipative forces and fluxes, questions that may perhaps be clarified by statistical micromechanics.

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Appendix: A source-flux construct for Edelen's formulae

As another derivation and extension of Edelen's formulae, we consider a source-flux construct explored in a previous work on continuum balances, which yields a flux field whose divergence is equal to a given source field [21]. This provides a constructive demonstration for Euclidean spaces of the existence proof of Segev and De Botton [45], representing an "inverse divergence" [1], more properly dubbed "pseudoinverse".

Path-moments of densities and a Gauss-Maxwell formula

Adopting the notation of (1) together with another slight departure from conventional notation,[9] we denote various derived tensor fields, the transpose, trace, gradient, divergence, and Curl of tensor field $\boldsymbol{\tau}(\mathbf{x})$, respectively, by

$$\begin{aligned} &\boldsymbol{\tau}^{\mathrm{T}} \hat{=} [\tau^{i_1,i_2,\ldots,i_{m-2},i_m,i_{m-1}}],\ \operatorname{tr}\boldsymbol{\tau} = \boldsymbol{\tau}:\mathbf{I} \hat{=} [\tau^{i_1,i_2,\ldots,i_{m-1},\cdot}{}_{i_{m-1}}],\ \ m \geq 2,\ \ \mathbf{I} \hat{=} [\,\delta_i^j\,],\\ &\operatorname{grad}\boldsymbol{\tau} \hat{=} [\tau^{i_1,i_2,\ldots,i_m}{}_{;i_{m+1}}],\ \ \operatorname{div}\boldsymbol{\tau} = \operatorname{grad}\boldsymbol{\tau}:\mathbf{I} = \hat{=} [\tau^{i_1,i_2,\ldots,i_m}{}_{;i_m}],\\ &\operatorname{Curl}\boldsymbol{\tau} = \operatorname{grad}\boldsymbol{\tau} - (\operatorname{grad}\boldsymbol{\tau})^{\mathrm{T}} \hat{=} [\tau^{i_1,i_2,\ldots,i_{m-1},\cdot}{}_{i_m;i_{m+1}} - \tau^{i_1,i_2,\ldots,i_{m-1},\cdot}{}_{i_{m+1};i_m}]. \end{aligned} \tag{57}$$

For notational convenience, we have adopted tensor forms appropriate to a Riemannian metric, with covariant derivative denoted by semicolon. This becomes somewhat superfluous for the present Euclidean framework, but could be useful for extensions to general manifolds.

If $\operatorname{div}\boldsymbol{\tau} \equiv \mathbf{0}$ we designate $\boldsymbol{\tau}(\mathbf{x})$ as *solenoidal* and if $\operatorname{Curl}\boldsymbol{\tau} \equiv \mathbf{0}$ as *irrotational*. Also, the field $\boldsymbol{\omega} = \operatorname{Curl}\boldsymbol{\tau}$ is can be regarded as skew symmetric in the sense that it is the negative of the transpose defined above, so that

$$\boldsymbol{\omega} : (\mathbf{a} \otimes \mathbf{b}) = -\boldsymbol{\omega} : (\mathbf{b} \otimes \mathbf{a}) \hat{=} [\omega^{i_1,i_2,\ldots,i_{m-1}}{}_{i_m,i_{m+1}}^{\cdot\ \ \cdot} a^{i_m} b^{i_{m+1}}]$$

defines an alternating bilinear form in $\mathbf{a}$, $\mathbf{b}$. Here, as below, we employ the conventional $\otimes$ for various tensor products. For tensors of rank $m > 2$, the leading $m-2$ indices can usually be disregarded in what follows.

As an extension to $\mathbb{X}$ of a previous result [21] for $\mathbb{R}^3$, we assume a suitably bounded tensor-valued density $\boldsymbol{\varrho}_{\mathcal{A}}(\mathbf{x}) \hat{=} [\varrho_{\mathcal{A}}^{i_1,i_2,\ldots,i_m}]$, $\mathbf{x} \in \mathbb{X}$, defining a rank-$m$ tensor-valued weight

$$\boldsymbol{\mathcal{A}}(V) = \int_V \boldsymbol{\varrho}_{\mathcal{A}}(\mathbf{x}) \mathrm{d}V(\mathbf{x}) \tag{58}$$

where $dV(\mathbf{x})$, $\mathbf{x} \in V$, is an appropriate volume measure. We suppress notation for possible dependence on parameters such as time [21] or on certain supplementary variables employed by Edelen and co-workers [4,9], and, for the sake of brevity, we drop the subscript $\mathcal{A}$ on various quantities.

Then, given a continuous directed curve $\mathbb{P}(\mathbf{a}, \mathbf{b})$ running from $\mathbf{a}$ to $\mathbf{b}$ in $\mathbb{X}$, the line integral

$$\boldsymbol{\tau}(\mathbf{x}\,;\mathbb{P}) = \int_{\mathbf{z} \in \mathbb{P}(\mathbf{0},\mathbf{y})} \boldsymbol{\varrho}(\mathbf{x} - \mathbf{z}) \otimes \mathrm{d}\mathbf{z} \tag{59}$$

defines a *path moment* $\boldsymbol{\tau} \hat{=} [\tau^{i_1,i_2,\ldots,i_{m+1}}]$ of rank $m+1$. We note that the integrand of (59) can also be expressed as a tensor-valued Lagrangian form

$$\mathcal{L}\{\mathbf{z}(\tau), \mathbf{z}'(\tau)\}\mathrm{d}\tau, \ \ \text{with } \mathbf{z}' = \mathrm{d}\mathbf{z}/\mathrm{d}\tau,$$

where $\mathcal{L}$ is linear in its second argument and τ parametrizes $\mathbb{P}$.

The relation (59) obviously defines a functional on[10] $\mathbb{P}(\mathbf{0}, \mathbf{y})$. Regarded as function of $\mathbf{x}$ with parameter $\mathbf{y} \in \mathbb{X}$, the corresponding field $\boldsymbol{\tau}(\mathbf{x}, \mathbf{y})$ is readily shown to satisfy

$$\operatorname{div}\boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varrho}(\mathbf{x}) - \boldsymbol{\varrho}(\mathbf{x} - \mathbf{y}) \tag{60}$$

independently of the curve $\mathbb{P}$. Hence, the integral around any closed curve, such that $\mathbf{y} = \mathbf{0}$ in (59), represents a solenoidal field.

Now, (60) amounts to a parametric generalization of the classical Gauss-Maxwell form, with density field

$$\tilde{\boldsymbol{\varrho}}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varrho}(\mathbf{x}) - \boldsymbol{\varrho}(\mathbf{x} - \mathbf{y})$$

parametrized by $\mathbf{y}$. For densities $\boldsymbol{\varrho}(\mathbf{x})$ such that $\boldsymbol{\varrho}(\mathbf{0}) = \mathbf{0}$, (60) yields

$$\operatorname{div}\boldsymbol{\tau}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} = \boldsymbol{\varrho}(\mathbf{x}). \tag{61}$$

As an alternative derivation [21], one may proceed from (60) and the assumption that $\boldsymbol{\varrho}(\mathbf{x})$ represents a field with finite support $\mathfrak{B}$, e.g., a finite material body. In that case, we may choose $|\mathbf{y}|$ sufficiently large that $\mathbf{x} - \mathbf{y}$ lies outside $\mathfrak{B}$ for all $\mathbf{x} \in \mathfrak{B}$ so that we may disregard the term $\boldsymbol{\varrho}(\mathbf{x} - \mathbf{y})$ in (60). The same result applies to an unbounded support $\mathfrak{B}$, provided $\boldsymbol{\varrho}(\mathbf{x}) \to \mathbf{0}$ for $|\mathbf{x}| \to \infty$ so as to ensure convergence of the integrals (65). In either case, we follow [21] and designate $\boldsymbol{\varrho}(\mathbf{x})$ as *spatially restricted*.

[9] Apart from the Curl, most of the notation reduces to the conventional form in the case of vectors and second-rank tensors.
[10] and *not* on $\mathbb{P}(\mathbf{x}, \mathbf{y})$, as mis-written in [21].

Spatially restricted densities

For such density fields, Fourier-space methods provide a convenient derivation of certain results. Thus, for real tensor fields over a Euclidean space $\mathbb{X}$, we may identify, as dual function space $\mathbb{K}$ homeomorphic with $\mathbb{C}^n$, the Fourier transform:

$$\hat{\boldsymbol{\tau}}(\mathbf{k}) = \int_{\mathbb{X}} e^{-\imath \mathbf{k}\cdot\mathbf{x}} \boldsymbol{\tau}(\mathbf{x}) \mathrm{d}V(\mathbf{x}), \tag{62}$$

where $\mathbb{C}$ is the field of complex numbers and $\mathbf{k} \in \mathbb{X}^*$. In this representation, (59) and (60) become [21]

$$\hat{\boldsymbol{\tau}}(\mathbf{k}\,;\mathbb{P}) = \hat{\boldsymbol{\varrho}}(\mathbf{k}) \otimes \int_{\mathbf{z}\in\mathbb{P}(\mathbf{0},\mathbf{y})} e^{-\imath \mathbf{k}\cdot\mathbf{z}} \mathrm{d}\mathbf{z},$$

with (63.1)

$$\imath\,\hat{\boldsymbol{\tau}} \cdot \mathbf{k} = \hat{\boldsymbol{\varrho}}(\mathbf{k}) \int_{\mathbf{z}\in\mathbb{P}(\mathbf{0},\mathbf{y})} e^{-\imath \mathbf{k}\cdot\mathbf{z}} \mathrm{d}(\imath \mathbf{k}\cdot\mathbf{z}) = (1 - e^{-\imath \mathbf{k}\cdot\mathbf{y}})\hat{\boldsymbol{\varrho}}(\mathbf{k}). \tag{63.2}$$

The last relation can be written as

$$\imath(1 - e^{-\imath \mathbf{k}\cdot\mathbf{y}})^{-1} \hat{\boldsymbol{\tau}} \cdot \mathbf{k} = \hat{\boldsymbol{\varrho}}(\mathbf{k}), \tag{64}$$

and the inverse Fourier transform of the binomial expansion of $(1 - e^{-\imath \mathbf{k}\cdot\mathbf{y}})^{-1}$ acting on the integral (59) formally shifts the upper limit to $|\mathbf{y}| = \infty$, leading to the following generalization of the classical Gauss-Maxwell formula:

$$\boldsymbol{\tau}(\mathbf{x}) = \int_{\mathbf{0}}^{\infty} \boldsymbol{\varrho}(\mathbf{x} - \mathbf{z}) \otimes \mathrm{d}\mathbf{z}, \quad \text{with div}\ \boldsymbol{\tau}(\mathbf{x}) = \boldsymbol{\varrho}(\mathbf{x}), \tag{65}$$

where the integral involves an arbitrary path, subject integrability and independence of $\mathbf{x}$.

As the present paper is mainly concerned with densities $\boldsymbol{\varrho}(\mathbf{x})$ that are not spatially restricted, we explore the significance and utility of the more general form (60) involving the supplementary parameter $\mathbf{y}$. First, we consider a nonparametric generalization of (60) for $\mathbf{y}$ belonging to a set $\mathbb{Y}$ independent of $\mathbf{x}$.

Parametric averages

As an arbitrary density $\boldsymbol{\varrho}(\mathbf{x})$ may fail to admit a regular Fourier transform, we assume for the present purposes that $\boldsymbol{\varrho}(\mathbf{x})$ exhibits (multivariate) polynomial behavior for $|\mathbf{x}| \to \infty$, with transform interpreted as generalized function or *distribution* [31], or else that forces or fluxes are restricted in norm within some bounded region containing the origin.

Straight-line paths

For Euclidean spaces, it is meaningful to replace the path $\mathbb{P}(\mathbf{0}, \mathbf{y})$ in (59) by the straight-line path $\mathbf{z} = \lambda\mathbf{y}$, $0 \le \lambda \le 1$, to give, *modulo* a solenoidal field,

$$\boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \int_0^1 \boldsymbol{\varrho}(\mathbf{x} - \lambda\mathbf{y}) \mathrm{d}\lambda \otimes \mathbf{y}. \tag{66}$$

Then, the Fourier space analog of Eq. (63.1) becomes:

$$\hat{\boldsymbol{\tau}}(\mathbf{k}, \mathbf{y}) = \frac{(1 - e^{-\imath \mathbf{k}\cdot\mathbf{y}})}{\imath \mathbf{k}\cdot\mathbf{y}} \hat{\boldsymbol{\varrho}}(\mathbf{k}) \otimes \mathbf{y}, \tag{67}$$

while Eqs. (63.2) and (64) remain unchanged.

Note that in the limit $|\mathbf{y}| = \infty$, we may neglect the term $e^{-\imath \mathbf{k}\cdot\mathbf{y}}$ in (67), given appropriate restrictions on $\hat{\boldsymbol{\varrho}}$, yielding the transform pair:

$$\boldsymbol{\tau}(\mathbf{x},\mathbf{u}) = \int_0^\infty \boldsymbol{\varrho}(\mathbf{x}-s\mathbf{u})\mathrm{d}s \otimes \mathbf{u} \quad \Leftrightarrow \quad \hat{\boldsymbol{\tau}}(\mathbf{k},\mathbf{u}) = -\frac{\imath}{\mathbf{k}\cdot\mathbf{u}}\hat{\boldsymbol{\varrho}}(\mathbf{k}) \otimes \mathbf{u} = -\frac{\imath}{2}\hat{\boldsymbol{\varrho}} \otimes \nabla_{\mathbf{k}} \log(\mathbf{k}\cdot\mathbf{u})^2 \tag{68}$$

where $\mathbf{u}$ is an arbitrary unit vector and the squared argument of the logarithm avoids an inessential negative sign. The relation (68) may be regarded as a singular representation in terms of a "ray" with direction $\mathbf{u}$.

A general form independent of $\mathbf{u}$ is given formally by the convex combination in terms of a measure $d\mu(\mathbf{u})$ on the unit sphere $\mathcal{S}$:

$$\hat{\boldsymbol{\tau}}(\mathbf{k}) = -\frac{\imath}{2}\hat{\boldsymbol{\varrho}} \otimes \nabla_{\mathbf{k}} \int_{\mathcal{S}} \log(\mathbf{k}\cdot\mathbf{u})^2 \mathrm{d}\mu(\mathbf{u}) \;\text{ where }\; \int_{\mathcal{S}} \mathrm{d}\mu(\mathbf{u}) = 1. \tag{69}$$

Since

$$\log(\mathbf{k}\cdot\mathbf{u})^2 = \log k^2 + \log(\mathbf{e}_k\cdot\mathbf{u})^2, \;\text{ with }\; k^2 = |\mathbf{k}|^2, \;\; \mathbf{e}_k = \mathbf{k}/|\mathbf{k}|,$$

it follows that, modulo terms $\nabla\mathbf{e}_k$ representing a solenoidal field, (68) can be written as

$$\hat{\boldsymbol{\tau}}(\mathbf{k}) = -\frac{\imath}{k^2}\boldsymbol{\varrho} \otimes \mathbf{k} \quad \Leftrightarrow \quad \boldsymbol{\tau}(\mathbf{x}) = \nabla_{\mathbf{x}} \int_{\mathbb{X}} G(\mathbf{x}-\mathbf{x}')\boldsymbol{\varrho}(\mathbf{x}')\mathrm{d}V(\mathbf{x}') \tag{70}$$

where $G(\mathbf{x})$ represents the free-space Green's function for the Laplacian ∇^2, with $\hat{G}(\mathbf{k}) = 1/k^2$. Thus, the representation given by standard potential theory emerges as an average over rays.

Arbitrary paths

As discussed in a previous work [21], certain integrals over the parameter $\mathbf{y}$ in (59) can be employed to derive the standard material stress tensor in $\mathbb{R}^3$, represented here by $\boldsymbol{\tau}$, from an internal force density $\boldsymbol{\varrho}$. This provides a mathematical connection to peridynamics [30], and, referring the reader to that work [21] for details on this particular application, we briefly note that one may introduce a more abstract integral average based on normalized measure $\mathrm{d}\mu(\mathbf{y}) = w(\mathbf{y})\mathrm{d}V(\mathbf{y})$ involving weight $w(\mathbf{y})$:

$$\langle\boldsymbol{\tau}\rangle(\mathbf{x}) = \int_{\mathbb{X}} \boldsymbol{\tau}(\mathbf{x},\mathbf{y})w(\mathbf{y})\mathrm{d}V(\mathbf{y}), \;\text{ with }\; \int_{\mathbb{X}} w(\mathbf{y})\mathrm{d}V(\mathbf{y}) = 1 \tag{71}$$

where $\boldsymbol{\tau}(\mathbf{x},\mathbf{y})$ is given by (66). Hence, the average of the second member of (63.2) gives

$$\imath\langle\hat{\boldsymbol{\tau}}\rangle(\mathbf{k})\cdot\mathbf{k} = [1-\hat{w}(\mathbf{k})]\hat{\boldsymbol{\varrho}}(\mathbf{k}), \;\text{ where }\; \hat{w}(\mathbf{k}) = \int_{\mathbb{X}} e^{-\imath\mathbf{k}\cdot\mathbf{y}} w(\mathbf{y})\mathrm{d}V(\mathbf{y}). \tag{72}$$

Binomial expansion of $[1-\hat{w}(\mathbf{k})]^{-1}$ followed by convolution of inverse Fourier transforms gives

$$\operatorname{div}\tilde{\boldsymbol{\tau}}(\mathbf{x}) = \boldsymbol{\varrho}(\mathbf{x}), \;\text{ where }\; \tilde{\boldsymbol{\tau}}(\mathbf{x}) = \sum_{p=0}^{\infty} w_p * \langle\boldsymbol{\tau}\rangle(\mathbf{x}) \tag{73}$$

where, respectively, $*$ denotes convolution and w_p the special *iterated kernel*:

$$f * g(\mathbf{x}) = \int_{\mathbb{X}} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\mathrm{d}V(\mathbf{x}'), \;\text{ with }\; w_p(\mathbf{x}) = w * w_{p-1}(\mathbf{x}),$$
$$\text{for }\; p = 1, 2, \ldots, \text{ and }\; w_0(\mathbf{x}) = \delta(\mathbf{x}), \tag{74}$$

with δ denoting the Dirac delta.

The convergence of the formal series expansion in (73) depends on the form of $w(\mathbf{y})$. Without a detailed exploration, it appears most expeditious to keep to Fourier space, where the convergence of the binomial expansion for $[1-\hat{w}(\mathbf{k})]^{-1}$ is guaranteed by the condition $|\hat{w}(\mathbf{k})| < 1$, implying conditional convergence at $\mathbf{k} = 0$ where $\hat{w}(\mathbf{0}) = 1$ according to the second members of (71)–(72).

A generalization of Edelen's formulae

The above results provide a slight extension of Edelen's formula to tensor fields. Thus, for tensors $\boldsymbol{\varrho}$ of rank $m \geq 1$, the relation (66) can be written:

$$\begin{aligned} &\operatorname{dev} \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) - \frac{1}{n}\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) \otimes \mathbf{I}, \quad \text{with } \operatorname{tr}(\operatorname{dev} \boldsymbol{\tau}) = \mathbf{0}, \\ &\text{and } \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \operatorname{tr} \boldsymbol{\tau}(\mathbf{x}, \mathbf{y}) = \left[\int_0^1 \boldsymbol{\varrho}(\mathbf{x} - \lambda \mathbf{y}) \mathrm{d}\lambda \right] \cdot \mathbf{y}, \end{aligned} \tag{75}$$

where $\boldsymbol{\varphi}$ is rank $m-1$. Distinguishing gradients with respect to $\mathbf{x}$ and $\mathbf{y}$ by subscripts, it is easy to derive the following relations from (75):

$$\begin{aligned} &\operatorname{grad}_{\mathbf{y}} \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varrho}(\mathbf{x} - \mathbf{y}) + \left[\int_0^1 \lambda \operatorname{Curl} \boldsymbol{\varrho}(\mathbf{x} - \lambda \mathbf{y}) \mathrm{d}\lambda \right] \cdot \mathbf{y}, \\ &\operatorname{grad}_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varrho}(\mathbf{x}) - \boldsymbol{\varrho}(\mathbf{x} - \mathbf{y}) - \left[\int_0^1 \operatorname{Curl} \boldsymbol{\varrho}(\mathbf{x} - \lambda \mathbf{y}) \mathrm{d}\lambda \right] \cdot \mathbf{y}, \end{aligned} \tag{76}$$

and hence

$$\begin{aligned} &\boldsymbol{\varrho}(\mathbf{x}) = \operatorname{grad}_{\mathbf{x}} \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) + \operatorname{grad}_{\mathbf{y}} \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) + \boldsymbol{\upsilon}(\mathbf{x}, \mathbf{y}), \\ &\text{where } \boldsymbol{\upsilon}(\mathbf{x}, \mathbf{y}) = \left[\int_0^1 (1 - \lambda) \operatorname{Curl} \boldsymbol{\varrho}(\mathbf{x} - \lambda \mathbf{y}) \mathrm{d}\lambda \right] \cdot \mathbf{y}, \quad \text{with } \boldsymbol{\upsilon} \cdot \mathbf{y} = \mathbf{0} \end{aligned} \tag{77}$$

where use has been made of the skew symmetry of Curl $\boldsymbol{\varrho}$ and the chain rule,

$$\frac{\partial \boldsymbol{\varrho}}{\partial \lambda}(\mathbf{x} - \lambda \mathbf{y}) = -[\operatorname{grad}_{\mathbf{x}} \boldsymbol{\varrho}(\mathbf{x} - \lambda \mathbf{y})] \cdot \mathbf{y},$$

which is a straightforward extension of a result given by Edelen [9].

Hence, an arbitrary rank-m tensor field $\boldsymbol{\varrho}$ is given by (77) as the extended gradient with respect to $\mathbf{x} \oplus \mathbf{y}$ of a rank-($m-1$) tensor field $\boldsymbol{\varphi}$, *modulo* an additive rank-m tensor field $\boldsymbol{\upsilon}$ that is orthogonal to the parametric vector $\mathbf{y}$, i.e., such that $\mathbf{y}$ is contained in the null space of the linear map $\boldsymbol{\upsilon}$. This represents a tensor generalization of the vector forms for $m = 1$ given by Edelen [9] (Eqs. (2.1)–(2.4)), which are obtained by the following transformation from his notation:

$$\{\mathbf{X}, \mathbf{V}, \mathbf{J}, \Psi, \mathbf{U}\} \to \{\mathbf{x}, \mathbf{y}, \boldsymbol{\varrho}, \boldsymbol{\varphi}, \boldsymbol{\upsilon}\}. \tag{78}$$

A special case of paramount interest [9] is $\mathbf{y} = \mathbf{x}$, for which (77) reduces to

$$\begin{aligned} &\boldsymbol{\varrho}(\mathbf{x}) = \operatorname{grad} \boldsymbol{\varphi}(\mathbf{x}) + \boldsymbol{\upsilon}(\mathbf{x}), \quad \text{with } \boldsymbol{\upsilon} \cdot \mathbf{x} = \mathbf{0} \\ &\text{where} \\ &\boldsymbol{\varphi}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} = \left[\int_0^1 \boldsymbol{\varrho}(\lambda \mathbf{x}) \mathrm{d}\lambda \right] \cdot \mathbf{x}, \\ &\boldsymbol{\upsilon}(\mathbf{x}) = \boldsymbol{\upsilon}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} = \left[\int_0^1 \lambda \operatorname{Curl} \boldsymbol{\varrho}(\lambda \mathbf{x}) \mathrm{d}\lambda \right] \cdot \mathbf{x} \end{aligned} \tag{79}$$

which represent a generalization of Edelen's [9] Eqs. (2.15)–(2.18) for $m = 1$, with $\therefore\ \boldsymbol{\varrho} \in \mathbb{X}^*$. Although not explored further here, it is plausible that Edelen's homotopy operation [11] (Eqs. (5–3.2)) can be employed to obtain many of the results presented in this Appendix. Indeed, one Referee indicates that (76) follows from the Stokes theorem.

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